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Solving nonlinear fractional differential equations via quadratic interpolation and Picard's iteration

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Abstract

In this paper, we present a novel and efficient numerical method for solving a broad class of nonlinear fractional differential equations. Our approach uses interpolation techniques in conjunction with Picard's iterations to solve an equivalent nonlinear Volterra integral equation. We prove a convergence theorem and validate our theoretical results through several numerical examples that demonstrate the method's accuracy. A key advantage of this method is that it does not require solving nonlinear or linear systems during its implementation, and the sufficient condition for convergence is solely the Lipschitz continuity of the function involved in the equation.

1 Introduction

Classical models often fail to capture the nonlinearities, memory effects, and nonlocal dependencies inherent in many real-world systems. Fractional-order calculus (FOC), a mathematical concept whose foundations were laid in correspondence between Leibniz and L'Höpital in 1695, provides a powerful framework to overcome these limitations by generalizing differentiation and integration to arbitrary real or complex orders. While developed over subsequent centuries by luminaries like Euler, Liouville, and Riemann, its practical application has flourished in recent decades. This extension beyond the integer-order paradigm is embodied in key operators such as the Riemann-Liouville fractional integral and derivative, and the Caputo fractional derivative, which have proven essential for modeling complex phenomena across various scientific disciplines [5, 10, 7].

The most crucial quality of fractional operators-whether differential or integral-is their inherent ability to retain the complete history of a phenomenon. Consequently, mathematical models of real-world problems are often expressed as fractional-order differential or integro-differential equations. These equations are powerful instruments for describing the memory and hereditary properties of various materials and processes. Due to these properties, fractional calculus has been widely used in recent decades to model phenomena in continuum and statistical mechanics, solid mechanics, bio-engineering, medicine, seismology, electromagnetics, and more. For further applications, we refer the interested reader to [16].

In general, finding analytical solutions for fractional differential equations is challenging, as exact solutions are often unavailable. Due to this inherent complexity, research into numerical methods for solving various types of these equations has been expanded significantly in recent years. Popular techniques include two-step spline collocation methods [2, 4], spline collocation methods [3], operational matrix-based approaches [1], the fuzzy transform method [9], the finite difference method [8], as well as wavelet-based methods such as those using Legendre wavelets [6], Haar wavelets [13], Chebyshev wavelets [17], and fractional-order Bernoulli wavelets [11].

To solve a broad class of nonlinear fractional differential equations, we are interested in developing an effective iterative numerical approach. This new iterative technique is constructed based on using a quadratic interpolation through the Picard's iteration method on an equivalent integral equation of the given fractional differential equation.

A significant advantage of this new iterative approach is that it circumvents the need to solve nonlinear or linear systems of algebraic equations, unlike previously mentioned methods.

The following are our paper's outlines: fractional calculus preliminaries are recalled in Section 2. Details of the new method are described in Section 3. We deal with a rigorous convergence of the method in Section 4. Demonstration of theoretical results and numerical clarification are given in Section 5. A brief summary of the work and future directions for this research are given as final section.

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2 Preliminaries

Here, we prepare the fractional calculus preliminaries that will be used throughout the paper. For more details we refer the interested reader to [5]. Throughout this section we suppose $a, b \in \mathbb{R}$ and $a < b$.

Definition 2.1. Suppose that $\alpha \in \mathbb{R}^+$ and $f \in L_1[a,b]$. Then the integral operator defined by

$$J_a^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x \in [a,b] \tag{1}$$

is called the Riemann-Liouville fractional integral of order α. For $\alpha = 0$, it is the identity operator $J_a^0 := I$.

Definition 2.2. For $\alpha \in \mathbb{R}^+$, $m = \lceil \alpha \rceil$(the smallest integer greater than or equal to α) and $x \in [a,b]$, we denote by D_a^α the Riemann-Liouville fractional derivative of order α which is defined by

$$\begin{cases} D_a^\alpha f(x) := D^m J_a^{m-\alpha} f(x), & \alpha > 0 \\ D_a^\alpha f(x) := f(x), & \alpha = 0, \end{cases}$$

where $D^m = \frac{d^m}{dx^m}$.

Definition 2.3. Let $\alpha \geqslant 0, m = \lceil \alpha \rceil$, and $f \in A^m[a,b]$ (the set of all absolutely continuous functions of order m). The operator D_{*a}^α defined by

$$\begin{cases} D_{*a}^\alpha f(x) := J_a^{m-\alpha} D^m f(x), & \alpha > 0 \\ D_{*a}^\alpha f(x) := f(x), & \alpha = 0 \end{cases}$$

is called the Caputo fractional differential operator of order α.

The following proposition is proved easily by using the above definitions.

Proposition 2.1. *Let* $f(x) = (x-a)^\nu$, $\nu > -1$. *Then*

(i) $J_a^\alpha f(x) = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)}(x-a)^{\nu+\alpha}$.

