Sparse Interpolation

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Motivation

Motivation

interpolate

$$f(x) = \alpha_1 + \alpha_2 x^{100}$$

- Newton/Lagrange interpolation: 101 samples
- only 4 unknowns: α_1, α_2, x^0, x^{100}!
- how to solve it from 4 samples?

Motivation

- exponential analysis
- generalized eigenvalue problems
- computer algebra
- orthogonal polynomials
- signal processing
- moment problems
- nonlinear approximation theory
- many applications ...

Motivation

$$x_s = s\Delta, \quad s = 0, 1, 2, \ldots$$

$$\sum_{i=1}^{n} \alpha_i x_s^{k_i} = f_s, \quad n \ll \max(k_i), \quad k_i \in \mathbb{N}$$

$$\sum_{i=1}^{n_1} \alpha_{i,1} \cos(\phi_{i,1} x_s) + \sum_{i=1}^{n_2} \alpha_{i,2} \sin(\phi_{i,2} x_s) = f_s$$

$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = f_s, \quad \phi_i \in \mathbb{C}$$

Motivation

1. Univariate exponential sparse interpolation
   (Exercise)
2. Multivariate polynomial sparse interpolation
   (Exercise)
3. Connection with rational approximation theory
   (Exercise)
4. Applications unlimited

Basics: exponential

Basics: exponential

Figure: Gaspard Riche de Prony [1795]

Basics: exponential

interpolation problem:

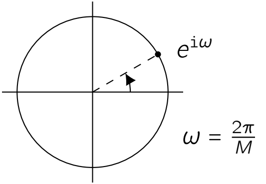
$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = f_s, \quad s = 0, \ldots, 2n-1$$

$$x_s = s\frac{2\pi}{M}, \quad \omega = 2\pi/M$$

$$|\Im(\phi_i)| < M/2, \quad \Omega_i = \exp(\phi_i \omega),$$

$$f_s = \sum_{i=1}^{n} \alpha_i \Omega_i^s, \quad s = 0, \ldots, 2n-1$$

$$\begin{cases} \alpha_1 + \cdots + \alpha_n = f_0 \\ \alpha_1 \Omega_1 + \cdots + \alpha_n \Omega_n = f_1 \\ \quad \vdots \\ \alpha_1 \Omega_1^{2n-1} + \cdots + \alpha_n \Omega_n^{2n-1} = f_{2n-1} \end{cases}$$



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Basics: exponential

finding Ω_i:

$$\prod_{i=1}^{n}(z-\Omega_i) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$$

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \alpha_i \Omega_i^s (\Omega_i^n + b_{n-1}\Omega_i^{n-1} + \cdots + b_0) \\ &= \sum_{i=1}^{n} \alpha_i \Omega_i^{n+s} + \sum_{j=0}^{n-1} b_j \left(\sum_{i=1}^{n} \alpha_i \Omega_i^{j+s} \right) \\ &= f_{s+n} + \sum_{j=0}^{n-1} b_j f_{s+j} \end{aligned}$$

Basics: exponential

$$\begin{pmatrix} f_0 & \cdots & f_{n-1} \\ \vdots & \iddots & \vdots \\ f_{n-1} & \cdots & f_{2n-2} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix} = - \begin{pmatrix} f_n \\ \vdots \\ f_{2n-1} \end{pmatrix}$$

Hankel matrix:

$$H_n^{(r)} = \begin{pmatrix} f_r & \cdots & f_{r+n-1} \\ \vdots & \iddots & \vdots \\ f_{r+n-1} & \cdots & f_{r+2n-2} \end{pmatrix}$$

Basics: exponential

Hadamard polynomial:

$$H_n^{(0)}(z) = \begin{vmatrix} f_0 & \dots & f_{n-1} & f_n \\ \vdots & \iddots & \vdots & \vdots \\ f_{n-1} & \dots & f_{2n-2} & f_{2n-1} \\ 1 & \dots & z^{n-1} & z^n \end{vmatrix}$$

$$\prod_{i=1}^{n} (z - \Omega_i) = \frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} = z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0$$

Basics: exponential

formally orthogonal polynomial:

$$\gamma : z^s \to f_s, \quad s = 0, 1, \ldots$$

$$\gamma : \exp(\phi_i x_s) = \Omega_i^s \to \sum_{i=1}^{n} \alpha_i \Omega_i^s = f_s$$

$$\gamma : z^i \frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} \to 0, \quad i = 0, \ldots, n-1$$

$$\frac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|} \perp_\gamma z^i, \quad i = 0, \ldots, n-1$$

[Henrici, 1974]

Basics: exponential

roots of $\dfrac{H_n^{(0)}(z)}{\left|H_n^{(0)}\right|}$ from GEP:

$$H_n^{(0)} = \begin{pmatrix} 1 & \dots & & 1 \\ \Omega_1 & \Omega_2 & \dots & \Omega_n \\ \vdots & & & \vdots \\ \Omega_1^{n-1} & \dots & & \Omega_n^{n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \begin{pmatrix} 1 & \Omega_1 & \dots & \Omega_1^{n-1} \\ \vdots & \Omega_2 & & \vdots \\ & \vdots & & \vdots \\ 1 & \Omega_n & \dots & \Omega_n^{n-1} \end{pmatrix}$$
$$= V_n^T D_\alpha V_n$$

$$H_n^{(1)} = V_n^T D_\alpha \begin{pmatrix} \Omega_1 & & \\ & \ddots & \\ & & \Omega_n \end{pmatrix} V_n$$

Basics: exponential

$$\det\left(H_n^{(1)} - \lambda H_n^{(0)}\right) = \det\left(V_n^T D_\alpha \begin{pmatrix} \Omega_1 - \lambda & & \\ & \ddots & \\ & & \Omega_n - \lambda \end{pmatrix} V_n \right)$$
$$= 0 \text{ for } \lambda = \Omega_i, \quad i = 1, \ldots, n$$

[Hua and Sarkar, 1990]

Basics: exponential

finding ϕ_i:

$$\exp(\phi_i) = \exp(\Re(\phi_i))e^{\mathrm{i}\Im(\phi_i)}$$

$$|\Im(\phi_i)| < \frac{M}{2}:$$

$$\begin{aligned}\arg(\Omega_i) &= \arg(\exp(\phi_i\omega)) \\ &= \Im(\phi_i)\,\frac{2\pi}{M} \in \left]-\pi, \pi\right[\end{aligned}$$

Basics: exponential

finding α_i:

$$\sum_{i=1}^{n} \alpha_i \Omega_i^{s+j} = f_{s+j}, \quad s = 0, \ldots, n-1, \quad 0 \le j \le n$$

$$\begin{pmatrix} \Omega_1^j & \ldots & \Omega_n^j \\ \vdots & & \vdots \\ \Omega_1^{j+n-1} & \ldots & \Omega_n^{j+n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_j \\ \vdots \\ f_{j+n-1} \end{pmatrix}$$

remaining interpolation conditions are linearly dependent

Basics: exponential

finding n:

$$N < n : \left| H_N^{(r)} \right| \not\equiv 0, \quad r = 0, 1, \ldots$$

$$N = n : \left| H_N^{(r)} \right| \neq 0 \quad \text{if } \Omega_i \neq \Omega_j \text{ for } i \neq j$$ [Kaltofen and Lee, 2003]

$$N > n : \left| H_N^{(r)} \right| \equiv 0, \quad r = 0, 1, \ldots$$

Example: exponential

$$\phi(x) = \sum_{i=1}^{4} \alpha_i \exp(\phi_i x)$$

$\alpha_1 = 1$	$\phi_1 = 0$
$\alpha_2 = 2.4$	$\phi_2 = -5 + 19.97\mathtt{i}$
$\alpha_3 = -2.1$	$\phi_3 = 3 + 45\mathtt{i}$
$\alpha_4 = 0.2$	$\phi_4 = 5.3\mathtt{i}$

