PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 17 · 2024 · Pages 82–87

On Stancu modification of (p, q)-Bernstein polynomials.

Aneta Sikorska-Nowak[a] · Grzegorz Nowak[b]

Communicated by S. De Marchi

Abstract

In this paper, we introduce the generalized (p, q)-Bernstein-Stancu polynomials based on the (p, q)-integers. We study approximation properties of these operators. In special cases, we obtain (p, q)-Bernstein operators, Stancu operators or Phillips polynomials.

1 Introduction

In recent years, (p, q) - integers have been introduced into classical linear positive operators to construct new approximation processes. A sequence of (p, q) - analogue of Bernstein operators was first introduced by Mursaleen [17, 18]. Besides, (p, q)-analogues of Szasz-Mirakyan [1], Baskakov Kantorovich [2], Bleimann-Butzer-Hahn [3] and Kantorovich type Bernstein-Stancu-Schurer [5] operators were also considered. For further developments, one can also refer to [19, 20, 21, 22, 4, 12]. These operators are double parameters corresponding to p and q versus single parameter q-Bernstein-type operators [26, 27, 28, 24, 25]. The aim of these generalizations is to provide appropriate and powerful tools for application areas such as numerical analysis, computer-aided geometric design and solutions of differential equations (see, [14, 15, 11, 31, 30]).

For any fixed real numbers $p > 0$, $q > 0$ and for nonnegative integer r, the (p, q)-integers of the number $[r]_{p,q}$ is defined by

$$[r]_{p,q} \equiv [r] = p^{r-1} + p^{r-2}q + p^{r-3}q^2 + \dots + pq^{r-2} + q^{r-1} = \begin{cases} \frac{p^r - q^r}{p-q}, & \text{when } p \neq q \\ np^{n-1}, & \text{when } p = q \end{cases}$$

The (p, q)-factorial $[r]_{p,q}!$ for $r \in N_0 = \{0, 1, 2, \dots\}$, is defined in the following way

$$[r]_{p,q}! = [1]_{p,q}[2]_{p,q} \dots [r]_{p,q} \qquad (r = 1, 2, \dots), \qquad [0]_{p,q}! = 1.$$

For the integers $n, k, (n \geq k \geq 0)$, the (p, q)-binomial or the Gaussian coefficients is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}$$

For $f \in C[0; 1]$, $0 < q < p < 1$, $\alpha \geq 0$ and each positive integer n, we introduce the following generalized (p, q)-Bernstein-Stancu operators:

$$B_n^{p,q,\alpha}(f; x) = \sum_{k=0}^{n} b_{n,k}^{p,q,\alpha}(x) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right), \tag{1}$$

where

$$b_{n,k}^{p,q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{\prod_{i=0}^{k-1}(p^i x + \alpha p[i]_{p,q}) \prod_{s=0}^{n-1-k}(p^s - q^s x + \alpha p[s]_{p,q})}{\prod_{j=0}^{n-1}(p^j + \alpha p[j]_{p,q})}. \tag{2}$$

Note, that an empty product in (2) denotes 1.

Let $j \in N_0$. For $f \in C[0, 1]$, we define the (p, q)-differences as follows

$$\Delta_{p,q}^0 f(x_j) = f(x_j), \tag{3}$$

and recursively,

$$\Delta_{p,q}^{k+1} f(x_j) = p^k \Delta_{p,q}^k f(x_{j+1}) - q^k \Delta_{p,q}^k f(x_j) \tag{4}$$

[a]Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poland
[b]Department of Mathematics, WSB Merito University, Poland

for $k \in N_0$ and $f_j = f\left(\frac{[j]_{p,q}}{p^{j-n}[n]_{p,q}}\right)$.

It is easily established by induction [23] that (p,q)-differences satisfy the relation

$$\Delta^k_{p,q} f(x_j) = \sum_{r=0}^{k} (-1)^r \left[\begin{array}{c} k \\ r \end{array} \right]_{p,q} p^{(k-r)(k-r-1)/2} q^{r(r-1)/2} f(x_{j+k-r}). \tag{5}$$

In this paper, we prove that the operators $B_n^{p,q,\alpha}(f;x)$ defined by the formula (1) can be expressed in terms of (p,q)-differences

$$B_n^{p,q,\alpha}(f;x) = \sum_{k=0}^{n} p^{-k(2n-k-1)/2} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} \Delta^k_{p,q} f_0 \prod_{i=0}^{k-1} \frac{p^i x + \alpha p[i]}{p^i + \alpha p[i]},$$

which generalizes the well known result [23] for the (p,q)-Bernstein polynomial. Moreover, we present some properties of these operators.

