PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 19 · 2026 · Pages 116–126

Rate of system pole detection using Hermite-Padé approximants to polynomial expansions

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Communicated by Wolfgang Erb

Abstract

In this paper, we study the rate at which system poles can be detected via Hermite-Padé approximants constructed from polynomial expansions based on orthogonal and Faber polynomials over a compact set E. Our analysis focuses on certain indicators introduced by Gonchar [13] which quantify the detection of poles by rows of the Padé table. We extend Gonchar's indicator formulas to our generalized Hermite-Padé approximants and explicitly compute the values of these indicators for the system poles of the vector of approximated functions.

Keywords: orthogonal polynomials, Faber polynomials, orthogonal Padé approximation, Padé-Faber approximation, Hermite-Padé approximation, rate of convergence

2020 Mathematics Subject Classification: 30E10, 41A21, 41A25

1 Introduction

Padé approximation is a powerful mathematical technique for approximating a function by rational functions with free poles. Unlike the Taylor series, which expresses a function as an infinite sum of polynomial terms, Padé approximation often provides more accurate and efficient representations, especially for functions with singularities. One of the main features of this method is its ability to detect the singularities of an approximated function. To illustrate the motivation behind this work, we first revisit the definition of classical Padé approximants:

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathbb{P}_n$ be the space of all polynomials of degree at most n. Consider

$$F(z) = \sum_{k=0}^{\infty} f_k z^k. \tag{1}$$

For given $n, m \in \mathbb{N}_0$, we can find $P \in \mathbb{P}_n$ and $Q \in \mathbb{P}_m, Q \not\equiv 0$, such that

$$(QF - P)(z) = \mathcal{O}(z^{n+m+1}), \qquad \text{as} \quad z \to 0.$$

The rational function

$$R_{n,m} := \frac{P}{Q} = \frac{P_{n,m}}{Q_{n,m}} \tag{2}$$

is called the (n, m) *classical Padé approximant of* F. The polynomials $P_{n,m}$ and $Q_{n,m}$ in the equation (2) are chosen such that $Q_{n,m}$ is monic, and $P_{n,m}$ and $Q_{n,m}$ have no common zeros. It is well-known that for any $n, m \in \mathbb{N}_0$, $R_{n,m}$ and $Q_{n,m}$ are uniquely determined. This approximation is named after Padé [18] who organized these rational functions into a table and explored the connection between these approximants and continued fractions in 1892. However, this approximation had been studied earlier by several mathematicians, for example by Lagrange [17] in 1776, Jacobi [14] in 1846, and Frobenius [11] in 1881.

For a function F as in the equation (1), we denote by $R_0(F)$ the radius of the largest open disk at the origin to which F can be extended analytically and by $R_m(F)$ the radius of the largest open disk at the origin to which F can be extended meromorphically with at most m poles counting multiplicities. Set

$$\mathbb{B}(a, R) := \{z \in \mathbb{C} : |z - a| < R\}.$$

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We denote by Q_m^F the monic polynomial whose zeros are the poles of F in $\mathbb{B}(0,R_m(F))$ counting their multiplicity and by $\mathcal{P}_m(F)$ the set of all distinct zeros of Q_m^F. In all what follows, m remains fixed and the sequence $\{R_{n,m}\}_{n\geq 0}$ is called the *m-th row sequence of classical Padé approximants of F*.

A cornerstone in the study of row sequences of Padé approximants is the Montessus de Ballore theorem [10] stated that if $R_0(F)>0$ and F has exactly m poles in $\mathbb{B}(0,R_m(F))$, then for each compact subset K of $\mathbb{B}(0,R_m(F))\setminus\mathcal{P}_m(F)$,

$$\limsup_{n\to\infty}\|F-R_{n,m}\|_K^{1/n}\leq\frac{\|z\|_K}{R_m(F)} \tag{3}$$

and

$$\limsup_{n\to\infty}\|Q_m^F-Q_{n,m}\|^{1/n}\leq\frac{\max\{|\lambda|:\lambda\in\mathcal{P}_m(F)\}}{R_m(F)}, \tag{4}$$

where $\|\cdot\|_K$ denotes a sup-norm on K and $\|\cdot\|$ denotes the coefficient norm in the space $\mathbb{P}_m$. Indeed, it was later proved in [12] that the above inequalities are actually equalities. As stated in (4), all poles of $R_{n,m}$ converge to those of F at a geometric rate as $n\to\infty$. Therefore, $R_{n,m}$ replicates F in $\mathbb{B}(0,R_m(F))$ at the rate given in (3).

In 1981, Gonchar [13] brought renewed attention to the behavior of row sequences in classical Padé approximation. His work established a significant result describing the meromorphic continuation of F in terms of the geometric asymptotic behavior of the poles of its approximants. This breakthrough also played a crucial role in shaping the study of inverse problems in Padé approximation and its extensions. Numerous later studies (see, e.g., [1, 2, 7, 8, 9, 15, 16, 22, 24]) were inspired by his findings. In the Gonchar paper, he introduced three key indicators to quantify the rate at which the poles of the approximants converge to the poles of F. Let $m\in\mathbb{N}$ be fixed. Let

$$\mathcal{P}_{n,m}:=\{\lambda_{n,1},\lambda_{n,2},\ldots,\lambda_{n,m_n}\},\qquad m_n\leq m,\qquad n\in\mathbb{N}_0$$

denote the collection of zeros of $Q_{n,m}$ (repeated according to their multiplicity). Define

$$|z-w|_1:=\min\{1,|z-w|\},\qquad z,w\in\mathbb{C}.$$

Fix $\lambda\in\mathbb{C}\setminus\{0\}$. The first indicator is defined by

$$\Delta(\lambda):=\limsup_{n\to\infty}\prod_{j=1}^{m_n}|\lambda_{n,j}-\lambda|_1^{1/n}=\limsup_{n\to\infty}\prod_{|\lambda_{n,j}-\lambda|<1}|\lambda_{n,j}-\lambda|^{1/n}.$$

Clearly, $0\leq\Delta(\lambda)\leq 1$ (when $m_n=0$ or $|\lambda_{n,j}-\lambda|\geq 1$ for all $j=1,2,\ldots,m_n$, the product is taken to be 1). To define the second indicator, we suppose that for each n, the points in

$$\mathcal{P}_{n,m}=\{\lambda_{n,1},\lambda_{n,2},\ldots,\lambda_{n,m_n}\}$$

are enumerated in nondecreasing distance to the point λ. We set

$$\delta_j(\lambda):=\limsup_{n\to\infty}|\lambda_{n,j}-\lambda|_1^{1/n},\qquad j=1,2,\ldots,m',$$

where $m':=\liminf_{n\to\infty}m_n$ and for $j=m'+1,\ldots,m$, we define $\delta_j(\lambda):=1$. Obviously, we have $0\leq\delta_j(\lambda)\leq 1$. The third indicator, a nonnegative integer $\sigma(\lambda)$, is defined as follows. If $\Delta(\lambda)=1$ (in that case, all $\delta_j(\lambda)=1$), then $\sigma(\lambda)=0$. If $\Delta(\lambda)<1$, then for some τ, $1\leq\tau\leq m$, we have that $\delta_1(\lambda)\leq\ldots\leq\delta_\tau(\lambda)<1$ and $\delta_{\tau+1}(\lambda)=1$ or $\tau=m$; in this case, we take $\sigma(\lambda)=\tau$. Basically, $\sigma(\lambda)$ counts the number of poles of $R_{n,m}$ that converge to λ as $n\to\infty$ at a geometric rate.