evaluate at $x_s = s\frac{2\pi}{100}$, $M = 100$, $|\Im(\phi_i)| < 50$

sequence $f_0, \ldots, f_7, \ldots$ is linearly generated

Example: exponential

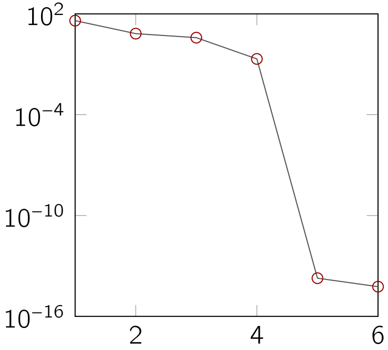


Figure: $H_N^{(0)}$ singular, $N = 6$

$$\frac{|\widetilde{\Omega}_j - \Omega_j|}{|\Omega_j|} \le 2 \times 10^{-12}, \qquad \frac{|\widetilde{\phi}_j - \phi_j|}{|\phi_j|} \le 2 \times 10^{-12}$$

Exercise: exponential

$$\phi(x) = \alpha_1 \exp(\phi_1 x) + \alpha_2 \exp(\phi_2 x)$$

4 unknowns $\phi_1, \phi_2, \alpha_1, \alpha_2$

identify $\phi(x)$ from

$$\begin{aligned}
\phi(0) &= \alpha_1 + \alpha_2 \\
\phi'(0) &= \alpha_1 \phi_1 + \alpha_2 \phi_2 \\
\phi''(0) &= \alpha_1 \phi_1^2 + \alpha_2 \phi_2^2 \\
\phi'''(0) &= \alpha_1 \phi_1^3 + \alpha_2 \phi_2^3
\end{aligned}$$

solution:

$$\left| H_3^{(0)} \right| = \begin{vmatrix} \phi_0 & \phi_0' & \phi_0'' \\ \phi_0' & \phi_0'' & \phi_0''' \\ \phi_0'' & \phi_0''' & \phi_0^{\prime v} \end{vmatrix} = 0 \quad \text{symbolically}$$

$n = 2$

```
> ##########################
  ## Exercise: exponential ##
  ##########################
> restart:
> with(LinearAlgebra):
> Phi[0] := alpha[1] * exp(phi[1]*x) + alpha[2] * exp(phi[2]*x);
```

$$\Phi_0 := \alpha_1 e^{\phi_1 x} + \alpha_2 e^{\phi_2 x} \quad (1$$

```
> for i from 1 to 4 do
   Phi[i] := diff(Phi[i-1], x);
  od;
```

$$\Phi_1 := \alpha_1 \phi_1 e^{\phi_1 x} + \alpha_2 \phi_2 e^{\phi_2 x}$$

$$\Phi_2 := \alpha_1 \phi_1^2 e^{\phi_1 x} + \alpha_2 \phi_2^2 e^{\phi_2 x}$$

$$\Phi_3 := \alpha_1 \phi_1^3 e^{\phi_1 x} + \alpha_2 \phi_2^3 e^{\phi_2 x}$$

$$\Phi_4 := \alpha_1 \phi_1^4 e^{\phi_1 x} + \alpha_2 \phi_2^4 e^{\phi_2 x} \quad (2$$

```
> for i from 0 to 4 do
   f[i] := simplify(subs(x=0, Phi[i]));
  od;
```

$$f_0 := \alpha_1 + \alpha_2$$

$$f_1 := \alpha_1 \phi_1 + \alpha_2 \phi_2$$

$$f_2 := \alpha_1 \phi_1^2 + \alpha_2 \phi_2^2$$

$$f_3 := \alpha_1 \phi_1^3 + \alpha_2 \phi_2^3$$

$$f_4 := \alpha_1 \phi_1^4 + \alpha_2 \phi_2^4 \quad (3$$

```
> H[3] := HankelMatrix([f[0],f[1],f[2],f[3],f[4]],3);
```

$$H_3 := \begin{bmatrix} \alpha_1 + \alpha_2 & \alpha_1 \phi_1 + \alpha_2 \phi_2 & \alpha_1 \phi_1^2 + \alpha_2 \phi_2^2 \\ \alpha_1 \phi_1 + \alpha_2 \phi_2 & \alpha_1 \phi_1^2 + \alpha_2 \phi_2^2 & \alpha_1 \phi_1^3 + \alpha_2 \phi_2^3 \\ \alpha_1 \phi_1^2 + \alpha_2 \phi_2^2 & \alpha_1 \phi_1^3 + \alpha_2 \phi_2^3 & \alpha_1 \phi_1^4 + \alpha_2 \phi_2^4 \end{bmatrix} \quad (4$$

```
> Determinant(H[3]);
```

$$0 \quad (5$$

```
> H[2] := HankelMatrix([f[0],f[1],f[2]],2);
```

$$H_2 := \begin{bmatrix} \alpha_1 + \alpha_2 & \alpha_1 \phi_1 + \alpha_2 \phi_2 \\ \alpha_1 \phi_1 + \alpha_2 \phi_2 & \alpha_1 \phi_1^2 + \alpha_2 \phi_2^2 \end{bmatrix} \quad (6$$

```
> F2 := Matrix([[f[2]],[f[3]]]);
```

$$F2 := \begin{bmatrix} \alpha_1 \phi_1^2 + \alpha_2 \phi_2^2 \\ \alpha_1 \phi_1^3 + \alpha_2 \phi_2^3 \end{bmatrix} \quad (7$$

```
> B:=LinearSolve(H[2],-F2);
```

$$B := \begin{bmatrix} \phi_1 \phi_2 \\ -\phi_1 - \phi_2 \end{bmatrix} \quad (8$$

```
> B_poly := z^2 + B(2)*z + B(1);
```

$$B\_poly := z^2 + \left(-\phi_1 - \phi_2\right) z + \phi_1 \phi_2 \quad (9$$

```
> z_root:=solve(B_poly=0, z);
```

$$z\_root := \phi_2, \phi_1 \quad (10$$

```
> V := Transpose(VandermondeMatrix([z_root[1],z_root[2]],2,2));
```

$$V := \begin{bmatrix} 1 & 1 \\ \phi_2 & \phi_1 \end{bmatrix} \quad (11$$

```
> F0 := Matrix([[f[0]],[f[1]]]);
```

$$F0 := \begin{bmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 \phi_1 + \alpha_2 \phi_2 \end{bmatrix} \quad (12$$

```
> A:=LinearSolve(V,F0);
```

$$A := \begin{bmatrix} \alpha_2 \\ \alpha_1 \end{bmatrix} \quad (13$$

```
> alpha[1] := Pi;
  alpha[2] := 5;
  phi[1] := 2;
  phi[2] := -1/7;
```

$$\alpha_1 := \pi$$

$$\alpha_2 := 5$$

$$\phi_1 := 2$$

$$\phi_2 := -\frac{1}{7} \quad (14$$

```
> for i from 0 to 4 do
   f[i] := simplify(subs(x=0, Phi[i]));
  od;
```

$$f_0 := \pi + 5$$

$$f_1 := 2\,\pi - \frac{5}{7}$$

$$f_2 := 4\,\pi + \frac{5}{49}$$

$$f_3 := 8\,\pi - \frac{5}{343}$$

$$f_4 := 16\,\pi + \frac{5}{2401} \quad (15$$

```
> H[3] := HankelMatrix([f[0],f[1],f[2],f[3],f[4]],3);
```

