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On discs contained in filled Julia sets

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Abstract

We present some results concerning closed discs contained in the filled Julia sets associated with polynomials $f_c : \mathbb{C} \ni z \longmapsto z^2 + c \in \mathbb{C}$ mostly for $c \in \left[-2, \frac{1}{4}\right]$ but also some other values. We investigate also a few non-autonomous Julia sets. As an application we prove that the pluricomplex Green function associated with the non-autonomous Julia set of the sequence $(f_{c_n})_{n=1}^{\infty}$ is Hölder continuous, provided $(c_n)_{n=1}^{\infty} \subset \overline{D}\left(0, \frac{1}{4}\right)$.

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1 Introduction

Let p be a polynomial. The associated *filled Julia set* is defined by the formula

$$\mathcal{K}[p] := \{z \in \mathbb{C} : \ (p^{\circ n}(z))_{n=1}^{\infty} \text{ is bounded}\},$$

where we use the notation $p^{\circ n}$ to denote the composition $p \circ \ldots \circ p$ of n copies of p. *The Julia set* of p is $\mathcal{J}[p] = \partial \mathcal{K}[p]$. If $\deg p \geq 2$, then $\mathcal{J}[p]$ and $\mathcal{K}[p]$ are non-empty, compact, perfect sets. Moreover they are completely invariant under the action of p, i.e.

$$p^{-1}(\mathcal{K}[p]) = \mathcal{K}[p] = p(\mathcal{K}[p]), \qquad p^{-1}(\mathcal{J}[p]) = \mathcal{J}[p] = p(\mathcal{J}[p]).$$

For the background we refer the reader e.g. to [7] or [5].

We will use the following equivalent characterisation of a Julia set in terms of the derivative of the associated polynomial.

Theorem 1.1. [7, Theorem 14.10] *Let p be a polynomial with $\deg p \geq 2$. Then*

$$\mathcal{J}[p] = \overline{\{z \in \mathbb{C} |\ \exists k \geq 0 : \ p^{\circ k}(z) = z \ \wedge \ |(p^{\circ k})'(z)| > 1\}}.$$

Recall another fact.

Theorem 1.2. [5, Theorem III.1.1] *Let p be a polynomial with $\deg p \geq 2$. Then*

$$\left\{z \in \mathbb{C} |\ \exists \theta \in \mathbb{Q} : \ p(z) = z \ \wedge \ p'(z) = e^{2\pi i \theta}\right\} \subset \mathcal{J}[p].$$

In what follows $\overline{D}(a, r) := \{z \in \mathbb{C} : |z - a| \leq r\}$ is the closed disc in the complex plane with center $a \in \mathbb{C}$ and radius $r > 0$. We will consider the family of polynomials $f_c : \mathbb{C} \ni z \longmapsto z^2 + c \in \mathbb{C}$ for $c \in \mathbb{C}$. We are primarily interested in the case $c \in \left[-2, \frac{1}{4}\right]$. Recall that this interval is the intersection of the Mandelbrot set with the real axis. It follows that the sets $\mathcal{K}[f_c]$ for such c are connected. Recall that $\mathcal{K}[f_c]$ is also connected for $c \in \overline{D}\left(0, \frac{1}{4}\right)$.

The starting points for our investigations were some results from [1] and [6]. The authors of [1] proved that $r_c := \frac{1}{2} + \sqrt{\frac{1}{4} - c}$ is the smallest radius of a closed disc containing $\mathcal{K}[f_c]$ for every $c \in (-\infty, 0]$. We are interested in the largest radii of some discs contained in the filled Julia set of f_c.

As a nice application of some of our results we prove the Hölder continuity of the Green function of some non-autonomous Julia sets.

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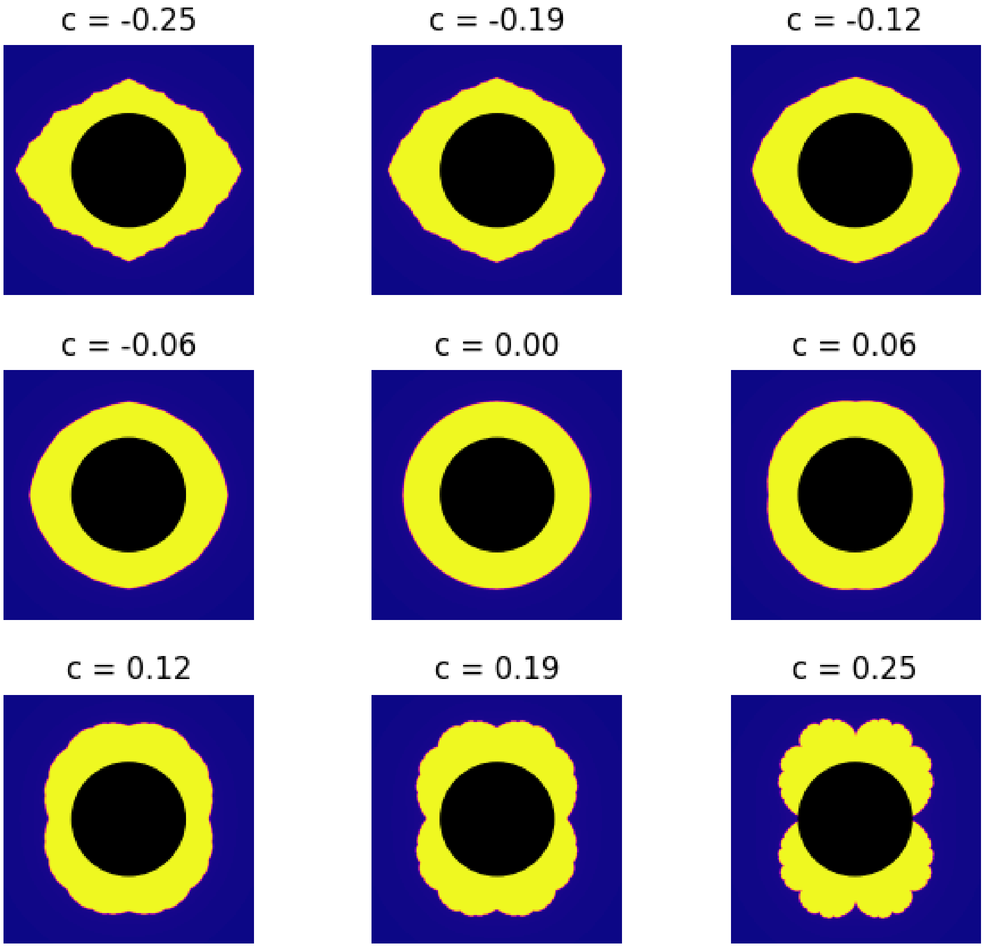
2 Discs and ellipses with center 0

In this section we establish a few results concerning closed discs with center 0 contained in some filled Julia sets. Recall first the following fact.

Proposition 2.1. [6, Proposition 2.1] $\overline{D}\left(0,\frac{1}{2}\right) \subset \mathcal{K}[f_c]$ *for every* $c \in \left[-\frac{1}{4},\frac{1}{4}\right]$.

Let us look at some illustrations.

Example 2.1. The filled Julia sets associated with f_c for some chosen c and the corresponding discs are shown in the images below.



We would like to find a radius larger than $\frac{1}{2}$. Define $\rho_c := \frac{1}{2} + \sqrt{\frac{1}{4} - c}$. We see that $\rho_c \in \left[\frac{1}{2}, 1\right]$ if $c \in \left[0, \frac{1}{4}\right]$.

Lemma 2.2.

$$\forall c \in \left(-\infty, \frac{1}{4}\right]: \quad \rho_c \in \mathcal{J}[f_c].$$

Proof. Note that ρ_c is a fixed point of f_c.

Fix first $c \in \left(-\infty, \frac{1}{4}\right)$. We obtain

$$|f_c'(\rho_c)| = |2\rho_c| = 1 + 2\sqrt{\frac{1}{4} - c} > 1,$$

since $c \in \left(-\infty, \frac{1}{4}\right)$. Therefore Theorem 1.1 yields $\rho_c \in \mathcal{J}[f_c]$.

Moreover, if $c = \frac{1}{4}$, then $f_{\frac{1}{4}}'(\rho_{\frac{1}{4}}) = 2\rho_{\frac{1}{4}} = 1 = e^{2\pi i \cdot 0}$, hence $\rho_{\frac{1}{4}} \in \mathcal{J}[f_{\frac{1}{4}}]$ by Theorem 1.2. □

Let $\mathcal{E}$ denote the family of ellipses with semi-axes contained in the axes of the coordinate system in $\mathbb{C}$. For an ellipse $E \in \mathcal{E}$ let $\mathrm{O}(E)$ denote the bounded connected component of $\mathbb{C} \setminus E$. We want to recall and use a result given in [1]. Since the authors did not prove it, we include here a proof for the convenience of the reader.