Gonchar's theorem [13] is the following.

Theorem 1.1 (Gonchar's theorem)**.** *Let F be defined as in the equation (1), $m\in\mathbb{N}$, and let $0\neq\lambda\in\mathbb{C}$. The following two assertions are equivalent:*

(a) F has a pole at $\lambda\in\mathbb{B}(0,R_m(F))$.

(b) $\Delta(\lambda)<1$ (or equivalently $\sigma(\lambda)\geq 1$).

If either (a) or (b) takes place, then

$$\Delta(\lambda)=\frac{|\lambda|}{R_m(F)}\qquad\text{and}\qquad\sigma(\lambda)=\tau, \tag{5}$$

where τ is the order of the pole at λ. Moreover, if we assume further that $\liminf_{n\to\infty}|\lambda-\lambda_{n,\tau+1}|>0$, then

$$\delta_1(\lambda)=\delta_2(\lambda)=\ldots=\delta_\tau(\lambda)=\left(\frac{|\lambda|}{R_m(F)}\right)^{1/\tau}. \tag{6}$$

In summary, λ is a pole of F of order τ in $\mathbb{B}(0, R_m(F))$ if and only if it attracts exactly τ zeros of the polynomial $Q_{n,m}$ with a geometric rate as $n \to \infty$. All three indicators are fully computed in (5) and (6).

In many applications, one encounters expansions that are beyond the Taylor series. Functions defined on more general domains are naturally expanded in terms of orthogonal or Faber polynomials. In such settings, the classical framework for pole detection is not directly applicable. However, recent work [24] has demonstrated that Gonchar's indicators can indeed be extended to Padé approximants constructed from orthogonal and Faber expansions. Building upon these developments, we investigate an analogue of Gonchar's theorem for Hermite-Padé approximants constructed from orthogonal and Faber polynomials. In the present work, the indicators Δ and σ corresponding to such Hermite-Padé approximants are estimated and computed.

The paper is organized as follows. In Section 2, we state definitions of orthogonal Hermite-Padé and Faber-Hermite-Padé approximants and recall some known results. We keep all main results in Section 3. Section 4 and Section 5 are devoted to auxiliary results and the proofs of the main results, respectively.

2 Background: Hermite-Padé Approximants to Polynomial Expansions

The construction of the first approximation is based on orthogonal polynomials. Let E be an infinite compact subset of the complex plane $\mathbb{C}$ such that $\overline{\mathbb{C}} \setminus E$ is simply connected. Throughout this paper, our set E is as expressed above. Let μ be a finite positive Borel measure with infinite support $\operatorname{supp}(\mu)$ contained in E. We say $\mu \in \mathcal{M}(E)$ when μ satisfies the above condition. Define the inner product

$$\langle g, h \rangle := \int g(\zeta)\overline{h(\zeta)}d\mu(\zeta), \quad g, h \in L_2(\mu).$$

For each $n \in \mathbb{N}_0$, let

$$p_n(z) := \kappa_n z^n + \cdots,$$

be the unique orthonormal polynomial of degree n with respect to μ with $\kappa_n > 0$; that is,

$$\langle p_n, p_m \rangle = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

Denote by $\mathcal{H}(E)$ the space of all functions holomorphic in some neighborhood of E. Set

$$\mathcal{H}(E)^d := \{(F_1, F_2, \ldots, F_d) : F_\alpha \in \mathcal{H}(E) \text{ for all } \alpha = 1, 2, \ldots, d\}.$$

Definition 2.1. Let $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ and $\mu \in \mathcal{M}(E)$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set

$$|\mathbf{m}| := m_1 + m_2 + \ldots + m_d.$$

Then, there exists $Q^\mu_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ such that $Q^\mu_{n,\mathbf{m}} \not\equiv 0$ and

$$\langle z^k Q^\mu_{n,\mathbf{m}} F_\alpha, p_n \rangle = 0 \qquad \text{for all } k = 0, 1, \ldots, m_\alpha - 1 \text{ and } \alpha = 1, 2, \ldots, d. \tag{7}$$

The corresponding vector of rational functions

$$\begin{aligned} \mathbf{R}^\mu_{n,\mathbf{m}} &:= (R^\mu_{n,\mathbf{m},1}, R^\mu_{n,\mathbf{m},2}, \ldots, R^\mu_{n,\mathbf{m},d}) \\ &= \left(\frac{\sum_{j=0}^{n-1} \langle Q^\mu_{n,\mathbf{m}} F_1, p_j \rangle p_j}{Q^\mu_{n,\mathbf{m}}}, \frac{\sum_{j=0}^{n-1} \langle Q^\mu_{n,\mathbf{m}} F_2, p_j \rangle p_j}{Q^\mu_{n,\mathbf{m}}}, \ldots, \frac{\sum_{j=0}^{n-1} \langle Q^\mu_{n,\mathbf{m}} F_d, p_j \rangle p_j}{Q^\mu_{n,\mathbf{m}}} \right) \end{aligned}$$

is called an $(n, \mathbf{m})$ *modified orthogonal Hermite–Padé (MOHP) approximant of* $\mathbf{F}$ *corresponding to* μ.

The second approximation is built from Faber polynomials defined as follows. Let Φ be the exterior conformal bijection from $\overline{\mathbb{C}} \setminus E$ to $\{w \in \overline{\mathbb{C}} : |w| > 1\}$ satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$. For each $\rho > 1$, we define

$$\Gamma_\rho := \{z \in \mathbb{C} : |\Phi(z)| = \rho\} \qquad \text{and} \qquad D_\rho := E \cup \{z \in \mathbb{C} : |\Phi(z)| < \rho\},$$

as the *level curve of index* ρ and the *canonical domain of index* ρ, respectively. Given $F \in \mathcal{H}(E)$, denote by $\rho_0(F)$ the index ρ of the largest canonical domain D_ρ to which F can be holomorphically extended and by $\rho_m(F)$ the index of the largest canonical domain D_ρ to which F can be meromorphically extended with at most m poles counting multiplicities. The indices $\rho_0(F)$ and $\rho_m(F)$ generalize $R_0(F)$ and $R_m(F)$, respectively.

The *Faber polynomial* corresponding to E of degree n is given by the following formula

$$\Phi_n(z) := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{\Phi^n(t)}{t - z} dt, \qquad z \in D_\rho, \qquad n = 0, 1, 2, \ldots.$$

The n-th *Faber coefficient* of $F \in \mathcal{H}(E)$ is defined by

$$[F]_n := \frac{1}{2\pi i} \int_{\Gamma_\rho} \frac{F(t)\Phi'(t)}{\Phi^{n+1}(t)} dt, \tag{8}$$

where $\rho \in (1, \rho_0(F))$.