$$H_3 := \begin{bmatrix} \pi + 5 & 2\pi - \frac{5}{7} & 4\pi + \frac{5}{49} \\ 2\pi - \frac{5}{7} & 4\pi + \frac{5}{49} & 8\pi - \frac{5}{343} \\ 4\pi + \frac{5}{49} & 8\pi - \frac{5}{343} & 16\pi + \frac{5}{2401} \end{bmatrix} \quad (16$$

```
> Determinant(H[3]);
```

$$0 \quad (17$$

```
> H[2] := HankelMatrix([f[0],f[1],f[2]],2);
```

$$H_2 := \begin{bmatrix} \pi + 5 & 2\pi - \frac{5}{7} \\ 2\pi - \frac{5}{7} & 4\pi + \frac{5}{49} \end{bmatrix} \quad (18$$

```
> F2 := Matrix([[f[2]],[f[3]]]);
```

$$F2 := \begin{bmatrix} 4\pi + \frac{5}{49} \\ 8\pi - \frac{5}{343} \end{bmatrix} \quad (19$$

```
> B:=LinearSolve(H[2],-F2);
```

$$B := \begin{bmatrix} -\frac{2}{7} \\ -\frac{13}{7} \end{bmatrix} \quad (20$$

```
> B_poly := z^2 + B(2)*z + B(1);
```

$$B\_poly := z^2 - \frac{13}{7} z - \frac{2}{7} \quad (21$$

```
> z_root:=solve(B_poly=0,z);
```

$$z\_root := 2, -\frac{1}{7} \quad (22$$

```
> V := Transpose(VandermondeMatrix([z_root[1],z_root[2]],2,2));
```

$$V := \begin{bmatrix} 1 & 1 \\ 2 & -\frac{1}{7} \end{bmatrix} \quad (23$$

```
> F0 := Matrix([[f[0]],[f[1]]]);
```

$$F0 := \begin{bmatrix} \pi + 5 \\ 2\pi - \frac{5}{7} \end{bmatrix} \quad (24$$

```
> A:=LinearSolve(V,F0);
```

$$A := \begin{bmatrix} \pi \\ 5 \end{bmatrix} \quad (25$$

Basics: polynomial

Basics: polynomial

Figure: Michael Ben-Or and Prasoon Tiwari [1988]

Basics: polynomial

interpolation problem:

$$\sum_{(k_1,\ldots,k_d)\in K} \alpha_{k_1,\ldots,k_d} x_1^{k_1}\cdots x_d^{k_d}, \quad \#K = n$$

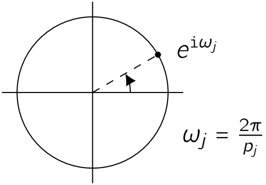
evaluate at

$$(x_1,\ldots,x_d) = (\omega_1^s,\ldots,\omega_d^s)$$

$$\omega_j = \exp(2\pi \mathtt{i}/p_j), \quad p_j > \partial_j p(x_1,\ldots,x_d), \quad p_j \text{ mutually prime}$$

$$p(\omega_1^s,\ldots,\omega_d^s) = f_s, \quad 0 \le s \le 2n-1$$

$$\Omega_i = \omega_1^{k_1^{(i)}}\cdots\omega_d^{k_d^{(i)}}, \quad i = 1,\ldots,n$$



[Giesbrecht, Labahn, and Lee, 2006]

Basics: polynomial

finding Ω_i:

$$\prod_{i=1}^{n}(z-\Omega_i) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$$

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \alpha_i \Omega_i^s (\Omega_i^n + b_{n-1}\Omega_i^{n-1} + \cdots + b_0) \\ &= \sum_{i=1}^{n} \alpha_i \Omega_i^{n+s} + \sum_{j=0}^{n-1} b_j \left(\sum_{i=1}^{n} \alpha_i \Omega_i^{j+s} \right) \\ &= f_{s+n} + \sum_{j=0}^{n-1} b_j f_{s+j} \end{aligned}$$

Basics: polynomial

$$\begin{pmatrix} f_0 & \dots & f_{n-1} \\ \vdots & \iddots & \vdots \\ f_{n-1} & \dots & f_{2n-2} \end{pmatrix} \begin{pmatrix} b_0 \\ \vdots \\ b_{n-1} \end{pmatrix} = - \begin{pmatrix} f_n \\ \vdots \\ f_{2n-1} \end{pmatrix}$$

Ω_i: zeros of formally orthogonal Hadamard polynomial

$$\prod_{i=1}^{n} (z - \Omega_i) = \frac{H_n^{(0)}(z)}{\left| H_n^{(0)} \right|}$$

Basics: polynomial

finding $(k_1^{(i)}, \dots, k_d^{(i)})$:

reverse Chinese remainder theorem

$$m = p_1 \cdots p_d$$

$$\Omega_i = \omega^{k(i)}, \quad \omega = \exp\left(\frac{2\pi \mathtt{i}}{\prod_{j=1}^{d} p_j}\right),$$

$$k(i) = k_1^{(i)} \frac{m}{p_1} + \cdots + k_d^{(i)} \frac{m}{p_d}$$

$$k_j^{(i)} \frac{m}{p_j} \bmod p_j = k(i) \bmod p_j,$$

$$k_j^{(i)} < p_j, \quad \gcd(p_j, m/p_j) = 1, \quad j = 1, \dots, d$$

Basics: polynomial

finding $\alpha_i = \alpha_{k_1^{(i)},\dots,k_d^{(i)}}$:

$$\sum_{i=1}^{n} \alpha_i \Omega_i^{s+j} = f_{s+j}, \quad s = 0, \dots, n-1, \quad 0 \le j \le n$$

$$\begin{pmatrix} \Omega_1^{j} & \dots & \Omega_n^{j} \\ \vdots & & \vdots \\ \Omega_1^{j+n-1} & \dots & \Omega_n^{j+n-1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} f_j \\ \vdots \\ f_{j+n-1} \end{pmatrix}$$

remaining interpolation conditions are linearly dependent

Basics: polynomial

finding n:

floating-point arithmetic:

$$\left|H_N^{(r)}\right| \equiv 0, \quad N > n, \quad r = 0, 1, \ldots$$

exact arithmetic:
increase n till

$$\delta_s := f_s + b_{n-1} f_{s-1} + \cdots + b_0 f_{s-n}$$

equals 0, $s \geq 2n$

[Massey, 1969]

Example: polynomial

$$p(x,y) = x^5y + 2.2x^4y^4 - 0.5xy^{11} + 0.1xy^{12}$$

$$p_1 = 6, \quad p_2 = 13, \quad \omega_1 = \exp(2\pi \mathtt{i}/6), \quad \omega_2 = \exp(2\pi \mathtt{i}/13),$$

$p(\omega_1^s, \omega_2^s), \quad s = 0, \ldots, 7 \quad$ (floating-point)

or

$p(p_1^s, p_2^s), \quad s = 0, \ldots, 7 \quad$ (exact arithmetic)

sequence $f_0, f_1, \ldots, f_7, \ldots$ is linearly generated and $\delta_8 = 0$

Hadamard polynomial:

$$z^4 + (-3.67 + 0.0799\mathtt{i})z^3 + (5.35 - 0.216\mathtt{i})z^2$$
$$(-3.67 + 0.216\mathtt{i})z + (0.997 - 0.0805\mathtt{i})$$