(ii) $D_a^\alpha f(x) = \begin{cases} 0, & \alpha - \nu \in \mathbb{N} \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)}(x-a)^{\nu-\alpha}, & \alpha - \nu \notin \mathbb{N}. \end{cases}$

(iii) $D_{*a}^\alpha f(x) = \begin{cases} 0, & \nu \in \{0,1,2,\dots,m-1\} \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)}(x-a)^{\nu-\alpha}, & \\ & \nu \in \mathbb{N} \text{ and } \nu \geqslant m, \text{ or } \nu \notin \mathbb{N} \text{ and } \nu > m-1. \end{cases}$

Theorem 2.2. *If* $\alpha, \nu \geqslant 0$ *and* $\Phi \in L_1[a,b]$, *then,*

$$J_a^\alpha J_a^\nu \Phi = J_a^\nu J_a^\alpha \Phi = J_a^{\alpha+\nu}\Phi, \quad a.e. \text{ on } [a,b]. \tag{2}$$

Moreover, if Φ *is continuous on [a,b] or* $\alpha + \nu \geqslant 1$, *then* (2) *holds everywhere on* $[a,b]$.

Proposition 2.3. *If* f *is continuous on [a, b] and* $\alpha, \nu \geqslant 0$, *and* $\nu \geqslant \alpha$, *then*

(i) $D_{*a}^\alpha J_a^\alpha f(x) = f(x), \quad x \in [a,b]$

(ii) $D_{*a}^\alpha J_a^\nu f(x) = J_a^{\nu-\alpha} f(x), \; x \in [a,b]$

Definition 2.4. For $n > 0$ and $z \in \mathbb{C}$, a one- parameter Mittag-Leffler function, E_n, is defined by

$$E_n(z) := \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(ln+1)}.$$

It is proved that the series is uniformly convergent.

The following is a generalization of this definition.

Definition 2.5. For the parameter $n_1, n_2 > 0$, a two-parameter Mittag-Leffler function, E_{n_1,n_2}, is defined by

$$E_{n_1,n_2}(z) := \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(ln_1+n_2)}.$$

3 Statement of the problem

We consider the fractional IVP

$$\begin{cases} D^{\alpha}_{*0}y(x) = f(x, y(x)), & x \in [0, b] \\ y^{(j)}(0) = d_j, & j = 0, \dots, m-1, \end{cases} \tag{3}$$

where $\alpha \in \mathbb{R}^+$, $b > 0$, $m = \lceil \alpha \rceil$ and d_j are given real numbers.

Below we state Theorems 3.1, 3.3 and Lemma 3.2 about the existence and uniqueness of solution, and an equivalent form of this problem.

Theorem 3.1. *(Existence)[5] Let $\alpha > 0$ and $m = \lceil \alpha \rceil$. Moreover, let $d_0, ..., d_{m-1} \in \mathbb{R}$, $B > 0$ and $b^* > 0$. Let $U := \{(x, y) : x \in [0, b^*], |y - \sum_{k=0}^{m-1} x^k d_k / k!| \leqslant B\}$ and $f : U \to \mathbb{R}$ be a continuous function. Furthermore, suppose that $M := \sup_{(x,t)\in U} |f(x, t)|$ and*

$$b := \begin{cases} b^* & if \;\; M = 0, \\ min\{b^*, (B\Gamma(\alpha+1)/M)^{1/\alpha}\} & else. \end{cases}$$

Then, there exists $y \in C[0, b]$ satisfying (3).

The following lemma states that the IVP (3) is equivalent to a nonlinear weakly singular Volterra integral equation.

Lemma 3.2. *(Equivalent form)[5] Assuming the hypotheses of Theorem 3.1, $y \in C[0, b]$ satisfies the problem (3) iff it is a solution of the following nonlinear VIE*

$$y(x) = \sum_{j=0}^{m-1} \frac{d_j}{j!} x^j + \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t, y(t)) dt \tag{4}$$

with $m = \lceil \alpha \rceil$.

Theorem 3.3. *(Uniqueness)[5] Let $\alpha > 0$, $m = \lceil \alpha \rceil, d_0, \dots, d_{m-1} \in \mathbb{R}, B > 0$ and $b^* > 0$. Let U be defined as in Theorem 3.1 and the continuous function $f : U \to \mathbb{R}$ satisfies the Lipschitz condition*

$$|f(x, y_1) - f(x, y_2)| \leqslant L|y_1 - y_2| \tag{5}$$

with Lipschitz constant $L > 0$ independent of x, y_1 and y_2. Then, using b as in Theorem 3.1, there exists a unique $y \in C[0, b]$ verifying the IVP (3).