Definition 2.2. Let $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$. Fix a multi-index $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}$. Set

$$|\mathbf{m}| := m_1 + m_2 + \ldots + m_d.$$

Then, there exists $Q^E_{n,\mathbf{m}} \in \mathbb{P}_{|\mathbf{m}|}$ such that $Q^E_{n,\mathbf{m}} \not\equiv 0$ and

$$[z^k Q^E_{n,\mathbf{m}} F_\alpha]_n = 0 \qquad \text{for all } k = 0, 1, \ldots, m_\alpha - 1 \text{ and } \alpha = 1, 2, \ldots, d. \tag{9}$$

The corresponding vector of rational functions

$$\mathbf{R}^E_{n,\mathbf{m}} := (R^E_{n,\mathbf{m},1}, R^E_{n,\mathbf{m},2}, \ldots, R^E_{n,\mathbf{m},d})$$
$$= \left(\frac{\sum_{j=0}^{n-1} [Q^E_{n,\mathbf{m}} F_1]_j \Phi_j}{Q^E_{n,\mathbf{m}}}, \frac{\sum_{j=0}^{n-1} [Q^E_{n,\mathbf{m}} F_2]_j \Phi_j}{Q^E_{n,\mathbf{m}}}, \ldots, \frac{\sum_{j=0}^{n-1} [Q^E_{n,\mathbf{m}} F_d]_j \Phi_j}{Q^E_{n,\mathbf{m}}} \right)$$

is called an $(n, \mathbf{m})$ *modified Faber–Hermite–Padé (MFHP) approximant of* $\mathbf{F}$ *corresponding to* E.

The *second kind function corresponding to* p_n defined by

$$s_n(z) := \int \frac{\overline{p_n(\zeta)}}{z - \zeta} d\mu(\zeta), \qquad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$$

plays an important role in our study. When we study asymptotic behaviors of MOHP approximation, we need to limit our interests to subclasses of $\mathcal{M}(E)$. Let us list needed subclasses in this paper in Definition 2.3.

Definition 2.3. Let $\mu \in \mathcal{M}(E)$.

(a) We say that $\mu \in \mathbf{Reg}(E)$ if and only if

$$\operatorname{cap}(\operatorname{supp}(\mu)) = \operatorname{cap}(E),$$

where $\operatorname{cap}(E)$ denotes the logarithmic capacity of E, and

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|,$$

uniformly on compact subsets of $\mathbb{C} \setminus \operatorname{Co}(E)$, where $\operatorname{Co}(E)$ denotes the convex hull of E.

(b) We say that $\mu \in \mathbf{Reg}_1(E)$ if and only if

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|,$$

uniformly on compact subsets of $\mathbb{C} \setminus E$.

(c) We say that $\mu \in \mathbf{Reg}_2(E)$ if and only if

$$\lim_{n\to\infty} |s_n(z)|^{1/n} = |\Phi(z)|^{-1}, \tag{10}$$

uniformly on compact subsets of $\mathbb{C} \setminus E$.

(d) We say that $\mu \in \mathbf{Reg}_{1,2}(E)$ if and only if $\mu \in \mathbf{Reg}_1(E) \cap \mathbf{Reg}_2(E)$.

(e) We say that $\mu \in \mathbf{Reg}^m_{1,2}(E)$ if and only if $\mu \in \mathbf{Reg}_{1,2}(E)$ and there exist a constant $c > 0$ and $n_0 \in \mathbb{N}$ such that the leading coefficients κ_n of the orthonormal polynomials p_n satisfy

$$\frac{\kappa_{n-m}}{\kappa_n} \geq c,$$

for all $n \geq n_0$.

The class $\mathbf{Reg}(E)$ is identical to the "regular" class described in Stahl and Totik's monograph [21]. A list of its defining properties can be found in [21, Theorem 3.1.1], while various regularity criteria are given in [21, Chapter 4]. When E is convex and $\operatorname{cap}(\operatorname{supp}(\mu)) = \operatorname{cap}(E)$, we have $\mathbf{Reg}(E) = \mathbf{Reg}_1(E) = \mathbf{Reg}_2(E) = \mathbf{Reg}_{1,2}(E)$ (see [5, pp. 20-21] for details). The classes in Definition 2.3 are extensively understood when μ is supported either on a real interval or on the unit circle (see [19, 20]). However, for a general compact set E, the picture is much less complete and the principal reference remains [21].

Finding $Q^\mu_{n,\mathbf{m}}$ or $Q^E_{n,\mathbf{m}}$ is equivalent to solving $|\mathbf{m}|+1$ unknowns from a system of $|\mathbf{m}|$ linear equations in (7) or (9), respectively. Because the number of unknowns is greater than the number of linear equations, $Q^\mu_{n,\mathbf{m}}$ and $Q^E_{n,\mathbf{m}}$ always exist but may not be unique. Since $Q^\mu_{n,\mathbf{m}} \not\equiv 0$ and $Q^E_{n,\mathbf{m}} \not\equiv 0$, we normalize them so that they are monic polynomials. Note that for any $(n, \mathbf{m}) \in \mathbb{N} \times \mathbb{N}^d$, $\mathbf{R}^\mu_{n,\mathbf{m}}$ and $\mathbf{R}^E_{n,\mathbf{m}}$ may not be unique. The approximations in Definitions 2.1 and 2.2 were recently introduced by Bosuwan and López Lagomasino in [5] and [6], where necessary and sufficient conditions for the geometric convergence of $Q^\mu_{n,\mathbf{m}}$ and $Q^E_{n,\mathbf{m}}$ along the $\mathbf{m}$-th row sequence were established. Additional results on MOHP and MFHP approximations and their applications can be found in [1, 2, 4, 24]. While Definitions 2.1 and 2.2 may initially appear unnatural (see the standard versions of orthogonal Hermite-Padé and Faber-Hermite-Padé approximants in Definitions 5 and 7 of [3]), their construction was inspired by the structure of type II Hermite-Padé approximants (see the discussion in [9]). Crucially, such modified structure leads to an inverse statement of Montessus de Ballore's theorem. The zeros of $Q^\mu_{n,\mathbf{m}}$ and $Q^E_{n,\mathbf{m}}$ can be used to localize system poles of a vector of functions $\mathbf{F}$ around the set E.

Definition 2.4. Given $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ and $\mathbf{m} = (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$, we say that $\lambda \in \mathbb{C}$ is a *system pole of order* τ *of* $\mathbf{F}$ *with respect to* $\mathbf{m}$ if τ is the largest positive integer such that for each $t = 1, 2, \ldots, \tau$, there exists at least one polynomial combination of the form

$$\sum_{\alpha=1}^{d} v_\alpha F_\alpha, \qquad \deg v_\alpha < m_\alpha, \qquad \alpha = 1, 2, \ldots, d, \tag{11}$$

which is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\lambda)|}$ except for a pole at $z = \lambda$ of exact order t.

Note that in the scalar case ($d = 1$), the concepts of a system pole and a pole coincide. Moreover, the definition of a system pole depends on the multi-index $\mathbf{m}$. However, since we will fix $\mathbf{m}$ throughout the paper, we may sometimes omit it for simplicity.

