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Some Integral Inequalities Involving a Fractional Integral Operator with Extended Hypergeometric Function

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Abstract

The present article deals with the generalized k-fractional integrals of a function with respect to the extended hypergeometric function which generalizes a variety of fractional integrals such as Riemann-Liouville, Hadamard fractional integrals, Katugampola fractional integrals, (k,s) fractional integral operators. Moreover, we derive the Hermite-Hadamard inequalities by employing k-fractional integrals of a function with respect to the extended hypergeometric function and the trapezoid type inequalities for the functions whose derivatives in absolute value are convex. Some special cases of these inequalities are also provided.

Keywords: Riemann-Liouville Fractional Integral Operator, Hermite-Hadamard Fractional Integrals Operator, Trapezoid Type Inequalities, Extended Gauss Hypergeometric Function.

1 Introduction

Integral inequalities play a crucial role in developing the qualitative and numerical aspects present in both applied and pure mathematics. This viewpoint aided in the development of unique and noteworthy results in numerous branches of the engineering and mathematical disciplines and offered a thorough platform for the investigation of numerous problems.
One of the most well-known inequality in the study of convex functions is the Hermite-Hadamard inequality with numerous implications and mathematical interpretations. Several mathematicians have dedicated their endeavour to generalizing, improving and upgrading it for varieties of functions including the use of convex mapping. Hermite and Hadamard made substantial contributions to the literature when they established inequalities for convex functions. For thorough examination one may follow [6, 13].

In accordance with these inequalities if $h : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $\varphi_1, \varphi_2 \in I$ with $\varphi_1 < \varphi_2$ then

$$h\left(\frac{\varphi_1+\varphi_2}{2}\right) \leq \frac{1}{\varphi_2-\varphi_1}\int_0^1 h(\nu)d\nu \leq \frac{h(\varphi_1)+h(\varphi_2)}{2}. \tag{1}$$

If h is concave then both inequalities hold in the opposite way. We point out that Hadamard's inequalities are a straightforward extension of the notion of convexity and may be inferred from Jensen's inequality. Recently, there has been a resurgence of interest in Hadamard's inequalities for convex functions and a wide range of improvements and generalizations have been investigated. It may be easily seen in [1, 2, 3] and [7, 9].

2 New Generalized Fractional Integral Operators

The theory of fractional calculus covers the investigation of the arbitrary order integration and derivative of a function. Fractional derivative provides a full grasp of memory and hereditary properties of many materials and processes, including certain natural and scientific phenomena. Particularly the study of fractal theory, theory of dynamic system control, theory of viscoelasticity, electrochemistry, diffusion processes, modelling and many others domains have benefited greatly from the idea of fractional

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derivatives. For further informations readers may follow [5, 8, 10, 11, 12, 15].
We illustrate the generalised k- fractional integrals of a function with respect to extended hypergeometric functions in this section. The function ${}_2F_1^{(p,k)}$ is defined [4, 14] as

$$ {}_2F_1^{(p,k)}(\Theta_1, \Theta_2;\ \Theta_3; t) = \sum_{s=0}^{\infty} \frac{(\Theta_1)_{s,k}(\Theta_2)_{s,k}}{(\Theta_3)_{s,k}} \frac{t^s}{ps!}, \tag{2} $$

where $k \in \mathbb{R}^+$ and $p > 0$, also Θ_1, Θ_2, $\Theta_3 \in \mathbb{C}$ s.t. $\Theta_3 \neq 0, -1, -2, -3, \cdots$

Definition 2.1. For $k, \Upsilon, \varrho > 0$ and $\delta \in \mathbb{R}$, suppose Φ:$[\varphi_1, \varphi_2] \rightarrow \mathbb{R}$ is a positive monotone and increasing function on $(\varphi_1, \varphi_2]$, containing a continuous derivative $\Phi'(\nu)$ on (φ_1, φ_2). Then, the left and right sided generalized k-fractional integral of ζ with respect to the function Φ on $[\varphi_1, \varphi_2]$ are defined correspondingly as follows:

$$ \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+,\delta}\zeta(\nu) = \int_{\varphi_1}^{\nu} \frac{\Phi'(t)}{(\Phi(\nu)-\Phi(t))^{1-\frac{\Upsilon}{\mathfrak{K}}}} {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\nu)-\Phi(t))^{\varrho}]\zeta(t)dt, \quad \nu > \varphi_1, \tag{3} $$

and

$$ \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-,\delta}\zeta(\nu) = \int_{\nu}^{\varphi_2} \frac{\Phi'(t)}{(\Phi(t)-\Phi(\nu))^{1-\frac{\Upsilon}{\mathfrak{K}}}} {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(t)-\Phi(\nu))^{\varrho}]\zeta(t)dt, \quad \nu < \varphi_2. \tag{4} $$

Here, we provide a few significant special cases related to the integral operators (3) and (4).

