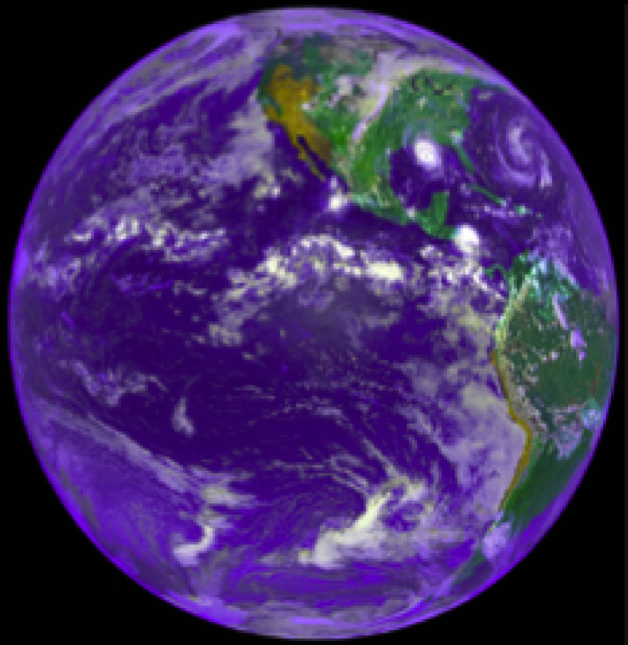
2013 Dolomites Research Week on Approximation

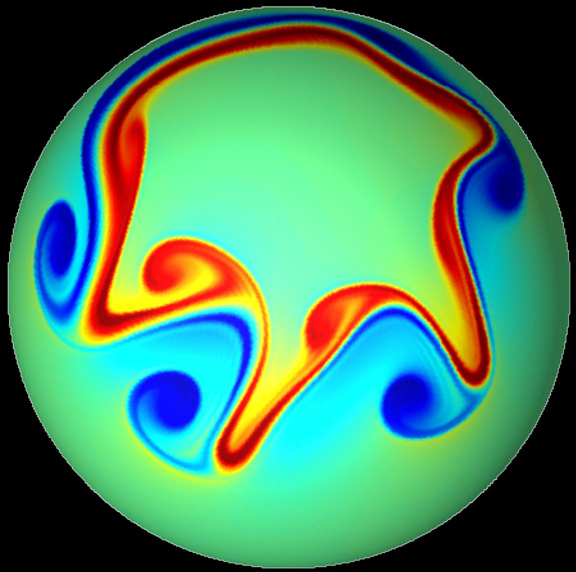


Lecture 5&6: Kernel methods for modeling geophysical fluids: shallow water equations on a sphere and mantle convection in a spherical shell.

Grady B. Wright

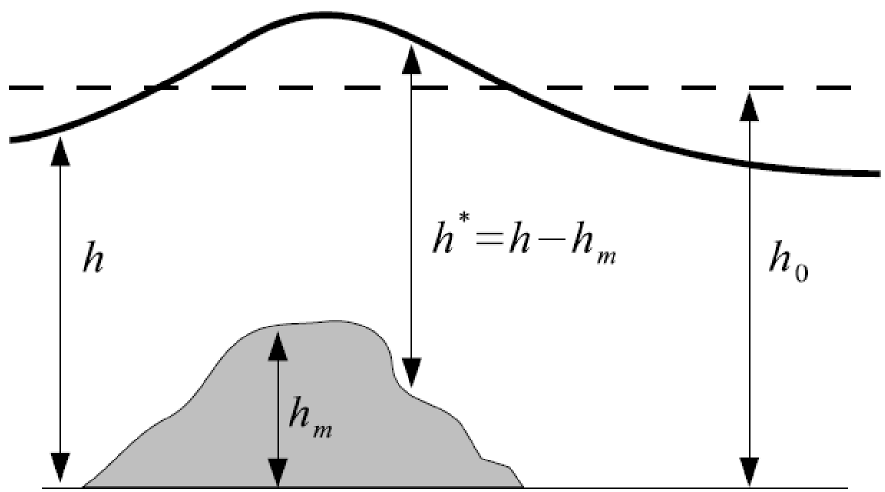
Boise State University

Shallow water wave equations on a rotating sphere



Shallow water equations (SWE) on a rotating sphere

- Model for the nonlinear dynamics of a shallow, hydrostatic, homogeneous, and inviscid fluid layer.



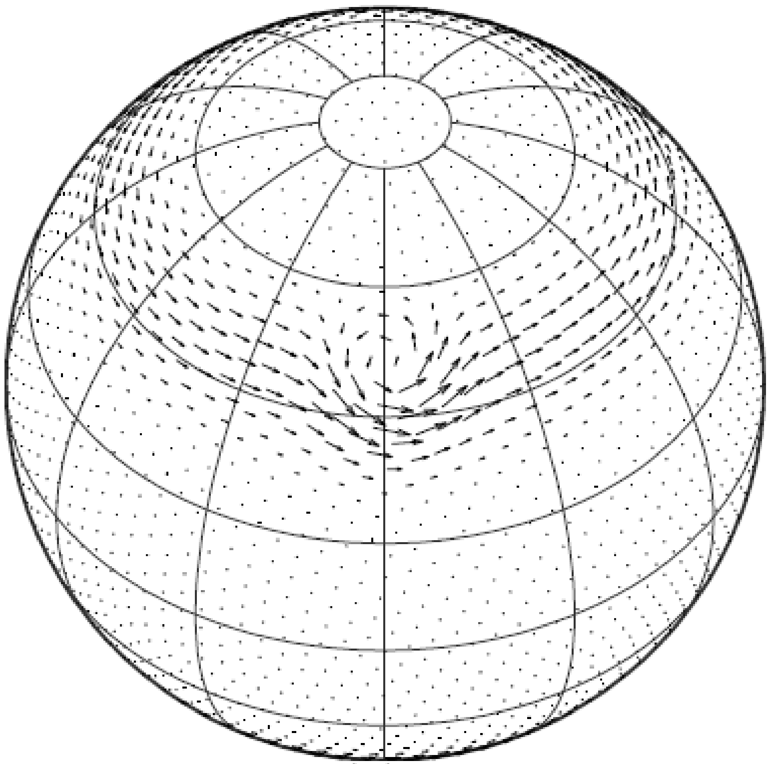
- Idealized test-bed for the horizontal dynamics of all 3-D global climate models.