Let τ be the order of λ as a system pole of $\mathbf{F}$ with respect to $\mathbf{m}$. For each $t = 1, 2, \ldots, \tau$, we let $\rho_{\lambda,t}(\mathbf{F}, \mathbf{m})$ be the largest index among all indices $\rho_t(G)$, where G is a polynomial combination of type (11) that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\lambda)|}$ except for a pole at $z = \lambda$ of order t. We define

$$\boldsymbol{\rho}_\lambda(\mathbf{F}, \mathbf{m}) := \min_{t=1,2,\ldots,\tau} \rho_{\lambda,t}(\mathbf{F}, \mathbf{m}). \tag{12}$$

Let $Q^{\mathbf{F}}_{\mathbf{m}}$ stand for the monic polynomial whose zeros are the system poles of $\mathbf{F}$ with respect to $\mathbf{m}$ taking account of their order and $\mathcal{P}_{\mathbf{m}}(\mathbf{F})$ stand for the set of all zeros of $Q^{\mathbf{F}}_{\mathbf{m}}$. The following is a combination of [5, Theorem 1.2] and [6, Theorem 1.4].

Theorem 2.1. *Let* $\mathbf{F} = (F_1, F_2, \ldots, F_d) \in \mathcal{H}(E)^d$ *and* $\mathbf{m} \in \mathbb{N}^d$ *be a fixed multi-index. Then, the following assertions are equivalent:*

(a) $\mathbf{F}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

(b) The polynomials $Q^{\mu}_{n,\mathbf{m}}$ *corresponding to* $\mu \in \mathbf{Reg}^{|\mathbf{m}|}_{1,2}(E)$ *are uniquely determined for all sufficiently large* n*, and there exists a polynomial* $\tilde{Q}_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$ *such that*

$$\limsup_{n\to\infty} \|Q^{\mu}_{n,\mathbf{m}} - \tilde{Q}_{|\mathbf{m}|}\|^{1/n} = \tilde{\theta} < 1. \tag{13}$$

(c) The polynomials $Q^{E}_{n,\mathbf{m}}$ *are uniquely determined for all sufficiently large* n*, and there exists a polynomial* $Q^{*}_{|\mathbf{m}|}$ *of degree* $|\mathbf{m}|$ *such that*

$$\limsup_{n\to\infty} \|Q^{E}_{n,\mathbf{m}} - Q^{*}_{|\mathbf{m}|}\|^{1/n} = \theta^* < 1. \tag{14}$$

Moreover, if one of the assertions (a)–(c) takes place, then $\tilde{Q}_{|\mathbf{m}|} = Q^{*}_{|\mathbf{m}|} = Q^{\mathbf{F}}_{\mathbf{m}}$ *and*

$$\tilde{\theta} = \theta^* = \max\left\{ \frac{|\Phi(\lambda)|}{\boldsymbol{\rho}_\lambda(\mathbf{F}, \mathbf{m})} : \lambda \in \mathcal{P}_{\mathbf{m}}(\mathbf{F}) \right\}.$$

Remark 1. Under the conditions that (a) holds and $\mu \in \mathbf{Reg}_2(E)$, the polynomials $Q^{\mu}_{n,\mathbf{m}}$ are uniquely determined for all sufficiently large n and

$$\limsup_{n\to\infty} \|Q^{\mu}_{n,\mathbf{m}} - Q^{\mathbf{F}}_{\mathbf{m}}\|^{1/n} \le \max\left\{ \frac{|\Phi(\lambda)|}{\boldsymbol{\rho}_\lambda(\mathbf{F}, \mathbf{m})} : \lambda \in \mathcal{P}_{\mathbf{m}}(\mathbf{F}) \right\}. \tag{15}$$

Although the authors of [5] did not explicitly mention Remark 1 in their paper, the proof of (15) under the condition $\mu \in \mathbf{Reg}_2(E)$ can be found on pages 24-28 of [5]. Furthermore, since $\dim(\mathbb{P}_{|\mathbf{m}|}) = |\mathbf{m}| + 1 < \infty$, all norms on the space $\mathbb{P}_{|\mathbf{m}|}$ are equivalent. This implies that the coefficient norm in (13) and (14) may be replaced by any other norm on $\mathbb{P}_{|\mathbf{m}|}$.

An analogue of (a)⇒(b) in Gonchar's theorem for MOHP and MFHP approximations were considered by Bosuwan [2]. Given $\lambda \in \mathbb{C}$ and $\mathbf{m} \in \mathbb{N}^d$, the notations $\Delta^{\mu}(\lambda)$, $\sigma^{\mu}(\lambda)$, $\Delta^{E}(\lambda)$, and $\sigma^{E}(\lambda)$ in the following theorem are defined as in Section 1 taking

$$\mathcal{P}^{\mu}_{n,\mathbf{m}} := \{\tilde{\lambda}_{n,1}, \tilde{\lambda}_{n,2}, \ldots, \tilde{\lambda}_{n,\beta_n}\}, \qquad \beta_n \le |\mathbf{m}|,$$
$$\mathcal{P}^{E}_{n,\mathbf{m}} := \{\lambda^*_{n,1}, \lambda^*_{n,2}, \ldots, \lambda^*_{n,\gamma_n}\}, \qquad \gamma_n \le |\mathbf{m}|,$$

to be the collections of zeros of $Q^{\mu}_{n,\mathbf{m}}$ and $Q^{E}_{n,\mathbf{m}}$ enumerated in nondecreasing distance to λ, respectively.

The rate of attraction of a system pole λ of $\mathbf{F}$ to the poles of MOHP and MFHP approximants is given in the following theorem (see [2]).

Theorem 2.2. *Let* $\mathbf{F} \in \mathcal{H}(E)^d$ *and fix* $\mathbf{m} \in \mathbb{N}^d$*. Assume that* λ *is a system pole of order* τ *of* $\mathbf{F}$ *with respect to* $\mathbf{m}$*. Then, the following are true.*

(a) If $\mu \in \mathbf{Reg}_2(E)$*, then*

$$\Delta^{\mu}(\lambda) \le \frac{|\Phi(\lambda)|}{\boldsymbol{\rho}_\lambda(\mathbf{F}, \mathbf{m})} \qquad \textit{and} \qquad \sigma^{\mu}(\lambda) \ge \tau. \tag{16}$$

(b)

$$\Delta^{E}(\lambda) \le \frac{|\Phi(\lambda)|}{\boldsymbol{\rho}_\lambda(\mathbf{F}, \mathbf{m})} \qquad \textit{and} \qquad \sigma^{E}(\lambda) \ge \tau. \tag{17}$$

The aim of this work is to refine the estimates of $\Delta^{\mu}(\lambda)$ and $\Delta^{E}(\lambda)$ in (16) and (17), and show that our new refined estimates cannot be improved.

3 Main Results

The weaker version of Gonchar's theorem for MOHP and MFHP approximants is stated in the following.