1. The operator (3) provides the generalized fractional integral of ζ with respect to Φ on $[\Phi_1, \Phi_2]$ for $\mathfrak{K} = 1$. This relation is explained by

$$ \xi^{p,k,\Phi}_{\varrho,\Upsilon,\varphi_1^+,\delta}\zeta(\nu) = \int_{\varphi_1}^{\nu} \frac{\Phi'(t)}{(\Phi(\nu)-\Phi(t))^{1-\Upsilon}} {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\nu)-\Phi(t))^{\varrho}]\zeta(t)dt, \quad \nu > \varphi_1. $$

2. The operator (3) yields the generalized k-fractional integral of ζ for $\Phi(t) = t$. This relation is provided by

$$ \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+,\delta}\zeta(\nu) = \int_{\varphi_1}^{\nu} (\nu - t)^{\frac{\Upsilon}{\mathfrak{K}}-1} {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\nu-t)^{\varrho}]\zeta(t)dt, \quad \nu > \varphi_1. $$

3. The operator (3) gives the generalized k-fractional integral of ζ for $\Phi(t) = \ln t$. This relation is obtained by

$$ \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+,\delta}\zeta(\nu) = \int_{\varphi_1}^{\nu} \Big(ln\frac{\nu}{t}\Big)^{\frac{\Upsilon}{\mathfrak{K}}-1} {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}\Big[\delta\Big(ln\frac{\nu}{t}\Big)^{\varrho}\Big]\zeta(t)dt, \quad \nu > \varphi_1. $$

4. The operator (3) leads to the generalized (k, s)-fractional integral of ζ for $\Phi(t) = \frac{t^{s+1}}{s+1}$, $s \in \mathbb{R} - \{-1\}$. This relation is given by

$$ \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+,\delta}\zeta(\nu) = (s+1)^{\frac{\Upsilon}{\mathfrak{K}}-1} \int_{\varphi_1}^{\nu} \Big(\nu^{s+1} - t^{s+1}\Big)^{\frac{\Upsilon}{\mathfrak{K}}-1} \times {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}\Big[\delta\Big(\frac{\nu^{s+1}-t^{s+1}}{s+1}\Big)^{\varrho}\Big]t^s\zeta(t)dt, \ \nu > \varphi_1. $$

In the same manner, for operator (4) all above special cases can also be obtained.

3 Hermite-Hadamard Inequalities for Generalized Fractional Integral Operators

Let us continue by going over some of the notations from [9, 16]. Consider a function $\zeta : I^0 \rightarrow \mathbb{R}$ with conditions $0 < \varphi_1 < \varphi_2 < \infty$ and $\varphi_1, \varphi_2 \in I^0$. Also, we assume that $\zeta \in L^{\infty}(\varphi_1, \varphi_2)$ such that $I^{\alpha}_{\varphi_1^+;\Phi}\zeta(\nu)$ and $I^{\alpha}_{\varphi_2^-;\Phi}\zeta(\nu)$ are well defined. We consider the function

$$ \hat{\zeta}(\nu) = \zeta(\varphi_1 + \varphi_2 - \nu), \quad \nu \in [\varphi_1, \varphi_2], \ \text{ and } \ F(\nu) = \zeta(\nu) + \hat{\zeta}(\nu), \quad \nu \in [\varphi_1, \varphi_2]. $$

Applying the variable $\varrho = \frac{\tau-\varphi_1}{\nu-\varphi_1}$ in eq. (3), we have

$$ \begin{aligned} \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+,\delta}\zeta(\nu) = \int_0^1 & \frac{(\nu-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\nu)}{[\Phi(\nu)-\Phi((1-\varrho)\varphi_1+\varrho\nu)]^{1-\frac{\Upsilon}{\mathfrak{K}}}} \\ & \times {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\nu)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\zeta(\varrho\nu+(1-\varrho)\varphi_1)d\varrho, \end{aligned} \tag{5} $$

Similarly applying the variable $\varrho = \frac{\tau-\nu}{\varphi_2-\nu}$, we have

$$ \begin{aligned} \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},b^-,\delta}\zeta(\nu) = \int_0^1 & \frac{(\varphi_2-\nu)\Phi'((1-\varrho)\nu+s\varphi_2)}{[\Phi((1-\varrho)\nu+\varrho\varphi_2)-\Phi(\nu)]^{1-\frac{\Upsilon}{\mathfrak{K}}}} \\ & \times {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi((1-\varrho)\nu+\varrho\varphi_2)-\Phi(\nu))^{\varrho}]\zeta(\varrho\varphi_2+(1-\varrho)\nu)d\varrho. \end{aligned} \tag{6} $$