2 Main result

Lemma 2.1. *For any fixed real numbers $p > 0$, $q > 0$, $\alpha \geq 0$ and for nonnegative integer n, we have*

$$\prod_{s=0}^{n-1} (p^s - q^s x + \alpha p[s]) = p^{-n(n-1)/2} \tag{6}$$

$$\cdot \sum_{s=0}^{n} (-1)^s q^{s(s-1)/2} p^{(n-s)(n-s-1)/2} \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} \prod_{i=0}^{s-1} (p^i x + \alpha p[i]) \prod_{j=s}^{n-1} (p^j + \alpha p[j]).$$

Proof . We use induction on n. The base case for $n = 1$ is evident. Assuming the lemma holds for a given $n \in N$, we then proceed to demonstrate it for $n+1$, following through the algebraic manipulations to uphold the induction principle. Then, using (6), we obtain

$$p^{n(n-1)/2} \prod_{s=0}^{n} (p^s - q^s x + \alpha p[s]) = (p^n - q^n x + \alpha p[n])$$

$$\cdot \sum_{s=0}^{n} (-1)^s q^{s(s-1)/2} p^{(n-s)(n-s-1)/2} \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} \prod_{i=0}^{s-1} (p^i x + \alpha p[i]) \prod_{j=s}^{n-1} (p^j + \alpha p[j])$$

$$= \sum_{s=0}^{n} (p^n + p\alpha[n] + q^n \alpha p^{-s+1}[s]) (-1)^s q^{s(s-1)/2} p^{(n-s)(n-s-1)/2}$$

$$\cdot \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} \prod_{i=0}^{s-1} (p^i x + \alpha p[i]) \prod_{j=s}^{n-1} (p^j + \alpha p[j])$$

$$-q^n \sum_{s=0}^{n} (-1)^s q^{s(s-1)/2} p^{(n-s)(n-s-1)/2-s} \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} \prod_{i=0}^{s} (p^i x + \alpha p[i]) \prod_{j=s}^{n-1} (p^j + \alpha p[j])$$

$$= \sum_{s=0}^{n} (p^n + p\alpha[n] + q^n \alpha p^{-s+1}[s]) (-1)^s q^{s(s-1)/2} p^{(n-s)(n-s-1)/2}$$

$$\cdot \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} \prod_{i=0}^{s-1} (p^i x + \alpha p[i]) \prod_{j=s}^{n-1} (p^j + \alpha p[j])$$

$$+q^n \sum_{s=1}^{n+1} (-1)^s q^{(s-1)(s-2)/2} p^{(n-s)(n-s+1)/2-s+1}$$

$$\cdot \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q} (p^{s-1} + \alpha p[s-1]) \prod_{i=0}^{s-1} (p^i x + \alpha p[i]) \prod_{j=s}^{n-1} (p^j + \alpha p[j])$$

$$= (p^n + \alpha p[n]) p^{-n(n-1)/2} \prod_{j=0}^{n-1} (p^j + \alpha p[j])$$

$$+(-1)^{n+1} q^{n(n+1)/2} p^n (p^n + \alpha p[n]) \prod_{i=0}^{n} (p^i x + \alpha p[i])$$

$$+\sum_{s=1}^{n} R_s (-1)^s q^{s(s-1)/2} p^{(n-s)(n-s-1)/2} \prod_{i=0}^{s-1} (p^i x + \alpha p[i]) \prod_{j=s}^{n-1} (p^j + \alpha p[j]),$$

where

$$R_s = (p^n + \alpha p[n] + \alpha q^n p^{-s+1}[s]) \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q}$$

$$+q^{n-s+1} p^{n-2s+1} \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q} (p^{s-1} + \alpha p[s-1])$$

Thanks to obvious equalities

$$q^{s-1}[s] \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} + p^{n-s}[s-1] \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q} = [n] \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q}$$

and

$$p^s \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} + q^{n-s+1} \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q} = \left[\begin{array}{c} n+1 \\ s \end{array} \right]_{p,q}$$

we have

$$R_s = (p^n + \alpha p[n]) \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q}$$

$$+\alpha \left(q^n p^{-s+1}[s] \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} + q^{n-s+1} p^{n-2s+2} \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q} p[s-1] \right) + q^{n-s+1} p^{n-s} \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q}$$

$$= (p^n + \alpha p[n]) \left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} + \alpha q^{n-s+1} p^{-s+1}[n] \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q} + q^{n-s+1} p^{n-s} \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q}$$

$$= (p^n + \alpha p[n]) \left(\left[\begin{array}{c} n \\ s \end{array} \right]_{p,q} + \left[\begin{array}{c} n \\ s-1 \end{array} \right]_{p,q} p^{-s} q^{n-s+1} \right)$$

$$= (p^n + \alpha p[n]) p^{-s} \left[\begin{array}{c} n+1 \\ s \end{array} \right]_{p,q} .$$

Collecting the results we obtain Lemma.