3.1 Method of solution

The Picard's iterative method corresponding to Eq.(4) is defined in the form:

$$\begin{cases} y_0(x) = \sum_{j=0}^{m-1} \frac{d_j}{j!} x^j, \\ y_{n+1}(x) = y_0(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t, y_n(t)) dt, & n \geqslant 0. \end{cases} \tag{6}$$

Using the transformation

$$t = xs \tag{7}$$

we get

$$y_{n+1}(x) = y_0(x) + \frac{x^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(xs, y_n(xs)) ds. \tag{8}$$

Now, let the function $K_n : [0, b] \times [0, 1] \longrightarrow \mathbb{R}$ for b given as in Theorem 3.1, be defined as

$$K_n(x, s) := f(xs, y_n(xs)), \quad n \geqslant 0 \tag{9}$$

(note that $y_n(x)$ is determined in the n^{th} step of iterations, hence K_n is a known function in terms of (x,s) in the $(n + 1)^{th}$ step). Then the iterative formual (8) is simplified as follows

$$y_{n+1}(x) = y_0(x) + \frac{x^\alpha}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} K_n(x, s) ds. \tag{10}$$

The integral on the right-hand side of Eq.(10) is a weakly singular integral whenever $0 < \alpha < 1$ and one may be unable to compute it easily and exactly. To make the integration easy, we suppose that the set $\{\xi_0, \xi_1, \dots, \xi_N\}$ is a partition of $[0, 1]$, such that $\xi_j = \frac{j}{N}, j = 0, 1, \dots, N$. Using this partition we can rewrite (10) in the form

$$y_{n+1}(x) = y_0(x) + \frac{x^\alpha}{\Gamma(\alpha)} \sum_{j=0}^{N-1} \int_{\xi_j}^{\xi_{j+1}} (1 - s)^{\alpha-1} K_n(x, s) ds. \tag{11}$$

Now, we approximate the function $K_n(x,s)$ on each interval $[\xi_j, \xi_{j+1}]$ by a second degree interpolation polynomial with respect to s, which interpolates it at the points $\xi_j, \xi_{j+\frac{1}{2}}, \xi_{j+1}$, i.e.,

$$
\begin{aligned}
K_n(x,s) \simeq \tilde{k}_n(x,s) = & \frac{(s-\xi_{j+1})(s-\xi_{j+\frac{1}{2}})}{(\xi_j-\xi_{j+1})(\xi_j-\xi_{j+\frac{1}{2}})} K_n(x,\xi_j) \\
& + \frac{(s-\xi_j)(s-\xi_{j+1})}{(\xi_{j+\frac{1}{2}}-\xi_j)(\xi_{j+\frac{1}{2}}-\xi_{j+1})} K_n(x,\xi_{j+\frac{1}{2}}) \\
& + \frac{(s-\xi_j)(s-\xi_{j+\frac{1}{2}})}{(\xi_{j+1}-\xi_j)(\xi_{j+1}-\xi_{j+\frac{1}{2}})} K_n(x,\xi_{j+1}) \\
& = \left(2N^2s^2-(4Nj+3N)s+2j^2+3j+1\right) K_n\left(x, \frac{j}{N}\right) \\
& - \left(4N^2s^2-(8Nj+4N)s+4j^2+4j\right) K_n\left(x, \frac{j+\frac{1}{2}}{N}\right) \\
& + \left(2N^2s^2-(4Nj+N)s+2j^2+j\right) K_n\left(x, \frac{j+1}{N}\right).
\end{aligned}
\tag{12}
$$

Substituting in (11) yields

$$
\begin{aligned}
y_{n+1}(x) \simeq y_{n+1,N}(x) = y_0(x) + & \frac{x^\alpha}{\Gamma(\alpha)} \sum_{j=0}^{N-1} \int_{\xi_j}^{\xi_{j+1}} (1-s)^{\alpha-1} \\
& \Big[\left(2N^2s^2-(4Nj+3N)s+2j^2+3j+1\right) K_n\left(x, \frac{j}{N}\right) \\
& - \left(4N^2s^2-(8Nj+4N)s+4j^2+4j\right) K_n\left(x, \frac{j+\frac{1}{2}}{N}\right) \\
& + \left(2N^2s^2-(4Nj+N)s+2j^2+j\right) K_n\left(x, \frac{j+1}{N}\right) \Big] ds,
\end{aligned}
\tag{13}
$$

where $y_{n,N}(x)$ is used as approximation to $y_n(x)$ after using interpolation approximation to $K_n(x,s)$. Now, the integrals on the right-hand side of (13) can be computed exactly and we find the following simplified formula of our new interpolation based iterative formula for $n \geq 0$

$$
\begin{aligned}
& y_{n+1,N}(x) = y_0(x) \\
& + \frac{x^\alpha}{\Gamma(\alpha+3)} \sum_{j=0}^{N-1} \left[W_{1j} K_n\left(x, \frac{j}{N}\right) + W_{2j} K_n\left(x, \frac{j+\frac{1}{2}}{N}\right) + W_{3j} K_n\left(x, \frac{j+1}{N}\right) \right],
\end{aligned}
\tag{14}
$$

where $K_n(x, \frac{j}{N})$, $K_n(x, \frac{j+\frac{1}{2}}{N})$ and $K_n\left(x, \frac{j+1}{N}\right)$ are computed via (9) and the integration weights are given as follows

$$
\begin{aligned}
W_{1j} := & 2N^2(H_1(\xi_{j+1})-H_1(\xi_j)) - (4Nj+3N)(H_2(\xi_{j+1})-H_2(\xi_j)) \\
& + (2j^2+3j+1)(H_3(\xi_{j+1})-H_3(\xi_j))
\end{aligned}
$$