Equations	Momentum	Transport
Spherical coordinates	$\frac{\partial \mathbf{u}_s}{\partial t} + \mathbf{u}_s \cdot \nabla_s \mathbf{u}_s + f\hat{\mathbf{k}} \times \mathbf{u}_s + g\nabla_s h = 0$	$\frac{\partial h^*}{\partial t} + \nabla_s \cdot (h^* \mathbf{u}_s) = 0$
	Singularity at poles!	
Cartesian coordinates	$\frac{\partial \mathbf{u}_c}{\partial t} + P \begin{bmatrix} (\mathbf{u}_c \cdot P\nabla_c)u_c + f(\mathbf{x} \times \mathbf{u}_c) \cdot \hat{\mathbf{i}} + g(P\hat{\mathbf{i}} \cdot \nabla_c)h \\ (\mathbf{u}_c \cdot P\nabla_c)v_c + f(\mathbf{x} \times \mathbf{u}_c) \cdot \hat{\mathbf{j}} + g(P\hat{\mathbf{j}} \cdot \nabla_c)h \\ (\mathbf{u}_c \cdot P\nabla_c)w_c + f(\mathbf{x} \times \mathbf{u}_c) \cdot \hat{\mathbf{k}} + g(P\hat{\mathbf{k}} \cdot \nabla_c)h \end{bmatrix} = 0$	$\frac{\partial h^*}{\partial t} + (P\nabla_c) \cdot (h^* \mathbf{u}_c) = 0$
	Smooth over entire sphere!	

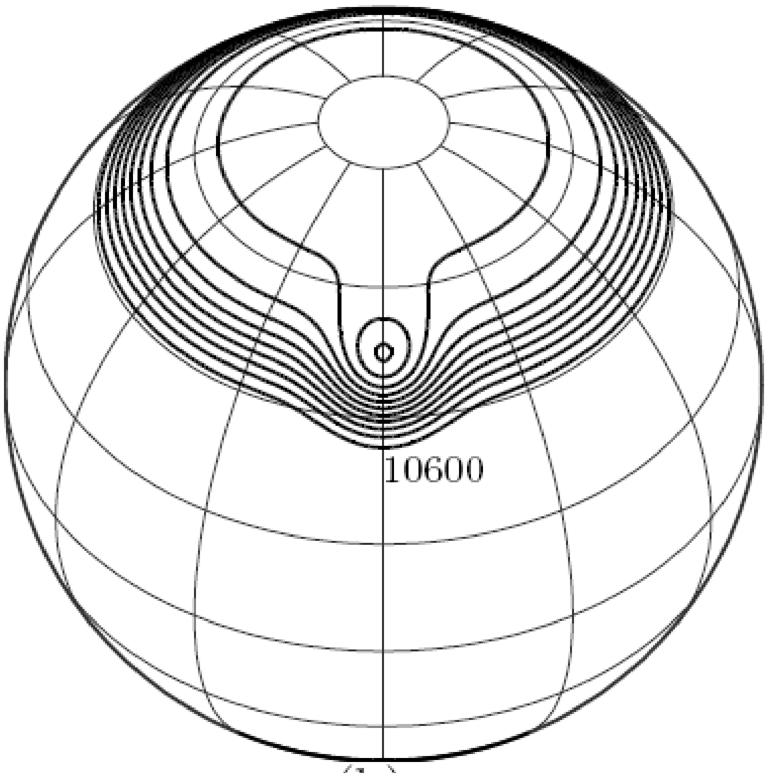
Numerical Example I: Global RBF collocation method

Forcing terms added to the shallow water equations to generate a flow that mimics a short wave trough embedded in a westerly jet.
(Test case 4 of Williamson *et. al.* 1992)