Theorem 2.2. *The generalized Bernstein-Stancu polynomial can be expressed in the form as follows:*

$$B_n^{p,q,\alpha}(f;x) = \sum_{k=0}^{n} p^{-k(2n-k-1)/2} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} \Delta_{p,q}^k f_0 \prod_{i=0}^{k-1} \frac{p^i x + \alpha p[i]}{p^i + \alpha p[i]}, \tag{7}$$

where $\Delta_{p,q}^k f_0$ *is defined by the earlier stated recursive formula* (4).

Proof. Using (1), (2) and (6) we have

$$B_n^{p,q,\alpha}(f;x) \prod_{i=0}^{n-1} (p^i + \alpha p[i]) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} f\left(\frac{[k]}{p^{k-n}[n]} \right) S_k, \tag{8}$$

where

$$S_k = p^{-(n-k)(n-k-1)/2} \sum_{s=0}^{n-k} (-1)^s q^{s(s-1)/2} p^{(n-k-s)(n-k-s-1)/2} \left[\begin{array}{c} n-k \\ s \end{array} \right]_{p,q}$$

$$\cdot \prod_{i=0}^{k-1} (p^i x + \alpha p[i]) \prod_{j=0}^{s-1} (p^j x + \alpha p[j]) \prod_{l=s}^{n-k-1} (p^l + \alpha p[l]).$$

First we prove that

$$S_k = p^{k(k-1)/2 - n(n-1)/2} \sum_{s=0}^{n-k} (-1)^s q^{s(s-1)/2} p^{(n-k-s)(n-k-s-1)/2} \tag{9}$$

$$\cdot \left[\begin{array}{c} n-k \\ s \end{array} \right]_{p,q} \prod_{i=0}^{k+s-1} (p^i x + \alpha p[i]) \prod_{j=k+s}^{n-1} (p^j + \alpha p[j])$$

for all $n \in N$ and $(x \in [0;1])$.

We use the induction on k. First, we see that equality (9) holds for $k = 0$ and $n \in N$. Let us assume that (9) holds for a given k, $0 \le k \le n$, and for all $n \in N$. Then from (9), we obtain

$$S_{k+1} = p^{-(n-k-1)(n-k-2)/2} \sum_{s=0}^{n-k-1} (-1)^s q^{s(s-1)/2} p^{(n-k-s-1)(n-k-s-2)/2} \left[\begin{array}{c} n-k-1 \\ s \end{array} \right]_{p,q}$$

$$\cdot\prod_{i=0}^{k}(p^i x+\alpha p[i])\prod_{j=0}^{s-1}(p^j x+\alpha p[j])\prod_{l=s}^{n-k-2}(p^l+\alpha p[l])$$

$$=(p^k x+\alpha p[k])p^{k(k-1)/2-(n-1)(n-2)/2}\sum_{s=0}^{n-k-1}(-1)^s q^{s(s-1)/2}p^{(n-k-s-1)(n-k-s-2)/2}$$

$$\cdot\left[\begin{array}{c} n-k-1 \\ s \end{array}\right]_{p,q}\prod_{i=0}^{k+s-1}(p^i x+\alpha p[i])\prod_{j=k+s}^{n-2}(p^j+\alpha p[j])$$

$$=p^{k(k-1)/2-(n-1)(n-2)/2}\sum_{s=0}^{n-k-1}(-1)^s q^{s(s-1)/2}p^{(n-k-s-1)(n-k-s-2)/2-s}\left[\begin{array}{c} n-k-1 \\ s \end{array}\right]_{p,q}$$

$$\cdot\prod_{i=0}^{k+s}(p^i x+\alpha p[i])\prod_{j=k+s}^{n-2}(p^j+\alpha p[j])$$

$$+p^{k(k-1)/2-(n-1)(n-2)/2}\alpha\sum_{s=0}^{n-k-1}(-1)^s q^{s(s-1)/2}p^{(n-k-s-1)(n-k-s-2)/2-s}\left[\begin{array}{c} n-k-1 \\ s \end{array}\right]_{p,q}$$

$$\cdot(p^{s+1}[k]-p[k+s])\prod_{i=0}^{k+s-1}(p^i x+\alpha p[i])\prod_{j=k+s}^{n-2}(p^j+\alpha p[j]).$$