$$
\begin{aligned}
W_{2j} := & -4N^2(H_1(\xi_{j+1})-H_1(\xi_j)) + (8Nj+4N)(H_2(\xi_{j+1})-H_2(\xi_j)) \\
& - (4j^2+4j)(H_3(\xi_{j+1})-H_3(\xi_j))
\end{aligned}
$$

$$
\begin{aligned}
W_{3j} := & 2N^2(H_1(\xi_{j+1})-H_1(\xi_j)) - (4Nj+N)(H_2(\xi_{j+1})-H_2(\xi_j)) \\
& + (2j^2+j)(H_3(\xi_{j+1})-H_3(\xi_j))
\end{aligned}
$$

with

$$
\begin{aligned}
H_1(s) &= -(\alpha+1)(\alpha+2)s^2(1-s)^\alpha - 2(\alpha+2)s(1-s)^{\alpha+1} - 2(1-s)^{\alpha+2} \\
H_2(s) &= -(\alpha+1)(\alpha+2)s(1-s)^\alpha - (\alpha+2)(1-s)^{\alpha+1}, \\
H_3(s) &= -(\alpha+1)(\alpha+2)(1-s)^\alpha.
\end{aligned}
$$

4 Convergence

In this section, we prove the following theorem, which addresses the uniform convergence of our iterative method under appropriate conditions on Eq. (3).

Theorem 4.1. *Let* $M = \max_{(x,y)\in U} |f(x,y)|$, *where* U *is defined as in Theorem 3.1, and* $f(x,y)$ *satisfies the condition (5). Then*

$$\lim_{n,N\to\infty} y_{n,N}(x) = y(x)$$

uniformly on $[0,b]$, *where* b *is defined as in Theorem 3.1.*

Proof. We can write

$$y(x) - y_{n,N}(x) = (y(x) - y_n(x)) + (y_n(x) - y_{n,N}(x)), \tag{15}$$

where $(y(x)-y_n(x))$ is the error in the n^{th} Picard's iteration, and $(y_n(x)-y_{n,N}(x))$ is the error that comes from using interpolation. It is sufficient to show that $\lim_{n\to\infty}(y(x)-y_n(x)) = 0$ and $\lim_{n,N\to\infty}(y_n(x)-y_{n,N}(x)) = 0$, uniformly.

To prove the first statement, we employ the mathematical induction on the main process of Picard's iterations to show that

$$|y(x) - y_n(x)| \leqslant \frac{L^n M x^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)}. \tag{16}$$

Let $n = 0$. Then by (4) and (6) we have

$$\begin{aligned} |y(x) - y_0(x)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t)) dt \right| \\ &\leqslant \frac{M x^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{17}$$

that is, (16) is true for $n = 0$. For $n = 1$, we use (4) and (6) again, and get

$$\begin{aligned} |y(x) - y_1(x)| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} [f(t, y(t)) - f(t, y_0(t))] dt \right| \\ &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |f(t, y(t)) - f(t, y_0(t))| dt. \end{aligned}$$

Since $f(t, y(t))$ satisfies (5), we have

$$|y(x) - y_1(x)| \leqslant \frac{L}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |y(x) - y_0(x)| dt$$

and using the inequality (17) for $|y(x) - y_0(x)|$, it yields

$$|y(x) - y_1(x)| \leqslant \frac{LM}{\Gamma(\alpha+1)} \int_0^x (x-t)^{\alpha-1} t^\alpha dt. \tag{18}$$

By using the transformation (7), the exact value of the integral on the right-hand side of (18) is obtained:

$$\int_0^x (x-t)^{\alpha-1} t^\alpha dt = x^{2\alpha} \int_0^1 (1-s)^{\alpha-1} s^\alpha ds = \frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} x^{2\alpha}, \tag{19}$$

where we used definition of Beta function and its relation with Euler gamma function, i.e.,

$$\int_0^1 (1-s)^{\alpha-1} s^\alpha ds = B(\alpha, \alpha+1) = \frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}.$$

Therefore,

$$|y(x) - y_1(x)| \leqslant \frac{LM x^{2\alpha}}{\Gamma(2\alpha+1)},$$

which proves (16) for $n = 1$. Let (16) be true for $n = k$, i.e.,

$$|y(x) - y_k(x)| \leqslant \frac{L^k M x^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)}. \tag{20}$$

For $n = k+1$, we have

$$|y(x) - y_{k+1}(x)| \leqslant \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} |f(t, y(t)) - f(t, y_k(t))| dt.$$

Then by using (5) and (20) we get

$$|y(x)-y_{k+1}(x)| \leqslant \frac{L}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}|y(t)-y_k(t)|dt \tag{21}$$
$$\leqslant \frac{L^{k+1}M}{\Gamma((k+1)\alpha+1)\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{(k+1)\alpha}dt$$

and using the transformation (7), it implies that

$$|y(x)-y_{k+1}(x)| \leqslant \frac{L^{k+1}Mx^{(k+2)\alpha}}{\Gamma((k+1)\alpha+1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}s^{(k+1)\alpha}ds,$$

where we have

$$\int_0^1 (1-s)^{\alpha-1}s^{(k+1)\alpha}ds = B(\alpha,(k+1)\alpha+1) = \frac{\Gamma(\alpha)\Gamma((k+1)\alpha+1)}{\Gamma((k+2)\alpha+1)}$$

and so

$$|y(x)-y_{k+1}(x)| \leqslant \frac{L^{k+1}Mx^{(k+2)\alpha}}{\Gamma((k+2)\alpha+1)},$$

which completes the induction for (16).