Theorem 3.1. *Let* $\mathbf{F} \in \mathcal{H}(E)^d$ *and* $\mathbf{m} \in \mathbb{N}^d$ *be a fixed multi-index. Then, the following are equivalent:*

(a) $\mathbf{F}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities.*

(b) The polynomials $Q^{\mu}_{n,\mathbf{m}}$ *corresponding to* $\mu \in \mathbf{Reg}^{|\mathbf{m}|}_{1,2}(E)$ *are uniquely determined for all sufficiently large* n*, and there exist* λ_j*,* $j = 1, 2, \ldots, \gamma$ *such that* $\Delta^{\mu}(\lambda_j) < 1$ *for all* $j = 1, 2, \ldots, \gamma$ *and* $\sum_{j=1}^{\gamma} \sigma^{\mu}(\lambda_j) = |\mathbf{m}|$.

(c) The polynomials $Q^{E}_{n,\mathbf{m}}$ *are uniquely determined for all sufficiently large* n*, and there exist* λ_j*,* $j = 1, 2, \ldots, \gamma$ *such that* $\Delta^{E}(\lambda_j) < 1$ *for all* $j = 1, 2, \ldots, \gamma$ *and* $\sum_{j=1}^{\gamma} \sigma^{E}(\lambda_j) = |\mathbf{m}|$.

Remark 2. (i) If one of (a), (b), or (c) holds, then for each $j = 1, 2, \ldots, \gamma$, the point $\lambda_j \in \mathbb{C} \setminus E$ is a system pole of $\mathbf{F}$ with respect to $\mathbf{m}$ and

$$\tau_j = \sigma^{\mu}(\lambda_j) = \sigma^{E}(\lambda_j),$$

where τ_j denotes the order of λ_j as a system pole of $\mathbf{F}$.

(ii) To show that (a) implies both (b) and the fact that each $\lambda_j \in \mathbb{C} \setminus E$ is a system pole of $\mathbf{F}$ with respect to $\mathbf{m}$, satisfying

$$\tau_j = \sigma^{\mu}(\lambda_j) = \sigma^{E}(\lambda_j),$$

where τ_j denotes the order of λ_j as a system pole of $\mathbf{F}$, we only need to assume that $\mu \in \mathbf{Reg}_2(E)$. Conversely, if $\mu \in \mathbf{Reg}^{|\mathbf{m}|}_{1,2}(E)$, then (b) implies (a).

A refinement of Theorem 2.2 on the estimates of $\Delta^{\mu}(\lambda)$ and $\Delta^{E}(\lambda)$ is in the following theorem.

Theorem 3.2. *Let* $\mathbf{F} \in \mathcal{H}(E)^d$ *and fix* $\mathbf{m} \in \mathbb{N}^d$*. Assume that* λ *is a system pole of order* τ *of* $\mathbf{F}$ *with respect to* $\mathbf{m}$*. Then, the following are true:*

(a) If $\mu \in \mathbf{Reg}_2(E)$*, then*

$$\Delta^{\mu}(\lambda) \leq \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})} \qquad \textit{and} \qquad \sigma^{\mu}(\lambda) \geq \tau. \tag{18}$$

(b)

$$\Delta^{E}(\lambda) \leq \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})} \qquad \textit{and} \qquad \sigma^{E}(\lambda) \geq \tau. \tag{19}$$

Now, if $\mathbf{F}$ in Theorem 3.2 has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$, then all inequalities in (18) and (19) become equalities. Therefore, the estimates in Theorem 3.2 are sharp.

Theorem 3.3. *Let* $\mathbf{F} \in \mathcal{H}(E)^d$ *and* $\mathbf{m} \in \mathbb{N}^d$ *be a fixed multi-index. Assume that* $\mathbf{F}$ *has exactly* $|\mathbf{m}|$ *system poles with respect to* $\mathbf{m}$ *counting multiplicities and* λ *is a system pole of order* τ *of* $\mathbf{F}$ *with respect to* $\mathbf{m}$*. Then, the following are true:*

(a) If $\mu \in \mathbf{Reg}_{1,2}(E)$*, then*

$$\Delta^{\mu}(\lambda) = \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})} \qquad \textit{and} \qquad \sigma^{\mu}(\lambda) = \tau. \tag{20}$$

(b)

$$\Delta^{E}(\lambda) = \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})} \qquad \textit{and} \qquad \sigma^{E}(\lambda) = \tau.$$

By the equation (12), we see that $\rho_{\lambda,1}(\mathbf{F},\mathbf{m}) \geq \boldsymbol{\rho}_{\lambda}(\mathbf{F},\mathbf{m})$. There are many examples of $\mathbf{F}$, $\mathbf{m}$, and λ that $\rho_{\lambda,1}(\mathbf{F},\mathbf{m}) > \boldsymbol{\rho}_{\lambda}(\mathbf{F},\mathbf{m})$. Therefore, our new estimates in Theorem 3.2 improve upon those in Theorem 2.2. Let us give some simple examples here.

Example 3.1. Let $\mathbf{m} = (1, 1)$, $E = [-1, 1]$, and $\mathbf{F} = (F_1, F_2)$, where

$$F_1(z) := \frac{1}{2-z} + \log(5-z) \qquad \text{and} \qquad F_2(z) := \frac{1}{(2-z)^2} + \log(4-z).$$

Clearly, $\Phi(z) = z + \sqrt{z^2 - 1}$, $\mathbf{F}$ has exactly $|\mathbf{m}| = 2$ system poles with respect to $\mathbf{m}$, and 2 is a system pole of order 2. Since $\rho_{2,1}(\mathbf{F},\mathbf{m}) = |\Phi(5)|$, $\rho_{2,2}(\mathbf{F},\mathbf{m}) = |\Phi(4)|$, and $\boldsymbol{\rho}_2(\mathbf{F},\mathbf{m}) = |\Phi(4)|$, $\boldsymbol{\rho}_2(\mathbf{F},\mathbf{m}) < \rho_{2,1}(\mathbf{F},\mathbf{m})$.

Notice that $\mathbf{F}$ and $\mathbf{m}$ in Example 3.1 satisfy the hypothesis of Theorem 3.3. One may ask, "Are the estimates in the first inequalities of (18) and (19) better than those in (16) and (17) when $\mathbf{F}$ does not have exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$?" The answer is yes, as shown in the following example.

Example 3.2. Let $\mathbf{m} = (2, 1, 1)$, $E = [-1, 1]$, and $\mathbf{F} = (F_1, F_2, F_3)$, where

$$F_1(z) := \frac{1}{2-z} + e^{1/(3-z)} + \frac{1}{4-z},$$
$$F_2(z) := \frac{1}{2-z} + e^{1/(3-z)} + \frac{1}{(4-z)^2} + \log(5-z),$$
$$F_3(z) := \frac{1}{2-z} + e^{1/(3-z)} + \frac{2}{4-z}.$$

Therefore, $\Phi(z) = z + \sqrt{z^2-1}$, $\mathbf{F}$ has 3 system poles with respect to $\mathbf{m}$, 2 is a system pole of order 1, and 4 is a system pole of order 2. Since $\rho_{4,1}(\mathbf{F}, \mathbf{m}) = \infty$, $\rho_{4,2}(\mathbf{F}, \mathbf{m}) = |\Phi(5)|$, and $\boldsymbol{\rho}_4(\mathbf{F}, \mathbf{m}) = |\Phi(5)|$, $\boldsymbol{\rho}_4(\mathbf{F}, \mathbf{m}) < \rho_{4,1}(\mathbf{F}, \mathbf{m})$.