We see that

$$\left[\begin{array}{c} n-k-1 \\ s \end{array}\right]_{p,q}(p^{s+1}[k]-p[k+s])=-\left[\begin{array}{c} n-k-1 \\ s-1 \end{array}\right]_{p,q}[n-k-s]pq^k,$$

and hence

$$S_{k+1}=p^{k(k-1)/2-(n-1)(n-2)/2}\sum_{s=0}^{n-k-1}(-1)^s q^{s(s-1)/2}p^{(n-k-s-1)(n-k-s-2)/2-s}$$

$$\cdot\left[\begin{array}{c} n-k-1 \\ s \end{array}\right]_{p,q}\prod_{i=0}^{k+s}(p^i x+\alpha p[i])\prod_{j=k+s}^{n-2}(p^j+\alpha p[j])$$

$$-\alpha p^{k(k-1)/2-(n-1)(n-2)/2}\sum_{s=1}^{n-k-1}(-1)^s q^{s(s-1)/2}p^{(n-k-s-1)(n-k-s-2)/2-s}$$

$$\cdot\left[\begin{array}{c} n-k-1 \\ s-1 \end{array}\right]_{p,q}[n-k-s]pq^k\prod_{i=0}^{k+s-1}(p^i x+\alpha p[i])\prod_{j=k+s}^{n-2}(p^j+\alpha p[j])$$

$$=(-1)^{n-k-1}q^{(n-k-1)(n-k-2)/2}p^{k(k+1)/2-n(n-1)/2}\prod_{i=0}^{n-1}(p^i x+\alpha p[i])$$

$$+p^{k(k-1)/2-(n-1)(n-2)/2}\sum_{s=0}^{n-k-2}(-1)^s q^{s(s-1)/2}p^{(n-k-s-1)(n-k-s-2)/2-s}\left[\begin{array}{c} n-k-1 \\ s \end{array}\right]_{p,q}$$

$$\cdot(p^{k+s}+\alpha p[k+s]+\alpha q^{k+s}p^{-n+k+s+2}[n-k-s-1])\prod_{i=0}^{k+s}(p^i x+\alpha p[i])\prod_{j=k+s+1}^{n-2}(p^j+\alpha p[j]).$$

It is easy to verify that

$$p^{k+s}+\alpha p[k+s]+\alpha q^{k+s}p^{-n+k+s+2}[n-k-s-1]=p^{-n+k+s+1}(p^{n-1}+\alpha p[n-1])$$

and thus

$$S_{k+1}=(-1)^{n-k-1}q^{(n-k-1)(n-k-2)/2}p^{k(k+1)/2-n(n-1)/2}\prod_{i=0}^{n-1}(p^i x+\alpha p[i])$$

$$+p^{k(k+1)/2-n(n-1)/2}\sum_{s=0}^{n-k-2}(-1)^s q^{s(s-1)/2}p^{(n-k-s-1)(n-k-s-2)/2}\left[\begin{array}{c} n-k-1 \\ s \end{array}\right]_{p,q}$$

$$\cdot\prod_{i=0}^{k+s}(p^i x+\alpha p[i])\prod_{j=k+s+1}^{n-1}(p^j+\alpha p[j])$$

$$=p^{k(k+1)/2-n(n-1)/2}\sum_{s=0}^{n-k-1}(-1)^s q^{s(s-1)/2}p^{(n-k-s-1)(n-k-s-2)/2}\left[\begin{array}{c} n-k-1 \\ s \end{array}\right]_{p,q}$$

$$\cdot \prod_{i=0}^{k+s}(p^i x+\alpha p[i]) \prod_{j=k+s+1}^{n-1} (p^j+\alpha p[j])$$

Therefore (9) is evident. Consequently, in view of (8) and (9) we get

$$B_n^{p,q,\alpha}(f;x)\prod_{i=0}^{n-1}(p^i+\alpha p[i])$$
$$=\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} f_k p^{k(k-1)/2-n(n-1)/2}\sum_{s=k}^{n}(-1)^{s-k}q^{(s-k)(s-k-1)/2}p^{(n-s)(n-s-1)/2}$$
$$\left[\begin{array}{c} n-k \\ s-k \end{array}\right]_{p,q}\prod_{j=0}^{s-1}(p^j x+\alpha p[j])\prod_{i=s}^{n-1}(p^i+\alpha p[i]).$$

