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Advanced Insights into Two-Dimensional Bell Polynomials through Convolution Structures with Appell Sequences

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Abstract

This study introduces $2D$ Bell polynomials alongside Appell polynomials, examining their properties through generating functions. We delve into various characteristics of these polynomials, such as explicit forms, summation formulae, recurrence relations, and addition formulas. Additionally, we introduce the $2D$ Bell-Appell-based Stirling polynomials of the second kind, detailing their associated outcomes. This research aims to deepen the understanding of $2D$ Bell-based Appell and $2D$ Bell-Appell-based Stirling polynomials within mathematical analysis, with significant implications for approximation techniques. The findings could inform the development of new mathematical theories and applications, enhancing methods for approximation in various contexts.

1 Introduction and preliminaries

Special polynomials offer powerful tools for approximation purposes, especially in contexts requiring high accuracy and computational efficiency. Their role in function approximation is rooted in their orthogonal properties, generating functions, and recurrence relations, making them highly suitable for constructing efficient polynomial expansions. Polynomials like Hermite, Legendre, Chebyshev, and others are particularly valuable because they allow functions to be approximated over specified intervals or domains, minimizing the overall error with respect to various norms.

Polynomials, especially families such as Bell, Laguerre, Hermite, Legendre, and Chebyshev, hold a prominent place in function approximation theory. Their utilization stems from unique properties-orthogonality, recurrence relations, and specific error-minimizing behaviors-that render them effective tools for approximating complex functions in various applied mathematical and physical contexts. In function approximation, polynomials are used to construct series expansions that can approximate target functions over specified intervals. Bell and Chebyshev polynomials, in particular, are known for their superior convergence rates and minimal error characteristics. For instance, Chebyshev polynomials are pivotal in Chebyshev series expansions, which are often preferred in practical applications for their optimal approximation properties on intervals. By minimizing the maximum error between the polynomial approximation and the target function, Chebyshev polynomials yield highly efficient representations, especially in numerical analysis and computational applications where reducing approximation errors is critical.

Further examples include “Bell polynomials", used in combinatorial mathematics and the analysis of recursive structures, where they capture properties of partitions and moments. Appell polynomials [18] provide powerful tools in the theory of special functions and modular forms due to their closed-form generating functions and recurrence relations, which allow for accurate approximations and fast convergence in series expansions.

Each of these polynomial families serves specialized functions and advances the field of approximation theory by contributing structures that improve convergence rates, enhance numerical stability, and optimize computational efficiency in practical applications. Each of these families of polynomials has its own unique properties and applications, making them important and interesting objects of study in mathematics and its applications. These polynomial families naturally appear in a wide range of scientific areas, including mathematical physics, engineering, computer science, and related disciplines; see, for example, [1, 2, 3, 4, 5].

Ongoing investigations into generalized classes of special polynomials have revealed several previously unexplored features of these mathematical objects. Researchers have uncovered new properties, interrelations, and potential applications that were not

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evident in earlier studies. Such findings deepen the theoretical understanding of special polynomials and may lead to meaningful advances in both pure and applied sciences. The importance of special polynomials within mathematics stems from their frequent role in the solution of differential equations, the theory of orthogonal polynomials, numerical analysis, and various computational problems. These polynomials are often characterized by well-defined algebraic structures and recurrence relations, which make them particularly suitable for systematic analysis. Examining these structural aspects also contributes to the development of broader algebraic frameworks. In addition, special polynomials exhibit strong links with other branches of mathematics, such as combinatorics, number theory, and mathematical analysis. These connections encourage interdisciplinary exploration and foster the formulation of new theoretical perspectives. Consequently, research on special polynomials not only enhances mathematical insight but also stimulates collaboration and innovation across diverse areas of study.

Exponential operators are emphasized as a potent tool, especially for handling differential equations. These operators often streamline the analysis of differential equations and provide an efficient method for expressing solutions. Bell polynomials [2], which arise in combinatorial mathematics, are powerful tools for expressing certain types of polynomial approximations. These polynomials are instrumental in function approximation due to their inherent combinatorial structure, which allows for an effective representation of polynomial expansions. For a given function $f(\nu_1)$, Bell polynomials can be employed to construct approximations based on its derivatives. By expressing $f(\nu_1)$ in terms of its Taylor series and using Bell polynomials, one can derive an approximation that captures the behavior of the function near a specified point. This is particularly useful when dealing with analytic functions, where the convergence properties of the series play a critical role. Further, When approximating functions using Bell polynomials, it is essential to understand the associated error bounds. The error $E_n(\nu_1)$ in approximating a function $f(\nu_1)$ by its Bell polynomial expansion can be characterized by the difference between the actual function and its polynomial representation:

$$E_n(\nu_1) = f(\nu_1) - P_n(\nu_1),$$

where $P_n(\nu_1)$ is the polynomial derived from the Bell polynomials. The error can often be expressed in terms of the $(n+1)$-th derivative of f evaluated at a point in the interval of interest, which yields a bound of the form:

$$|E_n(\nu_1)| \leq \frac{M}{(n+1)!}|\nu_1 - a|^{n+1}$$

where M is a bound on the $(n+1)$-th derivative of f in the interval containing a. This form of error estimation ensures that as n increases, the approximation becomes more accurate, demonstrating the convergence properties of Bell polynomials.

The convergence of the Bell polynomial approximation is a critical aspect that merits attention. For a well-behaved function $f(\nu_1)$, the series constructed from Bell polynomials will converge to $f(\nu_1)$ as the degree of the polynomial increases. This convergence is guaranteed under certain conditions, such as the continuity and differentiability of f. The speed of convergence can be influenced by the choice of the base point a and the nature of the function being approximated, see for example [15, 25].

In [2], it is emphasized that through a suitable change of variable, the application of an exponential operator to a function of the Bell polynomials can be analogous to the operation of a shift operator. This analogy enhances our understanding of how exponential operators can facilitate the manipulation of differential equations. By leveraging this relationship, one can derive explicit solutions to differential equations while maintaining control over the error bounds and convergence properties of the approximations involved.

Consequently, for any parameter μ, when the shift operator $\exp(\mu\partial_{\nu_1})$ is applied to any function of ν_1, it produces the following outcome:

$$\exp(\mu\partial_{\nu_1})f(\nu_1) = \sum_{n=0}^{\infty}\partial_{\nu_1}^n f(\nu_1)\frac{\mu^n}{n!} = \sum_{n=0}^{\infty} f^n(\nu_1)\frac{\mu^n}{n!} = f(\nu_1+\mu), \tag{1}$$

where $\partial_{\nu_1}^n = \frac{\partial^n}{\partial_{\nu_1}^n}$.

The following identities are derived from equation (1):

$$\exp(\mu\ \nu_1^2\partial_{\nu_1})\{f(\nu_1)\} = f\Big(\frac{\nu_1}{1-\mu\nu_1}\Big), \tag{2}$$

$$\exp(\mu\partial_{\nu_1})\{q_1^n\} = \Big(\nu_1+\mu\Big)^n, \tag{3}$$

$$\exp(\mu\partial_{\nu_1}^n)\{e^{\nu_1}\} = e^{\nu_1+\mu}, \tag{4}$$

$$\exp(\mu\nu_1\ \partial_{\nu_1})f\{\nu_1\} = f(e^{\nu_1}\mu). \tag{5}$$

One remarkable class within special polynomials is the Bell polynomials [1], attributed to mathematician Eric Temple Bell. These polynomials are not only central to combinatorial mathematics but are also instrumental in approximation theory and convergence analysis. Bell polynomials, denoted as $\mathbb{B}_n(\nu_1)$, have applications that extend beyond combinatorics, serving as a foundation for constructing various asymptotic expansions and providing approximations to complex functional forms. Their significance in convergence properties lies in their role within series expansions and their behavior under limiting processes, making them a versatile tool in numerical analysis. Bell polynomials are uniquely defined by their exponential generating function, which encapsulates their structure and enables their broad applicability in both theoretical and practical contexts and are defined by their unique exponential generating function:

$$\sum_{n=0}^{\infty} \mathbb{B}_n(\nu_1)\frac{\xi^n}{n!} = e^{\nu_1(e^{\xi}-1)}, \tag{6}$$

when $\nu_1 = 1$, the Bell polynomials simplify to the Bell numbers, expressed by the following relation:

$$\sum_{n=0}^{\infty} \mathbb{B}_n \frac{\xi^n}{n!} = e^{e^{\xi}-1}. \tag{7}$$

Bell polynomials, especially partial Bell polynomials, are instrumental in approximation theory due to their role in defining higher-order derivatives and structured expansions within exponential generating functions. These polynomials facilitate the representation of complex functions and are essential in the study of convergence properties across various series expansions. In approximation theory, Bell polynomials enable precise computation of function approximations by allowing terms to capture intricate dependency patterns within function derivatives. This makes them valuable for accurate approximations in combinatorial structures and probabilistic settings, as well as algorithmic analysis. They are also frequently applied in enumerating structures such as set partitions and integer compositions, offering insights that are foundational to both combinatorics and theoretical algorithms [12, 13, 14, 15, 19, 22, 23, 24, 25].