According to the Definition 2.5, the series

$$E_{\alpha,\alpha+1}(Lx^\alpha) = \sum_{n=0}^{\infty}\frac{(Lx^\alpha)^n}{\Gamma(n\alpha+\alpha+1)}$$

is uniformly convergent on [0, b], therefore its general term tends to zero uniformly, this means that

$$\lim_{n\to\infty}\frac{L^nMx^{(n+1)\alpha}}{\Gamma(\alpha(n+1)+1)} = xM\lim_{n\to\infty}\frac{(Lx^\alpha)^n}{\Gamma(n\alpha+\alpha+1)} = xM.0 = 0 \tag{22}$$

and so from (16) we have

$$\lim_{n\to\infty}|y(x)-y_n(x)| = 0. \tag{23}$$

To prove that $\lim_{n,N\to\infty}(y_n(x)-y_{n,N}(x)) = 0$, we have

$$|y_n(x)-y_{n,N}(x)| \leqslant \frac{x^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{N-1}\int_{\xi_j}^{\xi_{j+1}}(1-s)^{\alpha-1}\left|k_n(x,s)-\tilde{k}_n(x,s)\right|ds,$$

where $\tilde{k}_n$ is the polynomial interpolation of k_n as defined in (12). Thus,

$$|y_n(x)-y_{n,N}(x)| \leqslant \frac{x^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{N-1}\int_{\xi_j}^{\xi_{j+1}}(1-s)^{\alpha-1}\Big|k_n(x,s)-L_1(s)k_n(x,\xi_j)$$
$$-L_2(s)k_n(x,\xi_{j+\frac{1}{2}})-L_3(s)k_n(x,\xi_{j+1})\Big|ds$$
$$=\frac{x^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{N-1}\int_{\xi_j}^{\xi_{j+1}}(1-s)^{\alpha-1}\Big|k_n(x,s)(L_1(s)+L_2(s)+L_3(s))$$
$$-L_1(s)k_n(x,\xi_j)-L_2(s)k_n(x,\xi_{j+\frac{1}{2}})-L_3(s)k_n(x,\xi_{j+1})\Big|ds$$
$$\leqslant \frac{x^\alpha}{\Gamma(\alpha)}\sum_{j=0}^{N-1}\int_{\xi_j}^{\xi_{j+1}}(1-s)^{\alpha-1}\Big(|L_1(s)|\Big|k_n(x,s)-k_n(x,\xi_j)\Big|$$
$$+|L_2(s)|\Big|k_n(x,s)-k_n(x,\xi_{j+\frac{1}{2}})\Big|+|L_3(s)|\Big|k_n(x,s)-k_n(x,\xi_{j+1})\Big|\Big)ds$$
$$\leqslant \frac{x^\alpha}{\Gamma(\alpha)}\times\omega(k_n(x,.),\frac{1}{N})\times\Lambda\times\sum_{j=0}^{N-1}\int_{\xi_j}^{\xi_{j+1}}(1-s)^{\alpha-1}ds$$
$$=\frac{x^\alpha}{\Gamma(\alpha+1)}\times\omega(k_n(x,.),\frac{1}{N})\times\Lambda \tag{24}$$

where $\Lambda = \max_{s\in[0,1]} \sum_{i=0}^{3} |L_i(s)|$, and

$$\begin{aligned}\omega\big(K_n(x,.),\frac{1}{N}\big) &= \max_{s_1,s_2\in[0,1],|s_1-s_2|\leqslant \frac{1}{N}} |K_n(x,s_1)-K_n(x,s_2)| \\ &= \max_{s_1,s_2\in[0,1],|s_1-s_2|\leqslant \frac{1}{N}} \big|f\big(s_1x, y_n(s_1x)\big) \\ &\qquad - f\big(s_2x, y_n(s_2x)\big)\big| \\ &\leqslant \max_{x_1,x_2\in[0,b],\ |x_1-x_2|\leqslant \frac{b}{N}} |f(x_1,y_n(x_1)) - f(x_2,y_n(x_2))|\end{aligned}$$

where $x_1 = s_1x$, $x_2 = s_2x$. Thus, we can write

$$\omega\left(K_n(x,.),\frac{1}{N}\right) \leqslant \omega\left(g_n(x),\frac{b}{N}\right),$$

where

$$g_n(x) = f(x,y_n(x)), \quad x\in[0,b].$$

Due to the continuity of f and the result of the first part of proof (uniformly convergence), we have

$$\lim_{n\to\infty} g_n(x) = g(x) = f(x,y(x))$$

uniformly on $[0,b]$, and so

$$\lim_{n,N\to\infty} \omega\left(K_n(x,.),\frac{1}{N}\right) = 0. \tag{25}$$

From (24) and (25) we conclude

$$\lim_{n,N\to\infty} |y_n(x) - y_{n,N}(x)| = 0.$$

This completes the proof. □

5 Applications and numerical examples

In this part, we will evaluate the suggested approach using a number of examples in order to demonstrate numerical accuracy and validate the theoretical results. The maximum absolute error is denoted in this section by the notation $E_{max} := \max_{x\in[a,b]} |y(x) - y_{n,N}(x)|$.