Basics: polynomial

$$m = 78, \qquad \omega = \exp(2\pi \mathtt{i}/78),$$

$$\Omega_1 = \omega^{k(1)} = \omega^{71} = \omega^{5\times 13 + 1\times 6}$$

$$\Omega_2 = \omega^{k(2)} = \omega^{76} = \omega^{4\times 13 + 4\times 6}$$

$$\Omega_3 = \omega^{k(3)} = \omega^{79} = \omega^{1\times 13 + 11\times 6}$$

$$\Omega_4 = \omega^{k(4)} = \omega^{85} = \omega^{1\times 13 + 12\times 6}$$

$$K = \{(5,1),(4,4),(1,11),(1,12)\} \Rightarrow \text{ terms } x^5y, x^4y^4, xy^{11}, xy^{12}$$

Vandermonde system

$$\sum_{i=1}^{4} \alpha_i \Omega_i^{s+j} = f_{s+j}, \quad s = 0, \dots, n-1, \quad 0 \le j \le n$$

Exercise: polynomial

$$p(x,y) = (x-3)^5(y+5) + 2.2(x-3)^4(y+5)^4 - 0.5(x-3)(y+5)^{11} + 0.1(x-3)(y+5)^{12}$$
$$= u^5v + 2.2u^4v^4 - 0.5uv^{11} + 0.1uv^{12},$$
$$u = x-3,\ v = y+5$$

$$p_1 = 6, \qquad p_2 = 13,$$
$$\omega_1 = \exp(2\pi \mathtt{i}/6), \qquad \omega_2 = \exp(2\pi \mathtt{i}/13),$$
$$u = \omega_1^s, \qquad v = \omega_2^s$$

$$p(\omega_1^s + 3, \omega_2^s - 5), \quad s = 0, \dots, 7$$

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```
> ##########################
  ## Exercise: polynomial ##
  ##########################
> restart:
> with(LinearAlgebra):
> read "./ExpLog.mm":
  read "./RevChRem.mm":
> Digits:= 16:
> poly_uv := u^5*v + 2.2*u^4*v^4-0.5*u*v^11+0.1*u*v^12;
```

$$poly\_uv := u^5\, v + 2.2\, u^4\, v^4 - 0.5\, u\, v^{11} + 0.1\, u\, v^{12} \quad (1)$$

```
> poly_xy := subs(u=x-3,v=y+5, poly_uv);
```

$$poly\_xy := (x-3)^5\,(y+5) + 2.2\,(x-3)^4\,(y+5)^4 - 0.5\,(x-3)\,(y+5)^{11} + 0.1\,(x-3)\,(y+5)^{12} \quad (2)$$

```
> expand(poly_xy);
```

$$-1.464750\;10^5\, x - 1.45595805\;10^7\, y + 1.101600\;10^5 + 72900.0\, x^2 - 3.21998325\;10^7\, y^2 + x^5\, y + 1085.0\, x^4\, y - 13110.0\, x^3\, y + 59130.0\, x^2\, y + 4.7644175\;10^6\, x\, y + 2.2\, x^4\, y^4 + 44.0\, x^4\, y^3 - 26.4\, x^3\, y^4 + 330.0\, x^4\, y^2 - 528.0\, x^3\, y^3 + 118.8\, x^2\, y^4 - 3960.0\, x^3\, y^2 + 2376.0\, x^2\, y^3 + 6.4450749\;10^6\, x\, y^4 + 17820.0\, x^2\, y^2 + 1.07374355\;10^7\, x\, y^3 + 1.07065475\;10^7\, x\, y^2 + 5.5\, x\, y^{11} + 137.5\, x\, y^{10} + 2062.5\, x\, y^9 + 20625.0\, x\, y^8 + 1.443750\;10^5\, x\, y^7 + 7.218750\;10^5\, x\, y^6 + 2.5781250\;10^6\, x\, y^5 + 0.1\, x\, y^{12} + 5\, x^5 + 1300.0\, x^4 - 16050.0\, x^3 - 1.93357593\;10^7\, y^4 - 3.22229985\;10^7\, y^3 - 16.5\, y^{11} - 412.5\, y^{10} - 6187.5\, y^9 - 61875.0\, y^8 - 4.331250\;10^5\, y^7 - 2.1656250\;10^6\, y^6 - 7.7343750\;10^6\, y^5 - 0.3\, y^{12} \quad (3)$$

```
> p1 := 6;
  p2 := 13;
```

$$p1 := 6$$
$$p2 := 13 \quad (4)$$

```
> w1 := evalf(exp(2*Pi*I/p1));
  w2 := evalf(exp(2*Pi*I/p2));
```

$$w1 := 0.5000000000000000 + 0.8660254037844385\, I$$
$$w2 := 0.8854560256532100 + 0.4647231720437684\, I \quad (5)$$

```
> for i from 0 to 7 do
    f[i] := evalf(subs(x=w1^i+3, y=w2^i-5, poly_xy)):
  od;
```

$$f_0 := 2.8 + 0.\, I$$

$$f_1 := 2.602841017834476 - 0.8741573457604577\, I$$
$$f_2 := 2.064816858508122 - 1.589980617941464\, I$$
$$f_3 := 1.329941999443174 - 2.035486797328322\, I$$
$$f_4 := 0.5888159453592064 - 2.177111443869176\, I$$
$$f_5 := 0.02061376974983473 - 2.067629883507751\, I$$
$$f_6 := -0.2610215691867326 - 1.827541993129119\, I$$
$$f_7 := -0.2414681965457515 - 1.605741666597298\, I$$ (6

```
> H0[4] := HankelMatrix([f[0],f[1],f[2],f[3],f[4],f[5],f[6]],4);
```

$H0_4$:= [[2.8 + 0. I, 2.602841017834476 − 0.8741573457604577 I, 2.064816858508122 − 1.589980617941464 I, 1.329941999443174 − 2.035486797328322 I], (7

[2.602841017834476 − 0.8741573457604577 I, 2.064816858508122 − 1.589980617941464 I, 1.329941999443174 − 2.035486797328322 I, 0.5888159453592064 − 2.177111443869176 I],

[2.064816858508122 − 1.589980617941464 I, 1.329941999443174 − 2.035486797328322 I, 0.5888159453592064 − 2.177111443869176 I, 0.02061376974983473 − 2.067629883507751 I],

[1.329941999443174 − 2.035486797328322 I, 0.5888159453592064 − 2.177111443869176 I, 0.02061376974983473 − 2.067629883507751 I, -0.2610215691867326 − 1.827541993129119 I]]

```
> H1[4] := HankelMatrix([f[1],f[2],f[3],f[4],f[5],f[6],f[7]],4);
```

$H1_4$:= [[2.602841017834476 − 0.8741573457604577 I, 2.064816858508122 − 1.589980617941464 I, 1.329941999443174 − 2.035486797328322 I, 0.5888159453592064 − 2.177111443869176 I], (8

[2.064816858508122 − 1.589980617941464 I, 1.329941999443174 − 2.035486797328322 I, 0.5888159453592064 − 2.177111443869176 I, 0.02061376974983473 − 2.067629883507751 I],

[1.329941999443174 − 2.035486797328322 I, 0.5888159453592064 − 2.177111443869176 I, 0.02061376974983473 − 2.067629883507751 I, -0.2610215691867326 − 1.827541993129119 I],

[0.5888159453592064 − 2.177111443869176 I, 0.02061376974983473 − 2.067629883507751 I, -0.2610215691867326 − 1.827541993129119 I, -0.2414681965457515 − 1.605741666597298 I]]

```
> eigenvalue_list := Eigenvalues(H1[4],H0[4], output=list);
```

eigenvalue_list := [0.8451900855580868 + 0.5344658261253400 I, 0.8451900855413997 − 0.5344658261265820 I, 0.9870502626364418 − 0.1604112808397337 I, 0.9967573080994497 + 0.08046656865700708 I] **(9**