Another important family of numbers in combinatorics is formed by the Stirling numbers, which arise naturally in the study of permutations, combinations, and set partitions [6, 7, 8, 9, 10, 11, 16, 17]. Broadly, Stirling numbers are classified into two fundamental types: the Stirling numbers of the first kind, denoted by $S_1(n,\epsilon)$, and the Stirling numbers of the second kind, denoted by $S_2(n,\epsilon)$. The Stirling numbers of the first kind $S_1(n,\epsilon)$ count the number of permutations of n distinct elements that consist of exactly ϵ cycles. In other words, they enumerate the different ways in which n elements can be arranged into ϵ disjoint cyclic structures. In contrast, the Stirling numbers of the second kind $S_2(n,\epsilon)$ represent the number of ways to divide a set of n distinct elements into ϵ non-empty subsets, where the order of elements within each subset is irrelevant, but the partition structure itself is significant.

Due to their rich combinatorial interpretation, Stirling numbers play a central role in a wide range of problems involving permutations, combinations, and partitions. Beyond pure combinatorics, they also appear in the analysis of algorithms and in the investigation of diverse combinatorial and algebraic structures. It is worth noting that several notational conventions for Stirling numbers exist in the literature; the notation adopted here is among the most commonly used.

Stirling polynomials of the second kind, denoted by $S_2(n,\epsilon;\nu_1)$, are naturally associated with exponential generating functions. Their exponential generating function is given by

$$\sum_{n=0}^{\infty} \frac{S_2(n,\epsilon;\nu_1)\ \xi^n}{n!} = \frac{(e^{\xi\nu_1}-1)^{\epsilon}}{\epsilon!}. \tag{8}$$

In the particular case when $\nu_1 = 1$, this generating function reduces to the classical exponential generating function of the Stirling numbers of the second kind:

$$\sum_{n=o}^{\infty} \frac{S_2(n,\epsilon)\ \xi^n}{n!} = \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}. \tag{9}$$

Moreover, the Stirling numbers of the second kind $S_2(n,\epsilon)$ satisfy well-known recurrence relations, which can be expressed in the forms

$$\nu_1^n = \sum_{n=o}^{\infty} S_2(n,\epsilon)\,(\nu_1)_{\epsilon} \tag{10}$$

or equivalently,

$$(\nu_1)_n = \sum_{\epsilon=o}^{n} S_2(n,\epsilon)\ \nu_1^{\epsilon}. \tag{11}$$

Here, $(\nu_1)_{\epsilon} = \nu_1(\nu_1-1)(\nu_2-2)\cdots(\nu_1-(\epsilon-1))$ denotes the falling factorial.

In addition, for each integer $\epsilon \in \mathbb{N}_0$, the quantity

$$S_{\epsilon}(n) = \sum_{l=0}^{n} l^{\epsilon} \tag{12}$$

is known as the sum of integer powers. The corresponding exponential generating function for $S_{\epsilon}(n)$ is given by

$$\sum_{\epsilon=0}^{\infty} S_{\epsilon}(n)\ \frac{\xi^{\epsilon}}{\epsilon!} = \frac{e^{(n+1)\xi-1}}{e^{\xi}-1}. \tag{13}$$

These notions form a fundamental part of combinatorial theory and find extensive applications in counting problems involving partitions and subsets of distinct elements.

Appell polynomials, introduced by the French mathematician Paul Appell [18], form an important category of special functions in mathematical analysis. They are commonly characterized as solutions to specific differential equations that frequently emerge in the investigation of various mathematical and physical problems. Appell polynomials are notable for their unique features, such as recurrence relations, generating functions, and explicit expressions, which make them highly valuable in fields like probability theory, mathematical physics, and combinatorics. These polynomials demonstrate exceptional flexibility as fundamental tools for addressing differential equations, performing integral transforms, and solving other complex mathematical challenges. Their intricate structure and broad applications have made Appell polynomials a vital topic in contemporary mathematical research. The generating relation for Appell polynomials is as follows:

$$\mathbb{L}(\xi)e^{\nu_1\xi}=\sum_{r=0}^{\infty}\frac{\xi^r}{r!}\mathbb{L}_r(\nu_1), \tag{14}$$

where

$$\mathbb{L}(\xi)=\sum_{r=0}^{\infty}\frac{\xi^r}{r!}\mathbb{L}_r;\quad \mathbb{L}_0\neq 0. \tag{15}$$

