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Approximation error of generalized Shannon sampling operators with bandlimited kernels in terms of an averaged modulus of smoothness

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Abstract

The aim of this paper is to study the approximation properties of generalized sampling operators in terms of an averaged modulus of smoothness.

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1 Introduction

For the uniformly continuous and bounded functions $f \in C(\mathbb{R})$ the generalized sampling series are given by ($t \in \mathbb{R}$; $W > 0$)

$$(S_W f)(t) := \sum_{k=-\infty}^{\infty} f(\frac{k}{W}) s(Wt - k), \tag{1}$$

where the condition for the operator $S_W : C(\mathbb{R}) \to C(\mathbb{R})$ to be well-defined is

$$\sum_{k=-\infty}^{\infty} |s(u-k)| < \infty \quad (u \in \mathbb{R}), \tag{2}$$

the absolute convergence being uniform on compact intervals of $\mathbb{R}$.

If the kernel function is

$$s(t) = \operatorname{sinc}(t) := \frac{\sin \pi t}{\pi t},$$

which do not satisfy (2), we get the classical (Whittaker-Kotel'nikov-)Shannon operator,

$$(S_W^{\mathrm{sinc}} f)(t) := \sum_{k=-\infty}^{\infty} f(\frac{k}{W}) \operatorname{sinc}(Wt - k).$$

Because $\operatorname{sinc}(t) \notin L^1(\mathbb{R})$ the series $(S_W^{\mathrm{sinc}} f)$ for an arbitrary function $f \in C(\mathbb{R})$ may be divergent. A set of fixed points of the sampling operator S_W^{sinc} is equal to the Bernstein class $B^p_{\pi W}$ (if $p = \infty$, then B^p_σ with $\sigma < \pi W$) – the class of those bounded functions $f \in L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ which can be extended to an entire function $f(z)$ $(z \in \mathbb{C})$ of exponential type σ ([1] or [2], 4.3.1), i. e.,

$$|f(z)| \leqslant e^{\sigma |y|} \|f\|_C \quad (z = x + iy \in \mathbb{C}).$$

The idea to replace the sinc kernel $\operatorname{sinc}(\cdot) \notin L^1(\mathbb{R})$ by another kernel function $s \in L^1(\mathbb{R})$ appeared first in [3], where the case $s(t) = (\operatorname{sinc}(t))^2$ was considered. A systematic study of sampling operators (1) for arbitrary kernel functions s with (2) was initiated at RWTH Aachen by P. L. Butzer and his students since 1977 (see [4], [1], [5] and references cited there).

Since in practice signals are however often discontinuous, this paper is concerned with the convergence of $S_W f$ to f in the $L^p(\mathbb{R})$-norm for $1 < p < \infty$, the classical modulus of continuity being replaced by the averaged modulus of smoothness $\tau_k(f; 1/W)_p$. For the classical (Whittaker-Kotel'nikov-Shannon) operator this approach was introduced by P. L. Butzer, C. Bardaro, R. Stens and G. Vinti (2006) in [6] for $1 < p < \infty$. For time-limited kernels s this approach was applied for $1 \leqslant p < \infty$ in [7] and [8].

In this paper we study an even band-limited kernel s, i.e. $s \in B^1_\pi$, defined by an even window function $\lambda \in C_{[-1,1]}$, $\lambda(0) = 1$, $\lambda(u) = 0$ $(|u| \geqslant 1)$ by the equality

$$s(t) := s(\lambda; t) := \int_0^1 \lambda(u) \cos(\pi t u)\, du. \tag{3}$$

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In fact, this kernel is the Fourier transform of $\lambda \in L^1(\mathbb{R})$,

$$s(t) = \sqrt{\frac{\pi}{2}}\lambda^\wedge(\pi t). \tag{4}$$

We first used the band-limited kernel in general form (3) in [9], see also [10]. We studied the generalized sampling operators $S_W : C(\mathbb{R}) \to C(\mathbb{R})$ with the kernels in form (3) in [11], [12], [13], [14], [15] and [16]. We computed exact values of operator norms

$$\|S_W\| := \sup_{\|f\|_C \leqslant 1} \|S_W f\|_C$$

and estimated the order of approximation in terms of modulus of smoothness. In this paper we give similar results for $L^p(\mathbb{R})$ norm in terms of the averaged modulus of smoothness.

2 Preliminary results

In this section we follow the approach of Butzer et al [6] of convergence problems of Shannon sampling series in a suitable subspace of $L^p(\mathbb{R})$.