Proposition 2.3. [1, Remark 1] *For $c \in \left[0, \frac{1}{4}\right]$ we have*

$$E_c := \left\{ z \in \mathbb{C} : \frac{(\mathrm{Re}(z))^2}{\rho_c^2} + \frac{(\mathrm{Im}(z))^2}{\rho_c + c} = 1 \right\} \subset \mathcal{K}[f_c].$$

In addition

$$E_c = \bigcup \{ E \in \mathcal{E} : \overline{\mathrm{O}(E)} \subset \mathcal{K}[f_c] \}.$$

Proof. Fix $c \in \left[0, \frac{1}{4}\right]$. Recall that ρ_c is a fixed point of f_c, thus

$$\rho_c^2 + c = \rho_c, \tag{1}$$

and therefore

$$\rho_c + c = 2\rho_c - \rho_c^2 = \rho_c(2 - \rho_c) > 0.$$

In particular E_c is indeed an ellipse. We consider its parametrisation

$$\Gamma_{E_c} : [0, 2\pi) \ni t \longmapsto (\rho_c \cos t, \sqrt{\rho_c + c} \sin t) \in \mathbb{R}^2.$$

Fix $z_0 \in E_c$. We have $z_0 = \rho_c \cos\theta + i\sqrt{\rho_c + c}\sin\theta$ for some $\theta \in [0, 2\pi)$, hence by the definition of f_c and (1) we obtain

$$\begin{aligned} f_c(z_0) &= \rho_c^2 \cos^2\theta - (\rho_c + c)\sin^2\theta + c + i\rho_c\sqrt{\rho_c + c}\sin 2\theta \\ &= \rho_c \cos 2\theta + i\rho_c\sqrt{\rho_c + c}\sin 2\theta. \end{aligned}$$

Therefore

$$f_c(z_0) \in E_c' := \left\{ z \in \mathbb{C} : \frac{(\mathrm{Re}(z))^2}{\rho_c^2} + \frac{(\mathrm{Im}(z))^2}{\rho_c^2(\rho_c + c)} = 1 \right\}.$$

We have proved that $f_c(E_c) \subset E_c'$. Similarly, we show that $E_c' \subset f_c(E_c)$. Hence $f_c(E_c) = E_c'$.

Since f_c is holomorphic, it is an open mapping, thus

$$\partial f_c(\mathrm{O}(E_c)) \subset f_c(\partial(\mathrm{O}(E_c))) = f_c(E_c) = E_c'.$$

In addition, $\mathrm{O}(E_c') \subset \mathrm{O}(E_c)$, since $\rho_c \le 1$. Hence $f_c(\mathrm{O}(E_c)) \subset \mathrm{O}(E_c)$. Therefore $f_c^{\circ n}(\mathrm{O}(E_c)) \subset \mathrm{O}(E_c)$ for every integer n, which means that $\mathrm{O}(E_c) \subset \mathcal{K}[f_c]$.

In view of Lemma 2.2, we know that $\rho_c \in \mathcal{J}[f_c]$, hence by the invariance of the Julia set and the definition of f_c we obtain

$$\{\rho_c, -\rho_c\} = f_c^{-1}(\rho_c) \subset \mathcal{J}[f_c],$$

and also

$$\{i\sqrt{\rho_c + c}, -i\sqrt{\rho_c + c}\} = f_c^{-1}(-\rho_c) \subset \mathcal{J}[f_c].$$

Moreover $\{\rho_c, -\rho_c, i\sqrt{\rho_c + c}, -i\sqrt{\rho_c + c}\} \subset E_c$, so the proof is completed. □

Now we are able to prove that the radius in Proposition 2.1 can be enlarged.

Proposition 2.4. *Fix $c \in \left[0, \frac{1}{4}\right]$. Then $\overline{D}(0, \rho_c) \subset \mathcal{K}[f_c]$. Moreover*

$$\rho_c = \max\left\{ r > 0 : \overline{D}(0, r) \subset \mathcal{K}[f_c] \right\}.$$

Proof. Fix $c \in \left[0, \frac{1}{4}\right]$. In view of Proposition 2.3 it is sufficient to show that $\overline{D}(0, \rho_c) \subset \overline{\mathrm{O}(E_c)}$. By (1) we have $\rho_c^2 = \rho_c - c \le \rho_c + c$. Therefore for any $z \in \overline{D}(0, \rho_c)$

$$\frac{(\mathrm{Re}(z))^2}{\rho_c^2} + \frac{(\mathrm{Im}(z))^2}{\rho_c + c} \le \frac{(\mathrm{Re}(z))^2}{\rho_c^2} + \frac{(\mathrm{Im}(z))^2}{\rho_c^2} \le 1.$$

The additional assertion follows from Lemma 2.2. □

We can generalise the previous result.

Proposition 2.5. *$\overline{D}(0, \rho_{|c|}) \subset \mathcal{K}[f_c]$ for every $c \in \left[-\frac{1}{4}, \frac{1}{4}\right]$.*

Proof. Fix $c \in \left[-\frac{1}{4}, \frac{1}{4}\right]$ and $z_0 \in \overline{D}(0, \rho_{|c|})$. Then $z_0 = r\cos t + ir\sin t$ for some $r \in [0, \rho_{|c|}]$ and $t \in [0, 2\pi)$. Thus

$$f_c(z_0) = r^2\cos^2 t + 2ir^2\sin t\cos t - r^2\sin^2 t + c = r^2\cos 2t + c + ir^2\sin 2t,$$

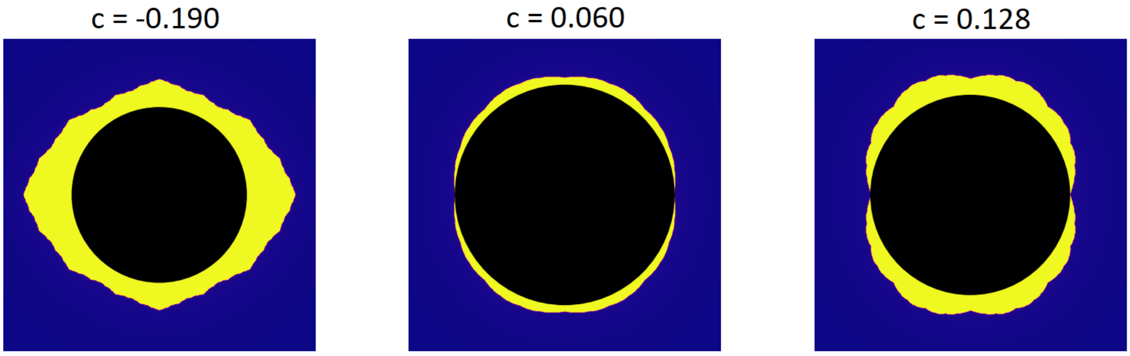
so $f_c(z_0) \in \overline{D}(c, \rho_{|c|}^2)$, since $r^2 \in [0, \rho_{|c|}^2]$.

Hence $f_c\left(\overline{D}(0, \rho_{|c|})\right) \subset \overline{D}(c, \rho_{|c|}^2)$. Moreover, $\overline{D}(c, \rho_{|c|}^2) \subset \overline{D}(0, \rho_{|c|})$, because $|c| = \rho_{|c|} - \rho_{|c|}^2$ by (1). We conclude that $f_c\left(\overline{D}(0, \rho_{|c|})\right) \subset \overline{D}(0, \rho_{|c|})$. Consequently $f_c^{\circ n}\left(\overline{D}(0, \rho_{|c|})\right) \subset \overline{D}(0, \rho_{|c|})$ for all integers $n \ge 1$, which completes the proof. □

Remark 1. Note that for $c \in \left[0, \frac{1}{4}\right]$ we get $\{\rho_c, -\rho_c\} \subset \overline{D}(0, \rho_c) \cap \mathcal{J}[f_c]$ by Lemma 2.2, Proposition 2.4 and the symmetry of $\mathcal{J}[f_c]$ with respect to the origin.

Let us look at some illustrations.

Example 2.2. The filled Julia sets $\mathcal{K}[f_c]$ for some chosen c and the corresponding discs are shown in the images below.



As a consequence we obtain a generalisation of Proposition 2.3.

Corollary 2.6.

$$\widetilde{E}_c := \left\{ z \in \mathbb{C} : \frac{(\mathrm{Re}(z))^2}{\rho_{|c|}^2} + \frac{(\mathrm{Im}(z))^2}{\rho_{|c|} + c} = 1 \right\} \subset \mathcal{K}[f_c]$$

for $c \in \left[-\frac{1}{4}, \frac{1}{4}\right]$.