Theorem 3.1. *Consider* $\Phi : [\varphi_1, \varphi_2] \to \mathbb{R}$ *is a positive monotone and increasing function on* $(\varphi_1, \varphi_2]$, *containing a continuous derivative* $\Phi'(\nu)$ *on* (φ_1, φ_2). *Then the following Hermite-Hadamard type inequalities for generalized k-fractional integrals of* ζ *with respect to the function* Φ *on* $[\varphi_1, \varphi_2]$ *hold whenever the function* ζ *is convex:*

$$\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big) \le \frac{1}{4\mathfrak{K}[\Phi(\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]} \times \Big[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1) + \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\Big] \le \frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}. \tag{7}$$

Proof. Since the function ζ is convex on $[\varphi_1,\varphi_2]$, we have

$$\zeta\Big(\frac{\nu_1+\nu_2}{2}\Big) \le \frac{\zeta(\nu_1)+\zeta(\nu_2)}{2}, \quad \forall\ \nu_1, \nu_2 \in [\varphi_1, \varphi_2]. \tag{8}$$

For $\varrho \in [0,1]$, consider $\nu_1 = \varrho\varphi_1 + (1-\varrho)\varphi_2$ and $\nu_2 = (1-\varrho)\varphi_1 + \varrho\varphi_2$. So we have

$$\zeta\Big(\frac{\nu_1+\nu_2}{2}\Big) \le \frac{1}{2}\zeta(\varrho\varphi_1 + (1-\varrho)\varphi_2) + \frac{1}{2}\zeta((1-\varrho)\varphi_1 + \varrho\varphi_2). \tag{9}$$

Multiplying both sides of the equation (9) by

$$\frac{(\varphi_2-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}],$$

and integrate with respect to ϱ over (0,1), we achieve

$$\begin{aligned}
\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)\int_0^1 &\frac{(\varphi_2-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2) \\
&\quad -\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]d\varrho \\
\le \Big(\frac{\varphi_2-\varphi_1}{2}\Big)\int_0^1 &\frac{\Phi'((1-\varrho)\varphi_1+s\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2) \\
&\quad -\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\zeta(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho \\
+\Big(\frac{\varphi_2-\varphi_1}{2}\Big)\int_0^1 &\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2) \\
&\quad -\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\zeta((1-\varrho)\varphi_1+\varrho\varphi_2)d\varrho.
\end{aligned} \tag{10}$$

Changing the variable $\nu = \Phi(\varphi_2) - \Phi((1-\varrho)\varphi_1+\varrho\varphi_2)$, we deduce that

$$\begin{aligned}
&\int_0^1 \frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]d\varrho \\
&= \frac{1}{\varphi_2-\varphi_1}\int_0^{\Phi(\varphi_2)-\Phi(\varphi_1)} \nu^{\frac{\Upsilon}{\mathfrak{K}}-1}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta\nu^{\varrho}]d\varrho, \\
&= \frac{\mathfrak{K}}{\varphi_2-\varphi_1}\nu^{\frac{\Upsilon}{\mathfrak{K}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta\nu^{\varrho}]\Big|_0^{\Phi(\varphi_2)-\Phi(\varphi_1)}, \\
&= \frac{\mathfrak{K}}{\varphi_2-\varphi_1}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}].
\end{aligned} \tag{11}$$

Also from the equation (5), we have

$$\begin{aligned}
(\varphi_2-\varphi_1)\int_0^1 &\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2) \\
&-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\zeta(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho = \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}\hat{\zeta}(\varphi_2),
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
(\varphi_2-\varphi_1)\int_0^1 &\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2) \\
&-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\zeta(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho = \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}\zeta(\varphi_2).
\end{aligned} \tag{13}$$

Substitute the equalities (11), (12) and (13) into (10), we have

$$\begin{aligned}
\mathfrak{K}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)&(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\, {}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}] \\
&\le \frac{1}{2}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}\zeta(\varphi_2) + \frac{1}{2}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}\hat{\zeta}(\varphi_2),
\end{aligned}$$

i.e,

$$\mathfrak{K}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}] \leq \frac{1}{2}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2). \tag{14}$$

Similarly, multiplying both sides of (9) by

$$\frac{(\varphi_2-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_{1\,\varrho,\Upsilon}^{(p,k)}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}],$$

we find the following equation after integrating over (0,1) with respect to ϱ

$$\begin{aligned}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)&\int_0^1\frac{(\varphi_2-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_{1\,\varrho,\Upsilon}^{(p,k)}[\delta(\Phi((1-\varrho)\varphi_1\\&\qquad+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]d\varrho\\
\leq\Big(\frac{\varphi_2-\varphi_1}{2}\Big)&\int_0^1\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_{1\,\varrho,\Upsilon}^{(p,k)}[\delta(\Phi((1-\varrho)\varphi_1\\&\qquad+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\zeta(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho\\
+\Big(\frac{\varphi_2-\varphi_1}{2}\Big)&\int_0^1\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_{1\,\varrho,\Upsilon}^{(p,k)}[\delta(\Phi((1-\varrho)\varphi_1\\&\qquad+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\zeta((1-\varrho)\varphi_1+\varrho\varphi_2)d\varrho.\end{aligned}\tag{15}$$