Initial velocity field



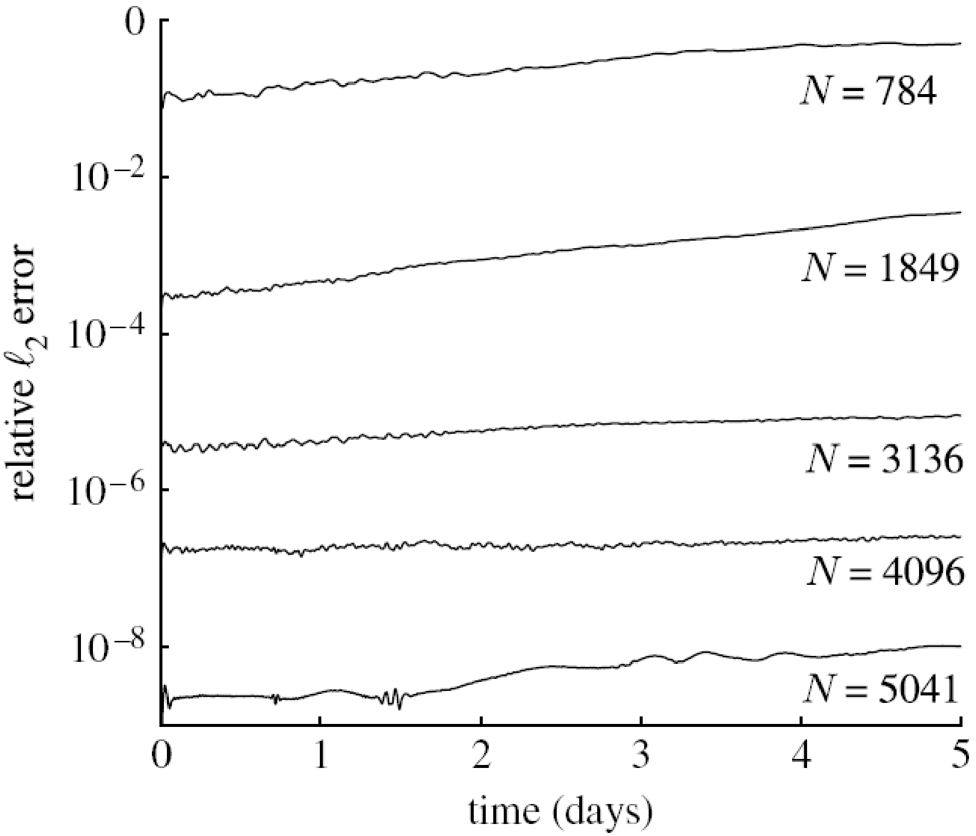
Initial geopotential height field



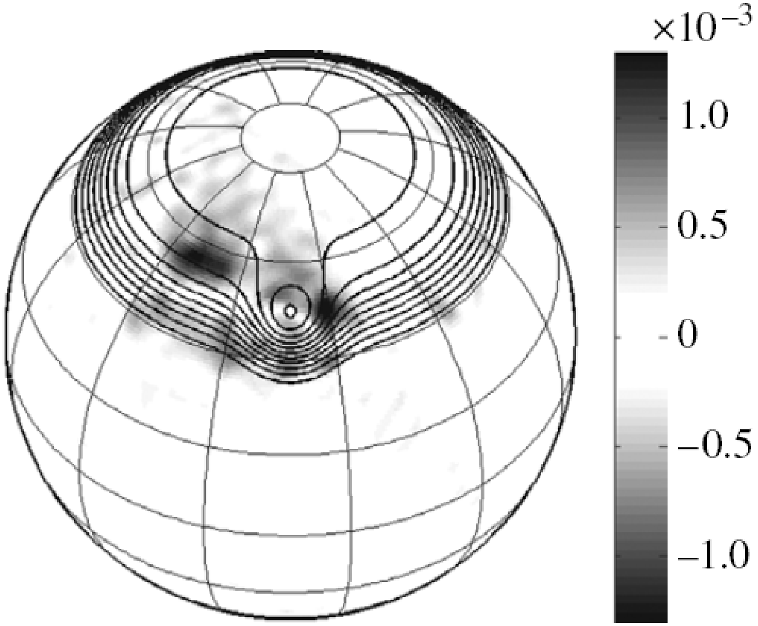
Errors after trough travels once around the sphere

- Results of the RBF Shallow Water Model:
  (Flyer & W, 2009)

Error as a function of time and N



Error height field, $t = 5$ days



N = 3136, white < 10^{-5}
Error (exact - numerical)

Comparison with commonly used methods

Method	N	Time step	Relative ℓ_2 error
RBF	4,096	8 minutes	2.5×10^{-6}
	5,041	6 minutes	1.0×10^{-8}
Sph. Harmonic	8,192	3 minutes	2.0×10^{-3}
Double Fourier	32,768	90 seconds	4.0×10^{-4}
Spect. Element	24,576	45 seconds	4.0×10^{-5}

Time-step for RBF method: Temporal Errors = Spatial Errors
Time-step for other methods: Limited by numerical stability

- RBF method runtime in MATLAB using 2.66 GHz Xeon Processor

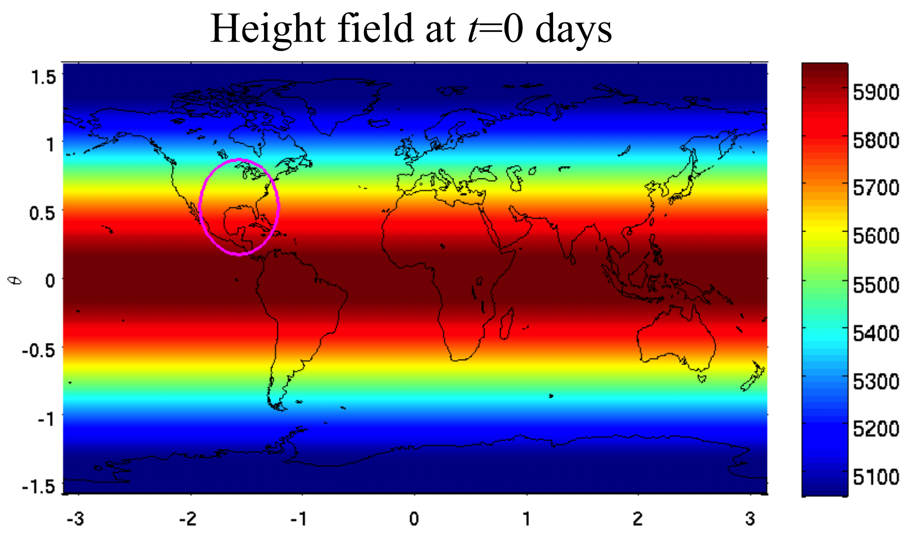
N	Runtime per time step (sec)	Total Runtime
4,096	0.41	6 minutes
5,041	0.60	12 minutes

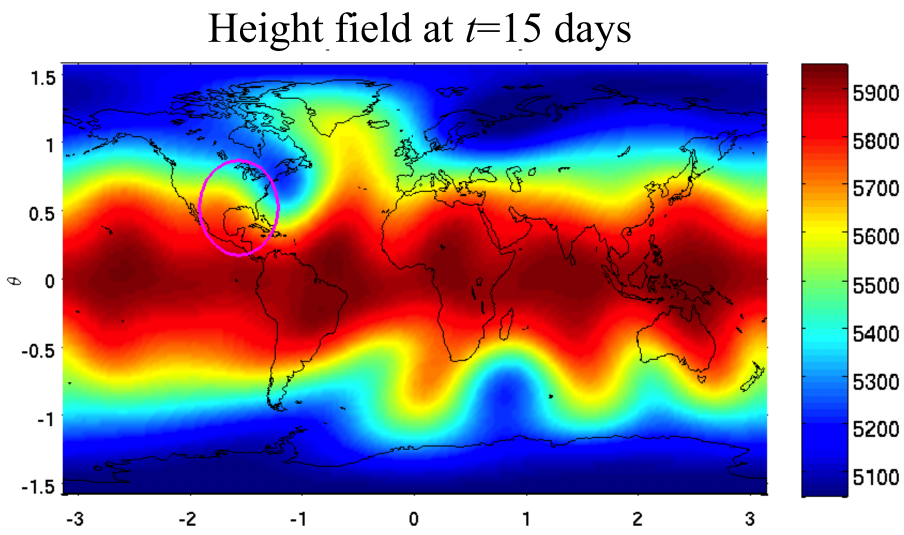
For much higher numerical accuracy, RBFs uses less nodes & larger time steps