The manuscript addresses a broad spectrum of engaging themes that are likely to attract and sustain the reader's attention. In Section 2, we focus on the rich structure of $2D$ Bell-based Appell polynomials and present a thorough investigation of their fundamental properties using generating function techniques. This section systematically develops explicit representations, summation formulas, recurrence relations, and addition theorems, thereby clarifying their close connections with Stirling polynomials of the second kind. Subsequently, Section 4 is devoted to the introduction of the $2D$ Bell–Appell-based Stirling polynomials of the second kind, where several noteworthy results are derived to deepen the understanding of their analytical behavior. Finally, we emphasize the overall importance of the present work, highlighting its contribution to the study of the structural properties and potential applications of $2D$ Bell-based Appell polynomials and $2D$ Bell–Appell-based Stirling polynomials within the broader framework of mathematical analysis.

2 $2D$ Bell based Appell polynomials

The $2D$ special Bell–Appell polynomials play an important role in both theoretical and applied mathematics, owing to their distinctive algebraic structure and wide range of applications. These polynomials arise as a natural generalization of the classical Bell polynomials through their convolution with Appell polynomial sequences. As a result, they form a class of hybrid polynomials that is particularly relevant in areas such as combinatorial analysis, number theory, and statistical mechanics.

A notable characteristic of the $2D$ special Bell–Appell polynomials is their capacity to represent multivariate exponential generating functions, which makes them a powerful tool for the analysis of combinatorial configurations and enumeration problems in discrete mathematics. In number theory, they are effectively employed to explore properties of partitions, compositions, and related combinatorial structures. Furthermore, in statistical mechanics, these polynomials are useful for examining systems with two-dimensional degrees of freedom, thereby offering valuable insights into thermodynamic behavior and phase transition phenomena. Beyond these areas, in applied mathematics and engineering, $2D$ special Bell–Appell polynomials provide a coherent and systematic framework for modeling and analyzing complex multivariable systems.

Generating functions are pivotal in mathematics, serving as powerful tools for enumerating combinatorial structures, solving recurrence relations, and analyzing probability distributions. By encoding sequences of numbers or objects into algebraic expressions, generating functions simplify complex counting problems, enabling the application of techniques from calculus and algebra. They provide a unified framework for studying sequences, facilitating the derivation of closed-form expressions, and analysing asymptotic behaviour. Moreover, in probability theory, generating functions aids in understanding distributions, moments, and characteristic functions of random variables, offering insights into various probabilistic phenomena. Overall, generating functions are indispensable in mathematical analysis, providing versatile methods for solving problems across diverse areas of mathematics. Thus, here we introduce the convoluted Bell-Appell polynomials, possessing the generating relation:

$$\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\vartheta_1,\vartheta_2)\frac{\xi^n}{n!}=\mathbb{L}(\xi)e^{\vartheta_1(e^{\xi}-1)+\vartheta_2(e^{\xi}-1)^j}. \tag{16}$$

For, $\mathbb{L}(\xi)=1$, these polynomials reduce to the $2D$ Bell polynomials represented by the generating relation:

$$\sum_{n=0}^{\infty}\mathbb{B}_n^{[j]}(\vartheta_1,\vartheta_2)\frac{\xi^n}{n!}=e^{\vartheta_1(e^{\xi}-1)+\vartheta_2(e^{\xi}-1)^j}. \tag{17}$$

Additionally, in this section, we present the explicit forms and key properties of the $2D$ Bell-Appell polynomials, denoted by ${}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\vartheta_1,\vartheta_2)$. The detailed expressions and summation formulae derived here are essential for advancing both the theoretical understanding and practical application of these polynomials in mathematical analysis. These explicit forms offer compact and systematic representations of 2*D* Bell-Appell polynomials, aiding their computational use and deepening their theoretical exploration. Additionally, the summation formulae provide efficient methods for calculating polynomial sums, allowing researchers to tackle complex mathematical challenges more effectively. By clarifying the explicit forms and summation properties of these polynomials, this work strengthens the theoretical framework and broadens the practical relevance of 2*D* Bell-Appell polynomials across a range of mathematical and scientific fields. The derived results are as follows:

Theorem 2.1. *The* 2*D Bell-Appell polynomials denoted by* ${}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\vartheta_1,\vartheta_2)$ *satisfy the listed series representation form:*

$$ {}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\vartheta_1,\vartheta_2)=\sum_{s=0}^{n}\binom{n}{s}\mathbb{L}_{n-s}\mathbb{B}_s^{[j]}(\vartheta_1,\vartheta_2). \tag{18}$$