A natural question is whether Theorem 3.3 remains valid if we remove the condition that $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$.

4 Auxiliary Results

We gather all useful known lemmas in this section. The Cauchy-Hadamard formulas for orthogonal and Faber polynomial expansions can be found in Lemmas 4.1 and 4.2. The reader can see the proofs of Lemmas 4.1 and 4.2 in [5, Lemma 2.1] and [23], respectively.

For a function $F \in \mathcal{H}(E)$, we define

$$\langle F \rangle_n := \langle F, p_n \rangle.$$

Lemma 4.1. *Let $F \in \mathcal{H}(E)$ and $\mu \in \mathbf{Reg}_1(E)$. Then,*

$$\rho_0(F) = \left(\limsup_{n\to\infty} |\langle F \rangle_n|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} \langle F \rangle_n p_n(z)$ converges to $F(z)$ uniformly on each compact subset of $D_{\rho_0(F)}$.

Lemma 4.2. *If $F \in \mathcal{H}(E)$, then*

$$\rho_0(F) = \left(\limsup_{n\to\infty} |[F]_n|^{1/n} \right)^{-1}.$$

Moreover, the series $\sum_{n=0}^{\infty} [F]_n \Phi_n$ converges to F uniformly on each compact subset of $D_{\rho_0(F)}$.

A simple relation (see [5, Lemma 2.2]) between the orthonormal polynomial p_n and the second type function s_n is stated in the following lemma.

Lemma 4.3. *Let $F \in \mathcal{H}(E)$, $n \in \mathbb{N}_0$, and $\rho \in (1, \rho_0(F))$. Then,*

$$\langle F \rangle_n = \langle F, p_n \rangle = \frac{1}{2\pi i} \int_{\Gamma_\rho} F(w) s_n(w) dw.$$

Let $\mathbf{m} \in \mathbb{N}^d$ be fixed and let $q^\mu_{n,\mathbf{m}}$ be the normalization of $Q^\mu_{n,\mathbf{m}}$ in terms of its zeros $\lambda_{n,j}$ so that

$$q^\mu_{n,\mathbf{m}}(z) := \prod_{|\lambda_{n,j}|\le 1} (z - \lambda_{n,j}) \prod_{|\lambda_{n,j}|>1} \left(1 - \frac{z}{\lambda_{n,j}}\right). \tag{21}$$

This normalization is known as the spherical normalization. Furthermore, it is not difficult to show that

$$\Delta^\mu(\lambda) = \limsup_{n\to\infty} |q^\mu_{n,\mathbf{m}}(\lambda)|^{1/n}$$

and the polynomials $q^\mu_{n,\mathbf{m}}$ are uniformly bounded on each compact subset of $\mathbb{C}$ as stated below.

Lemma 4.4. *Let $\mathbf{F} \in \mathcal{H}(E)^d$, $\mathbf{m} \in \mathbb{N}^d$, $\mu \in \mathcal{M}(E)$, and $K \subset \mathbb{C}$ be a compact set. Then, there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$,*

$$\|q^\mu_{n,\mathbf{m}}\|_K \le c,$$

where $\|\cdot\|_K$ is the sup-norm on K.

5 Proofs of Main Results

Proof of Theorem 3.1. We will only show that (a)⇔(b). The same reasoning can be applied to (a)⇔(c).

((a)⇒(b)): Let $\mu \in \mathbf{Reg}_2(E)$ and

$$Q^{\mathbf{F}}_{\mathbf{m}}(z) := \prod_{j=1}^{\gamma}(z-\lambda_j)^{\tau_j}.$$

Clearly, $\sum_{j=1}^{\gamma}\tau_j = |\mathbf{m}|$. The uniqueness of $Q^{\mu}_{n,\mathbf{m}}$ follows from Remark 1. By Theorem 2.2, for all $j=1,2,\dots,\gamma$, $\Delta^{\mu}(\lambda_j)<1$ and $\sigma^{\mu}(\lambda_j)\ge \tau_j$. Then,

$$|\mathbf{m}| = \sum_{j=1}^{\gamma}\tau_j \le \sum_{j=1}^{\gamma}\sigma^{\mu}(\lambda_j) \le |\mathbf{m}|.$$

Consequently, $\sum_{j=1}^{\gamma}\sigma^{\mu}(\lambda_j) = |\mathbf{m}|$ and $\sigma^{\mu}(\lambda_j)=\tau_j$ for all $j=1,2,\dots,\gamma$.

((b)⇒(a)): Let $\mu \in \mathbf{Reg}^{|\mathbf{m}|}_{1,2}(E)$. Assume that

$$\Delta^{\mu}(\lambda_j)<1, \quad j=1,2,\dots,\gamma, \qquad \text{and} \qquad \sum_{j=1}^{\gamma}\sigma^{\mu}(\lambda_j)=|\mathbf{m}|. \tag{22}$$

Set

$$\hat{Q}_{|\mathbf{m}|}(z) := \prod_{j=1}^{\gamma}(z-\lambda_j)^{\tau_j},$$

where $\tau_j := \sigma^{\mu}(\lambda_j)$. By (22), we have

$$\delta^{\mu}_k(\lambda_j)<1, \qquad k=1,2,\dots,\tau_j, \quad j=1,2,\dots,\gamma. \tag{23}$$

We want to use Theorem 2.1 by showing that

$$\limsup_{n\to\infty}\|Q^{\mu}_{n,\mathbf{m}}-\hat{Q}_{|\mathbf{m}|}\|^{1/n}<1. \tag{24}$$

Let $\varepsilon>0$ be sufficiently small such that for all distinct $j,k\in\{1,2,\dots,\gamma\}$, $\mathbb{B}(\lambda_j,\varepsilon)\cap\mathbb{B}(\lambda_k,\varepsilon)=\emptyset$. We name the zeros of $Q^{\mu}_{n,\mathbf{m}}$ in $\mathbb{B}(\lambda_j,\varepsilon)$ in nondecreasing distance to the point λ_j, namely

$$|\lambda_j-\lambda_{n,j,1}|\le|\lambda_j-\lambda_{n,j,2}|\le\ldots\le|\lambda_j-\lambda_{n,j,\tau_j}|.$$

By our assumptions,

$$Q^{\mu}_{n,\mathbf{m}}(z)=\prod_{j=1}^{\gamma}\prod_{k=1}^{\tau_j}(z-\lambda_{n,j,k}).$$