For the first two examples (Eqs. 5.1 and 5.2), exact solutions are unknown when $\alpha \neq 1$. To assess the accuracy of our method and compare it with the results from [15] and [14], we report the squared 2-norm $\|Res_n\|_2^2$ of the residual $Res_n(x) := D^{\alpha}_{*0}y_n(x) - f(x,y_n(x))$, where To compute the Caputo derivative $D^{\alpha}_{*0}y_n(x)$ in our numerical implementation and obtain the reported results, we used the Maple package (specifically, the fracdiff command). The values are computed for $\alpha = 0.65, 0.75, 0.85, 0.95$ in Table 3 and $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$ in Table 5, respectively.

Example 5.1. [15] Consider the fractional IVP

$$\begin{cases} D^{\alpha}_{*0}y(x) = 1 - y^2(x), & 0<\alpha\leqslant 1, \quad x\in[0,1], \\ y(0) = 0, \end{cases} \tag{26}$$

where $y(x) = \frac{e^{2x}-1}{e^{2x}+1}$ denotes the exact solution in the case $\alpha = 1$. According to Eq. (3), we have

$$f(x,y(x)) = 1 - y^2(x).$$

To solve this problem we use the proposed method for different values of n and N and we report E_{max} for the case $\alpha = 1$ in Table 1. Table 2 shows corresponding absolute errors (Abs.Err) for $x = 0, 0.2, 0.4, 0.6, 0.8, 1$. Figure 1 shows that the approximate solutions converge to the exact solution (case $\alpha = 1$), when α increases from 0.5 to 0.95. Notice that in Theorem 3.1, we have $b = min\left\{1, (\frac{B\Gamma(\alpha+1)}{M})^{\frac{1}{\alpha}}\right\}$, where M and b are defined as in Theorem 3.1. If $\frac{1}{\Gamma(1+\alpha)\leq B\leq\sqrt{2}}$, then $b = 1$.

Example 5.2. [14] For the fractional IVP

$$\begin{cases} D^{\alpha}_{*0}y(x) = 2y(x) - y^2(x) + 1, & 0<\alpha\leqslant 1, \quad x\in[0,1], \\ y(0) = 0, \end{cases} \tag{27}$$

Table 1: The values of E_{max} for Example 5.1 in the case $\alpha = 1$

n	N	E_{max}	$CPUtime$
3	30	2.2533e-02	0.157
5	30	6.9918e-04	0.281
10	30	8.2273e-09	1.063
15	30	2.7498e-10	1.562

Table 2: Absolute errors of Example 5.1 for $N = 30$

x	$Abs.Err$	$Abs.Err$	$Abs.Err$	$Abs.Err$
	$n = 3$	$n = 5$	$n = 10$	$n = 15$
0	0	0	0	0
0.2	4.7660e-07	3.2340e-11	1.9375e-12	1.9375e-12
0.4	5.7031e-05	5.6057e-08	4.2622e-11	4.2622e-11
0.6	8.7409e-04	4.0963e-06	1.7087e-10	1.7133e-10
0.8	5.6584e-03	7.7871e-05	1.4906e-10	2.7463e-10
1	2.2533e-02	6.9918e-04	8.2273e-09	1.4018e-10

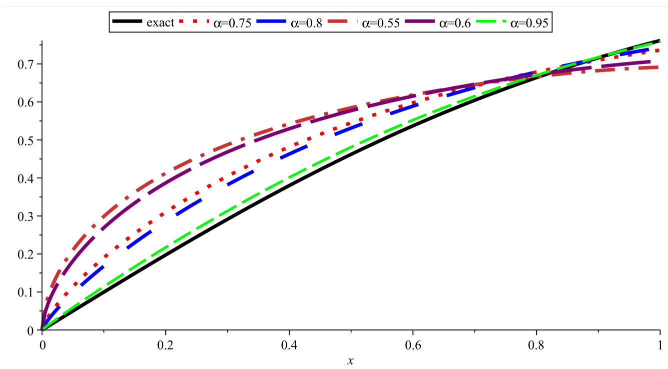

Figure 1: Comparing the exact and approximate solutions of Example 5.1 for $n = 8, N = 30$

Table 3: The $\| Res_n \|^2$ for Example 5.1

	$\alpha = 0.65$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$
$n = 6, k = 2$[15]	$8.03667e-8$	$3.12708e-9$	$5.15031e-10$	$3.98973e-11$
Our method (n=10, N=30)	1.2325e-7	9.5612e-10	5.2884e-12	2.2900e-14

Table 4: The values of E_{max} for Example 5.2 in the case $\alpha = 1$

n	N	E_{max}	$CPUtime$
3	100	8.6311e-03	0.250
5	100	1.0386e-04	0.562
10	100	2.5223e-10	6.140
15	100	8.7328e-11	16.532