Proof. In view of Proposition 2.3 it is sufficient to consider $c \in \left[-\frac{1}{4}, 0\right]$. It follows from (1) that $\rho_{|c|}^2 = \rho_{|c|} - |c| = \rho_{|c|} + c$, so $\widetilde{E}_c = \partial \overline{D}(0, \rho_{|c|})$. Proposition 2.5 yields $\overline{D}(0, \rho_{|c|}) \subset \mathcal{K}[f_c]$. □

Now we prove the following generalisation of Proposition 2.1.

Proposition 2.7. $\overline{D}\left(0, \frac{1}{2}\right) \subset \mathcal{K}[f_c]$ *for every* $c \in \overline{D}\left(0, \frac{1}{4}\right)$.

Proof. (cf. Proof of [6, Proposition 2.1]) Fix $z \in \overline{D}\left(0, \frac{1}{2}\right)$. Note that

$$|f_c(z)| = |z^2 + c| \leq |z|^2 + |c| \leq \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}.$$

Inductively we can prove that $|f_c^{\circ n}(z)| \leq \frac{1}{2}$ for all integers $n \geq 1$. □

Let us now recall a special example from [6].

Example 2.3. [6, Proposition 2.2]

1. $\overline{D}(0, R_1) \subset \mathcal{K}[f_{-1}]$, where $R_1 \in \left(0, \frac{1}{2}\right)$ is such that $R_1^3 + 2R_1 = 1$.
2. $\overline{D}(-1, R_2) \subset \mathcal{K}[f_{-1}]$, where $R_2 \in \left(0, \frac{1}{2}\right)$ is such that $R_2^3 + 4R_2^2 + 4R_2 = 1$.

The proof of these inclusions is based on the fact that the absolute term of $f_{-1}^{\circ 2}$ equals zero. One can apply a similar method to some other cases. Consider for instance

$$f_c^{\circ 3}(z) = z^8 + 4cz^6 + (6c^2 + 2c)z^4 + (4c^3 + 4c^2)z^2 + c^4 + 2c^3 + c^2 + c$$

for some $c \in \mathbb{C}$. Solving the equation $c^4 + 2c^3 + c^2 + c = 0$ we get polynomials f_c for which the absolute term of $f_c^{\circ 3}$ equals to zero. As a consequence, repeating similar reasoning as in the proof of [6, Proposition 2.2], we obtain the following result

Example 2.4.

1. $\overline{D}(0, R_3) \subset \mathcal{K}[f_c]$, where

$$c = \frac{1}{3}\left(-2 - \left(\frac{2}{25 - 3\sqrt{69}}\right)^{\frac{1}{3}} - \left(\frac{1}{2}\left(25 - 3\sqrt{69}\right)\right)^{\frac{1}{3}}\right) \approx -1.75$$

   and $R_3 \in \left(0, \frac{1}{5}\right)$ is such that $R_3^7 + 4|c|R_3^5 + |6c^2 + 2c|R_3^3 + |4c^3 + 4c^2|R_3 = 1$.
2. $\overline{D}(0, R_4) \subset \mathcal{K}[f_c]$, where c is one of the solutions of $c^4 + 2c^3 + c^2 + c = 0$ (we take $c \approx -0.12256 - 0.74486i$) and $R_4 \in (0.3, 0.4)$ is such that $R_4^7 + 4|c|R_4^5 + |6c^2 + 2c|R_4^3 + |4c^3 + 4c^2|R_4 = 1$.

Proof. 1. It can be shown that

$$f_c^{\circ 3}(z) = z^8 + 4cz^6 + (6c^2 + 2c)z^4 + (4c^3 + 4c^2)z^2.$$

We remark, by studying the sign of the polynomial

$$P(x) = x^7 + 4|c|x^5 + |6c^2 + 2c|x^3 + |4c^3 + 4c^2|x - 1$$

on the interval $\left[0, \frac{1}{5}\right]$, that there exists a unique $R_3 \in \left(0, \frac{1}{5}\right)$ such that $P(R_3) = 0$. In particular $R_3^7 + 4|c|R_3^5 + |6c^2 + 2c|R_3^3 + |4c^3 + 4c^2|R_3 = 1$.

Let now $|z| \leq R_3$. Then

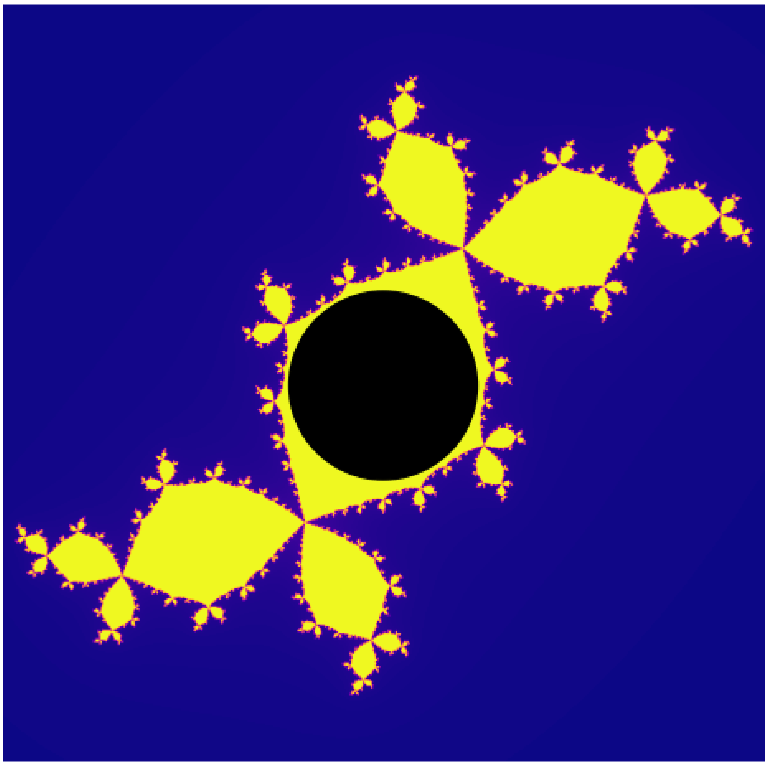
$$|f_c^{\circ 3}(z)| \leq |z|(|z|^7 + 4|c||z|^5 + |6c^2 + 2c||z|^3 + |4c^3 + 4c^2||z|) \leq R_3.$$

Thus inductively $|f_c^{\circ 3k}(z)| \leq R_3$ for every $k \geq 1$ and this implies $f_c^{\circ 3k}(z) \nrightarrow \infty$. Hence $\overline{D}(0, R_3) \subset \mathcal{K}[f_c]$.

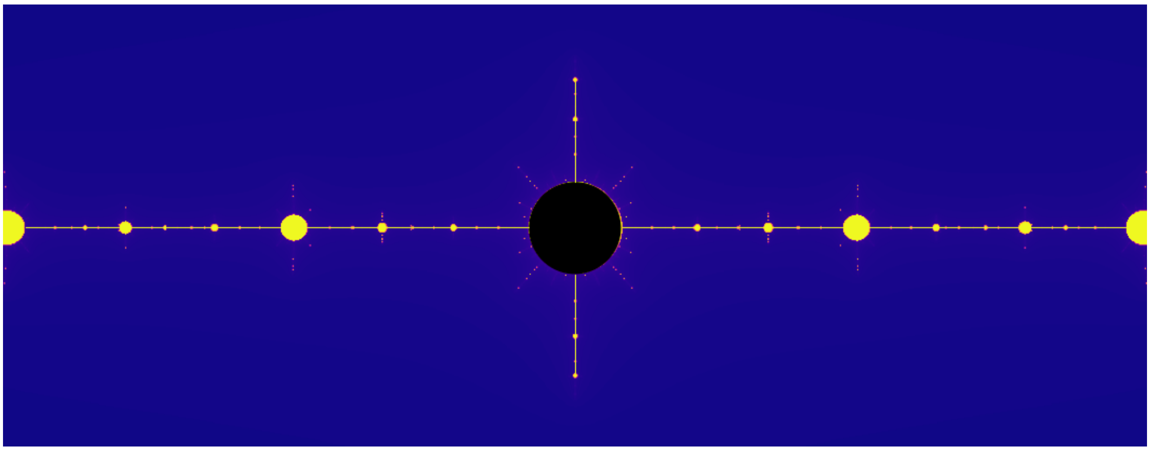
The proof of 2. is analogous: R_3 needs to be replaced by R_4 respectively. □

Let us look at some illustrations.