Proof. Starting from the generating relation (16) and using (15) and (17), we may rewrite it in the form

$$\mathbb{L}(\xi)e^{\vartheta_1(e^{\xi}-1)+\vartheta_2(e^{\xi}-1)^j}=\sum_{n=0}^{\infty}\mathbb{L}_n\frac{\xi^n}{n!}\sum_{s=0}^{\infty}\mathbb{B}_s^{[j]}(\vartheta_1,\vartheta_2)\frac{\xi^s}{s!}. \tag{19}$$

Now, substituting the right-hand side of (16) into the left-hand side of the preceding identity and then simplifying, we obtain

$$\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\vartheta_1,\vartheta_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\mathbb{L}_n\sum_{s=0}^{\infty}\mathscr{B}_s^{[j]}(\vartheta_1,\vartheta_2)\frac{\xi^{n+s}}{n!\,s!}. \tag{20}$$

Next, by applying a standard rearrangement of the double series and replacing n by $n-s$ on the right-hand side, we arrive at

$$\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\vartheta_1,\vartheta_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\binom{n}{s}\mathbb{L}_{n-s}\;\mathbb{B}_s(\vartheta_1,\vartheta_2)\frac{\xi^n}{n!}. \tag{21}$$

Finally, comparing the coefficients of $\frac{\xi^n}{n!}$ on both sides yields the desired identity (18). □

Theorem 2.2. *The* 2*D Bell-Appell polynomials denoted by* ${}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)$ *satisfy the listed explicit form:*

$$ {}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)=\sum_{s=0}^{[n]}\binom{n}{s}{}_{\mathbb{B}}\mathbb{L}_{n-s}^{[j]}(\nu_1)\frac{S_2(s,k)\,(e^{\xi}-1)^{jk-k}}{1-\nu_2}. \tag{22}$$

Proof. From (16), the generating function can be decomposed as

$$\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}=\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)}\exp(\nu_2(e^{\xi}-1)^j). \tag{23}$$

By substituting (6) and (9) into the right-hand side of the above expression, we obtain

$$\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}=\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1)\frac{\xi^n}{n!}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\nu_2^rS_2(s,k)\frac{\xi^s}{s!}(e^{\xi}-1)^{jk-k}. \tag{24}$$

Next, inserting the right-hand side of (16) into the left-hand side of (24) and simplifying the resulting series, we get

$$\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1)\sum_{s=0}^{\infty}\frac{S_2(s,k)\,(e^{\xi}-1)^{jk-k}}{1-\nu_2}\frac{\xi^{n+s}}{n!\,s!}. \tag{25}$$

Rearranging the double series and replacing n by $n-s$ on the right-hand side gives

$$\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\sum_{s=0}^{[n]}\binom{n}{s}{}_{\mathbb{B}}\mathbb{L}_{n-s}^{[j]}(\nu_1)\frac{S_2(s,k)(e^{\xi}-1)^{jk-k}}{1-\nu_2}\frac{\xi^n}{n!}. \tag{26}$$

Comparing the coefficients of $\frac{\xi^n}{n!}$ on both sides proves (22). □

Theorem 2.3. *The* 2*D Bell-Appell polynomials denoted by* ${}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)$ *satisfy the listed series representation form:*

$$ {}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)=\sum_{s=0}^{[n]}\sum_{p=0}^{[s]}\binom{n}{s}\binom{s}{p}\frac{S_2(n-s,l)}{1-\nu_1}\frac{S_2(s-p,k)\,(e^{\xi}-1)^{jk-k}}{1-\nu_2}. \tag{27}$$

Proof. We begin again with (16), and rewrite its generating function as

$$\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}=\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)}e^{\nu_2(e^{\xi}-1)^j}. \tag{28}$$

Employing (9) together with (15) on the right-hand side, we find

$$\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}=\sum_{p=0}^{\infty}\mathbb{L}\frac{\xi^p}{p!}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty}\nu_1^l S_2(n,l)\frac{\xi^n}{n!}(e^{\xi}-1)^{jk-k}n!\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\nu_2^r S_2(s,k)\frac{\xi^s}{s!}(e^{\xi}-1)^{jk-k}. \tag{29}$$

Substituting the right-hand side of (16) into the left-hand side of (29) and simplifying yields

$$\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\left[\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\mathbb{L}\frac{S_2(n,l)}{1-\nu_1}\frac{S_2(s,k)\,(e^{\xi}-1)^{jk-k}}{1-\nu_2}\frac{\xi^{s+p}}{p!\,s!}\right]\frac{\xi^n}{n!}. \tag{30}$$