Next, in view of the equality

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}\left[\begin{array}{c} n-k \\ s-k \end{array}\right]_{p,q}=\left[\begin{array}{c} n \\ s \end{array}\right]_{p,q}\left[\begin{array}{c} s \\ k \end{array}\right]_{p,q}$$

we obtain

$$B_n^{p,q,\alpha}f(x)\prod_{i=0}^{n-1}(p^i+\alpha p[i])$$
$$=\sum_{k=0}^{n}f_k p^{k(k-1)/2-n(n-1)/2}\sum_{s=k}^{n}\left[\begin{array}{c} n \\ s \end{array}\right]_{p,q}\left[\begin{array}{c} s \\ k \end{array}\right]_{p,q}(-1)^{s-k}q^{(s-k)(s-k-1)/2}p^{(n-s)(n-s-1)/2}$$
$$\cdot\prod_{j=0}^{s-1}(p^j x+\alpha p[j])\prod_{i=s}^{n-1}(p^i+\alpha p[i])$$
$$=\sum_{s=0}^{n}p^{-s(2n-s-1)/2}\left[\begin{array}{c} n \\ s \end{array}\right]_{p,q}\prod_{j=0}^{s-1}(x+\alpha[j])\prod_{i=s}^{n-1}(1+\alpha[i])$$
$$\cdot\sum_{k=0}^{s}(-1)^{s-k}q^{(s-k)(s-k-1)/2}p^{k(k-1)/2}\left[\begin{array}{c} s \\ k \end{array}\right]_{p,q}f_k.$$

The condition (6) completes the proof.

Remark 1. It should be noted that when $\alpha=0$, this result coincides with that obtained by Mursaleen [23] for the (p,q)-Bernstein polynomial. Moreover, when $p=1$, this result corresponds to what Nowak [24] obtained for the q-Bernstein-Stancu operators.

Theorem 2.3. *For $f(x)=x^s$, $s\in\{0,1,2\}$, we have*

$$B_n^{p,q,\alpha}(1;x)=1,\qquad B_n^{p,q,\alpha}(t;x)=x, \tag{10}$$

and

$$B_n^{p,q,\alpha}(t^2;x)=\frac{1}{1+\alpha}\left(x(x+\alpha)+\frac{p^{n-1}}{[n]}x(1-x)\right),$$

for all $n\in N$ and $x\in[0;1]$.

Proof. This proof involves estimating $B_n^{p,q,\alpha}(t^s;x)$ for $s=0,1,2$. The result can be easily verified for $s=0$ based on the last theorem. Utilizing the definition of $B_n^{p,q,\alpha}(t^s;x)$, the properties of (p,q)-differences, and previous theorem results, one can show that these identities hold. For $s=2$, the detailed computation of $B_n^{p,q,\alpha}(t^2;x)$ involves using the (p,q)-difference formulas to arrive at the stated expression.

$$\Delta^0_{p,q}f_0=f_0=0,\qquad \Delta^1_{p,q}f_0=f_1-f_0=\frac{1}{p^{2-2n}[n]^2},$$
$$\Delta^2_{p,q}f_0=pf_2-(p+q)f_1+qf_0$$
$$=p\left(\frac{[2]}{p^{2-n}[n]}\right)^2-(p+q)\left(\frac{1}{p^{1-n}[n]}\right)^2=\frac{p^{2n-3}q[2]}{[n]^2}.$$

Consequently

$$B_n^{p,q,\alpha}(t^2;x)=\left[\begin{array}{c} n \\ 0 \end{array}\right]_{p,q}\Delta^0_{p,q}f_0+\left[\begin{array}{c} n \\ 1 \end{array}\right]_{p,q}\frac{1}{p^{n-1}}\Delta^1_{p,q}f_0x+\left[\begin{array}{c} n \\ 2 \end{array}\right]_{p,q}\frac{1}{p^{2n-3}}\Delta^2_{p,q}f_0x\frac{x(x+\alpha)}{1+\alpha}$$
$$=\frac{p^{n-1}}{[n]}x+\frac{q[n-1]}{[n]}\frac{x(x+\alpha)}{1+\alpha}$$
$$=\frac{1}{1+\alpha}\left(x(x+\alpha)+\frac{p^{n-1}}{[n]}x(1-x)\right).$$

The following remark is a consequence of Bohman and Korovkin Theorem

Remark 2. Let the sequences (p_n), (q_n) and (α_n) be such that $0 < q_n < p_n \leq 1, \alpha_n \geq 0$. Then, for any $f \in C[0;1]$, the sequence $B_n^{p,q,\alpha}(f;x)$ converges uniformly to $f(x)$ on $[0;1]$, if and only if $lim_{n\to\infty} q_n = lim_{n\to\infty} p_n = 1$ and $lim_{n\to\infty} \alpha_n = 0$, where $B_n^{q,\alpha}(f;x)$ is defined by (1) with $p = p_n$, $q = q_n$ and $\alpha = \alpha_n$.

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