```
> total_exp_list := []:
  for i from 1 to 4 do
    total_exp_list := [op(total_exp_list), round(ExpLog(eigenvalue_list[i],6*13))]:
  od:
> total_exp_list;
```

[7, 71, 76, 1] **(10**

```
> exp_list:=[]:
  for i from 1 to 4 do
    exp_list := [op(exp_list), [RevChRem(total_exp_list[i], 6, 6*13), RevChRem(total_exp_list[i], 13,
  6*13)]]:
  od:
> exp_list;
```

[[1, 12], [5, 1], [4, 4], [1, 11]] **(11**

```
> V := Transpose(VandermondeMatrix(eigenvalue_list));
```

V := [[1. + 0. I, 1. + 0. I, 1. + 0. I, 1. + 0. I], **(12**
[0.8451900855580868 + 0.5344658261253400 I, 0.8451900855413997 − 0.5344658261265820 I, 0.9870502626364418 − 0.1604112808397337 I, 0.9967573080994497 + 0.08046656865700708 I],
[0.4286925614298439 + 0.9034504346214993 I, 0.4286925614003088 − 0.9034504346057614 I, 0.9485364419500248 − 0.3166679937654143 I, 0.9870502625782285 + 0.1604112807331159 I],
[−0.1205366802302719 + 0.9927088741336253 I, −0.1205366802550991 − 0.9927088740899947 I, 0.8854560256661491 − 0.4647231719910745 I, 0.9709418173518603 + 0.2393156640939936 I]]

```
> F := Matrix([[f[0]],[f[1]],[f[2]],[f[3]]]);
```

$$F := \begin{bmatrix} 2.8 + 0.\,I \\ 2.602841017834476 - 0.8741573457604577\,I \\ 2.064816858508122 - 1.589980617941464\,I \\ 1.329941999443174 - 2.035486797328322\,I \end{bmatrix}$$ **(13**

```
> A := LinearSolve(V, F);
```

$$A := \begin{bmatrix} 0.1000000000041224 + 1.216000331768808\ 10^{-11}\ \mathrm{I} \\ 1.000000000007150 + 2.565636441003167\ 10^{-11}\ \mathrm{I} \\ 2.200000000261579 - 7.965585370092284\ 10^{-11}\ \mathrm{I} \\ -0.500000002728518 + 4.183975685423307\ 10^{-11}\ \mathrm{I} \end{bmatrix} \quad (14)$$

```
> var_list := [u,v]:

  p := 0:

  for i from 1 to 4 do
     term := A[i,1]:

     for j from 1 to 2 do
        term := term*var_list[j]^exp_list[i,j]:
     od:

     p := p + term:
  od:
```

```
> p;
```

$$(0.1000000000041224 + 1.216000331768808\ 10^{-11}\ \mathrm{I})\ u\ v^{12} + (1.000000000007150 + 2.565636441003167\ 10^{-11}\ \mathrm{I})\ u^5\ v + (2.200000000261579 - 7.965585370092284\ 10^{-11}\ \mathrm{I})\ u^4\ v^4 + (-0.500000002728518 + 4.183975685423307\ 10^{-11}\ \mathrm{I})\ u\ v^{11} \quad (15)$$