Numerical Example II: RBF-FD method

(Flyer, Lehto, Blaise, Wright, and St-Cyr. 2012)

Flow over a conical mountain (Test case 5 of Williamson *et. al.* 1992)



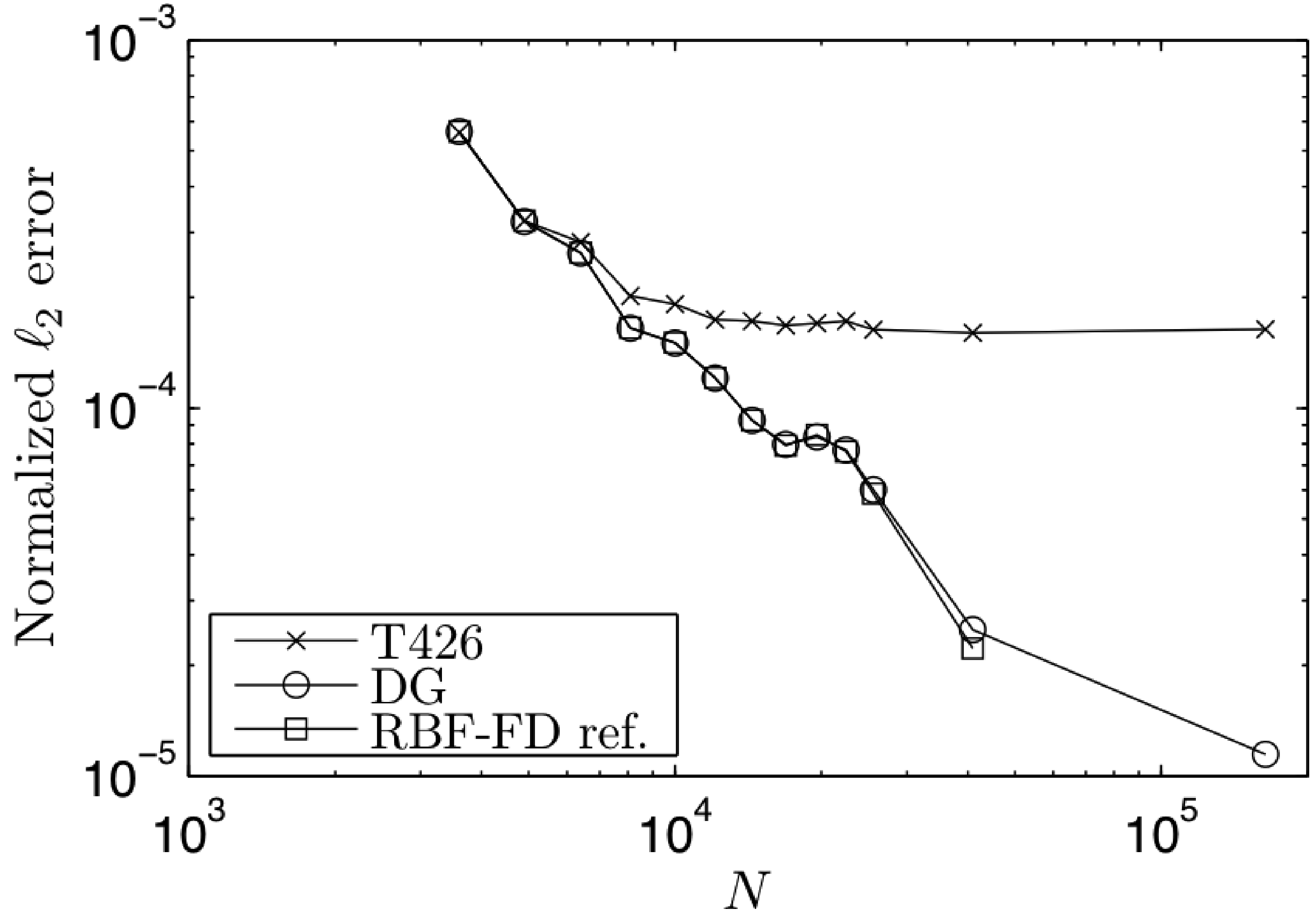


Remarks:

- The mountain is only continuous, not differentiable.
- No analytical solution.
- Comparisons in numerical solutions are done against some reference numerical solutions at a high resolution.