2.1 Averaged modulus of smoothness

The Bulgarian school under Sendov [17] has introduced a so-called averaged modulus of smoothness $\tau_k(f;\delta)_p$. However, in Sendov and Popov [17] this modulus is only studied for bounded, measurable functions $f : [a,b] \to \mathbb{R}$, whereas (at least) in sampling analysis one needs signals $f : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$). For this purpose Butzer, Bardaro, Stens and Vinti [6] extended the concept of this averaged modulus to functions belonging to the space

$$M(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C}; f \text{ measurable and bounded on } \mathbb{R}\}.$$

Let $f \in M(\mathbb{R})$ and $\delta \geqslant 0$. The k-th averaged modulus of smoothness for $1 \leqslant p \leqslant \infty$ is defined as ([6], Def. 1)

$$\tau_k(f;\delta)_p := \|\omega_k(f;\cdot;\delta)\|_p, \tag{5}$$

where $\omega_k(f;t;\delta)$ is a local modulus of smoothness of order $k \in \mathbb{N}$ at $t \in \mathbb{R}$

$$\omega_k(f;t;\delta) := \sup\{|\mathring{\Delta}_h^k f(x)|; x - \frac{kh}{2}, x + \frac{kh}{2} \in [t - \frac{k\delta}{2}, t + \frac{k\delta}{2}]\},$$

where the central difference is given by

$$\mathring{\Delta}_h^k f(x) = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} f(x + (\frac{k}{2} - \ell)h). \tag{6}$$

Classical modulus of smoothness ([18], p.76) is defined for $f \in C(\mathbb{R})$ and $\delta \geqslant 0$ by

$$\omega_k(f;\delta)_C := \sup_{|h|\leqslant\delta} \|\mathring{\Delta}_h^k f(\cdot)\|_C,$$

and for $f \in L^p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ by

$$\omega_k(f;\delta)_p := \sup_{|h|\leqslant\delta} \|\mathring{\Delta}_h^k f(\cdot)\|_p.$$

The averaged modulus of smoothness has the following properties ([6], Proposition 4, [19], 4.6.6):

$$\begin{array}{ll} \tau_k(f;\delta)_C & = \omega_k(f;\delta)_C, \\ \tau_k(f;\delta)_\infty & = \omega_k(f;\delta)_\infty, \\ \omega_k(f;\delta)_p & \leqslant \tau_k(f;\delta)_p \quad (1 \leqslant p < \infty), \\ \omega_k(f,h\delta)_p & \leqslant \lfloor 1+h \rfloor^k \omega_k(f,\delta)_p \text{ for any } h > 0 \ (1 \leqslant p < \infty), \end{array} \tag{7}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to $x \in \mathbb{R}$.

2.2 The space Λ^p

Since the sampling series $S_W f$ of (1) of an arbitrary L^p-function f may be divergent, we have to restrict the matter to a suitable subspace. Further, since we want to use the τ- modulus as a measure for the approximation error, we have to ensure that it is finite for all functions under consideration. In [6] was proved that we can define a suitable subspace as follows

Definition 2.1 ([6], Def. 10, [7], Def. 2.1).

(a) A sequence $\Sigma := (x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ is called an admissible partition of $\mathbb{R}$ or an admissible sequence, if it satisfies

$$0 < \inf_{j\in\mathbb{Z}} \Delta_j \leqslant \sup_{j\in\mathbb{Z}} \Delta_j < \infty, \quad \Delta_j := x_j - x_{j-1}.$$

(b) Let $\Sigma := (x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ be an admissible partition of $\mathbb{R}$. The discrete $\ell^p(\Sigma)$-norm of a sequence of function values f_Σ on Σ of a function $f : \mathbb{R} \to \mathbb{C}$ is defined for $1 \leqslant p < \infty$ by

$$\|f\|_{\ell^p(\Sigma)} := \left\{ \sum_{j\in\mathbb{Z}} |f(x_j)|^p \Delta_j \right\}^{1/p}.$$

(c) The space Λ^p for $1 \leqslant p < \infty$ is defined by

$$\Lambda^p := \{f \in M(\mathbb{R}); \|f\|_{\ell^p(\Sigma)} < \infty \text{ for each admissible sequence } \Sigma\}.$$

$\|\cdot\|_{\ell^p(\Sigma)}$ is a seminorm on Λ^p.

It can be shown (see [6], Proposition 18) that if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 \leqslant p < \infty$ we have

$$\lim_{\delta\to 0} \tau_k(f;\delta)_p = 0, \tag{8}$$

where $R_{loc}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C},$ is locally Riemann integrable on $\mathbb{R}\}$. We have for $1 \leqslant p < \infty$ that $B_W^p \subsetneq W_p^r \subsetneq \Lambda^p \subsetneq L^p$, where

$$W_p^r := \{f \in L^p; f \in AC_{loc}^r, f^{(r)} \in L^p\}$$

is the classical Sobolev space.

In the following we consider the uniform partitions $\Sigma_W := (j/W)_{j\in\mathbb{Z}} \subset \mathbb{R}$ for $W > 0$ only. For these partitions we have

$$\|f\|_{\ell^p(W)} := \left\{ \frac{1}{W} \sum_{j\in\mathbb{Z}} \left| f\left(\frac{j}{W}\right)\right|^p \right\}^{1/p} \leqslant \|f\|_p + \frac{1}{W}\|f'\|_p, \quad f \in W_p^r. \tag{9}$$

2.3 Sampling operators

For the classical (Whittaker-Kotel'nikov-)Shannon operator we have (see [6], Corollary 33) that if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 < p < \infty$ we have

$$\lim_{W\to\infty} \|S_W^{\text{sinc}} f - f\|_p = 0. \tag{10}$$

Holds the following theorem:

Theorem 2.1 ([6], Th. 32). *Let $f \in \Lambda^p$ for $1 < p < \infty$, any $r \in \mathbb{N}$. Then*

$$\|S_W^{\text{sinc}} f - f\|_p \leqslant c_r\, \tau_r(f; \frac{1}{W})_p. \tag{11}$$

The constants c_r are independent of f and W.