```
> poly_uv;
```

$$u^5\ v + 2.2\ u^4\ v^4 - 0.5\ u\ v^{11} + 0.1\ u\ v^{12} \quad (16)$$

Approximation theory

Approximation theory

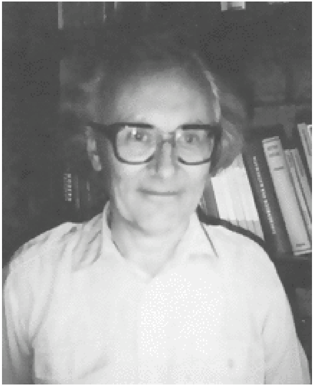


Figure: Henri Padé [1892] and Christian Pommerenke [1973]

Approximation theory

$$f_s = \sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s), \qquad s = 0, 1, \ldots, 2n-1$$

$$f(z) = \sum_{j=0}^{\infty} f_j z^j, \qquad f_j = 0, \qquad j < 0$$

$$p(z) = \sum_{i=0}^{\ell} a_i z^i,$$

$$q(z) = \sum_{i=0}^{m} b_i z^i$$

$$\left(\sum_{j=0}^{\infty} f_j z^j\right) q(z) - p(z) = \sum_{i \geq \ell+m+1} c_i z^i$$

Approximation theory

$$\begin{cases} f_0 b_0 = a_0 \\ f_1 b_0 + f_0 b_1 = a_1 \\ \quad \vdots \\ f_\ell b_0 + \cdots + f_{\ell-m} b_m = a_\ell \end{cases}$$

$$b_0 = 1$$

$$\begin{cases} f_{\ell+1} b_0 + \cdots + f_{\ell-m+1} b_m = 0 \\ \quad \vdots \\ f_{\ell+m} b_0 + \cdots + f_\ell b_m = 0 \end{cases}$$

$$H_m^{(\ell+1-m)} \begin{pmatrix} b_m \\ \vdots \\ b_1 \end{pmatrix} = - \begin{pmatrix} f_{\ell+1} \\ \vdots \\ f_{\ell+m} \end{pmatrix}$$

$$[\ell/m](z) := p(z)/q(z)$$

Approximation theory

$$f_s = \sum_{i=1}^{n} \alpha_i \exp(\phi_i x_s) = \sum_{i=1}^{n} \alpha_i \Omega_i^s$$

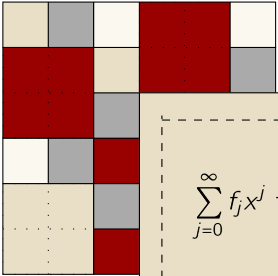
$$f(z) = \sum_{j=0}^{\infty} f_j z^j = \sum_{i=1}^{n} \frac{\alpha_i}{1 - z\Omega_i} = \text{Laplace transform of } \sum_{i=1}^{n} \alpha_i \exp(\phi_i x)$$

Approximation theory

$$[n-1/n](z) = p(z)/q(z)$$

$$\begin{aligned} q(z) &= \prod_{i=1}^{n}(1 - z\Omega_i) \\ &= z^n \frac{H_n^{(0)}(1/z)}{\left|H_n^{(0)}\right|} \\ &= b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + 1 \end{aligned}$$

Approximation theory



$\rightarrow H_m^{(n-m)}$ regular, $m \geq n$

$\sum_{j=0}^{\infty} f_j x^j$ from rational function

$H_\nu^{(r)}$ singular, $\nu > n$, $r > n - \nu$

$\downarrow$

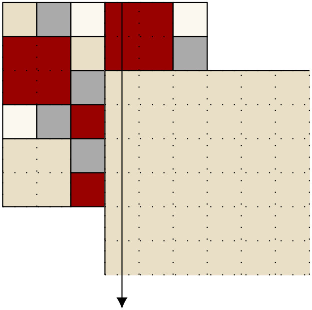
$H_n^{(\ell+1-n)}$ regular, $\ell + 1 \geq n$

Approximation theory

$f(z) + \varepsilon(z)$ meromorphic with poles in $0 \leq |z| < R$ of total multiplicity n [de Montessus de Ballore, 1905]

$$\Downarrow$$

$[\nu/n](z) \to f(z) + \varepsilon(z)$ uniformly on compact sets excluding poles, with poles of $f(z) + \varepsilon(z)$ attracting poles of $[\nu/n](z)$ according to their multiplicity



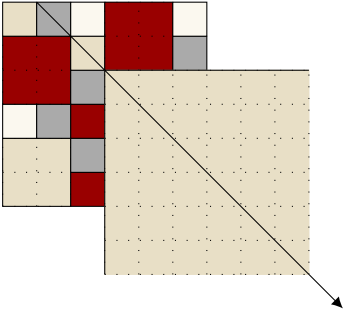
Approximation theory

$f(z)+\varepsilon(z)$ analytic except for a countable number of poles [Nuttall, 1970] and essential singularities [Pommerenke, 1973]

$$\Downarrow$$

$[\nu-1/\nu](z) \to f(z)+\varepsilon(z)$ in measure on compact sets, i.e.

$$\Lambda_2\left(\{z : |f(z)+\varepsilon(z)-[\nu-1/\nu](z)| \geq \tau\}\right) \to 0$$



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Approximation theory

mathematical (noise free):

1. build $H_\nu^{(0)}, \quad \nu = 0, 1, 2, \dots$
2. $H_\nu^{(0)} = U \Sigma V^T$ singular value decomposition
3. $\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_\nu \end{pmatrix}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > \sigma_{n+1} = \cdots = \sigma_\nu = 0$
4. find $\Omega_i, \phi_i, \alpha_i, i = 1, \dots, n$

Approximation theory

numerical (with noise):

1. take ν large enough so that noise is clearly separated from n
2. solve $H_\nu^{(1)} v_i = \lambda_i H_\nu^{(0)} v_i, \quad i = 1, \ldots, \nu, \quad \lambda_i = \Omega_i, \quad i = 1, \ldots, n$
3. find ϕ_i
4. solve $\sum_{i=1}^{n} \alpha_i \exp(\phi_i x_j) = f_j, \quad 0 \le j \le 2\nu - 1$

Example: noise

$$\phi_1 = 0, \qquad \alpha_1 = 1,$$

$$\phi_2 = -0.2 + 39.5\mathtt{i}, \qquad \alpha_2 = 2, \qquad x_s = s\frac{2\pi}{100},$$

$$\phi_3 = -0.5 + 40\mathtt{i}, \qquad \alpha_3 = 4, \qquad M = 100$$

$$\phi_4 = -1, \qquad \alpha_4 = 8,$$

$$\|\varepsilon(z)\|_\infty = 10^{-2}, \qquad \text{uniform random noise}$$

Example: noise

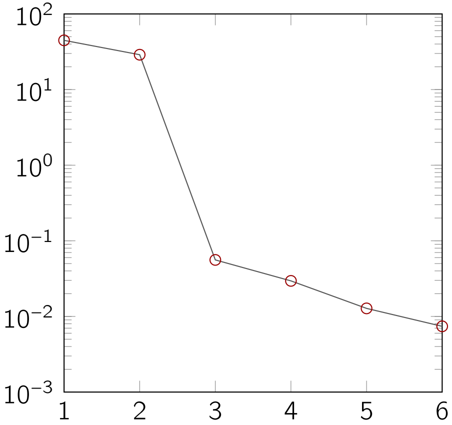


Figure: Singular values $H_\nu^{(0)}$ with $n = 4, \nu = 6$

Example: noise

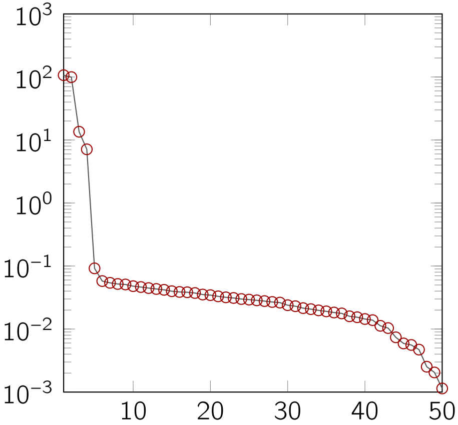


Figure: Singular values $H_\nu^{(0)}$ with $n = 4, \nu = 50$

Exercise: approximation

$$\phi_1 = 1.5\mathtt{i}, \qquad \alpha_1 = 10^{-3},$$

$$\phi_2 = 12.7\mathtt{i}, \qquad \alpha_2 = 2, \qquad x_s = s\frac{2\pi}{100},$$

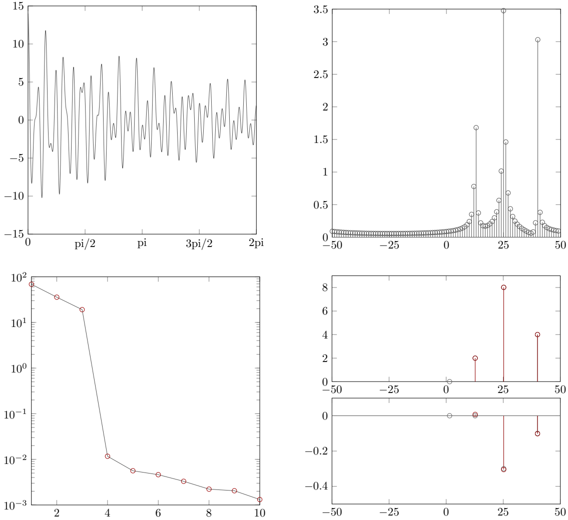
$$\phi_3 = -0.1 + 40\mathtt{i}, \qquad \alpha_3 = 4, \qquad M = 100$$

$$\phi_4 = -0.3 + 25.2\mathtt{i}, \qquad \alpha_4 = 8,$$

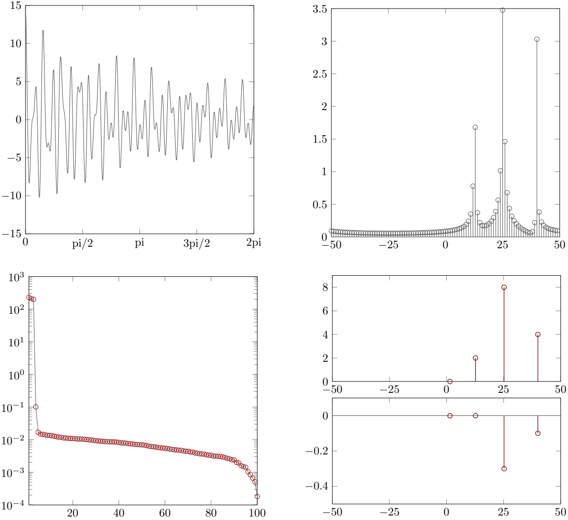
$\|\varepsilon(z)\|_\infty = 2 \times 10^{-3}$, uniform random noise

```
format long;

phi = [1.5*1i, 12.7*1i, -0.1+40*1i, -0.3+25.2*1i];
alpha = [10^(-3), 2, 4, 8];

eps = 2*10^(-3);

M = 100;

plot_signal

pause

plot_fft

pause

% synthesized input data with added noise
N = input('Enter the dimension for SVD: ');
% (10, 6, 3), (100, 100, 4)

randn('seed',0);

omega = 2*pi/M*(0:2*N-1);
f = syn_exp(alpha, phi, omega);

v = randn(size(f))+randn(size(f))*1i;
vv = v/norm(v,Inf);
f = f + eps*vv;

% form Hankel matrices H0 and H1 from y sequence
[H0,H1] = mat_ge(f);

plot_svd

pause

% reconstruct the parameters via generalized eigenvalues
n = input('Size of the model: ');
```

```
% compute the generalized eigenvalues and form the Vandermonde system
E = eig(H1(1:n,1:n),H0(1:n,1:n));
V = rot90(vander(E));

% amplitudes
A = V\f(1:n).';