Example 2.5. The filled Julia sets $\mathcal{K}[f_c]$ for some chosen c and the corresponding discs are shown in the images below.



$\mathcal{K}[f_c]$ for $c = -0.12 - 0.74i$ (in yellow) and $\overline{D}(0, R_4)$ (in black).



$\mathcal{K}[f_c]$ for $c \approx -1.75$ (in yellow) and $\overline{D}(0, R_3)$ (in black).

3 Discs with other centers

Note first that $\rho_c - \sqrt{c} > 0$ for $c \in \left(0, \frac{1}{4}\right]$. Now we prove the following

Proposition 3.1. *Fix $c \in \left(0, \frac{1}{4}\right]$ and $z_0 \in \{\sqrt{-c}, -\sqrt{-c}\}$. Then*

$$\rho_c - \sqrt{c} \le \max\left\{r > 0 : \overline{D}(z_0, r) \subset \mathcal{K}[f_c]\right\} \le \frac{\rho_c}{\sqrt{c}}.$$

Proof. We prove the first inequality. It is elementary to see that $\overline{D}(z_0, \rho_c - \sqrt{c}) \subset \overline{D}(0, \rho_c)$. In addition, Proposition 2.5 implies that $\overline{D}(0, \rho_c) \subset \mathcal{K}[f_c]$. Therefore we obtain $\overline{D}(z_0, \rho_c - \sqrt{c}) \subset \mathcal{K}[f_c]$, which yields the claim.

Now we prove the second inequality. In view of Proposition 2.6 we know that $0 \in \mathcal{K}[f_c]$, hence by the invariance of the Julia set and the definition of f_c we obtain

$$\{\sqrt{-c}, -\sqrt{-c}\} \subset \mathcal{K}[f_c].$$

Hence $z_0 \in \mathcal{K}[f_c]$. Let $r_0 = \max\left\{r > 0 : \overline{D}(z_0, r) \subset \mathcal{K}[f_c]\right\}$.

Next, we prove the inclusion $\overline{D}(0, |z_0|r_0) \subset f\left(\overline{D}(z_0, r_0)\right)$ following some ideas from [17]. Let $w \in \overline{D}(0, |z_0|r_0)$. Then

$$\left|z_0 + \sqrt{w-c}\right| \left|z_0 - \sqrt{w-c}\right| = |f(z_0) - w| = |w| \le |z_0| r_0,$$

so by the triangle inequality

$$\left|\frac{1}{z_0 + \sqrt{w-c}}\right| + \left|\frac{1}{z_0 - \sqrt{w-c}}\right| \ge \left|\frac{2z_0}{(z_0 + \sqrt{w-c})(z_0 - \sqrt{w-c})}\right| \ge \frac{2}{r_0}.$$

Hence

$$\left|\frac{1}{z_0 - \sqrt{w-c}}\right| \ge \frac{1}{r_0} \quad \text{or} \quad \left|\frac{1}{z_0 + \sqrt{w-c}}\right| \ge \frac{1}{r_0}.$$

Without loss of generality we assume that the first inequality holds. Then $\sqrt{w-c} \in \overline{D}(z_0, r_0)$, which implies that $w = f\left(\sqrt{w-c}\right) \in f\left(\overline{D}(z_0, r_0)\right)$ and the claim follows.

Moreover, $f\left(\overline{D}(z_0, r_0)\right) \subset \mathcal{K}[f_c]$ by the invariance of the Julia set. We conclude that $\overline{D}(0, |z_0|r_0) \subset \mathcal{K}[f_c]$. Consequently $|z_0|r_0 \le \rho_c$ by Proposition 2.4. □

We note here that using Koebe's one-quarter theorem it can be shown that if $c \in \left(0, \frac{1}{4}\right]$ and $z_0 \in \{-\sqrt{-c}, \sqrt{-c}\}$, then

$$\max\left\{r \in (0, \sqrt{c}] : \overline{D}(z_0, r) \subset \mathcal{K}[f_c]\right\} \le \frac{2\rho_c}{\sqrt{c}}.$$

However our result is stronger. Let us further remark that for the given parameter range the centers $-\sqrt{-c}$ and $\sqrt{-c}$ lie on the imaginary axis.

Note that $z(c) := \frac{1-\sqrt{1-4c}}{2}$ is a fixed point of f_c. Recall now the following result.

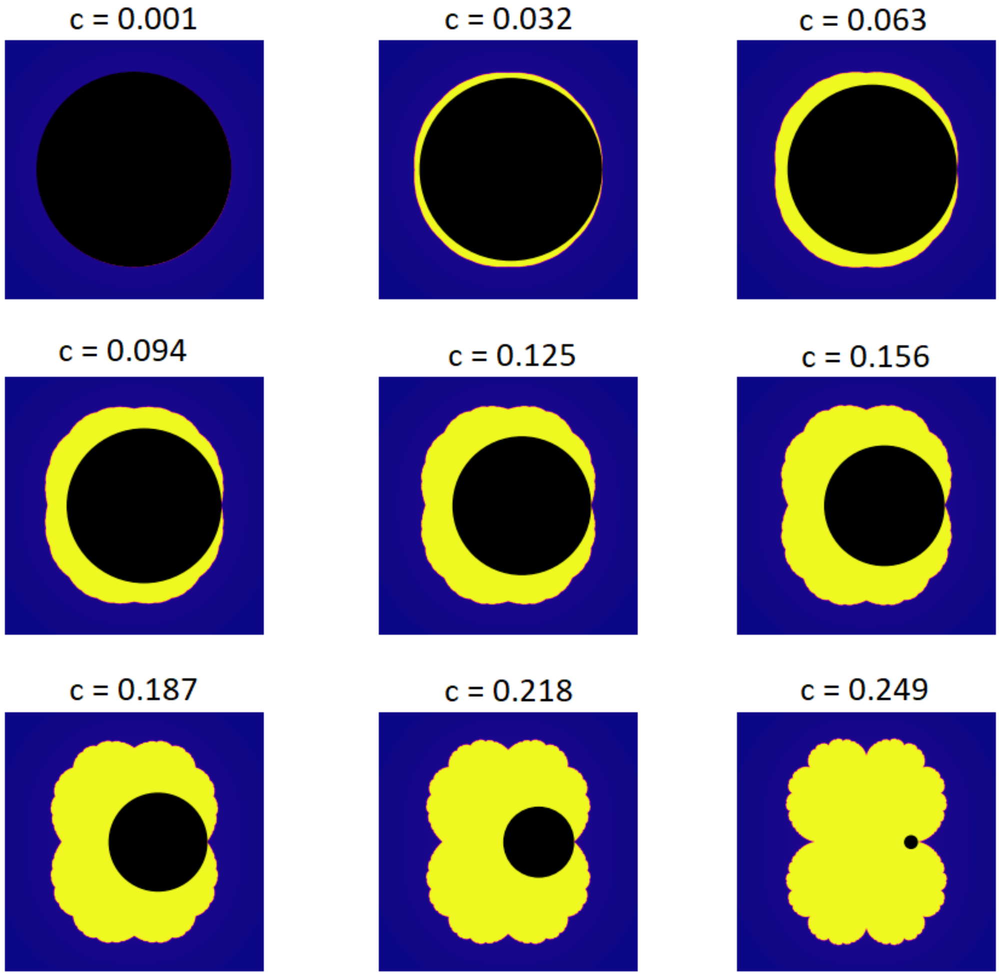
Proposition 3.2. [6, Proposition 2.1] $\overline{D}\left(\frac{1-\sqrt{1-4c}}{2}, \sqrt{1-4c}\right) \subset \mathcal{K}[f_c]$ *for every* $c \in \left(0, \frac{1}{4}\right)$.

Let us look at the boundary values of the interval in Proposition 3.2. In this section we expand the definition of the closed disc to the case of $r = 0$, putting $\overline{D}(a, 0) := \{a\}$.

Remark 2. Note that for $c = 0$ we get $\overline{D}(0, 1) = \mathcal{K}[f_0]$ and for $c = \frac{1}{4}$ we can write $\overline{D}\left(\frac{1}{2}, 0\right) = \{\frac{1}{2}\} \subset \mathcal{K}[f_{\frac{1}{4}}]$, since $\frac{1}{2}$ is a fixed point of $f_{\frac{1}{4}}$.

Let us look at some illustrations.

Example 3.1. The filled Julia sets $\mathcal{K}[f_c]$ for some chosen c and the corresponding discs are shown in the images below.



The illustrations allow to suspect that the following result is true.