Changing the variable $\nu=\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)$, we deduce that

$$\begin{aligned}&\int_0^1\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_{1\,\varrho,\Upsilon}^{(p,k)}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]d\varrho\\&\qquad=\frac{\mathfrak{K}}{\varphi_2-\varphi_1}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}].\end{aligned}\tag{16}$$

Also from the equation (6), we have

$$\begin{aligned}(\varphi_2-\varphi_1)&\int_0^1\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_{1\,\varrho,\Upsilon}^{(p,k)}[\delta(\Phi((1-\varrho)\varphi_1\\&+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\zeta(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho=\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}\hat{\zeta}(\varphi_1),\end{aligned}\tag{17}$$

and

$$\begin{aligned}(\varphi_2-\varphi_1)&\int_0^1\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_{1\,\varrho,\Upsilon}^{(p,k)}[\delta(\Phi((1-\varrho)\varphi_1\\&+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\zeta((1-\varrho)\varphi_1+\varrho\varphi_2)d\varrho=\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}\zeta(\varphi_1).\end{aligned}\tag{18}$$

Substitute the equalities (16), (17) and (18) into (15), we have

$$\begin{aligned}\mathfrak{K}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)&(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]\\&\leq\frac{1}{2}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}\zeta(\varphi_1)+\frac{1}{2}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}\hat{\zeta}(\varphi_1),\end{aligned}$$

i.e,

$$\mathfrak{K}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}] \leq \frac{1}{2}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1). \tag{19}$$

On combining (14) and (19), we get

$$\begin{aligned}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)&\leq\frac{1}{4\mathfrak{K}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\\&\qquad\times\Big[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\Big].\end{aligned}$$

The first part of inequality is completed.
Also for the second part of equality of (7), since ζ is the convex function, so

$$\zeta(\varrho\varphi_1+(1-\varrho)\varphi_2)\leq\varrho\zeta(\varphi_1)+(1-\varrho)\zeta(\varphi_2),$$
$$\text{and }\zeta((1-\varrho)\varphi_1+\varrho\varphi_2)\leq(1-\varrho)\zeta(\varphi_1)+\varrho\zeta(\varphi_2).$$

With the two inequalities above added together, we obtain

$$\zeta(\varrho\varphi_1+(1-\varrho)\varphi_2)+\zeta((1-\varrho)\varphi_1+\varrho\varphi_2)\le\zeta(\varphi_1)+\zeta(\varphi_2). \tag{20}$$

Multiplying both sides of (20) by

$$(\varphi_2-\varphi_1)\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}],$$

and integrating the resulting inequality over (0,1) with respect to ϱ, we have

$$\begin{aligned}(\varphi_2-\varphi_1)\int_0^1&\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1\\&+\varrho\varphi_2))^{\varrho}]\zeta(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho\\+(\varphi_2-\varphi_1)\int_0^1&\frac{\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2)\\&-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\zeta((1-\varrho)\varphi_1+\varrho\varphi_2)d\varrho\\\le\big(f(\varphi_1)+f(\varphi_2)\big)\int_0^1&\frac{(\varphi_2-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\\&{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]d\varrho.\end{aligned} \tag{21}$$

Substitute the inequalities (11), (12) and (13) into (21) to obtain

$$\begin{aligned}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\le\mathfrak{K}[\zeta(\varphi_1)+\zeta(\varphi_2)][\Phi(\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\\\times{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}].\end{aligned} \tag{22}$$

Similarly, multiplying both sides of (20) by,

$$\frac{(\varphi_2-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}],$$

We obtain the following equation after integrating over (0,1) with respect to ϱ

$$\begin{aligned}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)\le\mathfrak{K}[\zeta(\varphi_1)+\zeta(\varphi_2)][\Phi(\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\\\times{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}].\end{aligned} \tag{23}$$

On adding the equality (22) and (23), we get

$$\begin{aligned}\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\le&2\mathfrak{K}(\zeta(\varphi_1)+\zeta(\varphi_2))(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{k}}\\&\times{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}],\end{aligned}$$

i.e,

$$\begin{aligned}\frac{1}{4\mathfrak{K}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\Big[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)\\+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\Big]\le\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}.\end{aligned}$$