We write

$$\begin{aligned}\hat{Q}_{|\mathbf{m}|}(z)-Q^{\mu}_{n,\mathbf{m}}(z) &= \hat{Q}_{|\mathbf{m}|}(z)-\frac{(z-\lambda_{n,1,1})\hat{Q}_{|\mathbf{m}|}(z)}{(z-\lambda_1)}+\frac{(z-\lambda_{n,1,1})\hat{Q}_{|\mathbf{m}|}(z)}{(z-\lambda_1)}-\cdots\\ &+\frac{(z-\lambda_{n,\gamma,1})\ldots(z-\lambda_{n,\gamma,\tau_\gamma-1})\prod_{j=1}^{\gamma-1}\prod_{k=1}^{\tau_j}(z-\lambda_{n,j,k})\hat{Q}_{|\mathbf{m}|}(z)}{(z-\lambda_\gamma)^{\tau_\gamma-1}\prod_{j=1}^{\gamma-1}(z-\lambda_j)^{\tau_j}}-Q^{\mu}_{n,\mathbf{m}}(z).\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\hat{Q}_{|\mathbf{m}|}-Q^{\mu}_{n,\mathbf{m}}\right|(z)\\ &\le\sum_{j=1}^{\gamma}\sum_{k=1}^{\tau_j}\left|\frac{(z-\lambda_{n,j,1})\ldots(z-\lambda_{n,j,k-1})\prod_{\beta=1}^{j-1}\prod_{\alpha=1}^{\tau_j}(z-\lambda_{n,\beta,\alpha})\hat{Q}_{|\mathbf{m}|}(z)}{(z-\lambda_j)^k\prod_{\beta=1}^{j-1}(z-\lambda_\beta)^{\tau_\beta}}\right||\lambda_j-\lambda_{n,j,k}|.\end{aligned} \tag{25}$$

Since

$$\lim_{n\to\infty}\frac{(z-\lambda_{n,j,1})\ldots(z-\lambda_{n,j,k-1})\prod_{\beta=1}^{j-1}\prod_{\alpha=1}^{\tau_j}(z-\lambda_{n,\beta,\alpha})\hat{Q}_{|\mathbf{m}|}(z)}{(z-\lambda_j)^k\prod_{\beta=1}^{j-1}(z-\lambda_\beta)^{\tau_\beta}}=\frac{\hat{Q}_{|\mathbf{m}|}}{(z-\lambda_j)},$$

uniformly on each compact subset of $\mathbb{C}$, it follows from (23) and (25) that (24) holds. By Theorem 2.1, $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and $\hat{Q}_{|\mathbf{m}|}=Q^{\mathbf{F}}_{\mathbf{m}}$. □

Proof of Theorem 3.2. (a): Let $\mu \in \mathbf{Reg}_2(E)$. For each $n \in \mathbb{N}$, let $q^{\mu}_{n,\mathbf{m}}$ be the denominator of an $(n,\mathbf{m})$ MOHP approximant with the spherical normalization defined in (21). By Lemma 4.4, this normalization implies that the polynomials $q^{\mu}_{n,\mathbf{m}}$ are uniformly bounded with respect to n on each compact subset of $\mathbb{C}$. Let λ be a system pole of order τ of $\mathbf{F}$ with respect to $\mathbf{m}$. We consider a polynomial combination G of type (11) that is holomorphic on a neighborhood of $\overline{D}_{|\Phi(\lambda)|}$ except for a simple pole at $z = \lambda$ and verifies that $\rho_1(G) = \rho_{\lambda,1}(\mathbf{F},\mathbf{m})$. Define

$$H(z) := (z-\lambda)G(z),$$

$$a_n := \langle G q^{\mu}_{n,\mathbf{m}} \rangle_n = \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} G(z) q^{\mu}_{n,\mathbf{m}}(z) s_n(z)dz, \tag{26}$$

where $1 < \rho_1 < |\Phi(\lambda)|$ and the second equality (26) follows from Lemma 4.3, and

$$\beta_n := \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} G(z) q^{\mu}_{n,\mathbf{m}}(z) s_n(z)dz,$$

where $|\Phi(\lambda)| < \rho_2 < \rho_{\lambda,1}(\mathbf{F},\mathbf{m})$. By the Cauchy residue theorem,

$$\beta_n - a_n = \operatorname{res}(G q^{\mu}_{n,\mathbf{m}} s_n, \lambda) = H(\lambda) q^{\mu}_{n,\mathbf{m}}(\lambda) s_n(\lambda). \tag{27}$$

Using (7), we obtain $a_n = 0$. Then, we have

$$\beta_n = H(\lambda) q^{\mu}_{n,\mathbf{m}}(\lambda) s_n(\lambda). \tag{28}$$

Let $\varepsilon > 0$. It is not difficult to check that by (10) and the uniform boundedness of $q^{\mu}_{n,\mathbf{m}}$ on Γ_{ρ_2}, there exists $c_1 > 0$ such that for n sufficiently large,

$$|\beta_n| = \left| \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} G(z) q^{\mu}_{n,\mathbf{m}}(z) s_n(z)dz \right| \le \frac{c_1}{(\rho_2 - \varepsilon)^n}. \tag{29}$$

Then, by (10) and (29), it follows from (28) that

$$\limsup_{n\to\infty} |q^{\mu}_{n,\mathbf{m}}(\lambda)|^{1/n} = \limsup_{n\to\infty} \frac{|\beta_n|^{1/n}}{|H(\lambda)|^{1/n}|s_n(\lambda)|^{1/n}} \le \frac{|\Phi(\lambda)|}{\rho_2 - \varepsilon}.$$

Letting $\varepsilon \to 0$ and $\rho_2 \to \rho_{\lambda,1}(\mathbf{F},\mathbf{m})$, by the normalization of $q^{\mu}_{n,\mathbf{m}}$, we obtain

$$\Delta^{\mu}(\lambda) = \limsup_{n\to\infty} |q^{\mu}_{n,\mathbf{m}}(\lambda)|^{1/n} \le \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})}.$$

The second inequality in (18) directly follows from Theorem 2.2.

(b): To show the inequalities in (19), we can use the same scheme of proof as above by replacing

$$q^{\mu}_{n,\mathbf{m}} \quad \text{by} \quad q^{E}_{n,\mathbf{m}}$$

where $q^{E}_{n,\mathbf{m}}$ is the spherical normalization of $Q^{E}_{n,\mathbf{m}}$, and

$$s_n(z) \quad \text{by} \quad \frac{\Phi'(z)}{\Phi^{n+1}(z)},$$

and making use of (8), (9), and Theorem 2.2. Note that the statement of Lemma 4 also holds for $q^{E}_{n,\mathbf{m}}$. So, we leave the details for the reader. □

Proof of Theorem 3.3. (a): Let $\mu \in \mathbf{Reg}_{1,2}(E)$. Assume that $\mathbf{F}$ has exactly $|\mathbf{m}|$ system poles with respect to $\mathbf{m}$ and λ is a system pole of $\mathbf{F}$ with respect to $\mathbf{m}$ of order τ. By Remark 2, $\sigma^{\mu}(\lambda) = \tau$. Furthermore, by Remark 1,

$$\lim_{n\to\infty} Q^{\mu}_{n,\mathbf{m}} = Q^{\mathbf{F}}_{\mathbf{m}}.$$

Consequently,

$$\Delta^{\mu}(\lambda) = \limsup_{n\to\infty} |Q^{\mu}_{n,\mathbf{m}}(\lambda)|^{1/n}.$$