Proposition 3.3. *For every* $c \in \left[0, \frac{1}{4}\right]$ *we have*

$$\max\left\{r > 0 : \overline{D}\left(\frac{1-\sqrt{1-4c}}{2}, r\right) \subset \mathcal{K}[f_c]\right\} = \sqrt{1-4c}.$$

Proof. Fix $c \in \left[0, \frac{1}{4}\right]$. By Lemma 2.2 we have $\rho_c = \frac{1}{2} + \sqrt{\frac{1}{4} - c} \in \mathcal{J}[f_c] = \partial\mathcal{K}[f_c]$. But $\rho_c \in \partial\overline{D}\left(\frac{1-\sqrt{1-4c}}{2}, \sqrt{1-4c}\right)$. Proposition 3.2 together with Remark 2 completes the proof. □

Recall now another fact from [6].

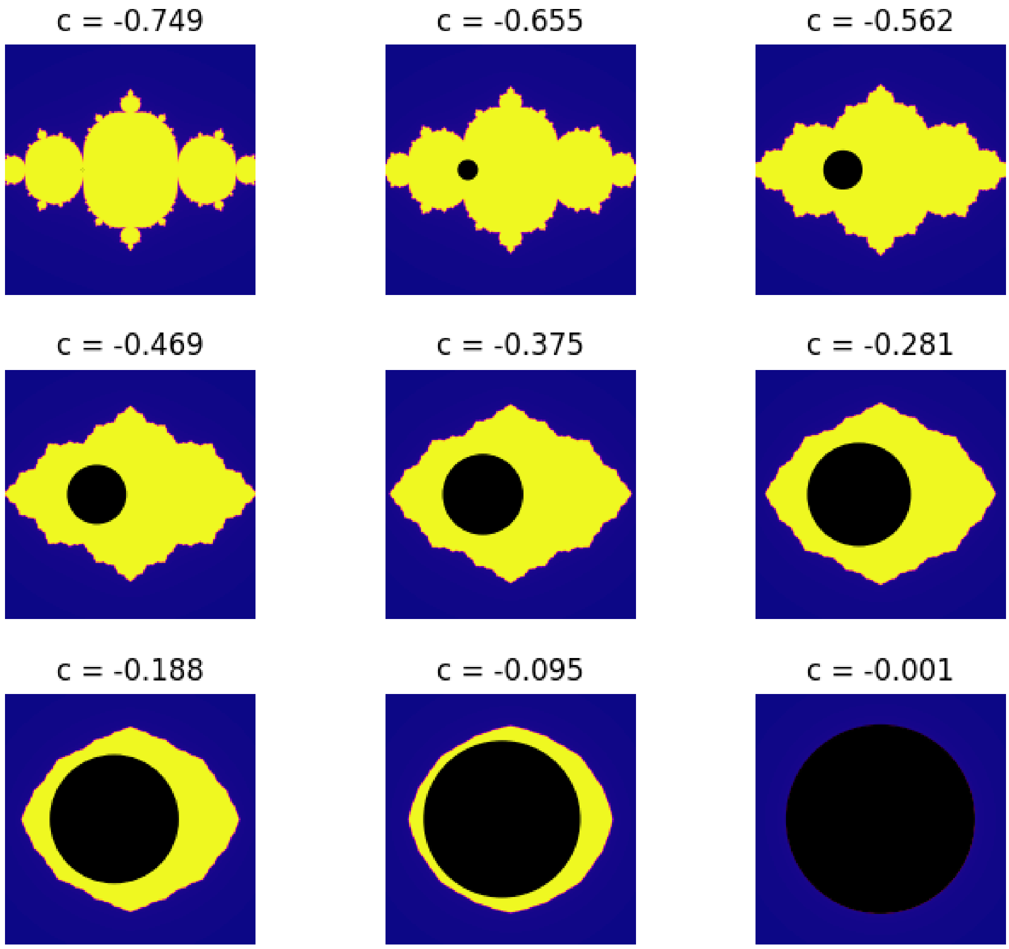
Proposition 3.4. [6, Proposition 2.1] $\overline{D}\left(\frac{1-\sqrt{1-4c}}{2}, 2-\sqrt{1-4c}\right) \subset \mathcal{K}[f_c]$ *for every* $c \in \left(-\frac{3}{4}, 0\right)$.

Once again let us look at the boundary points of the interval from the Proposition 3.4.

Remark 3. Once again for $c = 0$ we get $\overline{D}(0,1) = \mathcal{K}[f_0]$ and for $c = -\frac{3}{4}$ we can write $\overline{D}\left(-\frac{1}{2}, 0\right) = \{-\frac{1}{2}\} \subset \mathcal{K}[f_{-\frac{3}{4}}]$, since $-\frac{1}{2}$ is a fixed point of $f_{-\frac{3}{4}}$.

Let us look at some illustrations.

Example 3.2. The filled Julia sets $\mathcal{K}[f_c]$ for some chosen c and the corresponding discs are shown in the images below.



Lemma 3.5. *For every* $c \in \left[-2, -\frac{3}{4}\right]$ *we have* $\frac{1-\sqrt{1-4c}}{2} \in \mathcal{J}[f_c]$.

Proof. Fix first $c \in \left[-2, -\frac{3}{4}\right)$. Recall that $z(c) := \frac{1-\sqrt{1-4c}}{2}$ is a fixed point of f_c. And since $c \in \left[-2, -\frac{3}{4}\right)$, we obtain

$$|f_c'(z(c))| = |2z(c)| = \sqrt{1-4c} - 1 > 1.$$

Hence $z(c) \in \mathcal{J}[f_c]$ by Theorem 1.1.

Moreover $f'_{-\frac{3}{4}}\left(z\left(-\frac{3}{4}\right)\right) = -1 = e^{2\pi i \cdot \frac{1}{2}}$ and $z\left(-\frac{3}{4}\right)$ is a fixed point of $f_{-\frac{3}{4}}$, so $z\left(-\frac{3}{4}\right) \in \mathcal{J}[f_{-\frac{3}{4}}]$ by Theorem 1.2. □

Lemma 3.5 shows that Proposition 3.4 can not be generalised for $c \in \left[-2, -\frac{3}{4}\right]$.

Let us finally observe the following corollary.

Corollary 3.6. *1.* $\overline{D}\left(\frac{\sqrt{1-4c}-1}{2}, \sqrt{1-4c}\right) \subset \mathcal{K}[f_c]$ *for every* $c \in \left[0, \frac{1}{4}\right]$.

2. $\overline{D}\left(\frac{\sqrt{1-4c}-1}{2}, 2-\sqrt{1-4c}\right) \subset \mathcal{K}[f_c]$ *for every* $c \in \left[-\frac{3}{4}, 0\right]$.

Proof. It follows from Propositions 3.2 and 3.4, Remarks 2 and 3 and the symmetry of $\mathcal{K}[f_c]$ with respect to the origin. □

4 Examples of non-autonomous Julia sets

The notion of the filled Julia set can be generalised. Consider an arbitrary sequence of polynomials $(p_n)_{n=1}^{\infty}$ of degree at least two. The associated *non-autonomous filled Julia set* of that sequence is defined by

$$\mathcal{K}[(p_n)_{n=1}^{\infty}] := \{z \in \mathbb{C} : \ ((p_n \circ \ldots \circ p_1)(z))_{n=1}^{\infty} \ \text{ is bounded}\}.$$

Remark 4. Note that it follows from the previous definition that

$$\mathcal{K}[(p_n)_{n=1}^{\infty}] = \bigcup_{r\in\mathbb{N}} \bigcap_{n\geq 1} (p_n \circ \ldots \circ p_1)^{-1} \left(\overline{D}(0,r)\right).$$

The *non-autonomous Julia set* of a sequence $(p_n)_{n=1}^{\infty}$ is defined by $\mathcal{J}[(p_n)_{n=1}^{\infty}] := \partial\mathcal{K}[(p_n)_{n=1}^{\infty}]$. The complement of the non-autonomous Julia set to the Riemann sphere $\mathbb{C}_{\infty}$, denoted by $\mathcal{F}[(p_n)_{n=1}^{\infty}]$, is called the *Fatou set*, in accordance with the classical iteration theory. Connected components of the Fatou set are called *stable domains*. If ∞ belongs to the Fatou set, we denote the corresponding stable domain by $A(\infty)$. A domain $M \subset \mathbb{C}_{\infty}$ is called *invariant*, if $p_n(M) \subset M$ for all $n \in \mathbb{N}$. For some further information see e.g. [2], [3] and [9]. The author of [4] has shown that non-autonomous Julia sets can be finite, unlike the filled Julia set of one polynomial.

Example 4.1. [4] $\mathcal{K}[(z \longmapsto n^{2^n} z^2)_{n=1}^{\infty}] = \{0\}$.