This is the right side of equality (7), so the proof is accomplished. □

Corollary 3.2. *In the previous Theorem 3.1, if we select* $\mathfrak{K}=1$ *then we achieve following inequality:*

$$\begin{aligned}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)\le&\frac{1}{4\mathfrak{K}(\varphi_2-\varphi_1)^{\Upsilon}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+1}[\delta(\varphi_2-\varphi_1)^{\varrho}]}\\&\times\Big[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\varphi_1^+;\delta}F(\varphi_2)\Big]\le\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}.\end{aligned}$$

Corollary 3.3. *In Theorem 3.1, If we choose* $\Phi(t)=t$ *then we obtain the following inequality:*

$$\begin{aligned}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)\le&\frac{1}{4\mathfrak{K}(\varphi_2-\varphi_1)^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\varphi_2-\varphi_1)^{\varrho}]}\\&\times\Big[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\Big]\le\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}.\end{aligned}$$

Corollary 3.4. *In Theorem 3.1, If we choose* $\Phi(t)=\ln t$ *then we obtain following inequality:*

$$\begin{aligned}\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)\le&\frac{1}{4\mathfrak{K}(ln\frac{\varphi_2}{\varphi_1})^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(ln\frac{\varphi_2}{\varphi_1})^{\varrho}]}\\&\times\Big[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\Big]\le\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}.\end{aligned}$$

Corollary 3.5. *The following inequality for generalized (k,s)-fractional integrals are obtained if we choose $\Phi(t)=\frac{t^{s+1}}{s+1}$ in Theorem 3.1.*

$$\zeta\Big(\frac{\varphi_1+\varphi_2}{2}\Big)\leq\frac{(s+1)^{\frac{\Upsilon}{\mathfrak{K}}}}{4\mathfrak{K}(\varphi_2^{s+1}-\varphi_1^{s+1})^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\frac{\varphi_2^{s+1}-\varphi_1^{s+1}}{s+1})^{\varrho}]}$$
$$\times\Big[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\Big]\leq\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}.$$

4 Trapezoid Type Inequalities for Generalized Fractional Integral Operators

The trapezoid type inequalities have been established in this section by employing generalized k-fractional integrals and extended hypergeometric function. For more detail informations one may read [17].

Lemma 4.1. *Suppose $\zeta:[\varphi_1,\varphi_2]\to\mathbb{R}$ is a function which is differentiable on (φ_1,φ_2) with $\varphi_1<\varphi_2$. If $\zeta'\in L[\varphi_1,\varphi_2]$ then the following equality holds:*

$$\Bigg(\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}-\frac{1}{4\mathfrak{K}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\times\Big(\xi^{p,k,\Phi}_{\varrho,\Upsilon,\varphi_2^-,\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\mathfrak{K},\varphi_1^+,\delta}F(\varphi_2)\Big)\Bigg)$$
$$=\frac{(\varphi_2-\varphi_1)}{4\mathfrak{K}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\int_0^1\chi^{p,k}_{\varrho,\Upsilon,\Phi}(s)\times\zeta'(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho,$$

where $\chi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}:[0,1]\to\mathbb{R}$ is defined by

$$\chi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}=[\Phi(\varrho\varphi_1+(1-\varrho)\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((\varphi_1)+(1-\varrho)\varphi_2)-\Phi(\varphi_2))^{\varrho}]$$
$$-[\Phi(\varrho\varphi_2+(1-\varrho)\varphi_1)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((\varphi_2)+(1-\varrho)\varphi_1)-\Phi(\varphi_1))^{\varrho}]$$
$$-[\Phi(\varrho\varphi_2)-\Phi(\varrho\varphi_1+(1-\varrho)\varphi_2)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((\varphi_2)-\Phi(\varrho\varphi_1+(1-\varrho)\varphi_2))^{\varrho}]$$
$$+[\Phi(\varrho\varphi_2)-\Phi(\varrho\varphi_2+(1-\varrho)\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((\varphi_2)-\Phi(\varrho\varphi_2+(1-\varrho)\varphi_1))^{\varrho}].$$

Proof. Since from relation (5), we have

$$\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+,\delta}F(\varphi_2)=\int_0^1\frac{(\varphi_2-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}$$
$$\times\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]F(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho,$$

Applying integration by parts, we have

$$\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+,\delta}F(\varphi_2)=\mathfrak{K}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]F(\varphi_1)$$
$$+\mathfrak{K}(\varphi_2-\varphi_1)\int_0^1[\Phi(\varphi_2)-\Phi((1-s)\varphi_1+s\varphi_2)]^{\frac{\Upsilon}{\mathfrak{K}}}\tag{24}$$
$$\times\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]F'(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho.$$

Similarly, from relation (6) we have

$$\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-,\delta}F(\varphi_1)=\int_0^1\frac{(\varphi_2-\varphi_1)\Phi'((1-\varrho)\varphi_1+\varrho\varphi_2)}{[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{1-\frac{\Upsilon}{\mathfrak{K}}}}$$
$$\times\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]F(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho,$$