Convergence comparison: 3 reference solutions

Convergence plot RBF-FD with stencil size of m=31

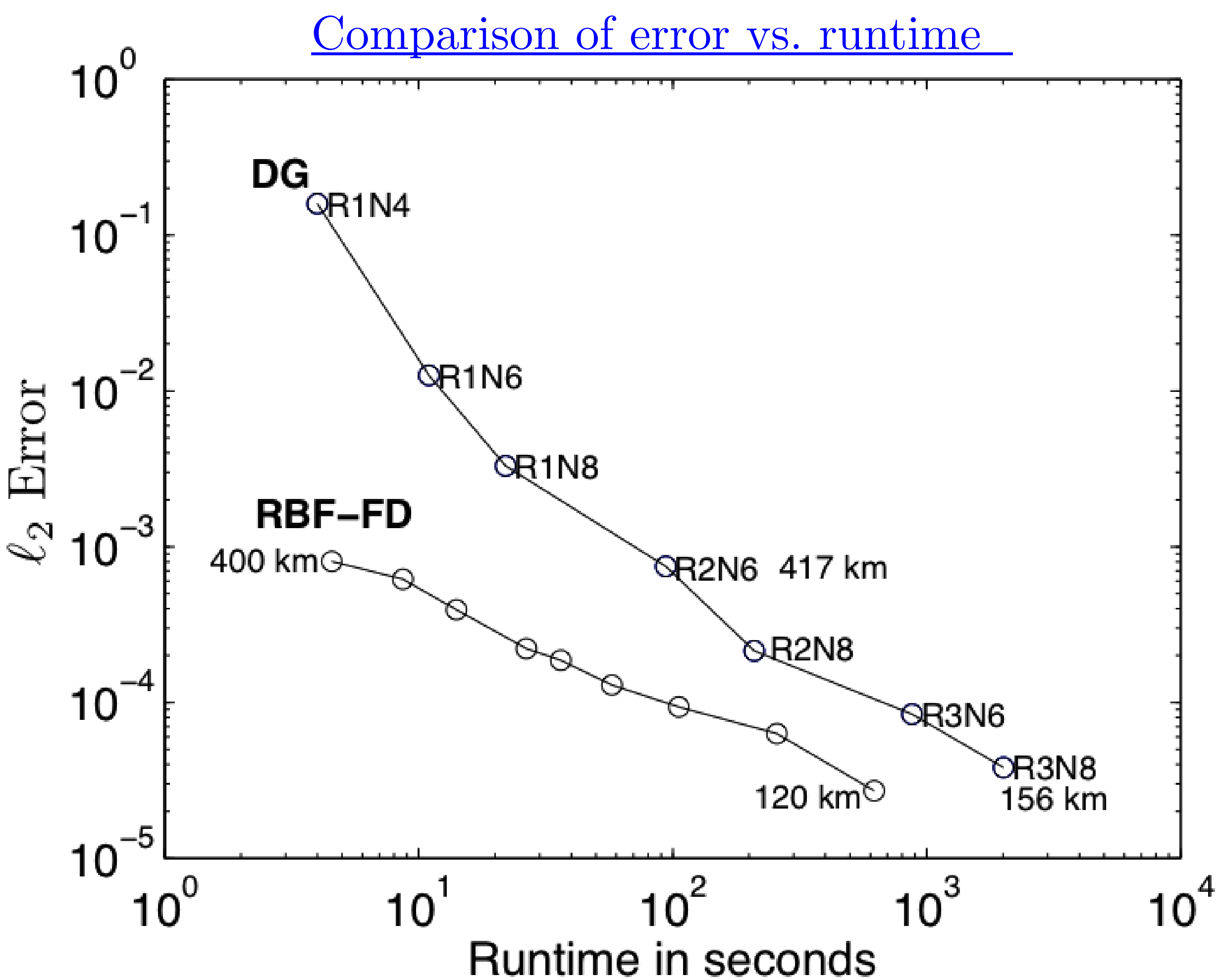


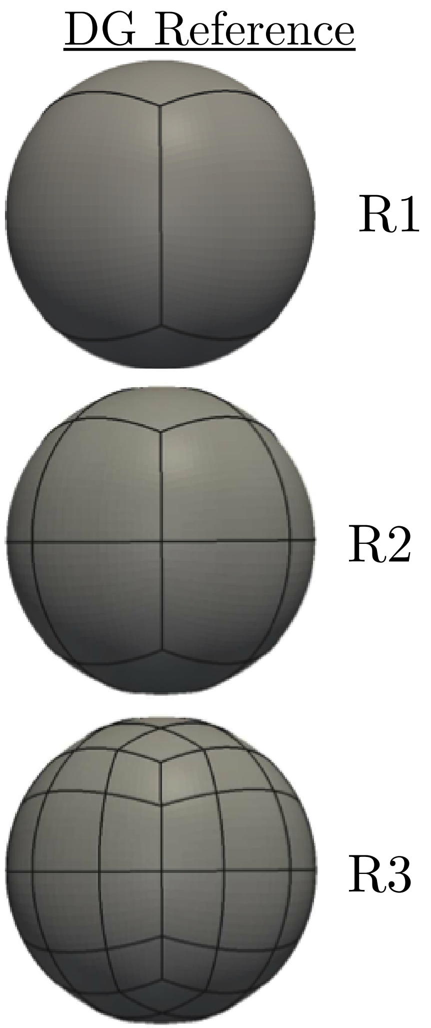
× Standard Literature/Comparison: NCAR's Sph. Har. T426, Resolution ≈ 30 km at equator

○ New Model at NCAR Discontinuous Galerkin – Spectral Element, Resolution ≈ 30 km

□ RBF-FD model, Resolution ≈ 60 km

Error vs. runtime comparison





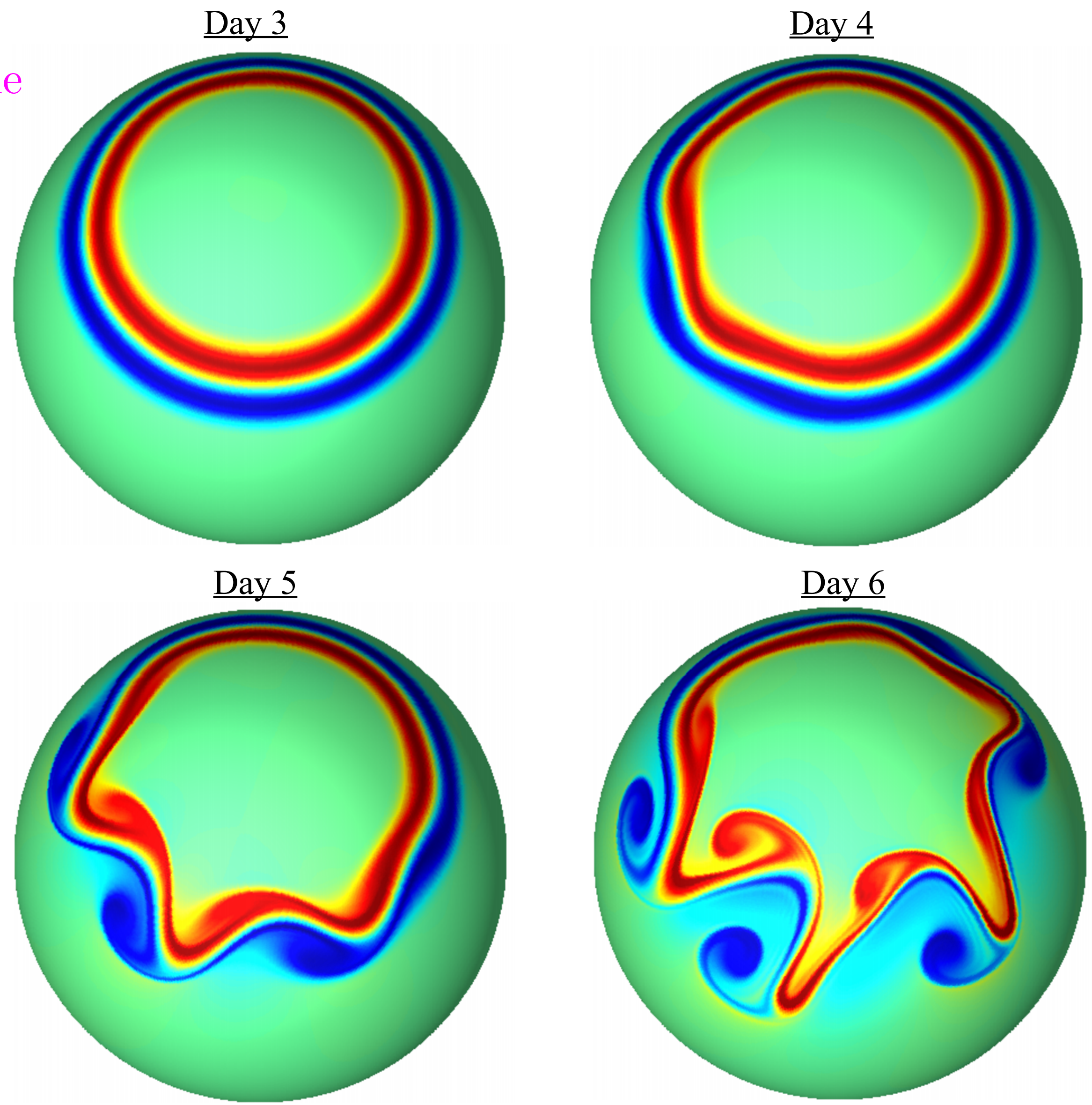
Machine: MacBook Pro, Intel i7 2.2 GHz, 8 GB Memory