% frequencies and damping factors
alpha_rec = A;
phi_rec = log(E)*M/(2*pi);

pause

% extract the non-zero terms
extract

% plot computed parameters
plot_recontructed_parameters
```



15
10
5
0
−5
−10
−15
0
pi/2
pi
3pi/2
2pi
3.5
3
2.5
2
1.5
1
0.5
0
−50
−25
0
25
50
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
2
4
6
8
10
8
6
4
2
0
−0.2
−0.4


0
pi/2
pi
3pi/2
2pi

Applications

Applications

Exponential analysis in physical phenomena:

- power system transient detection
- motor fault diagnosis
- drug clearance / glucose tolerance
- magnetic resonance / infrared spectroscopy
- vibration analysis
- seismic data analysis
- music signal processing
- corrosion rate / crack initiation
- odour recognition with electronic nose
- typed keystroke recognition
- liquid (explosive) identification
- ...

Transients

Figure: Transient detection and characterization

Transients

short lived high frequency signal:

- speech processing
- turbulent flow
- power lines
- ...

Transients

- model with $\phi_i = 120\pi\mathtt{i}$,

$$\sum_{i=1}^{n} \alpha_i \cos(120\pi x + \gamma_i)\mathbf{1}_{[A_i, Z_i[}$$

- $n = 3, \alpha_i = 1, \gamma_{1,3} = -\pi/2, \gamma_2 = 3\pi/4$
- $[A_1, Z_1[= [0, 0.0308[$
  $[A_2, Z_2[= [0.0308, 0.0625[$
  $[A_3, Z_3[= [0.0625, 0.1058[$
- $M = 1200$
- uniformly distributed noise in $[-0.05, 0.05]$

Transients

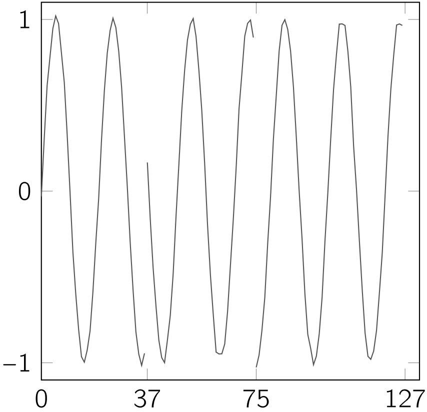


Figure: Given transient signal

Transients

- at each instance: 2 exponential terms
- characteristics of terms change
- inspect rank of

$$H_4^{(1)} = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \\ f_2 & f_3 & f_4 & f_5 \\ f_3 & f_4 & f_5 & f_6 \\ f_4 & f_5 & f_6 & f_7 \end{pmatrix}$$

$$[A_1, Z_1[= [0/M, 37/M[,$$
$$[A_2, Z_2[= [37/M, 75/M[,$$
$$[A_3, Z_3[= [75/M, 127/M[$$

Transients

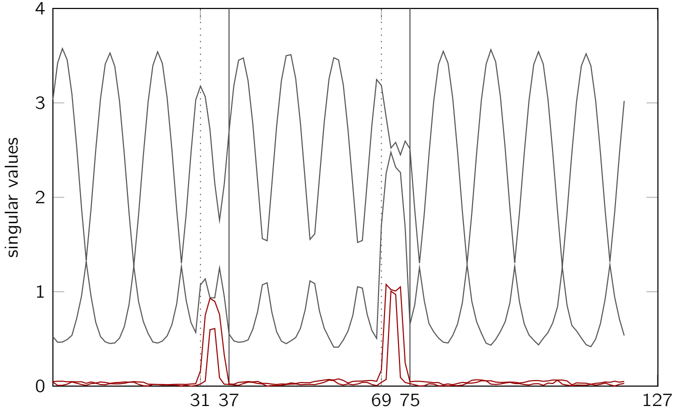


Figure: Numerical rank of $H_4^{(r)}$ evolving over time x

Audio signals

Figure: Reconstructing undersampled audio signals

Audio signals

song containing 29 notes of 0.25 seconds each:

- $M = 44100$ (Hz)
- 11025 samples per note, 319725 in total
- $16.35 \leq \phi_i \leq 4978.03, \; i = 1, \ldots, 100$
- complex exponential model

Audio signals

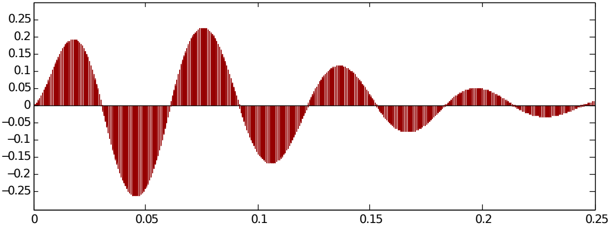


Figure: Sampled signal produced by 1 note

Audio signals

compressive sensing (optimisation, probabilistic)

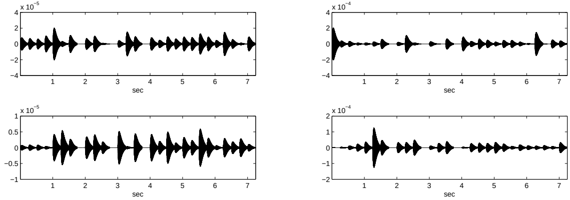


Figure: 4 runs with 1229 samples

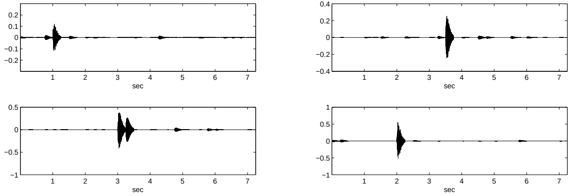


Figure: 4 runs with 456 samples

Audio signals