We will show below that some non-autonomous Julia sets contain some discs, which means that they are infinite.

Proposition 4.1. *Fix a sequence* $(c_n)_{n=1}^{\infty} \subset \overline{D}\left(0, \frac{1}{4}\right)$. *Then* $\overline{D}\left(0, \frac{1}{2}\right) \subset \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$.

Proof. (cf. Proof of Proposition 2.7) Fix $z \in \overline{D}\left(0, \frac{1}{2}\right)$. Note that

$$|f_{c_1}(z)| = |z^2 + c_1| \leq |z|^2 + |c_1| \leq \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}.$$

Inductively we can prove that $|(f_{c_n} \circ \ldots \circ f_{c_1})(z)| \leq \frac{1}{2}$ for all integers $n \geq 1$. □

The following result is known in much more general settings (see [8, Theorem 2.1] or [9, Section 4]). Our case is really elementary and we include the proof to make the paper more self-contained.

Proposition 4.2. $\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ *is compact for a fixed sequence* $(c_n)_{n=1}^{\infty} \subset \overline{D}\left(0, \frac{1}{4}\right)$.

Proof. Fix $(c_n)_{n=1}^{\infty} \subset (c_n)_{n=1}^{\infty} \subset \overline{D}\left(0, \frac{1}{4}\right)$. We prove first that for every integer $n \geq 1$ we have

$$|z| \geq 3 \quad \Longrightarrow \quad |f_{c_n}(z)| > 2|z|.$$

Fix $z \in \mathbb{C} \setminus D(0,3)$ and $n \geq 1$. Then

$$|f_{c_n}(z)| = |z^2 + c_n| \geq |z|^2 - |c_n| = |z|\left(|z| - \frac{|c_n|}{|z|}\right) > 2|z|.$$

Consequently $\mathcal{K}[(f_{c_n})_{n=1}^{\infty}] = \bigcap_{n=1}^{\infty} (f_{c_n} \circ \ldots \circ f_{c_1})^{-1} \left(\overline{D}(0,3)\right)$ in view of Remark 4. It follows that $\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ is compact. □

In what follows let us denote $F_n = f_{c_n} \circ f_{c_{n-1}} \circ \ldots f_{c_2} \circ f_{c_1}$ for a fixed sequence $(c_n)_{n=1}^{\infty} \subset \mathbb{C}$. Now it will be useful to recall the following classification.

Definition 4.1. (cf. [3, Definition 1]) We say that a sequence $(c_n)_{n=1}^{\infty}$ belongs to

- class $\mathcal{P}_I$, if there is an invariant domain M, $\infty \in M$, such that $F_n \xrightarrow{n\to\infty} \infty$ locally uniformly in M,
- class $\mathcal{P}_{II}$, if $F_n \xrightarrow{n\to\infty} \infty$ locally uniformly in some neighbourhood of ∞, although there is no invariant domain M such that $\infty \in M$,
- class $\mathcal{P}_{III}$, if $\infty \in \mathcal{F}[(f_{c_n})_{n=1}^{\infty}]$.

Note that it is possible to classify a sequence $(c_n)_{n=1}^{\infty}$ according to its growth.

Theorem 4.3. (cf. [3, Theorem 1]) *A sequence* $(c_n)_{n=1}^{\infty}$ *belongs to*

- *class* $\mathcal{P}_I$ *if and only if* $(c_n)_{n=1}^{\infty}$ *is bounded,*
- *class* $\mathcal{P}_{II}$ *if and only if* $(c_n)_{n=1}^{\infty}$ *is not bounded, but* $\log^+|c_n| = O(2^n)$,
- *class* $\mathcal{P}_{III}$ *if and only if* $\limsup_{n\to\infty} \frac{\log^+|c_n|}{2^n} = +\infty$.

In the rest of the chapter we will mainly deal with facts about the "size" of certain sets. The useful notion here is the diameter of a domain, i.e.

$$\operatorname{diam}(A) := \sup\{|z-w| : z, w \in A\}.$$

Let us firstly discribe the maximal range of the diameter of a stable domain.

Theorem 4.4. (cf. [3, Theorem 5 & 9]) *1. Let $(c_n)_{n=1}^{\infty} \in \mathcal{P}_I \cup \mathcal{P}_{II}$ and $V \neq A(\infty)$ be a stable domain. Then*

$$\operatorname{diam}(V) < 4.$$

2. For every positive $d < 4$ there is a sequence $(c_n)_{n=1}^{\infty} \in \mathcal{P}_I \cup \mathcal{P}_{II}$ whose Fatou set has a stable domain $V \neq A(\infty)$ such that

$$\operatorname{diam}(V) \geq d.$$

In particular, the above theorem gives an upper bound on the diameter of the filled Julia set if that set is the only stable domain. This result, together with Proposition 4.1, yields

Remark 5. For every sequence $(c_n)_{n=1}^{\infty} \subset \overline{D}\left(0, \frac{1}{4}\right)$ we have

$$\frac{1}{2} \leq \operatorname{diam}\left(V_{(c_n)_{n=1}^{\infty}}\right) < 4,$$

where $V_{(c_n)_{n=1}^{\infty}}$ is the stable domain containing 0.

Now we find a disc contained in a non-autonomous filled Julia set in a special case, which gives a lower bound on the diameter of this set. However, we need to make little modifications to Lemma 2.4 from [6] beforehand.

Lemma 4.5. *Let $n \geq 2$. For any $a \in (0,1]$ define $P^a : [0,1] \longrightarrow \mathbb{R}$ by the formula $P^a(x) := x(x+2a)^{2n} - 1$. Then*

1. *P^a is strictly increasing and thus injective;*
2. *There exists a unique $r_a \in (0,1)$ such that $P^a(r_a) = 0$;*
3. *Given $a_1, a_2 \in (0,1]$, we have*

$$a_1 \leq a_2 \Longrightarrow r_{a_1} \geq r_{a_2}.$$

Proof. *1.* We have $(P^a)'(x) = (x+2a)^{2n} + 2xn(x+2a)^{2n-1} > 0$ for $x \in [0,1]$. Therefore P^a is strictly increasing on the interval $[0,1]$. It follows that P^a is injective there.

2. We have $P^a(0) = -1 < 0$ and $P^a(1) = (1+2a)^{2n} - 1 > 0$. Hence in view of the intermediate value theorem there exists a point $r_a \in (0,1)$ such that $P^a(r_a) = 0$. It is unique, since P^a is injective. In particular $r_a(r_a+2a)^{2n} = 1$.

3. Fix $a_1, a_2 \in (0,1]$ such that $a_1 \leq a_2$. Then by the definition of P^a we have $P^{a_1}(x) \leq P^{a_2}(x)$ for every $x \in [0,1]$. Since P^{a_1} and P^{a_2} are strictly increasing, $(P^{a_1})^{-1}(y) \geq (P^{a_2})^{-1}(y)$ for every $y \in [-1, P^{a_1}(1)]$. In particular, since $P^{a_1}(1) > 0$, we obtain $r_{a_1} = (P^{a_1})^{-1}(0) \geq (P^{a_2})^{-1}(0) = r_{a_2}$. □

Recall now another result from [6].

Proposition 4.6. [6, Proposition 2.6] *$\overline{D}(-1, r_n) \subset \mathcal{K}[z \mapsto z^{2n} - 1]$ where $r_n \in (0,1)$ is such that $r_n(r_n+2)^{2n} = 1$ for every $n \geq 2$.*

Let us define $p_k(z) := z^{2n} - a^{(2n)^k}$ for $a \in (0,1]$ and integers $n \geq 2$ and $k \geq 0$. We give some generalisation of Proposition 4.6.

Proposition 4.7. *Fix $a \in (0,1]$ and $n \geq 2$. Then $\overline{D}(-\sqrt[2n]{a}, r) \subset \mathcal{K}[(p_k^{\circ 2})_{k=0}^{\infty}]$, where $r \in (0,1)$ is such that $r(r + 2\sqrt[2n]{a})^{2n} = 1$.*

Proof. Fix $a \in (0,1]$ and $n \geq 2$. Recall that $p_0(z) = z^{2n} - a$. Then $p_0^{\circ 2}(z) = (z^{2n} - a)^{2n} - a$.

Fix $z \in \overline{D}(-\sqrt[2n]{a}, r_{\sqrt[2n]{a}})$, where $r_{\sqrt[2n]{a}} \in (0,1)$ is such that

$$r_{\sqrt[2n]{a}}(r_{\sqrt[2n]{a}} + 2\sqrt[2n]{a})^{2n} = 1$$

(the existence of $r_{\sqrt[2n]{a}}$ is assured by Lemma 4.5). Then $|z - w| \leq r_{\sqrt[2n]{a}} + 2\sqrt[2n]{a}$ for every $w \in \{w_1, \ldots, w_{2n-1}\}$ where $\{w_1, \ldots, w_{2n-1}\}$ is the set of the roots of the equation $z^{2n} = a$ different from $-\sqrt[2n]{a}$.