The most general kernel for the sampling operators S_W in (1) is defined in the following way.

Definition 2.2 ([5], Def. 6.3). If $s : \mathbb{R} \to \mathbb{C}$ is a bounded function such that

$$\sum_{k=-\infty}^{\infty} |s(u-k)| < \infty \quad (u \in \mathbb{R}), \tag{12}$$

the absolute convergence being uniform on compact subsets of $\mathbb{R}$, and

$$\sum_{k=-\infty}^{\infty} s(u-k) = 1 \quad (u \in \mathbb{R}), \tag{13}$$

then s is said to be a kernel for sampling operators (1).

The absolute moment of order $r = 0, 1, 2, \ldots$ of a kernel s is defined by

$$m_r(s) := \sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{\infty} |u-k|^r |s(u-k)|. \tag{14}$$

The definition formulated above guarantees that operators (1) give approximations for continuous functions $f \in C(\mathbb{R})$.

Theorem 2.2 ([1], Th. 4.1). *Let $s \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ be a kernel. Then $\{S_W\}_{W>0}$ defines a family of bounded linear operators from $C(\mathbb{R})$ into itself, satisfying*

$$\|S_W\| = \sup_{u\in\mathbb{R}} \sum_{k=-\infty}^{\infty} |s(u-k)| =: m_0(s) \quad (W > 0). \tag{15}$$

For $f \in \Lambda^p$ we have:

Proposition 2.3 (cf [7], Proposition 3.2). *Let $s \in M(\mathbb{R}) \cap L^1(\mathbb{R})$ be a kernel. Then $\{S_W\}_{W>0}$ defines a family of bounded linear operators from Λ^p into L^p, $1 \leqslant p < \infty$, satisfying* $(1/p + 1/q = 1)$

$$\|S_W f\|_p \leqslant m_0^{1/q}(s)\|s\|_1^{1/p} \|f\|_{\ell^p(W)} \quad (W > 0). \tag{16}$$

If the kernel s is time-limited, i.e. there exists $T_0, T_1 \in \mathbb{R}$, $T_0 < T_1$ such that $s(t) = 0$ for $t \notin [T_0, T_1]$, then if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 \leqslant p < \infty$, we have (see [7], Th. 4.4)

$$\lim_{W\to\infty} \|S_W f - f\|_p = 0. \tag{17}$$

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3 Bandlimited kernels

In the following we assume that our kernel s in (3) belongs to $L^1(\mathbb{R})$, which yields $s \in B^1_\pi$, because the Fourier transform of s,

$$s^\wedge(x) = \frac{1}{\sqrt{2\pi}}\lambda\left(\frac{x}{\pi}\right) \text{ implies } s^\wedge(x) = 0 \text{ for } |x| \geqslant \pi. \tag{18}$$

For the band-limited functions $s \in B^p_\pi \subset L^p(\mathbb{R})$ the norm (15) is related to the norm $\|s\|_p$ as shown in following.

Theorem 3.1 (Nikolskii inequality; [20], p.124, [21], Th. 6.8)**.** *Let $1 \leqslant p \leqslant \infty$. Then, for every $s \in B^p_\sigma$,*

$$\|s\|_p \leqslant \sup_{u\in\mathbb{R}} \left\{ \sum_{k=-\infty}^{\infty} |s(u-k)|^p \right\}^{1/p} \leqslant (1+\sigma)\|s\|_p.$$

From the Nikolskii inequality we see that our assumption $s \in L^1(\mathbb{R})$ is sufficient for (12) and thus s in (3) is indeed a kernel in the sense of Definition 2.2.

These types of kernels arise in conjunction with window functions widely used in applications (e.g. [22], [23], [24], [25]), in Signal Analysis in particular. Unfortunately bandlimited kernels do not have compact support. Many kernels can be defined by (3).

1) $\lambda(u) = 1$ defines the sinc function.

2) $\lambda(u) = 1 - u$ defines the Fejér kernel $s_F(t) = \frac{1}{2}\mathrm{sinc}^2\frac{t}{2}$ (cf. [3]).