Applying integration by parts, we have

$$\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-,\delta}F(\varphi_1)=\mathfrak{K}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]F(\varphi_2)$$
$$+\mathfrak{K}(\varphi_2-\varphi_1)\int_0^1[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\tag{25}$$
$$\times\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]F'(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho.$$

On employing the relation $F(\nu)=f(\nu)+f(\varphi_1+\varphi_2-\nu)$ and adding (24), (25), we have

$$\begin{aligned}
&\frac{4(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+k}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}{(\varphi_2-\varphi_1)}\Bigg[\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}\\
&-\frac{1}{4\mathfrak{K}(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\Big(\xi^{p,k,\Phi}_{\varrho,\Upsilon,\varphi_2^-,\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\varphi_1^+,\delta}F(\varphi_2)\Big)\Bigg]\\
&=\int_0^1[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\\
&\qquad\times F'(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho\\
&-\int_0^1[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\\
&\qquad\times F'(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho.
\end{aligned}\tag{26}$$

Also $F'(\nu)=f'(\nu)-f'(\varphi_1+\varphi_2-\nu)$, so we have
$F'(\varrho\varphi_2+(1-\varrho)\varphi_1)=f'(\varrho\varphi_2+(1-\varrho)\varphi_1)-f'(\varrho\varphi_1+(1-\varrho)\varphi_2)$,
Using this relation, we get

$$\begin{aligned}
&\int_0^1[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\\
&\qquad\times F'(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho\\
&=\int_0^1[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\\
&\qquad\times\zeta'(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho\\
&-\int_0^1[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\\
&\qquad\times\zeta'(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho,\\
&=\int_0^1[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((1-\varrho)\varphi_2+\varrho\varphi_1)-\Phi(\varphi_1))^{\varrho}]\\
&\qquad\times\zeta'(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho\\
&-\int_0^1[\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)-\Phi(\varphi_1))^{\varrho}]\\
&\qquad\times\zeta'(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho.
\end{aligned}\tag{27}$$

and

$$\begin{aligned}
&\int_0^1[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\\
&\qquad\times F'(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho\\
&=\int_0^1[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\\
&\qquad\times\zeta'(\varrho\varphi_2+(1-\varrho)\varphi_1)d\varrho
\end{aligned}\tag{28}$$

$$\begin{aligned}
&-\int_0^1[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\\
&\qquad\times\zeta'(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho,\\
&=\int_0^1[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_2+\varrho\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_2+\varrho\varphi_1))^{\varrho}]\\
&\qquad\times\zeta'(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho\\
&-\int_0^1[\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_1{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi((1-\varrho)\varphi_1+\varrho\varphi_2))^{\varrho}]\\
&\qquad\times\zeta'(\varrho\varphi_1+(1-\varrho)\varphi_2)d\varrho.
\end{aligned}\tag{29}$$

On substituting the values (27), (28) into (26), we get the required result. □

Theorem 4.2. *Suppose $\Phi : [\varphi_1, \varphi_2] \to \mathbb{R}$ is a positive monotone and increasing function on $(\varphi_1, \varphi_2]$, containing a continuous derivative $\Phi'(x)$ on (φ_1, φ_2). If $\zeta : [\varphi_1, \varphi_2] \to \mathbb{R}$ be a differentiable mapping on (φ_1, φ_2) with $\varphi_1 < \varphi_2$ and $|\zeta'|$ is a convex function on $[\varphi_1, \varphi_2]$ then the following inequality hold:*

$$\left|\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2} - \frac{1}{4\mathfrak{K}[\Phi(\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]} \times \left[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1) + \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\right]\right|$$
$$\leq \frac{I^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}}{4(\varphi_2-\varphi_1)(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\left[|\zeta'(\varphi_1)|+|\zeta'(\varphi_2)|\right].$$

Where
$I^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi} = L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_2) + L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_1,\varphi_2) - L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_1) - L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_1,\varphi_1)$,

Also, $L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}$ is defined as-

$$L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\nu_1,\nu_2) = \int_0^{\frac{\varphi_1+\varphi_2}{2}} |\nu_1-u||\Phi(\nu_2)-\Phi(u)|^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\nu_2)-\Phi(u))^{\varrho}]du$$
$$- \int_{\frac{\varphi_1+\varphi_2}{2}}^{\varphi_2} |\nu_1-u||\Phi(\nu_2)-\Phi(u)|^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\nu_2)-\Phi(u))^{\varrho}]du.$$