- Further improvements for both methods may be possible using local mesh/node refinement near the mountain.

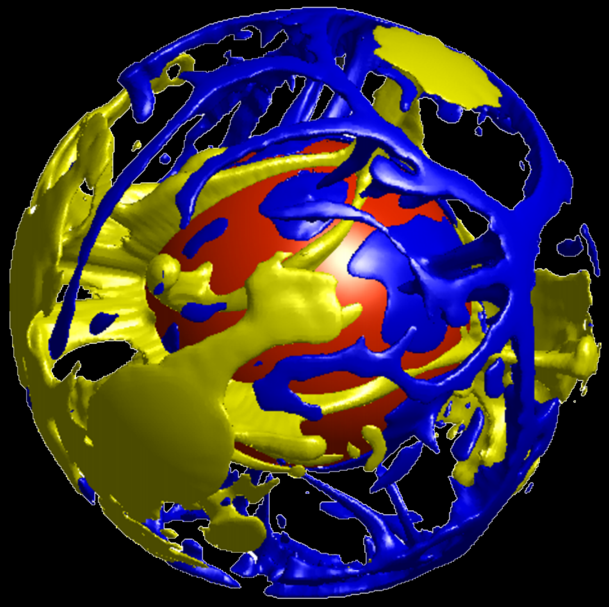
Numerical Example III: RBF-FD method

- Evolution of a highly non-linear wave: (Test case from Galewsky et. al. *Tellus*, 2004)
  Rapid cascade of energy from large to small scales resulting in sharp vorticity gradients
- RBF-FD method with N=163,842 nodes and m=31 point stencil.

Visualization of the relative vorticity



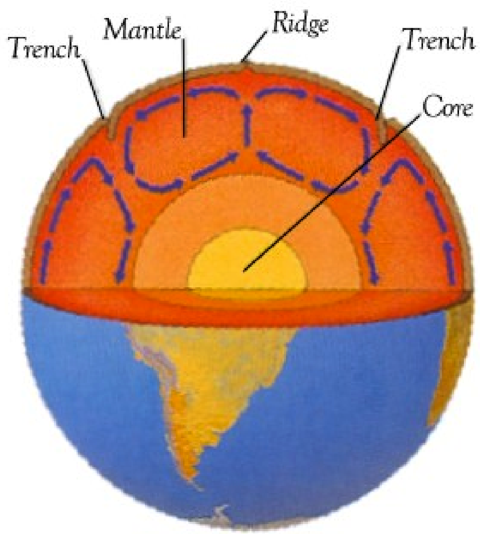
Thermal convection in a 3D spherical shell with applications to the Earth's mantle.



Simulating convection in the Earth's mantle

(Wright, Flyer, and Yuen. *Geochem. Geophys. Geosyst.*, 2010)

- Model assumptions:
  1. Fluid is incompressible
  2. Viscosity of the fluid is constant
  3. Boussinesq approximation
  4. Infinite Prandtl number, $\mathrm{Pr} = \dfrac{\text{kinematic viscosity}}{\text{thermal diffusivity}} \to \infty$



- Non-dimensional Equations:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \quad \text{(continuity)}, \\ \nabla^2 \mathbf{u} + \mathrm{Ra}\, T\, \hat{\mathbf{r}} - \nabla p &= 0 \quad \text{(momentum)}, \\ \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T - \nabla^2 T &= 0 \quad \text{(energy)}. \end{aligned}$$

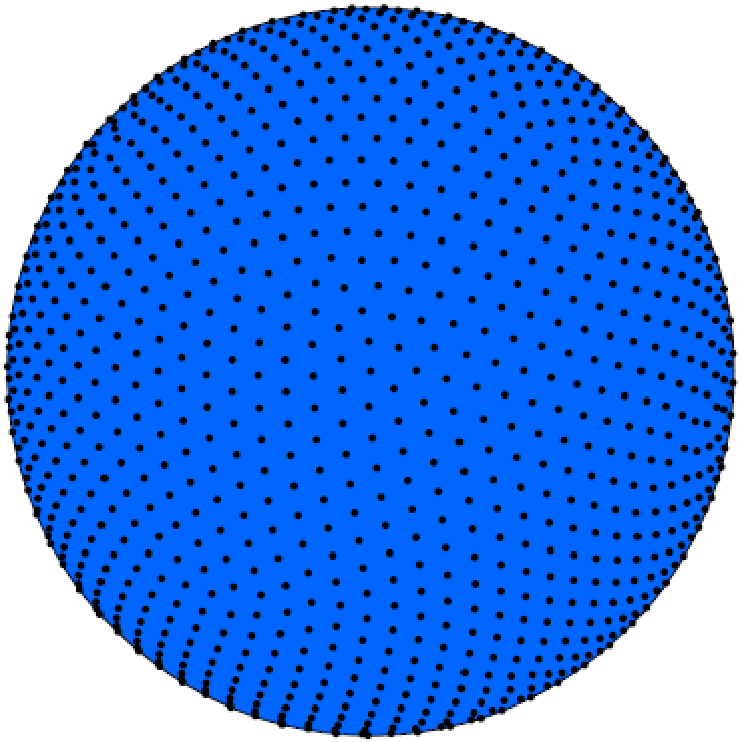
- Boundary conditions:

  Velocity: impermeable and shear-stress free
  Temperature (isothermal): $T = 1$ at core mantle bndry., $T = 0$ at crust mantle bndry.