3) $\lambda_j(u) := \cos\pi(j+1/2)u$, $j = 0, 1, 2, \ldots$ defines the Rogosinski-type kernel (see [9]) in the form

$$r_j(t) := \frac{1}{2}\left(\mathrm{sinc}(t+j+\frac{1}{2}) + \mathrm{sinc}(t-j-\frac{1}{2})\right). \tag{19}$$

4) $\lambda_H(u) := \cos^2\frac{\pi u}{2} = \frac{1}{2}(1+\cos\pi u)$ defines the Hann[1] kernel (see [14])

$$s_H(t) := \frac{1}{2}\frac{\mathrm{sinc}\, t}{1-t^2}. \tag{20}$$

5) The general cosine window

$$\lambda_{C,\mathbf{a}}(u) := \sum_{k=0}^{m} a_k \cos k\pi u \tag{21}$$

defines the Blackman-Harris kernel (see [15])

$$s_{C,\mathbf{a}}(t) := \frac{1}{2}\sum_{k=0}^{m} a_k\left(\mathrm{sinc}(t-k) + \mathrm{sinc}(t+k)\right), \tag{22}$$

provided (here and following $\lfloor x \rfloor$ is the largest integer less than or equal to $x \in \mathbb{R}$)

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} a_{2k} = \sum_{k=1}^{\lfloor \frac{m+1}{2} \rfloor} a_{2k-1} = \frac{1}{2}. \tag{23}$$

We get Hann kernel (20) if we take $m = 1$ in (22).

6) Powers of the Hann window (see [24], formula(25a))

$$\lambda_{H,m}(u) := \cos^m\left(\frac{\pi u}{2}\right) \tag{24}$$

$$= \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\cos\left((k-\frac{m}{2})\pi u\right), \tag{25}$$

give a general Hann kernel in the form

$$s_{H,m}(t) = 2^{-m}\frac{\Gamma(1+m)}{\Gamma(1+\frac{m}{2}-t)\Gamma(1+\frac{m}{2}+t)}. \tag{26}$$

From ([14], Prorposition 2) we have that for $m = 0, 1, 2, \ldots$, and $\ell \leqslant m$

$$s_{H,m}(t) = \frac{1}{2^{m-\ell}}\sum_{k=0}^{m-\ell}\binom{m-\ell}{k} s_{H,\ell}(t+k-\frac{m-\ell}{2}). \tag{27}$$

Comparing the window function $\lambda_{H,m}$ in (25) and the general cosine window $\lambda_{C,\mathbf{a}}$ in (21) we see that the general Hann kernel in case of $m = 2n$ ($n \in \mathbb{N}$) is a special case of the Blackman-Harris kernel. Indeed $s_{H,2n} = s_{C,\mathbf{a}^*}$, where the parameter vector $\mathbf{a}^* \in \mathbb{R}^{n+1}$ has components $a^*_0 = \frac{1}{2^{2n}}\binom{2n}{n}$ and $a^*_k = \frac{1}{2^{2n-1}}\binom{2n}{n-k}$ for $k = 1, 2, \ldots, n$.

[1] "Hann" is the correct name of this window, although the conventional usage is "Hanning". It is named after the well-known Austrian meteorologist Julius Ferdinand von Hann (1839-1921) (see [23], pp. 95–100, [24])

4 Sampling operators with kernels, which are linear combinations of translated sinc functions

Many kernels, we considered for sampling operators $S_W f : C(\mathbb{R}) \to C(\mathbb{R})$ are in fact linear combinations of translated sinc-functions. In this case we can use Theorem 2.1 for the estimates of speed of approximation.

4.1 Hann sampling operators

Let us first consider Hann sampling operators $H_{W,m}$ $(m = 0, 1, 2, \ldots)$. The Hann kernel $s_{H,m}(t) = O(|t|^{-m-1})$ as $|t| \to \infty$ (cf. [26]) and we have rapidly decreasing kernels with small truncation error. If we take in (27) $\ell = 0$ we have a linear combination of sinc-functions because $s_{H,0} = \text{sinc}$.

Theorem 4.1. *Let $H_{W,m}$ $(m = 1, 2, \ldots)$ be the Hann sampling operator defined by (1) with the kernel (26). Then for $f \in \Lambda^p$ $(1 < p < \infty)$*

$$\|H_{W,m} f - f\|_p \leqslant M_m \tau_2(f; \frac{1}{W})_p. \tag{28}$$

The constant M_m is independent of f and W. Moreover, if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 < p < \infty$, we have

$$\lim_{W\to\infty} \|H_{W,m} f - f\|_p = 0.$$

Proof: According to (27) the sampling series $H_{W,m} f$ has the form

$$(H_{W,m} f)(t) = \frac{1}{2}\left[(H_{W,m-1} f)(t - \frac{1}{2W}) + (H_{W,m-1} f)(t + \frac{1}{2W})\right].$$

We obtain

$$\begin{aligned}(H_{W,m} f)(t) - f(t) &= \frac{1}{2}\Big[(H_{W,m-1} f)(t - \frac{1}{2W}) - f(t - \frac{1}{2W}) + (H_{W,m-1} f)(t + \frac{1}{2W}) - f(t + \frac{1}{2W})\\ &+ f(t - \frac{1}{2W}) - 2f(t) + f(t + \frac{1}{2W})\Big],\end{aligned}$$

which gives

$$\|H_{W,m} f - f\|_p \leqslant \|H_{W,m-1} f - f\|_p + \frac{1}{2}\omega_2(f; \frac{1}{2W})_p.$$