Therefore

$$\begin{aligned}
|p_0^{\circ 2}(z) + a| &= \left|(z^{2n} - a)^{2n} - a + a\right| = \left|(z^{2n} - a)^{2n}\right| = \\
&= \left|z + \sqrt[2n]{a}\right|^{2n} \cdot \prod_{i=1}^{2n-1} |z - w_i|^{2n} \leq r_{\sqrt[2n]{a}}^{2n} (r_{\sqrt[2n]{a}} + 2\sqrt[2n]{a})^{(2n-1)2n} = \\
&= r_{\sqrt[2n]{a}} \left[r_{\sqrt[2n]{a}} (r_{\sqrt[2n]{a}} + 2\sqrt[2n]{a})^{2n} \right]^{2n-1} = r_{\sqrt[2n]{a}}.
\end{aligned}$$

We have proved that $p_0^{\circ 2}(z) \in \overline{D}(-a, r_{\sqrt[2n]{a}})$.

Take $r_a \in (0,1)$ such that $r_a(r_a + 2a)^{2n} = 1$. Then since $a \leq \sqrt[2n]{a}$, by Lemma 4.5 we obtain $r_a \geq r_{\sqrt[2n]{a}}$. Hence $\overline{D}(-a, r_{\sqrt[2n]{a}}) \subset \overline{D}(-a, r_a)$. Therefore we have $p_0^{\circ 2}(z) \in \overline{D}(-a, r_a)$.

Now we consider $p_1(z) = z^{2n} - a^{2n}$. Repeating the same reasoning, we will show that $p_1^{\circ 2}(p_0^{\circ 2}(z)) \in \overline{D}(-a^{2n}, r_a)$.

Note that $p_1^{\circ 2}(z) = (z^{2n} - a^{2n})^{2n} - a^{2n}$. Therefore

$$|p_1^{\circ 2}(p_0^{\circ 2}(z)) + a^{2n}| = \left|((p_0^{\circ 2}(z))^{2n} - a^{2n})^{2n} - a^{2n} + a^{2n}\right| = \left|((p_0^{\circ 2}(z))^{2n} - a^{2n})^{2n}\right| =$$

$$= |p_0^{\circ 2}(z) + a|^{2n} \cdot \prod_{i=1}^{2n-1} |p_0^{\circ 2}(z) - \widetilde{w}_i|^{2n} \leq r_a,$$

where $\{\widetilde{w}_1, \dots, \widetilde{w}_{2n-1}\}$ is the set of the roots of the equation $\zeta^{2n} = a^{2n}$ different from $-a$.

Inductively we can prove that

$$\forall k \geq 0 \; \exists r_{a^{(2n)^{k-1}}} \in (0,1): \; (p_k^{\circ 2} \circ p_{k-1}^{\circ 2} \circ \dots \circ p_0^{\circ 2})(z) \in \overline{D}(-a^{(2n)^k}, r_{a^{(2n)^{k-1}}}).$$

But since $a \in (0,1]$, we have $\overline{D}(-a^{(2n)^k}, r_{a^{(2n)^{k-1}}}) \subset \overline{D}(0,3)$ for every integer $k \geq 1$. □

We already know that $\mathcal{K}[(p_k^{\circ 2})_{k=0}^{\infty}]$ is infinite. However, we can say more about it.

Proposition 4.8. *$\mathcal{K}[(p_k^{\circ 2})_{k=0}^{\infty}]$ is compact.*

Proof. Fix $a \in (0,1]$ and integer $n \geq 2$. Firstly we prove that for every integer $k \geq 0$ the following implication holds

$$|z| \geq 2 \quad \Longrightarrow \quad |p_k(z)| > 7|z|.$$

Fix $z \in \mathbb{C} \setminus D(0,2)$ and $k \geq 0$. Then

$$|p_k(z)| = \left|z^{2n} - a^{(2n)^k}\right| \geq |z|^{2n} - a^{(2n)^k} = |z|\left(|z|^{2n-1} - \frac{a^{(2n)^k}}{|z|}\right) > 7|z|.$$

Consequently $|p_k^{\circ 2}(z)| > 7|p_k(z)| > 49|z|$. In view of Remark 4

$$\mathcal{K}[(p_k^{\circ 2})_{k=0}^{\infty}] = \bigcap_{k=0}^{\infty} \left(p_k^{\circ 2} \circ \dots \circ p_0^{\circ 2}\right)^{-1} (\overline{D}(0,2)).$$

It follows that $\mathcal{K}[(p_k^{\circ 2})_{k=0}^{\infty}]$ is compact. □

To conclude this subsection let us recall that in [2] non-autonomous Julia sets associated with some special sequences of polynomials are considered. The authors define namely a class $\mathcal{B}$ of sequences $(f_n)_{n=1}^{\infty}$, where f_n is a polynomial of degree $d_n \geq 2$ such that there is some $R > 0$ with

$$f_n(\{z \in \mathbb{C}: \; |z| \geq R\}) \subset \{z \in \mathbb{C}: \; |z| > R\}$$

for all $n \geq 1$ and $(f_n \circ \dots \circ f_1)|_{\{z \in \mathbb{C}: \; |z| > R\}} \xrightarrow{n \to \infty} \infty$ locally uniformly.

Example 4.2.

$$(p_k^{\circ 2})_{k=0}^{\infty} \in \mathcal{B}.$$

Proof. It is a direct consequence of the implication

$$\forall k \geq 0: \quad |z| \geq 2 \quad \Longrightarrow \quad |p_k^{\circ 2}(z)| > 49|z|$$

from the proof of Proposition 4.8. □

Example 4.3. Fix a sequence $(c_n)_{n=1}^{\infty} \subset \overline{D}\left(0, \frac{1}{4}\right)$. Then $(f_{c_n})_{n=1}^{\infty} \in \mathcal{B}$.

Proof. It is a direct consequence of the implication

$$\forall n \geq 1: \quad |z| \geq 3 \quad \Longrightarrow \quad |f_{c_n}(z)| > 2|z|$$

from the proof of Proposition 4.2. □

Therefore, for both of these sequences we can apply a range of results collected in [2].

5 Hölder Continuity Property of non-autonomous Julia sets

In this section we are interested in potential-theoretic properties of non-autonomous filled Julia sets.

Denote by $\mathcal{SH}(U)$ the set of all subharmonic functions in an open subset $U \subset \mathbb{C}$. Consider the family

$$\mathcal{L} := \{u \in \mathcal{SH}(\mathbb{C}) | \ \exists \beta \in \mathbb{R} : \ u(z) \leq \beta + \log^+ |z| \quad \text{for } z \in \mathbb{C}\}.$$

Recall that for a compact subset $E \subset \mathbb{C}$ the function

$$V_E(z) := \sup\{u(z) : \ u \in \mathcal{L} \text{ and } u|_E \leq 0\}, \quad z \in \mathbb{C},$$

is called the *L-extremal function* corresponding to E. A compact set E is said to be *regular* if V_E is continuous. One can prove that if $E \subset \mathbb{C}$ is a non-polar compact set, then $V_E|_{\mathbb{C}\setminus\widehat{E}}$ coincides with the generalised complex Green function for $\mathbb{C} \setminus \widehat{E}$ with pole at infinity (cf. [10, p. 182]).

The set

$$\widehat{E} := \{z \in \mathbb{C} | \ \forall p \in \mathcal{P}(\mathbb{C}) : \quad |p(z)| \leq \|p\|_E\},$$

where $\mathcal{P}(\mathbb{C})$ denotes the space of polynomials, is called the *polynomial hull* of the set $E \subset \mathbb{C}$. A set E is called *polynomially convex*, if $E = \widehat{E}$.

The following properties follow directly from the definition of the L-extremal function.

Proposition 5.1. [15, 2.4, 2.18, 4.14]

1.

$$\forall A, B \subset \mathbb{C} : \quad A \subset B \implies V_A \geq V_B$$

2. If $E \subset \mathbb{C}$ is a compact set, then

$$V_E(z) = 0 \iff z \in \widehat{E}.$$

3. If $E \subset \mathbb{C}$ is a compact set, then $V_E \equiv V_{\widehat{E}}$.

We recall a result proved in [8], which we state in a slightly different form.