Now, by rearranging the series and replacing s with $s-p$ on the right-hand side, we obtain

$$\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{p=0}^{[s]}\binom{s}{p}\frac{S_2(n,l)}{1-\nu_1}\frac{S_2(s-p,k)(e^{\xi}-1)^{jk-k}}{1-\nu_2}\frac{\xi^{n+s}}{n!\,s!}. \tag{31}$$

Finally, performing one more rearrangement (taking $n-s$ in place of n) and then comparing coefficients of $\frac{\xi^n}{n!}$ on both sides leads directly to (27). □

Theorem 2.4. *The 2D Bell-Appell polynomials denoted by ${}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)$. Then, the following summation formulas hold.*

$${}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1+\nu_3,\nu_2)=\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_3){}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_2). \tag{32}$$

Proof. By (16) and (6), we have

$$\begin{aligned}\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1+\nu_3,\nu_2)\frac{\xi^n}{n!} &= \mathbb{L}(\xi)e^{(\nu_1+\nu_2)(e^{\xi}-1)+\nu_3(e^{\xi}-1)^j}\\ &= \sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_3)\frac{\xi^n}{n!}\sum_{k=0}^{\infty}\mathbb{B}_k^{[j]}(\nu_2)\frac{\xi^k}{k!}\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_3)\mathbb{B}_k^{[j]}(\nu_2)\right]\frac{\xi^n}{n!}.\end{aligned}$$

Equating the coefficients of $\dfrac{\xi^n}{n!}$ on both sides completes the proof and establishes Theorem 2.4. □

Theorem 2.5. *For any arbitrary $n\in\mathbb{N}$, the following relation hold true:*

$${}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1+1,\nu_2)-{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)=\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2)\mathbb{B}_k-{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2). \tag{33}$$

Proof. Using (16), we compute

$$\begin{aligned}\sum_{n=0}^{\infty}\left[{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1+1,\nu_2)-{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\right]\frac{\xi^n}{n!} &= \mathbb{L}(\xi)e^{(\nu_1+1)(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}-e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}\\ &= e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}\left[e^{e^{\xi}-1}-1\right]\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2)B_n(\nu_1,\nu_2)B_k-{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\right]\frac{\xi^n}{n!}.\end{aligned}$$

Comparing coefficients on both sides gives the claimed identity (33). □

Theorem 2.6. *For $n\geq 1$, let ${}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)$ be the 2D Bell-Appell polynomials, then we have*

$$\frac{\partial}{\partial\nu_1}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)=\frac{1}{(n^2+n)}\sum_{k=0}^{n}\binom{n+1}{k}{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1,\nu_2) \tag{34}$$

and

$$\frac{\partial}{\partial\nu_2}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)=j!\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2)S_2(k,j), \tag{35}$$

respectively.

Proof. Differentiating both sides of (16) with respect to ν_1 gives

$$\begin{aligned}
\frac{\partial}{\partial \nu_1}\left[\sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}\right] &= \frac{\partial}{\partial \nu_1}\left[\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}\right] \\
&= \mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}(e^{\xi}-1) \\
&= \sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}\sum_{n=0}^{\infty}\frac{\xi^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1,\nu_2)\frac{\xi^{n+1}}{(n+1)!}
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial \nu_1}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)=\frac{1}{(n^2+n)}\sum_{k=0}^{n}\binom{n+1}{k}{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1,\nu_2).$$

Next, differentiating both sides of (16) with respect to ν_2, we get

$$\begin{aligned}
\frac{\partial}{\partial \nu_2}\left[\sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}\right] &= \frac{\partial}{\partial \nu_2}\left[\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}\right] \\
&= \mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}(e^{\xi}-1)^j \\
&= \sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}\sum_{n=0}^{\infty} j!\,S_2(n,j)\frac{\xi^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} j!\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2)S_2(k,j)\frac{\xi^n}{n!}
\end{aligned}$$

Hence,

$$\frac{\partial}{\partial \nu_2}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)=j!\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2)S_2(k,j).$$

□

Theorem 2.7. *For $n\geq 0$, let $\left\{{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\right\}_{n\geq 0}$ be the sequences of 2D Bell-Appell polynomials in the variable ν_1, ν_2 and ν_3, they satisfy the following relation*

$$\sum_{k=0}^{n}\binom{n}{k}\left[{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1+\nu_3,\nu_2){}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(2\nu_2)-{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2){}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_3,\nu_2)\right]=0.$$

Proof. Let's consider the following expressions

$$\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}=\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!} \tag{36}$$

and

$$\mathbb{L}(\xi)e^{\nu_3(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}=\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_3,\nu_2)\frac{\xi^n}{n!}. \tag{37}$$