The proof by induction shows that

$$\|H_{W,m} f - f\|_p \leqslant \|H_{W,0} f - f\|_p + \frac{m}{2}\omega_2(f; \frac{1}{W})_p.$$

Since $H_{W,0} = S_W^{sinc}$ the Theorem 2.1, taking into account the properties of the averaged modulus of smoothness, implies the following

$$\|H_{W,m} f - f\|_p \leqslant \tau_2(f; \frac{1}{W})_p \Big(c_2 + \frac{m}{2}\Big).$$

The last assertion follows from (28) and (8). ∎

4.2 Blackman-Harris sampling operators

For the general Blackman-Harris sampling operator $C_{W,\mathbf{a}}$ we have the estimate of speed of approximation via averaged modulus of smoothness of order 2.

Theorem 4.2. *Let $C_{W,\mathbf{a}}$ be the Blackman-Harris sampling operator defined by (1) with the kernel (22), then for $f \in \Lambda^p$ $(1 < p < \infty)$*

$$\|C_{W,\mathbf{a}} f - f\|_p \leqslant M_{\mathbf{a}} \tau_2(f; \frac{1}{W})_p. \tag{29}$$

The constant $M_{\mathbf{a}}$ is independent of f and W. Moreover, if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 < p < \infty$, we have

$$\lim_{W\to\infty} \|C_{W,\mathbf{a}} f - f\|_p = 0.$$

Proof: The Blackman-Harris kernel (22) is a linear combination of translated sinc-functions. This allows us to give for the corresponding operator $C_{W,\mathbf{a}}$ the representation

$$\begin{aligned}(C_{W,\mathbf{a}} f)(t) &= \frac{1}{2}\sum_{j\in\mathbb{Z}} f(\frac{j}{W}) \sum_{k=0}^{m} a_k \Big(\text{sinc}(Wt - j + k) + \text{sinc}(Wt - j - k)\Big)\\ &= \frac{1}{2}\sum_{k=0}^{m} a_k \Big(\sum_{j\in\mathbb{Z}} f(\frac{j}{W})\,\text{sinc}(Wt - j + k) + \sum_{j\in\mathbb{Z}} f(\frac{j}{W})\,\text{sinc}(Wt - j - k)\Big)\\ &= \frac{1}{2}\sum_{k=0}^{m} a_k \Big((S_W^{sinc} f)(t + \frac{k}{W}) + (S_W^{sinc} f)(t - \frac{k}{W})\Big),\end{aligned}$$

which results

$$(C_{W,\mathbf{a}}f)(t)-f(t) = \frac{1}{2}\sum_{k=0}^{m} a_k\left[\left((S_W^{sinc}f)(t+\frac{k}{W})-f(t+\frac{k}{W})\right)+\left((S_W^{sinc}f)(t-\frac{k}{W})-f(t-\frac{k}{W})\right)\right.$$
$$\left. + \left(f(t-\frac{k}{W})-2f(t)+f(t+\frac{k}{W})\right)\right]. \tag{30}$$

If we take L^p norm of (31), we get

$$\|C_{W,\mathbf{a}}f-f\|_p \leqslant \sum_{k=0}^{m}|a_k|\left(\|S_W^{sinc}f-f\|_p+\frac{1}{2}\|\mathring{\Delta}_{k/W}^2 f\|_p\right).$$

Now we can use Theorem 2.1, the definition and properties of modulus of smoothness and the properties of the averaged modulus of smoothness:

$$\|C_{W,\mathbf{a}}f-f\|_p \leqslant \sum_{k=0}^{m}|a_k|\left(c_2\tau_2(f;\frac{1}{W})_p+\frac{1}{2}\omega_2(f;\frac{k}{W})_p\right) \leqslant \tau_2(f;\frac{1}{W})_p\sum_{k=0}^{m}|a_k|\left(c_2+\frac{k^2}{2}\right).$$

The last assertion follows from (29) and (8). ∎

Some special choices of the parameter vector $\mathbf{a}$ for the general Blackman-Harris sampling operator $C_{W,\mathbf{a}}$ allow us to estimate of speed of approximation via averaged modulus of smoothness of higher order than 2.

Proposition 4.3. *For $m\in\mathbb{N}$, $1\leqslant\ell\leqslant m$ the kernel*

$$s(t)=\mathrm{sinc}(t)-\frac{1}{2^{2\ell+1}}\sum_{j=0}^{m-\ell}(-1)^{j+\ell}q_j[\mathring{\Delta}_1^{2\ell}\,\mathrm{sinc}(t-j)+\mathring{\Delta}_1^{2\ell}\,\mathrm{sinc}(t+j)] \tag{31}$$

with $\mathbf{q}\in\mathbb{R}^{m-\ell+1}$, $\sum_{j=0}^{m-\ell}q_j=1$ is a Blackman-Harris kernel $s_{C,\mathbf{a}(\mathbf{q})}$ with parameter vector $\mathbf{a}(\mathbf{q})\in\mathbb{R}^{m+1}$.