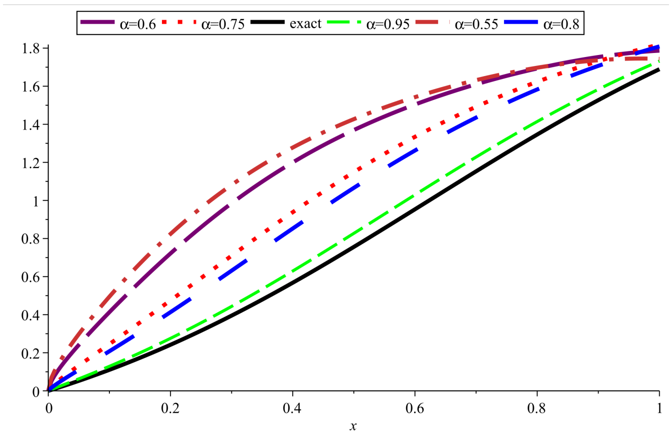

Figure 2: The plots of the exact and approximate solutions for Example 5.2 in the case $n = 5, N = 30$

the exact solution for $\alpha = 1$ is given by

$$y(x) = 1 + \sqrt{2}tanh\left(\sqrt{2}x + \frac{1}{2}Log(\frac{\sqrt{2}-1}{\sqrt{2}-1})\right) \tag{28}$$

and the right-hand side is

$$f(x, y(x)) = 2y(x) - y^2(x) + 1.$$

We used our method to solve this problem for different values of n and N and presented the results in Table 4.

As a result of Example 5.2, Figure 2 shows that, the approximate solutions approach to the exact solution as α increases from 0.55 to 0.95.

Example 5.3. In this example, we are going to show that our method effectively works for fractional IVP for which the given functions and exact solution are not sufficiently smooth, i.e.,

$$\begin{cases} D^{\alpha}_{*0}y(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}x^{\lambda-\alpha} - x^{\lambda}y(x) + y^2(x), & x \in [0, 1], 0 < \alpha < \lambda \leqslant 1, \\ y(0) = 0 \end{cases} \tag{29}$$

with the given exact solution $y(x) = x^{\lambda}$.

Comparing with (3), we have

$$f(x, y(x)) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}x^{\lambda-\alpha} - x^{\lambda}y(x) + y^2(x). \tag{30}$$

We used our method to this problem and obtained numerical results for some choices of n, N, α and λ, as shown in Tables 6, 7, 8.

Table 5: The $\| Res_n \|^2$ for Example 5.2

	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$
$n = 6, k = 2$[14]	1.49106e-3	1.3251e-3	5.86622e-4	1.98272e-4	5.83018e-5
Our method (n=6, N=10)	5.1239e-3	1.8054e-4	1.4281e-6	5.1977e-7	3.3507e-08

Table 6: The values of E_{max} for Example 5.3 ($\lambda = 1$)

n	N	E_{max}	E_{max}	E_{max}	E_{max}	E_{max}
		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 0.95$
2	100	6.4704e-07	9.8839e-06	7.8366e-05	4.5869e-04	1.5191e-03
2	200	1.8600e-07	3.2617e-06	2.9699e-05	1.9967e-04	7.3379e-04
2	300	8.9680e-08	1.7051e-06	1.6836e-05	1.2275e-04	4.7940e-04
4	300	6.4704e-07	2.2907e-06	2.0080e-05	1.3575e-04	5.1138e-04
8	300	1.8600e-07	2.4421e-06	2.0408e-05	1.3627e-04	5.1200e-04
12	300	8.9680e-08	2.4454e-06	2.0409e-05	1.3627e-04	5.1200e-04

Table 7: The values of E_{max} for Example 5.3 ($\lambda = \frac{3}{4}$)

n	N	E_{max}	E_{max}	E_{max}	E_{max}	E_{max}
		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.74$
2	100	7.7327e-06	8.1899e-05	5.4325e-04	1.2679e-03	1.7547e-03
2	200	2.6420e-06	3.2134e-05	2.4483e-04	6.1246e-04	8.7147e-04
2	300	1.4095e-06	1.8590e-05	1.5360e-04	4.0013e-04	5.7867e-04
4	300	2.3254e-06	2.6050e-05	1.9029e-04	4.7308e-04	6.7316e-04
8	300	3.0403e-06	2.8360e-05	1.9492e-04	4.7897e-04	6.7954e-04
12	300	3.1737e-06	2.8428e-05	1.9494e-04	4.7898e-04	6.7955e-04

Table 8: The values of E_{max} for the Example 5.3 ($\lambda = \frac{1}{2}$)

n	N	E_{max}	E_{max}	E_{max}	E_{max}	E_{max}
		$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.45$
2	100	1.3817e-05	6.3014e-05	2.0600e-04	5.7598e-04	9.2631e-04
2	200	5.2376e-06	2.5597e-05	8.9677e-05	2.6873e-04	4.4744e-04
2	300	5.2376e-06	1.5111e-05	5.5130e-05	1.7204e-04	2.9232e-04
4	300	5.5250e-06	2.6062e-05	8.8151e-05	2.5588e-04	4.2014e-04
8	300	9.0230e-06	3.6024e-05	1.0785e-04	2.8855e-04	4.6040e-04
12	300	1.0785e-05	3.8367e-05	1.0989e-04	2.8996e-04	4.6150e-04