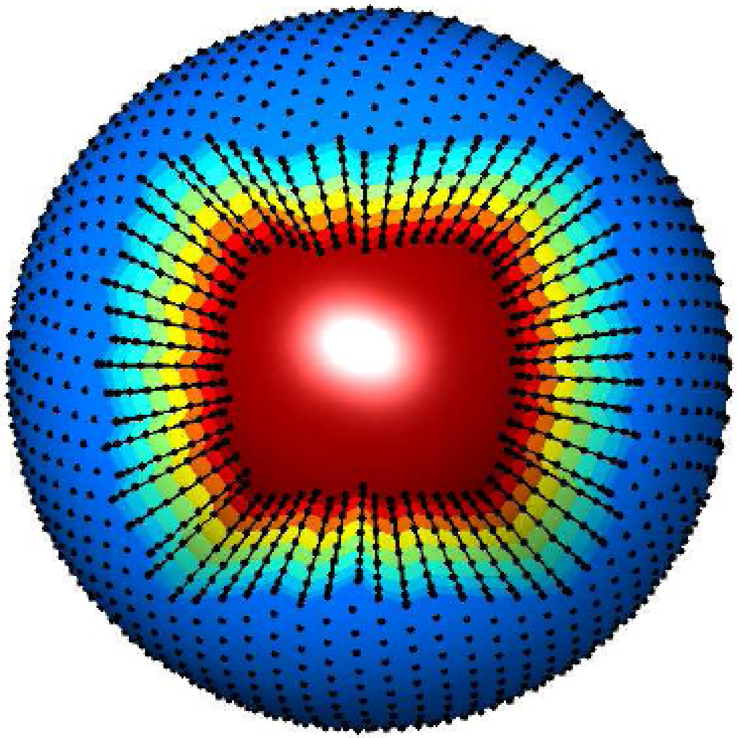
- Rayleigh, Ra, number governs the dynamics.
- Model for Rayleigh-Bénard convection

Global method for discretizing the equations

- Use a hybrid RBF-Pseudospectral method
- Collocation procedure using a 2+1 approach with
  - N RBF nodes on each spherical surface (angular directions) and
  - M Chebyshev nodes in the radial direction.



N RBF nodes (ME) on a spherical surface



3-D node layout showing *M* Chebyshev nodes in radial direction

Global method for discretizing the equations

- Rewrite the momentum equation using poloidal potential Φ:

$$\Delta_{\mathcal{S}}\Omega + \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Omega}{\partial r}\right) = \mathrm{Ra}\, r\, T$$

$$\Delta_{\mathcal{S}}\Phi + \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = r^2\Omega,$$

$$\mathbf{u} = \nabla \times \nabla \times ((\Phi r)\hat{\mathbf{r}})$$

$$\frac{\partial T}{\partial t} = -\left(u_r \frac{\partial T}{\partial r} + \mathbf{u}_{\mathcal{S}} \cdot (P\nabla T)\right) + \frac{1}{r^2}\Delta_{\mathcal{S}} T + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$$

- We have seen how to create a discrete representation for $P\nabla$ using RBFs.
- Need a method to create a discrete representation of $\Delta_{\mathcal{S}}$:
- A similar procedure can be used to $P\nabla$, by noting that

$$\Delta_{\mathcal{S}}\phi(\|\mathbf{x}-\mathbf{x}_j\|) = \frac{1}{4}\Big[(4-\|\mathbf{x}-\mathbf{x}_j\|^2)\phi''(\|\mathbf{x}-\mathbf{x}_j\|) + \frac{4-3\|\mathbf{x}-\mathbf{x}_j\|^2}{\|\mathbf{x}-\mathbf{x}_j\|}\phi'(\|\mathbf{x}-\mathbf{x}_j\|)\Big]$$

Steps of computational algorithm

$$\Delta_{\mathcal{S}}\Omega + \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Omega}{\partial r}\right) = \mathrm{Ra}\, r\, T$$

$$\Delta_{\mathcal{S}}\Phi + \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = r^2 \Omega,$$

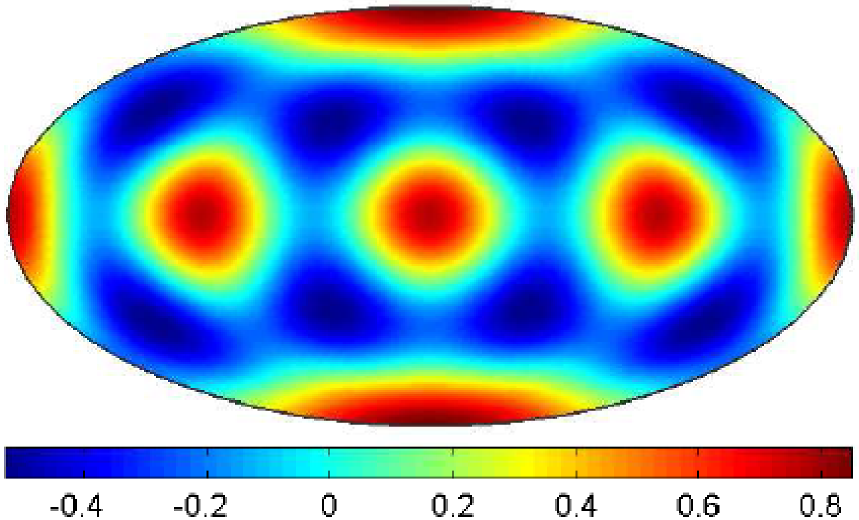
$$\mathbf{u} = \nabla \times \nabla \times ((\Phi r)\hat{\mathbf{r}})$$

$$\frac{\partial T}{\partial t} = -\left(u_r \frac{\partial T}{\partial r} + \mathbf{u}_{\mathcal{S}} \cdot (P\nabla T)\right) + \frac{1}{r^2}\Delta_{\mathcal{S}} T + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)$$