Proof. By using convexity of $|\zeta'|$ and Lemma 4.1, we obtain

$$\left|\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2} - \frac{1}{4\mathfrak{K}[\Phi(\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]} \times \left[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1) + \xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\right]\right|$$
$$\leq \frac{I^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}}{4(\varphi_2-\varphi_1)(\Phi(\varphi_2)-\Phi(\varphi_1))^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\int_0^1 |\chi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varrho)| \times |\zeta'(\varrho\varphi_1+(1-\varrho)\varphi_2)|d\varrho$$

$$\leq \frac{I^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}}{4(\varphi_2-\varphi_1)\Phi(\varphi_2)-\Phi(\varphi_1)^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\Bigg[|\zeta'(\varphi_1)|$$
$$\times \int_0^1 \varrho|\chi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varrho)|d\varrho + |\zeta'(\varphi_2)|\int_0^1(1-\varrho)|\chi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varrho)|d\varrho\Bigg]. \quad (30)$$

Here, we have

$$\int_0^1 \varrho|\chi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varrho)|d\varrho = \frac{1}{(\varphi_2-\varphi_1)^2}\int_{\varphi_1}^{\varphi_2}(\varphi_2-u)|\phi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(u)|dt,$$

where,

$$\phi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(u) = [\Phi(u)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}] - [\Phi(\varphi_1+\varphi_2-u)$$
$$-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_1+\varphi_2-u)-\Phi(\varphi_1))^{\varrho}] - [\Phi(\varphi_2)-\Phi(u)]^{\frac{\Upsilon}{\mathfrak{K}}}$$
$$\times {}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}] + [\Phi(\varphi_1+\varphi_2-u)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}$$
$$\times {}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1+\varphi_2-u))^{\varrho}].$$

As Φ is increasing and $\phi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}$ is non decreasing function respectively on $[\varphi_1, \varphi_2]$.

Also, $\phi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_1) = -2[\Phi(\varphi_2)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}] < 0$, and $\phi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\frac{\varphi_1+\varphi_2}{2}) = 0$.

Consequently, we get

$\phi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(u) \leq 0,\ if\ \ a \leq u \leq \frac{\varphi_1+\varphi_2}{2}$, and $\phi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(u) > 0,\ if\ \ \frac{\varphi_1+\varphi_2}{2} \leq u \leq \varphi_2$.

Therefore, we have

$$(\varphi_2-\varphi_1)^2\int_0^1 \varrho|\chi^{p,\mathfrak{K}}_{\varrho,\Upsilon,\Phi}(\varrho)|d\varrho = \int_{\varphi_1}^{\varphi_2}(\varphi_2-u)|\phi^{p,\mathfrak{K}}_{\varrho,\Upsilon,\Phi}|dt = I_1+I_2+I_3+I_4,$$

where

$$I_1 = \int_{\varphi_1}^{\frac{\varphi_1+\varphi_2}{2}}(\varphi_2-u)[\Phi(\varphi_2)-\Phi(u)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(u))^{\varrho}]du$$
$$-\int_{\frac{\varphi_1+\varphi_2}{2}}^{\varphi_2}(\varphi_2-u)[\Phi(\varphi_2)-\Phi(u)]^{\frac{\Upsilon}{\mathfrak{K}}}\,{}_2F_{1}{}^{(p,k)}_{\varrho,\Upsilon+\mathfrak{K}}[\delta(\Phi(\varphi_2)-\Phi(u))^{\varrho}]du = L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_2),$$

$$I_2=\int_{\varphi_1}^{\frac{\varphi_1+\varphi_2}{2}}(\varphi_2-u)[\Phi(u)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(u)-\Phi(\mu))^{\varrho}]du$$
$$+\int_{\frac{\varphi_1+\varphi_2}{2}}^{\varphi_2}(\varphi_2-u)[\Phi(u)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(u)-\Phi(\varphi_1))^{\varrho}]du=-L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_1),$$
$$I_3=\int_{\varphi_1}^{\frac{\varphi_1+\varphi_2}{2}}(\varphi_2-u)[\Phi(\varphi_1+\varphi_2-u)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_1+\varphi_2-u)-\Phi(\varphi_1))^{\varrho}]du$$
$$-\int_{\frac{\varphi_1+\varphi_2}{2}}^{\varphi_2}(\varphi_2-u)[g(\varphi_1+\varphi_2-u)-\Phi(\varphi_1)]^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_1+\varphi_2-u)-\Phi(\varphi_1))^{\varrho}]du$$
$$=-L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_1,\varphi_1),$$
$$I_4=\int_{\varphi_1}^{\frac{\varphi_1+\varphi_2}{2}}(\varphi_2-u)[\Phi(\varphi_2)-\Phi(\varphi_1+\varphi_2-u)]^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1+\varphi_2-u))^{\varrho}]du$$
$$+\int_{\frac{\varphi_1+\varphi_2}{2}}^{\varphi_2}(\varphi_2-u)[\Phi(\varphi_2)-\Phi(\varphi_1+\varphi_2-u)]^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1+\varphi_2-u))^{\varrho}]du$$
$$=-L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_1,\varphi_2).$$