Using Theorem 3.2, we obtain

$$\Delta^{\mu}(\lambda) = \limsup_{n\to\infty} \left|Q^{\mu}_{n,\mathbf{m}}(\lambda)\right|^{1/n} \le \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})}.$$

To show the first equality of (20), we need to show that

$$\limsup_{n\to\infty} \left|Q^{\mu}_{n,\mathbf{m}}(\lambda)\right|^{1/n} = \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})}.$$

Now, to the contrary, we assume that

$$\limsup_{n\to\infty}\left|Q^{\mu}_{n,\mathbf{m}}(\lambda)\right|^{1/n} < \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})}. \tag{30}$$

Clearly, this implies that $\rho_{\lambda,1}(\mathbf{F},\mathbf{m}) < \infty$. Choose a polynomial combination G of type (11) such that G is analytic on a neighborhood of $\overline{D}_{|\Phi(\lambda)|}$ except for a simple pole at $z=\lambda$ with $\rho_1(G)=\rho_{\lambda,1}(\mathbf{F},\mathbf{m})$. Furthermore, on the boundary of $D_{\rho_1(G)}$, the function G must have a singularity which is not a system pole. Otherwise, there exists another polynomial combination G_1 of type (11) such that $\rho_1(G_1)>\rho_1(G)$ which contradicts to the definition of $\rho_{\lambda,1}(\mathbf{F},\mathbf{m})$. By Lemmas 4.1 and 4.3, we obtain

$$\frac{1}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})} = \frac{1}{\rho_1(G)} = \limsup_{n\to\infty}\left(\left|\langle Q^{\mathbf{F}}_{\mathbf{m}}G\rangle_n\right|\right)^{1/n} = \limsup_{n\to\infty}\left(\left|\int_{\Gamma_{\rho_2}} Q^{\mathbf{F}}_{\mathbf{m}}(z)G(z)s_n(z)dz\right|\right)^{1/n}, \tag{31}$$

where $|\Phi(\lambda)| < \rho_2 < \rho_{\lambda,1}(\mathbf{F},\mathbf{m})$.

Consider

$$\int_{\Gamma_{\rho_2}} Q^{\mathbf{F}}_{\mathbf{m}}(z)G(z)s_n(z)dz = \int_{\Gamma_{\rho_2}}\left(Q^{\mathbf{F}}_{\mathbf{m}}(z)-Q^{\mu}_{n,\mathbf{m}}(z)\right)G(z)s_n(z)dz + \int_{\Gamma_{\rho_2}} Q^{\mu}_{n,\mathbf{m}}(z)G(z)s_n(z)dz \tag{32}$$

Using the same kind of reasoning as that applied to show (27) and (28), we arrive at

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q^{\mu}_{n,\mathbf{m}}(z)G(z)s_n(z)dz &= \frac{1}{2\pi i}\int_{\Gamma_{\rho_2}} Q^{\mu}_{n,\mathbf{m}}(z)G(z)s_n(z)dz - \frac{1}{2\pi i}\int_{\Gamma_{\rho_1}} Q^{\mu}_{n,\mathbf{m}}(z)G(z)s_n(z)dz\\ &= AQ^{\mu}_{n,\mathbf{m}}(\lambda)s_n(\lambda),\end{aligned} \tag{33}$$

where $A := \lim_{z\to\lambda}(z-\lambda)G(z)\neq 0$ and $1<\rho_1<|\Phi(\lambda)|<\rho_2<\rho_{\lambda,1}(\mathbf{F},\mathbf{m})$. By (10) and (30), the equation (33) implies

$$\limsup_{n\to\infty}\left|\int_{\Gamma_{\rho_2}} Q^{\mu}_{n,\mathbf{m}}(z)G(z)s_n(z)dz\right|^{1/n} < \frac{1}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})}. \tag{34}$$

On the other hand, using (10) and (15), we have

$$\begin{aligned}\limsup_{n\to\infty}\left(\left|\int_{\Gamma_{\rho_2}}\left(Q^{\mathbf{F}}_{\mathbf{m}}(z)-Q^{\mu}_{n,\mathbf{m}}(z)\right)G(z)s_n(z)dz\right|\right)^{1/n} &\le \limsup_{n\to\infty}\left(c_1\|Q^{\mathbf{F}}_{\mathbf{m}}-Q^{\mu}_{n,\mathbf{m}}\|_{\Gamma_{\rho_2}}\|s_n\|_{\Gamma_{\rho_2}}\right)^{1/n}\\ &\le \max\left\{\frac{|\Phi(\lambda)|}{\rho_{\lambda}(\mathbf{F},\mathbf{m})}:\lambda\in\mathcal{P}_{\mathbf{m}}(\mathbf{F})\right\}\frac{1}{\rho_2},\end{aligned}$$

where $c_1>0$ does not depend on n. Letting $\rho_2\to\rho_{\lambda,1}(\mathbf{F},\mathbf{m})$, we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\left(\left|\int_{\Gamma_{\rho_2}}\left(Q^{\mathbf{F}}_{\mathbf{m}}(z)-Q^{\mu}_{n,\mathbf{m}}(z)\right)G(z)s_n(z)dz\right|\right)^{1/n}\\ &\le \max\left\{\frac{|\Phi(\lambda)|}{\rho_{\lambda}(\mathbf{F},\mathbf{m})}:\lambda\in\mathcal{P}_{\mathbf{m}}(\mathbf{F})\right\}\frac{1}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})} < \frac{1}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})}.\end{aligned} \tag{35}$$

Combining (31), (34), and (35), it follows from (32) that

$$\frac{1}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})} = \limsup_{n\to\infty}\left(\left|\langle Q^{\mathbf{F}}_{\mathbf{m}}G\rangle_n\right|\right)^{1/n} < \frac{1}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})},$$

which is impossible. This completes the proof of

$$\Delta^{\mu}(\lambda) = \frac{|\Phi(\lambda)|}{\rho_{\lambda,1}(\mathbf{F},\mathbf{m})}.$$

(b): We can use the same scheme of proof as above by replacing

$$Q^{\mu}_{n,\mathbf{m}} \quad \text{by} \quad Q^{E}_{n,\mathbf{m}}$$

and

$$s_n(z) \quad \text{by} \quad \frac{\Phi'(z)}{\Phi^{n+1}(z)},$$

and making use of (8), Lemma 4.2, Theorem 2.1, Remark 2, and Theorem 3.2. So, we leave the details for the reader. □

6 Conclusions

In this paper, we investigated the rate at which system poles of a vector (or system) of functions can be detected by Hermite-Padé approximants constructed from orthogonal and Faber polynomials. By extending Gonchar's classical indicator approach, we derived explicit formulas that describe how rapidly the approximants reveal the system poles of the given function system. Our results generalize and improve previous estimates by incorporating geometric features of the compact set E and its associated conformal mapping. These findings not only deepen the theoretical understanding of Padé approximation on complex domains but also provide a foundation for further developments in rational approximation and analytic continuation.

7 Acknowledgments

I wish to express my gratitude toward the anonymous referees and the editor for helpful comments and suggestions leading to improvements of this work. This research project is supported by National Research Council of Thailand (NRCT): (Contract No. N41A650113) and is partially supported by the Faculty of Science, Mahidol University.

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