Table 9: The values of E_{max} for Example 5.4

		Our method		Method of Ref. [14]	
n	N	E_{max}	cpu time	J	E_{max}
8	20	1.6928e-02	0.094	4	7.8e-03
16	20	7.4291e-06	0.234	5	2.5e-03
24	20	5.1870e-10	0.266	6	7.7e-04
24	40	5.1813e-10	0.406	7	2.5e-04
24	80	5.1809e-10	0.671	8	7.8e-05
24	120	5.1809e-10	0.750	-	-

Table 10: Absolute errors of Example 5.5

x	$Abs.Err$	$Abs.Err$	$Abs.Err$	$Abs.Err$	$Abs.Err$
	$\alpha = 0.3$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 0.95$	$\alpha = 1$
0	0	0	0	0	0
0.2	9.6598e-14	1.1857e-19	9.6151e-24	1.8769e-23	9.2477e-36
0.4	1.5945e-08	3.5423e-13	1.9092e-19	7.1597e-23	3.9744e-26
0.6	1.8088e-05	2.1960e-09	9.8561e-15	2.6054e-19	1.7176e-20
0.8	2.6216e-03	1.0829e-06	2.1855e-11	1.9263e-15	1.7154e-16
1	1.1213e-01	1.3274e-04	8.6417e-09	1.9351e-12	2.1761e-13

Example 5.4. [14] Consider the following fractional IVP with a non-differentiable right-hand side and non-differentiable solution at origin:

$$\begin{cases} D_{*0}^{0.5} y(x) = \sqrt{x} + \dfrac{\sqrt{\pi}}{2} - y(x), & x \in [0,1] \\ y(0) = 0. \end{cases} \tag{31}$$

The exact solution and the right-hand side are respectively $y(x) = \sqrt{x}$ and $f(x, y(x)) = \sqrt{x} + \dfrac{\sqrt{\pi}}{2} - y(x)$.

We used our method to solve this problem for different values of n and N and compared our results with those of reference [14] in Table 9.

Example 5.5. [12] We consider another fractional IVP in the form

$$\begin{cases} D_{*0}^{\alpha} y(x) = -y^2(x) + x + \left(\dfrac{x^{\alpha+1}}{\Gamma(\alpha+2)} \right)^2, & x \in [0,1], \quad 0 < \alpha \leqslant 1, \\ y(0) = 0 \end{cases}$$

where $y(x) = \dfrac{x^{\alpha+1}}{\Gamma(\alpha+2)}$ denotes the exact solution.

According to the Eq.(3), we have

$$f(x, y(x)) = -y^2(x) + x + \left(\frac{x^{\alpha+1}}{\Gamma(\alpha+2)} \right)^2$$

For this example we report our numerical results (Abs.Err) via Table 10 for $n = 10$ and $N = 30$.

Example 5.6. As a final example we consider another fractional IVP in the form

$$\begin{cases} D_{*0}^{\alpha} y(x) = y(x) + y^2(x) + f(x), & 1 < \alpha \leqslant 2 \\ y(0) = 0, \quad y'(0) = 0 \end{cases} \tag{32}$$

where $y(x) = x^3$ is the exact solution and $f(x) = \frac{6}{\Gamma(4-\alpha)} x^{3-\alpha} - x^3 - x^6$. According to Eq. (3), we have

$$f(x, y(x)) = y(x) + y^2(x) + \frac{6}{\Gamma(4-\alpha)} x^{3-\alpha} - x^3 - x^6.$$

To solve this problem we used the proposed method for different values of n and N and reported E_{max} in Table 11.

Table 11: The values of E_{max} for the Example 5.6

n	N	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$	
		E_{max}	$CPUtime$	E_{max}	$CPUtime$	E_{max}	$CPUtime$
3	10	1.2519e-02	4.563	1.7518e-03	2.891	1.3609e-04	4.187
5	10	2.0376e-04	17.094	4.6471e-05	4.313	1.0169e-04	10.015
8	10	9.9520e-06	14.625	5.1732e-05	3.344	1.0180e-04	7.141
10	20	1.4269e-06	35.031	8.9862e-06	7.016	2.1828e-05	13.281
10	30	4.5574e-07	46.625	3.2416e-06	8.485	8.9003e-06	17.453
10	50	2.0281e-07	57.921	8.9957e-07	12.031	2.8811e-06	27.312

Conclusion

In this paper, we developed a novel and efficient numerical method for solving a class of nonlinear fractional differential equations. Our proposed approach employs quadratic interpolation within each iteration of Picard's method, offering following significant advantages:

- The implementation of our method does not require solving any algebraic systems of equations;
- Convergence of the method is guaranteed under the sole condition of Lipschitz continuity;
- Our method exhibits superior accuracy compared to most existing numerical techniques;
- As demonstrated in Section 5, the method exhibits rapid convergence.

These key attributes highlight the effectiveness and robustness of our proposed method for solving nonlinear fractional differential equations.

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