Theorem 5.2. (cf. [8, Theorem 2.1], [12, Proposition 3]) *Let $(c_n)_{n=1}^{\infty} \subset \mathbb{C}$ be a bounded sequence. Then the function*

$$g_{(c_n)_{n=1}^{\infty}} : \mathbb{C} \setminus \mathcal{K}[(f_{c_n})_{n=1}^{\infty}] \ni z \longmapsto \lim_{n\to\infty} \frac{1}{2^n} \log |F_n(z)| \in \mathbb{R}$$

is the complex Green function for $\mathbb{C} \setminus \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ with pole at infinity. Putting $g_{(c_n)_{n=1}^{\infty}}|_{\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]} \equiv 0$, the mapping $g_{(c_n)_{n=1}^{\infty}}$ extends continuously to $\mathbb{C}$.

Let us further note that the following occurs.

Proposition 5.3. [12, Corollary 4] *Fix a bounded sequence $(c_n)_{n=1}^{\infty} \subset \mathbb{C}$. Then*

$$\forall z \in \mathbb{C} : \quad V_{\mathcal{K}[(f_{c_n})_{n=2}^{\infty}]}(f_{c_1}(z)) = 2V_{\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]}(z).$$

Proof. This follows directly from the formula defining the Green function $g_{(c_n)_{n=1}^{\infty}}$. □

Recall the following definition.

Definition 5.1. We say that a set $E \subset \mathbb{C}$ satisfies *HCP* (*Hölder Continuity Property*) if

$$\exists M > 0, \ \alpha > 0 : \quad \operatorname{dist}(z, E) \leq 1 \quad \implies \quad V_E(z) \leq M \operatorname{dist}(z, E)^{\alpha}.$$

Błocki showed that HCP implies the Hölder continuity of the Green function in the whole $\mathbb{C}$ (see [16, Proposition 3.5]). Recall that Sibony proved that $\mathcal{K}[p]$ has HCP for any polynomial p of degree at least two (see [5, Theorem VIII.3.2 and the comment after the proof]).

We will also use

Example 5.1. (cf. [10, Example 5.1.1]) Fix $a \in \mathbb{C}$ and $r > 0$. Then

$$\forall z \in \mathbb{C} : \quad V_{\overline{D}(a,r)}(z) = \log^+ \frac{|z-a|}{r}.$$

The proof of the following theorem is based on the proof of [11, Theorem 1.2].

Theorem 5.4. *Let* $(c_n)_{n=1}^{\infty} \subset \overline{D}\left(0, \frac{1}{4}\right)$. *Then the set* $\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ *has HCP.*

Proof. In view of Proposition 4.2 the set $\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ is compact. Thus we can fix an $r > 1$ such that $\mathcal{K}[(f_{c_n})_{n=1}^{\infty}] \subset D(0,r) := \{z \in \mathbb{C} : |z| < r\}$.

If $z \in \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$, we have $V_{\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]}(z) = \operatorname{dist}(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]) = 0$ and the inequality is obvious. Now, let $z \notin \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ with $|z| < 2r$. Since $\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ is compact, there exists $z_0 \in \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ such that $|z - z_0| = \operatorname{dist}\left(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]\right) > 0$. Compactness of $\overline{D}\left(0, \frac{1}{4}\right)$ implies the existence of a point $\tilde{z_0} \in \overline{D}\left(0, \frac{1}{4}\right)$ satisfying $|z_0 - \tilde{z_0}| = \operatorname{dist}\left(z_0, \overline{D}\left(0, \frac{1}{4}\right)\right) > 0$. Next, let us define $c := \frac{|z_0 - \tilde{z_0}|}{|z - z_0|}$. Recall that $\overline{D}\left(0, \frac{1}{4}\right) \subsetneq \overline{D}\left(0, \frac{1}{2}\right) \subset \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ by Proposition 4.1. By the proof of Proposition 4.1 we have $F_j(\tilde{z_0}) \in \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ for every $j \in \mathbb{N}^\star$. Now, we consider a sequence of sets defined recursively:

$$K_0 := [\tilde{z_0}, z], \quad K_j := F_j(K_0) \quad \text{for } j \in \mathbb{N}^\star.$$

Since $z \notin \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$, we have $F_{m_0}(z) \notin \overline{D}(0, 2r)$ for some $m_0 \in \mathbb{N}^\star$. Let $m := \min\{n \in \mathbb{N}^\star : K_n \not\subset \overline{D}(0, 2r)\}$. Then there exists $z_1 \in [\tilde{z_0}, z] \setminus \{\tilde{z_0}\}$ such that $|F_m(z_1)| > 2r$.

Let $w \in K_0$. By the definition of m we conclude that

$$\begin{aligned}|(F_m)'(w)| = |(f'_{c_m} \circ f_{c_{m-1}} \circ \ldots \circ f_{c_1})(w) \cdot (f'_{c_{m-1}} \circ f_{c_{m-2}} \circ \ldots \circ f_{c_1})(w) \cdot \ldots \cdot f'_{c_1}(w)| \leq \\ \leq \max\{|f'_{c_j}(z)| : |z| \leq 2r, \ j \in \{1, \ldots, m\}\}^m = (4r)^m\end{aligned}$$

Hence, in view of the Mean Value Theorem for $[\tilde{z_0}, z_1]$ we get $|F_m(\tilde{z_0}) - F_m(z_1)| \leq (4r)^m |\tilde{z_0} - z_1|$. Therefore by the choice of $\tilde{z_0}$ and z_1 we obtain

$$\begin{aligned}2r < |F_m(z_1)| &\leq (4r)^m |\tilde{z_0} - z_1| + |F_m(\tilde{z_0})| \leq (4r)^m |\tilde{z_0} - z| + r \leq \\ &\leq (4r)^m (|\tilde{z_0} - z_0| + |z_0 - z|) + r = (4r)^m (c+1)|z_0 - z| + r = \\ &= (4r)^m (c+1) \operatorname{dist}(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]) + r \leq (4r)^m (c+1)^m \operatorname{dist}(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]) + r\end{aligned}$$

and consequently $(4r)^m (c+1)^m \operatorname{dist}(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]) > 1$. Let $\alpha := \log_{(4r(c+1))} 2$. Then

$$\operatorname{dist}(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}])^\alpha > (((4r(c+1))^m)^\alpha)^{-1} = (2^m)^{-1}. \tag{2}$$

Now, using Proposition 5.3 repeatedly we get

$$V_{\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]}(z) = \frac{1}{2^m} V_{\mathcal{K}[(f_{c_n})_{n=m+1}^{\infty}]}(F_m(z)). \tag{3}$$

By the definition of m we have $|F_{m-1}(z)| \leq 2r$. And since $\overline{D}\left(0, \frac{1}{2}\right) \subset \mathcal{K}[(f_{c_n})_{n=m+1}^{\infty}]$ by Propostion 4.1, Proposition 5.1 and Example 5.1 yield

$$\begin{aligned}V_{\mathcal{K}[(f_{c_n})_{n=m+1}^{\infty}]}(F_m(z)) &\leq \sup\{V_{\mathcal{K}[(f_{c_n})_{n=m+1}^{\infty}]}\left(f_{c_m}(w)\right) : |w| \leq 2r\} = \\ &= \sup\{V_{\mathcal{K}[(f_{c_n})_{n=m+1}^{\infty}]}\left(w^2 + c_m\right) : |w| \leq 2r\} \leq \\ &\leq \sup\left\{V_{\mathcal{K}[(f_{c_n})_{n=m+1}^{\infty}]}(v) : |v| \leq 4r^2 + \frac{1}{4}\right\} \leq \\ &\leq \sup\left\{V_{\overline{D}(0,\frac{1}{2})}(v) : |v| \leq 4r^2 + \frac{1}{4}\right\} = \\ &= \sup\left\{\log^+ \frac{|v|}{\frac{1}{2}} : |v| \leq 4r^2 + \frac{1}{4}\right\} = \\ &= \log^+\left(8r^2 + \frac{1}{2}\right) =: M > 0.\end{aligned} \tag{4}$$

Combining (2), (3) and (4) we obtain

$$V_{\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]}(z) \leq M \operatorname{dist}(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}])^\alpha.$$

Moreover, if $\operatorname{dist}(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]) \leq 1$, then

$$|z| \leq \operatorname{dist}(z, \mathcal{K}[(f_{c_n})_{n=1}^{\infty}]) + |z_0| \leq 1 + r < 2r$$

and this completes the proof of our assertion. □

It is well-known that the Hölder Continuity Property (Definition 5.1) is sufficient for a compact set to preserve Markov's inequality (cf. [14, Remark after Lemma 1] and [13]). Hence under the assumptions of the previous theorem the set $\mathcal{K}[(f_{c_n})_{n=1}^{\infty}]$ satisfies this inequality.

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