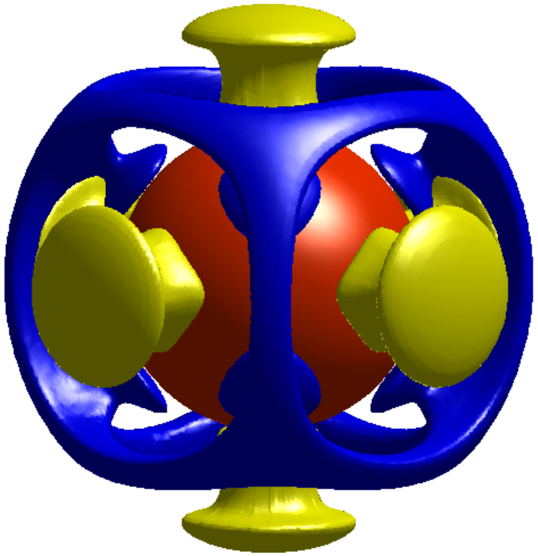
1. Discretize $\Delta_{\mathcal{S}}$ and $P\nabla$ for the unit sphere using N RBFs.
2. Discretize $\frac{\partial}{\partial r}$ and $\frac{\partial^2}{\partial r^2}$ using M Chebyshev polynomials.
3. Use T initial condition to solve for Ω.
4. Use Ω solution to solve for Φ.
5. Use Φ to compute the velocity $\mathbf{u}$
6. Discretize energy equation in time using an implicit/explicit scheme
   (a) Use trapezoidal rule for diffusion operator.
   (b) Use 3$^{\text{rd}}$ order Adams-Bashforth for the advection operator.
7. Time-step the energy equation to get a new T, go back to step 3

Ra=7000 benchmark: validation of method

Perturbation initial condition: $0.01\left[Y_4^0(\theta,\lambda)+\frac{5}{7}Y_4^4(\theta,\lambda)\right]$



Steady solution:



N = 1600 nodes on each spherical shell
M = 23 shells
Blue=downwelling, Yellow= upwelling, Red=core

- Comparisons against main previous results from the literature:

Method	No of nodes	Nu_{outer}	Nu_{innner}	$<V_{RMS}>$	$<T>$
Finite volume	663,552	3.5983	3.5984	31.0226	0.21594
Finite elements (CitCom)	393,216	3.6254	3.6016	31.09	0.2176
Finite differences (Japan)	12,582,912	3.6083		31.0741	0.21639
Spherical harmonics -FD	552,960	3.6086		31.0765	0.21582
Spherical harmonics -FD	Extrapolated	3.6096		31.0821	0.21577
RBF-Chebyshev	36,800	3.6096	3.6096	31.0820	0.21578

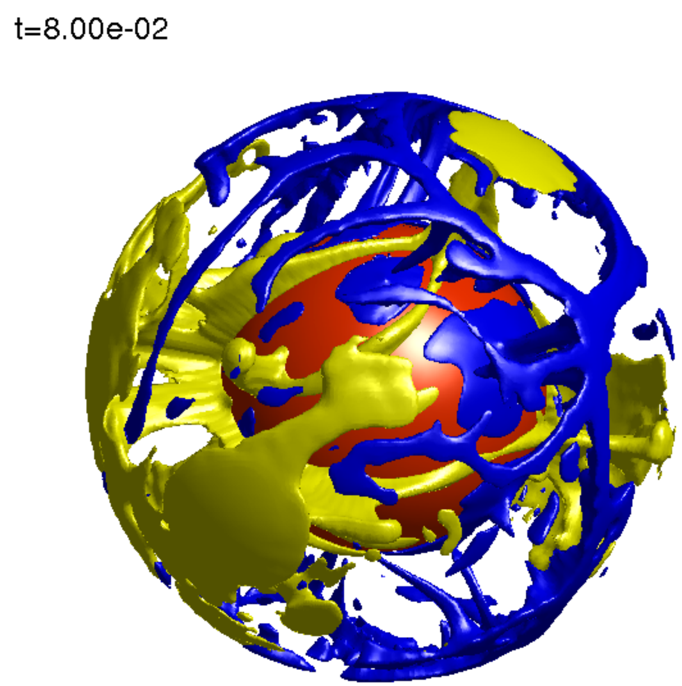
Nu = ratio of convective to conductive heat transfer across a boundary

Fully convective simulation: Ra=10^6

Model setup:

- Convection dominated flow
- N = 6561 RBF nodes, M = 81 Chebyshev nodes
- Time-step $O(10^{-7})$, which is about 34,000 years
- Simulation time to t=0.08 (4.5 times the age of the earth)

Results:



G. B. Wright, N. Flyer, and D. A. Yuen, 2010

Blue=downwelling,
Yellow= upwelling,
Red=core

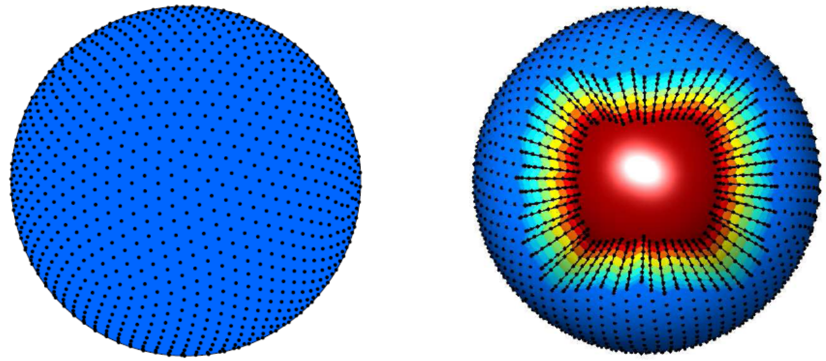
Simulation:

Current focus

- Improving computational efficiency using RBF-FD.

- First step is to do RBF-FD on each spherical surface instead of global RBFs.

Flyer, W, & Fornberg (2013)



- Ultimate goal is to go to fully 3D RBF-FD formulas (no tensor-product structure):