Proof. The Blackman-Harris kernels are combinations of translated sinc functions. The coefficients of positive and negative translated sinc functions are equal and the sum of coefficients of both even- and odd-translated sinc functions are equal to 1/2. We show that all the assertions hold for (31).

Denote

$$s_{m,\ell,j}(t):=\mathrm{sinc}(t)-\frac{(-1)^{j+\ell}}{2^{2\ell+1}}[\mathring{\Delta}_1^{2\ell}\,\mathrm{sinc}(t-j)+\mathring{\Delta}_1^{2\ell}\,\mathrm{sinc}(t+j)], \tag{32}$$

which allows us to represent the kernel (31) in the form

$$s(t)=\sum_{j=0}^{m-\ell}q_j s_{m,\ell,j}(t).$$

We get from (32), using the central differences in form (6), the representation

$$s_{m,\ell,j}(t)=\mathrm{sinc}(t)-\frac{(-1)^{j+\ell}}{2^{2\ell+1}}\mathring{\Delta}_1^{2\ell}[\mathrm{sinc}(t-j)+\mathrm{sinc}(t+j)]=\mathrm{sinc}(t)-\frac{(-1)^{j+\ell}}{2^{2\ell+1}}\sum_{k=0}^{2\ell}(-1)^k\binom{2\ell}{k}[\mathrm{sinc}(t-j+\ell-k)+\mathrm{sinc}(t+j+\ell-k)].$$

It is well known that sums of binomial coefficients $\binom{2\ell}{k}$ with both even and odd k are equal to $2^{2\ell-1}$. Using this result we can see that the sum of coefficients of sinc functions with odd translates is equal to

$$\frac{1}{2^{2\ell+1}}2^{2\ell-1}2=\frac{1}{2}$$

and the sum of coefficients of sinc functions with odd translates is equal to

$$1-\frac{1}{2^{2\ell+1}}2^{2\ell-1}2=\frac{1}{2},$$

which indicates that the kernel $s_{m,\ell,j}$ in (32) is a Blackman-Harris kernel and the kernel (31) is Blackman-Harris kernel if

$$\sum_{j=0}^{n-\ell}q_j=1.$$

□

Now we are able to prove the following theorem:

Theorem 4.4. *For $C_{W,\mathbf{a}}$ ($\mathbf{a}\in\mathbb{R}^{m+1}$) let ℓ, $1\leqslant\ell\leqslant m$ be fixed. If there exists a parameter vector $\mathbf{q}\in\mathbb{R}^{m-\ell+1}$, such that we have for the kernel (22) a representation via central differences (6) in form (31), then for $f\in\Lambda^p$ $(1<p<\infty)$*

$$\|C_{W,\mathbf{a}}f-f\|_p\leqslant M_{\mathbf{a},\ell}\tau_{2\ell}(f;\frac{1}{W})_p. \tag{33}$$

The constant $M_{\mathbf{a},\ell}$ is independent of f and W. Moreover, if $f\in\Lambda^p\cap R_{loc}(\mathbb{R})$ for $1<p<\infty$, we have

$$\lim_{W\to\infty}\|C_{W,\mathbf{a}}f-f\|_p=0.$$

Proof: The representation (31) allows us to give for the corresponding operator $C_{W,\mathbf{a}}$ the representation

$$(C_{W,\mathbf{a}}f)(t)=(S_W^{sinc}f)(t)-\frac{1}{2^{2\ell+1}}\sum_{k=0}^{m-\ell}(-1)^{k+\ell}q_k\left(\mathring{\Delta}^{2\ell}_{1/W}(S_W^{sinc}f)(t+\frac{k}{W})+\mathring{\Delta}^{2\ell}_{1/W}(S_W^{sinc}f)(t-\frac{k}{W})\right),$$

which results

$$\begin{aligned}(C_{W,\mathbf{a}}f)(t)-f(t)&=(S_W^{sinc}f)(t)-f(t)-\frac{1}{2^{2\ell+1}}\sum_{k=0}^{m-\ell}(-1)^{k+\ell}q_k\left[\mathring{\Delta}^{2\ell}_{1/W}\left((S_W^{sinc}f)(t+\frac{k}{W})-f(t+\frac{k}{W})\right)\right.\\&+\left.\mathring{\Delta}^{2\ell}_{1/W}\left((S_W^{sinc}f)(t-\frac{k}{W})-f(t-\frac{k}{W})\right)\right]-\frac{1}{2^{2\ell+1}}\sum_{k=0}^{m-\ell}(-1)^{k+\ell}q_k\left(\mathring{\Delta}^{2\ell}_{1/W}f(t-\frac{k}{W})+\mathring{\Delta}^{2\ell}_{1/W}f(t+\frac{k}{W})\right).\end{aligned} \tag{34}$$

If we take L^p norm of (35), we get

$$\|C_{W,\mathbf{a}}f-f\|_p\leqslant\|S_W^{sinc}f-f\|_p\left(1+\sum_{k=0}^{m-\ell}|q_k|\right)+\|\mathring{\Delta}^{2\ell}_{1/W}f\|_p\frac{1}{2^{2\ell}}\sum_{k=0}^{m-\ell}|q_k|.$$