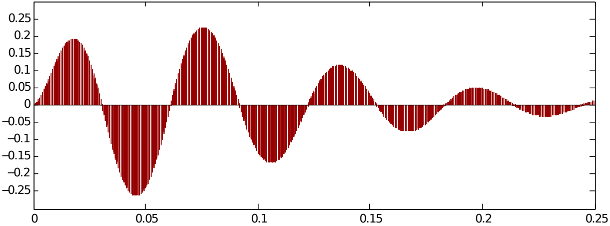


Figure: Full set of samples per note

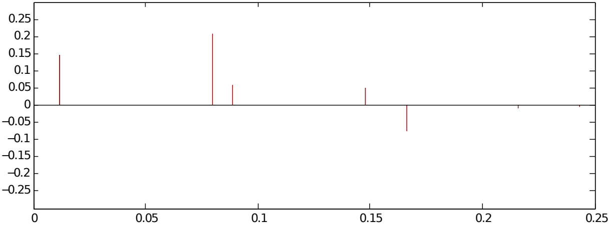


Figure: Sparse interpolation with 7 samples per note

MCSA

Figure: Preventive diagnosis of a broken rotor bar

MCSA

3-phase induction motors:

- consume 40 – 50% of all electricity in industrialized countries
- rotor made up metal bars
- stator current signal analysed
- broken bar(s) characterized by sideband frequencies
- difficult to diagnoze under low or no load

Figure: Stator and rotor

MCSA

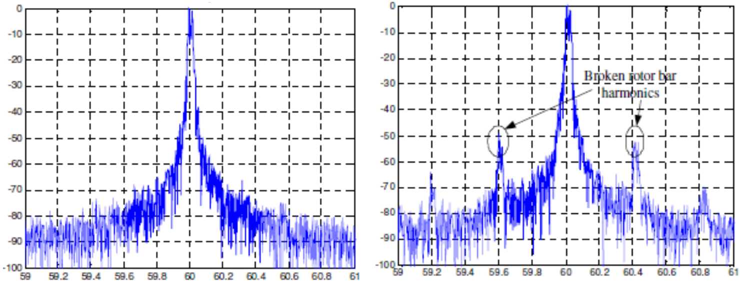


Figure: Stator current FFT spectra: healthy and with 1 broken bar

MCSA

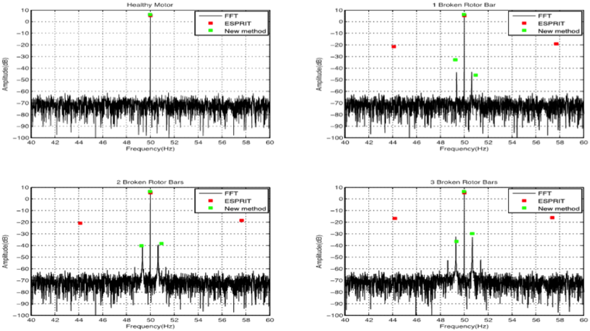


Figure: 10% load, 16dB noise, $\nu = 400$

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Bio-electrical

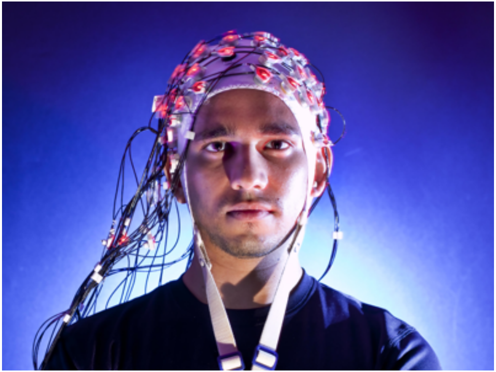

Figure: Sparse EEG approximation

Bio-electrical

Bio-electrical signals:

- electrical activity of cells and tissues
- clinical studies of health status
- ECG, EEG, EMG, EOG, ...
- sparse model ($n = 8$) is approximate

Bio-electrical

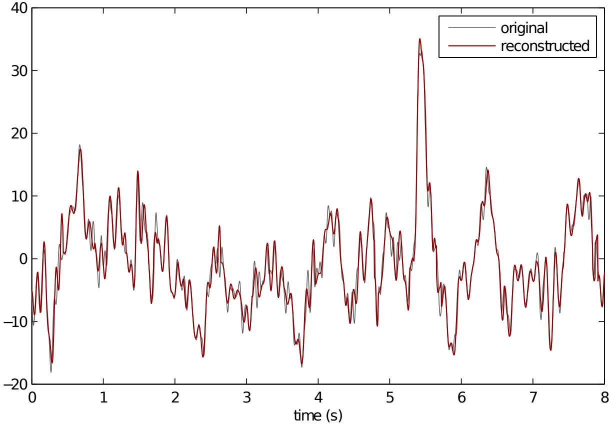

Figure: Reconstruction of 8 second [1 – 20] Hz bandpass filtered EEG

Bio-electrical

Figure: Sparse EOG approximation

Bio-electrical

Polysomnogram:

- 12 channels
- 22 wire attachments to patient
- heart rate, leg measurement, airflow (chest, abdomen), chin muscle, EEG, EOG, ...

Bio-electrical

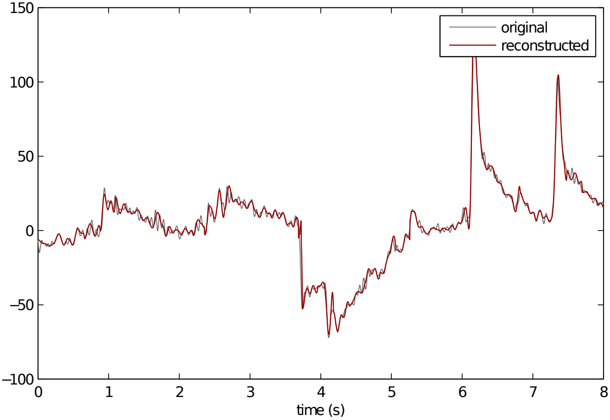

Figure: Reconstruction of 8 second EOG (CC = 99.2%)

Spectroscopy

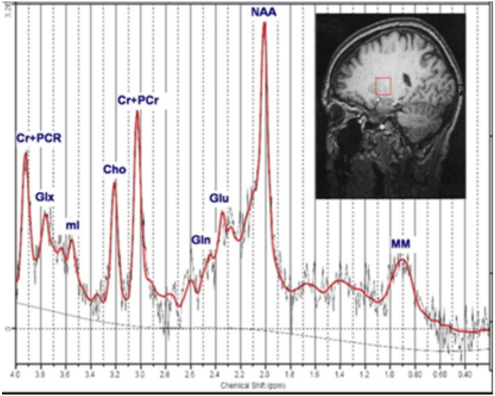

Figure: Spectral analysis of FID

Spectroscopy

Magnetic resonance spectroscopy:

- physical and chemical properties of molecules
- a.o. concentration of metabolites in the brain
- frequencies clustered → high frequence resolution
- free induction decay → time constraint
- Fourier methods need additional tools

Spectroscopy

$$\phi(x) = 5 \times 10^{-2} + 2e^{(-0.97+\mathrm{i}79.94\pi)x} + 4e^{(-1+\mathrm{i}80\pi)x} + 8e^{-1.1x} + \varepsilon(x)$$

$\|\varepsilon(x)\|_\infty = 10^{-3}$, circular Gaussian noise

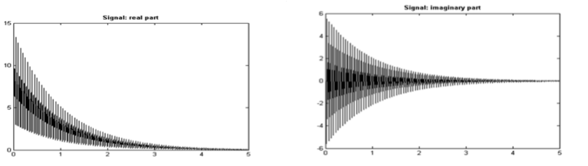

Figure: The real (left) and imaginary (right) part of $\phi(x)$

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Spectroscopy

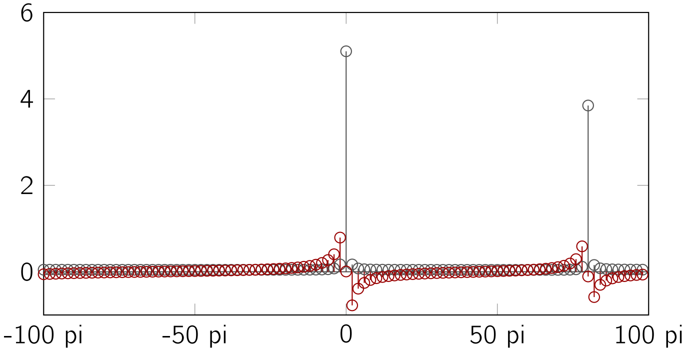

Figure: Real (black) and imaginary (red) parts of FFT

Spectroscopy

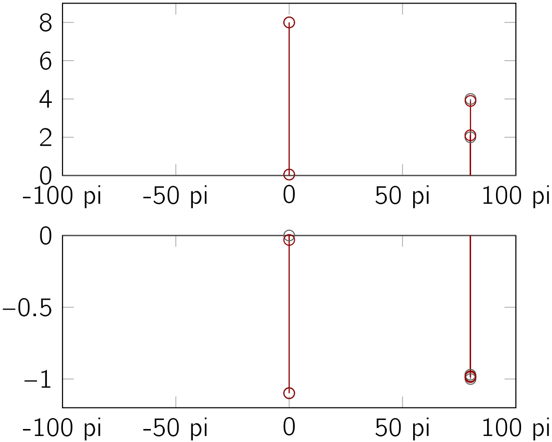

Figure: Amplitudes (top) and damping factors (bottom) of $\phi(x)$

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