From (36) and (37), we have

$$\begin{aligned}
\mathbb{L}(\xi)e^{(\nu_1+\nu_3)(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}e^{2\nu_2(e^{\xi}-1)^j} &= \left(\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_3,\nu_2)\frac{\xi^n}{n!}\right) \\
\left(\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1+\nu_3,\nu_2)\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(0,2\nu_2)\frac{\xi^n}{n!}\right) &= \left(\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}\right)\left(\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_3,\nu_2)\frac{\xi^n}{n!}\right) \\
\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1+\nu_3,\nu_2){}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(2\nu_2)\frac{\xi^n}{n!} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2){}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_3,\nu_2)\frac{\xi^n}{n!} \\
\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1+\nu_3,\nu_2){}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(2\nu_2) &= \sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2){}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_3,\nu_2)
\end{aligned}$$

Consequently,

$$\sum_{k=0}^{n}\binom{n}{k}\left[{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_1+\nu_3,\nu_2){}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(2\nu_2)-{}_{\mathbb{B}}\mathbb{L}_{n-k}^{[j]}(\nu_1,\nu_2){}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_3,\nu_2)\right]=0.$$

□

3 The $2D$ Bell-Appell based Stirling polynomials of the second kind

In this section, we introduce and investigate a new family of polynomials, referred to as the $2D$ Bell-Appell-based Stirling polynomials of the second kind. Our emphasis is on describing their fundamental structure and highlighting the main properties that characterize this class. Owing to their rich algebraic form, these polynomials are of interest both from a theoretical perspective and for potential applications. The primary objective here is to clearly present their defining framework and to provide a systematic basis for understanding their behavior. To this end, we begin by giving a precise definition, which forms the cornerstone for all subsequent results and discussions.

Definition 3.1.

$$\sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1,\nu_2)\frac{\xi^n}{n!} = \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\,\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}. \tag{38}$$

This definition provides the basic generating function representation and serves as the starting point for deriving several important identities and properties, as well as for exploring possible applications in wider mathematical settings.

Remark 1. If we set $\nu_2 = 0$ in equation (38), the resulting family reduces to the Bell-Stirling polynomials of the second kind, which are defined by

$$\sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1)\frac{\xi^n}{n!} = \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\,\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)}. \tag{39}$$

Remark 2. Furthermore, by choosing $\nu_1 = \nu_2 = 0$ in (38), we recover a sequence that coincides with the classical Stirling numbers of the second kind, as given in (9).

Theorem 3.1. *For a non-negative integer n, the* $2D$ *Bell-Appell based Stirling polynomials of the second kind satisfy the following convolution-type relation:*

$$\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}_2(l,\epsilon)\mathbb{B}_{n-l}(\nu_1,\nu_2) = {}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1,\nu_2). \tag{40}$$

Proof. Starting from the generating function in (38), we write

$$\sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1,\nu_2)\frac{\xi^n}{n!} = \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\,\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j} = \sum_{n=\epsilon}^{\infty}\mathcal{S}_2(n,\epsilon)\frac{\xi^n}{n!}\sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathbb{L}_n^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}. \tag{41}$$

Applying the Cauchy product to the series on the right-hand side yields

$$\sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1,\nu_2)\frac{\xi^n}{n!} = \sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}_2(l,\epsilon)\,{}_{\mathbb{B}}\mathbb{L}_{n-l}^{[j]}(\nu_1,\nu_2)\frac{\xi^n}{n!}. \tag{42}$$

Comparing the coefficients of equal powers of ξ on both sides completes the proof. □

Remark 3. In the special case $\nu_2 = 0$, relation (43) reduces to the corresponding identity for the Bell-based Stirling polynomials of the second kind:

$$\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}_2(l,\epsilon)\mathbb{B}_{n-l}(\nu_1) = {}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1), \tag{43}$$

for all non-negative integers n.

Theorem 3.2. *For any non-negative integer n, the following addition formula holds for the two-dimensional Bell-Appell-based Stirling polynomials of the second kind:*

$${}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1+\nu_3,\nu_2+\nu_4) = \sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;\nu_1,\nu_2)\,{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_3,\nu_4). \tag{44}$$

Proof. Using the generating functions in (38) and (16), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} {}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1+\nu_3,\nu_2+\nu_4)\frac{\xi^n}{n!} &= \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\,\mathbb{L}(\xi)e^{(\nu_1+\nu_3)(e^{\xi}-1)+(\nu_2+\nu_4)(e^{\xi}-1)^j}\\ &= \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^j}\,\mathbb{L}(\xi)e^{\nu_3(e^{\xi}-1)+\nu_4(e^{\xi}-1)^j}.\end{aligned}$$