Now we can use Theorem 2.1, the definition and properties of modulus of smoothness and the properties of the averaged modulus of smoothness:

$$\|C_{W,\mathbf{a}}f-f\|_p\leqslant\tau_{2\ell}(f;\frac{1}{W})_p\left(c_{2\ell}\left(1+\sum_{k=0}^{m-\ell}|q_k|\right)+\frac{1}{2^{2\ell}}\sum_{k=0}^{m-\ell}|q_k|\right).$$

The last assertion follows from (33) and (8). ∎

We give some examples, how to apply Theorem 4.4. If we take $m=3$, then there exist one Blackman-Harris operator, for which we have the estimate of order of approximation via τ-modulus of smoothness of order 6.

Corollary 4.5. *Take $m=3$ and $\ell=3$ in Theorem 4.4. Then we have the estimate of order of approximation of the corresponding sampling operator $C_{W,\mathbf{a}^*}$ in terms of τ-modulus of smoothness of order 6. The corresponding parameter vector $\mathbf{a}^*$ is in form:*

$$\mathbf{a}^*=\frac{1}{32}(22,15,-6,1).$$

For $m=3$ and $\ell=2$ we have a family, depending on one parameter q, of sampling operators.

Corollary 4.6. *Take $m=3$ and $\ell=2$ in Theorem 4.4. Then we have the estimate of order of approximation of the corresponding sampling operator $C_{W,\mathbf{a}_q}$ in terms of τ-modulus of smoothness of order 4. The corresponding parameter vector, depending on a parameter $q\in\mathbb{R}$, is in form*

$$\mathbf{a}_q=\frac{1}{16}(10+2q,8-q,-2-2q,q).$$

Some choices of the parameter q in Corollary 4.6 give us sampling operators with special properties.

Remark. If we take $q=0$ in Corollary 4.6, then we have the case, corresponding to $m=2$ and $\ell=2$ in Theorem 4.4. If we take $q=-1$ in Corollary 4.6, then the sampling operator $\overline{C}_{W,\mathbf{a}_{-1}}$, defined by (1) with the kernel $s(t)=2s_{C,\mathbf{a}_{-1}}(2t)$ is interpolating (see [16]).

4.3 Subordination by Rogosinski-type sampling operators

The Rogosinski-type sampling operators give us the opportunity to represent other sampling operators with even bandlimited kernels $s\in B^1_\pi$. Indeed, in [9] we proved the following subordination equalities

$$S_Wf=2\sum_{j=0}^{\infty}s(j+1/2)R_{W,j}f, \tag{35}$$

$$S_Wf-f=2\sum_{j=0}^{\infty}s(j+1/2)(R_{W,j}f-f). \tag{36}$$

By (16) we have for $f\in\Lambda^p$, $1\leqslant p<\infty$, satisfying $(1/p+1/q=1)$

$$\|R_{W,j}f\|_p\leqslant m_0^{1/q}(r_j)\|r_j\|_1^{1/p}\|f\|_{\ell^p(W)}\quad(W>0). \tag{37}$$

For the operator norm we proved (see [12] or [9], Th. 3), that

$$m_0(r_j) = \|R_{W,j}\| = \frac{4}{\pi}\sum_{\ell=0}^{2j}\frac{1}{2\ell+1} = \frac{2}{\pi}\ln(j+1) + O(1) \quad (j=0,1,\ldots).$$

We can show, that

$$\|r_j\|_1 = 2\sum_{\ell=0}^{2j}(-1)^k\Big(\operatorname{Sci}(\ell+1)-\operatorname{Sci}(\ell)\Big) = \frac{2}{\pi}\ln(j+1)+O(1) \quad (j=0,1,\ldots),$$

where the integral sinc is defined by

$$\operatorname{Sci}(x) := \int_0^x \operatorname{sinc}(v)dv.$$

So we need

$$\sum_{j=0}^{\infty}|s(j+1/2)|\log(j+1) < \infty$$

for (35). To use (36) we need

$$\sum_{j=0}^{\infty}|s(j+1/2)|(j+1)^2 < \infty,$$

as the proof of the following theorem suggest.

Theorem 4.7. *Let $R_{W,j}$ $(j=0,1,2,\ldots)$ be the Rogosinski-type sampling operator defined by (1) with the kernel (26), then for $f \in \Lambda^p$ $(1 < p < \infty)$*

$$\|R_{W,j}f - f\|_p \leqslant M_j\tau_2(f;\frac{1}{W})_p. \tag{38}$$

The constant M_j is independent of f and W. Moreover, if $f \in \Lambda^p \cap R_{loc}(\mathbb{R})$ for $1 < p < \infty$, we have

$$\lim_{W\to\infty}\|R_{W,j}f - f\|_p = 0.$$

Proof: The Rogosinski-type kernel (19) is a aritheoremetic mean of two translated sinc-functions. This allows us to give for the corresponding operator $R_{W,j}$ the representation

$$(R_{W,j}f)(t) = \frac{1}{2}\Big((S_W^{sinc}f)(t+\frac{2j+1}{2W}) + (S_W^{sinc}f)(t-\frac{2j+1}{2W})\Big).$$

The rest of the proof is analogous to the proof of Theorem 4.2. ■

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References

[1] P. L. Butzer, W. Splettstößer, R. L. Stens, The sampling theorems and linear prediction in signal analysis., Jahresber. Deutsch. Math-Verein 90 (1988) 1–70.