Thus from the previous equalities it follow that

$$\int_0^1\varrho|\chi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varrho)|d\varrho=\frac{L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_2)+L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_1,\varphi_2)-L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_1)-L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_1)}{(\varphi_2-\varphi_1)^2},\tag{31}$$

Similarly, it is clear that

$$\int_0^1(1-\varrho)|\chi^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varrho)|d\varrho=\frac{L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},g\Phi}(\varphi_2,\varphi_2)+L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_1,\varphi_2)-L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_1)-L^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}(\varphi_2,\varphi_1)}{(\varphi_2-\varphi_1)^2}.\tag{32}$$

We get the intended results if equalities (31) and (32) are substituted in (30). □

Corollary 4.3. *Whenever we choose k=1, Theorem 4.2 yields Â the following inequality for a generalised fractional integrals:*

$$\left|\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}-\frac{1}{4[\Phi(\varphi_2)-\Phi(\varphi_1)]^{\Upsilon}{}_2F_{1\,\varrho,\Upsilon+1}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\left[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\varphi_1^+;\delta}F(\varphi_2)\right]\right|$$
$$\leq\frac{I^{p,k,}_{\varrho,\Upsilon,\Phi}}{4(\varphi_2-\varphi_1)(\Phi(\varphi_2)-\Phi(\varphi_1))^{\Upsilon}{}_2F_{1\,\varrho,\Upsilon+1}^{(p,k)}[\delta(\Phi(\varphi_2)-\Phi(\varphi_1))^{\varrho}]}\Big[|\zeta'(\varphi_1)|+|\zeta'(\varphi_2)|\Big].$$

Corollary 4.4. *If we choose $\Phi(t)=t$ in Theorem 4.2, then we obtain following inequality for generalized k-fractional integrals:*

$$\left|\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}-\frac{1}{4\mathfrak{K}[\varphi_2-\varphi_1]^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\varphi_2-\varphi_1)^{\varrho}]}\left[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\right]\right|$$
$$\leq\frac{I^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}}{4(\varphi_2-\varphi_1)^{\frac{\Upsilon}{\mathfrak{K}}+1}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\varphi_2-\varphi_1)^{\varrho}]}\Big[|\zeta'(\varphi_1)|+|\zeta'(\varphi_2)|\Big].$$

Corollary 4.5. *If we choose $\Phi(t)=\ln t$ in Theorem 4.2, then we obtain following inequality for generalized k-fractional integrals:*

$$\left|\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}-\frac{1}{4\mathfrak{K}(ln\frac{\varphi_2}{\varphi_1})^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(ln\frac{\varphi_2}{\varphi_1})^{\varrho}]}\left[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\right]\right|$$
$$\leq\frac{I^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}}{4(\varphi_2-\varphi_1)^{\frac{\Upsilon}{\mathfrak{K}}+1}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\varphi_2-\varphi_1)^{\varrho}]}\Big[|\zeta'(\varphi_1)|+|\zeta'(\varphi_2)|\Big].$$

Corollary 4.6. *If we choose $\Phi(t)=\frac{t^{s+1}}{s+1}$ in Theorem 4.2, then we obtain the following inequality for the generalized (k,s)-fractional integral:*

$$\left|\frac{\zeta(\varphi_1)+\zeta(\varphi_2)}{2}-\frac{(s+1)^{\frac{\Upsilon}{\mathfrak{K}}}}{4\mathfrak{K}[\varphi_2^{s+1}-\varphi_1^{s+1}]^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\frac{\varphi_2^{s+1}-\varphi_1^{s+1}}{s+1})]}\left[\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_2^-;\delta}F(\varphi_1)+\xi^{p,k,\Phi}_{\varrho,\Upsilon,\mathfrak{K},\varphi_1^+;\delta}F(\varphi_2)\right]\right|$$
$$\leq\frac{(s+1)^{\frac{\Upsilon}{\mathfrak{K}}}I^{p,k}_{\varrho,\Upsilon,\mathfrak{K},\Phi}}{4(\varphi_2-\varphi_1)(\varphi_2^{s+1}-\varphi_1^{s+1})^{\frac{\Upsilon}{\mathfrak{K}}}{}_2F_{1\,\varrho,\Upsilon+\mathfrak{K}}^{(p,k)}[\delta(\frac{\varphi_2^{s+1}-\varphi_1^{s+1}}{s+1})]}\Big[|\zeta'(\varphi_1)|+|\zeta'(\varphi_2)|\Big].$$

5 Conclusion

With the use of an extended hypergeometric function, a new fractional operator has been defined in this article. The use of this fractional operator leads to the development of certain new theorems and inequalities. Several corollaries are also found when certain values of the parameters employed in this fractional operator are applied. Future applications of this newly generalised fractional operator include the ability to develop various inequalities.

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