Expressing both exponential factors in terms of their series expansions and applying the Cauchy product, we arrive at

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;\nu_1,\nu_2)\,{}_{\mathbb{B}}\mathbb{L}_k^{[j]}(\nu_3,\nu_4)\frac{\xi^n}{n!}.$$

Equating the coefficients of $\xi^n/n!$ on both sides proves the result. □

Theorem 3.3. *For any non-negative integer n, the 2D Bell-Appell-based Stirling polynomials of the second kind satisfy*

$$ {}_{\mathbb{B}}\mathcal{S}_2^{[j+\beta]}(n,\epsilon;\nu_1,\nu_2)=\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;\nu_1,\nu_2)\mathbb{B}_k^{[\beta]}. \tag{45}$$

Proof. From the defining generating function (38), we have

$$\sum_{n=0}^{\infty}{}_{\mathbb{B}}\mathcal{S}_2^{[j+\beta]}(n,\epsilon;\nu_1,\nu_2)\frac{\xi^n}{n!} = \frac{(e^{\xi}-1)^{\epsilon}}{\epsilon!}\,\mathbb{L}(\xi)e^{\nu_1(e^{\xi}-1)+\nu_2(e^{\xi}-1)^{j+\beta}}.$$

Separating the exponential term involving $(e^{\xi}-1)^{j+\beta}$ and expanding the resulting factors into series, the Cauchy product leads directly to the desired summation formula. Matching coefficients of $\xi^n/n!$ establishes the theorem. ☐

Theorem 3.4. *For $n\geq 1$, let ${}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1,\nu_2)$ denote the 2D Bell-Appell-based Stirling polynomials of the second kind. Then the following partial derivative relations hold:*

$$\frac{\partial}{\partial\nu_1}{}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1,\nu_2)=\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;\nu_1,\nu_2)-{}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1,\nu_2), \tag{46}$$

and

$$\frac{\partial}{\partial\nu_2}{}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n,\epsilon;\nu_1,\nu_2)=j!\sum_{k=0}^{n}\binom{n}{k}{}_{\mathbb{B}}\mathcal{S}_2^{[j]}(n-k,\epsilon;\nu_1,\nu_2)\mathcal{S}_2(k,j), \tag{47}$$

respectively.

Proof. Differentiating both sides of the generating function (38) with respect to ν_1 and simplifying yields the first identity. A similar differentiation with respect to ν_2, together with the series expansion of $(e^{\xi}-1)^j$, leads to the second relation. In both cases, equating coefficients of $\xi^n/n!$ completes the proof. ☐

4 Conclusion

In this work, we have developed a systematic study of 2*D* Bell-based Appell polynomials through the lens of generating functions. A range of structural properties has been derived, including explicit representations, summation identities, recurrence relations, and addition formulas. We have also discussed matrix formulations and product-type expressions, which enhance both the analytical and computational treatment of these polynomials. Furthermore, the introduction of the 2*D* Bell-Appell-based Stirling polynomials of the second kind broadens the scope of the theory and connects it with well-known combinatorial sequences. Overall, the results presented here contribute to a deeper understanding of Bell-Appell-type polynomials and provide a solid foundation for future investigations and applications across various areas of mathematical analysis.

In approximation theory and numerical analysis, the exploration of 2D Bell polynomials offers a promising avenue for advancing both theoretical understanding and practical applications. One critical area of future research could involve studying the convergence properties of these polynomials in various settings see for example [14, 15], thus will provide valuable insights into their potential in error estimation, stability analysis, and refinement of existing approximation techniques.

Another exciting direction is extending Bell polynomials beyond the two-dimensional case to explore their behavior in higher-dimensional spaces. This would allow for a broader understanding of their structural properties, leading to advancements in multi-variable function approximation. In doing so, researchers may uncover new connections between Bell polynomials and other special functions used in numerical methods, further expanding their role in solving complex systems.

Additionally, a significant focus of future research could involve the development of efficient computational algorithms to enhance the usability of 2D Bell-Appell polynomials in various applications. The challenge of handling large datasets and complex systems requires novel numerical techniques that ensure fast convergence and reduced computational costs. This is particularly relevant in fields like data analysis, optimization, and machine learning, where the ability to perform high-dimensional computations efficiently is paramount. These efforts could lead to more robust algorithms for applications such as high-dimensional approximation, image processing, and signal analysis, where precision and efficiency are essential.Thus, the intersection of approximation theory, convergence analysis, and computational efficiency will play a vital role in shaping the future development and application of 2D Bell polynomials across both theoretical and practical domains.

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