[2] A. F. Timan, Theory of Approximation of Functions of a Real Variable, MacMillan, New York, 1965, (Orig. Russian ed. Moscow, 1960).

[3] M. Theis, Über eine interpolationsformel von de la Vallee-Poussin., Math. Z. 3 (1919) 93–113.

[4] P. L. Butzer, G. Schmeisser, R. L. Stens, An introduction to sampling analysis, in: F. Marvasti (Ed.), Nonuniform Sampling, Theory and Practice, Kluwer, New York, 2001, pp. 17–121.

[5] R. L. Stens, Sampling with generalized kernels, in: J. R. Higgins, R. L. Stens (Eds.), Sampling Theory in Fourier and Signal Analysis: Advanced Topics, Clarendon Press, Oxford, 1999.

[6] P. L. Butzer, C. Bardaro, R. L. Stens, G. Vinti, Approximation error of the Whittaker cardinal series in terms of an averaged modulus of smoothness covering discontinuous signals., J. Math. Anal. Appl. 316 (2006) 269–306.

[7] P. L. Butzer, R. L. Stens, Reconstruction of signals in $L^p(\mathbb{R})$-space by generalized sampling series based on linear combinations of B-splines., Integral Transforms Spec. Funct. 19 (2008) 35–58.

[8] P. L. Butzer, C. Bardaro, R. L. Stens, G. Vinti, Prediction by samples from the past with error estimates covering discontinious signals., IEEE Transactions on Information Theory 56 (2010) 614–633.

[9] A. Kivinukk, G. Tamberg, On sampling series based on some combinations of sinc functions., Proc. of the Estonian Academy of Sciences. Physics Mathematics 51 (2002) 203–220.

[10] Z. Burinska, K. Runovski, H.-J. Schmeisser, On the approximation by generalized sampling series in L_p-metrics., Sampling Theory in Signal and Image Processing 5 (2006) 59–87.

[11] A. Kivinukk, Approximation by typical sampling series, in: Proc. 1999 Intern. Workshop on Sampling Theory and Applications, Loen, Norway, Norwegian Univ. Sci. and Technology, 1999, pp. 161–166.

[12] A. Kivinukk, G. Tamberg, Subordination in generalized sampling series by Rogosinski-type sampling series, in: Proc. 1997 Intern. Workshop on Sampling Theory and Applications, Aveiro, Portugal, 1997, Univ. Aveiro, 1997, pp. 397–402.

[13] A. Kivinukk, G. Tamberg, Blackman-type windows for sampling series., J. of Comp. Analysis and Applications 7 (2005) 361–372.

[14] A. Kivinukk, G. Tamberg, On sampling operators defined by the Hann window and some of their extensions., Sampling Theory in Signal and Image Processing 2 (2003) 235–258.

[15] A. Kivinukk, G. Tamberg, On Blackman-Harris windows for Shannon sampling series., Sampling Theory in Signal and Image Processing 6 (2007) 87–108.

[16] A. Kivinukk, G. Tamberg, Interpolating generalized Shannon sampling operators, their norms and approximation properties., Sampling Theory in Signal and Image Processing 8 (2009) 77–95.

[17] B. Sendov, V. Popov, The Averaged Moduli of Smoothness, Wiley, Chichester, 1988.

[18] P. L. Butzer, R. J. Nessel, Fourier Analysis and Approximation, Vol. 1, Birkhäuser Verlag, Basel, Stuttgart, 1971.

[19] R. Trigub, E. Belinsky, Fourier Analysis and Approximation of Functions, Kluwer, Dordrecht, 2004.

[20] S. M. Nikolskii, Approximation of Functions of Several Variables and Imbedding Theorems, Springer, Berlin, 1975, (Orig. Russian ed. Moscow, 1969).

[21] J. R. Higgins, Sampling Theory in Fourier and Signal Analysis, Clarendon Press, Oxford, 1996.

[22] H. H. Albrecht, A family of cosine-sum windows for high resolution measurements, in: IEEE International Conference on Acoustics, Speech and Signal Processing, Salt Lake City, Mai 2001, IEEE, Salt Lake City, 2001, pp. 3081–3084.

[23] R. B. Blackman, J. W. Tukey, The measurement of power spectra, Wiley-VCH, New York, 1958.

[24] F. J. Harris, On the use of windows for harmonic analysis., Proc. of the IEEE 66 (1978) 51–83.

[25] H. D. Meikle, A New Twist to Fourier Transforms, Dover, Berlin, 2004.

[26] G. Tamberg, On truncation error of some generalized Shannon sampling operators., Numerical Algorithms 55 (2010) 367–382.