



# Truncation of computational domains as an error control strategy for approximating option pricing involving PIDEs

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## Abstract

Pricing of option contracts where the underlying asset follows a jump diffusion process leads to a partial integro-differential equation. Due to the integral term, an exact or closed form solution to the resulting equation is impossible in general. In this paper, we have investigated the localization error when the finite difference method is applied to approximate the solution of the resulting equation. The main focus of this paper is to reduce the complexity in implementation of the integral term by truncating of computational domain in which the localization error is controlled. The numerical results present the behavior of the localization error with respect to the computational domain.

**Keywords:** Localization error; Computational domain; Option pricing; Merton model.

## 1 Introduction

Option pricing where the asset price involves jump diffusion process leads to a parabolic, partial integro-differential equation (PIDEs), which is an extension of the Black–Scholes PDE with a new integral term. Due to the drawbacks in computing of the closed form solution, a number of numerical methods have been studied in literature to approximate the solution of resulting equation [6, 13, 7]. The resulting PIDE is initially produced in an unbounded domain which should be localized into the finite domain due to the numerical computational purposes. Truncating large jumps and neglecting larger values for asset price induces the localization error. The payoff function of vanilla options is asymptotically close to the option price as asset price goes to infinity at maturity which leads to an exponentially decaying localization error.

Option pricing with exponential Lévy models has been studied in recent literature such as [14, 5]. Although the finite difference method is relatively efficient for a single asset option pricing [1, 2, 8], radial basis function approximation methods could provide a more general approach for tackling a multi-asset pricing problem [11, 15]. The pricing of European option using the domain truncating of PIDEs was introduced by Cont et al. [5] and the performance of the finite difference method as numerical scheme was investigated for a single asset case in bounded domain. The radial basis function approximation methods have been studied for Merton PIDE by truncated integral domain [5, 4, 3]. Both the truncation of the domain and the truncation of large jumps lead to errors that are exponentially small in terms of the truncating parameters. Computation of the integral term in the PIDE would involve the solution of a dense linear system. This can be remedied by approximating the integral term only on a bounded domain by truncating large jumps [12]. It is important to reduce the complexity when we tackle the multi-dimensional PIDEs corresponded to the basket option pricing.

In this paper, we construct the bounded computational domain to the PDE term proportional to the truncated integral term where the resulting error can be controlled. We also provide the upper bound for localization error which depends on the distribution of the jump magnitudes and the payoff function. Numerical experiments show the localization error behavior in terms of the computational domain. Moreover, they demonstrate it in terms of the truncating of integral term so that the maximum incurred localization error to would be less than a given tolerance.

Some details of the exponential Lévy model and basket option pricing are presented in section 2. Section 3 is dealing with computational domain and error estimation of the localization which depends on payoff function and magnitude of jump distribution. Constructing of computational domain and localization in two-asset and three-asset cases are presented in section 4.

## 2 Option pricing under jump diffusion processes

In exponential Lévy model, the evolution of risky basket asset  $S(t) = (S_1(t), \dots, S_d(t))$  is represented by

$$S_i(t) = S_i(0)e^{rt+L_i(t)}, \quad i = 1, 2, \dots, d \quad (1)$$

where  $S(0) = (S_1(0), \dots, S_d(0))$  is the basket price at the initial time,  $r$  is the risk-free interest rate and  $L(t) = (L_1(t), \dots, L_d(t))$  is  $d$ -dimensional Lévy process under the risk-neutral probability  $\mathbb{Q}$ . Then the price process  $S(t)$  follows the exponential jump diffusion model

$$L_i(t) = \left(-\lambda\kappa_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_i(t) + \sum_{k=1}^{N_i} Y_i^{(k)}, \quad i = 1, \dots, d, \quad (2)$$

where  $\sigma_i$  is the volatility of the underlying  $i$ th-asset,  $W_i$  is a standard Brownian motion where  $\rho_{ij}$  is correlation between  $W_i, W_j$ ,  $N_i$  is a Poisson process with intensity  $\lambda$ ,  $Y_i^{(k)}$  is an iid sequence of random variables related to  $i$ th-asset and  $\kappa_i = \mathbb{E}(e^{L_i(t)} - 1)$ . In this paper, we focus our attention on the classical Merton jump diffusion model [10] where the density function for the jump magnitudes follows the normal distribution with mean vector  $\tilde{\mu}$  and covariance matrix  $\Sigma$  as

$$g(y) = (2\pi)^{-d/2} (\det \Sigma)^{-1/2} \exp\left(-\frac{1}{2}(y - \tilde{\mu})^t \Sigma^{-1} (y - \tilde{\mu})\right). \quad (3)$$

We have supposed that there is a single Poisson process which derives correlated jumps in all assets. This corresponds to the single market which affects all prices [5].

Let  $T$  be the maturity and  $\tau = T - t$  denotes the time-to-maturity for the contingent claim  $V(x, \tau)$  where  $x := x(t)$  is the underlying basket price in log-price scaling

$$x = (x_1, \dots, x_d) = (\log(S_1), \dots, \log(S_d)) \in \Omega_\infty := \mathbb{R}^d. \quad (4)$$

Using the Itô's formula for finite activity jump processes, and taking expectations under the risk-neutral price process [14, 5], we obtain the following partial integro-differential equation (PIDE)

$$\begin{aligned} \frac{\partial V}{\partial \tau}(x, \tau) = & \underbrace{\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \rho_{ij} \sigma_i \sigma_j \frac{\partial^2 V}{\partial x_i \partial x_j}(x, \tau) + \sum_{i=1}^d \left(r - \frac{\sigma_i^2}{2} - \lambda \kappa_i\right) \frac{\partial V}{\partial x_i}(x, \tau) - (r + \lambda)V(x, \tau)}_{\mathcal{L}V} \\ & + \lambda \underbrace{\int_{\Omega} V(x + y, \tau) g(y) dy}_{\mathcal{I}V}, \quad (x, \tau) \in \Omega_\infty \times (0, T], \end{aligned} \quad (5)$$

where the jump magnitudes  $y = (y_1, \dots, y_d)$  in Merton model have Gaussian probability density  $g(y)$ . The payoff function of the European basket option is given by

$$F(x) = \max\left(\theta\left(E - \sum_{i=1}^d w_i e^{x_i}\right), 0\right), \quad (6)$$

where  $E$  is the strike price of the option,  $w_i$  is portion of  $i$ th-asset which is involved on basket and  $\theta$  takes value  $\theta = 1$  in put option and  $\theta = -1$  in call option respectively. The initial condition for (5) can be rewritten by

$$V(x, 0) = F(x). \quad (7)$$

The boundary of the computational domain can be divided into two parts. The near-field boundary, where one or more asset prices are zero, and the far-field boundary, where one or more asset-prices tend to infinity. For put options, the contract becomes worthless as the price of any of the underlying assets tends to infinity. For the near-field boundary, it can be noted that once  $S_i$  reaches zero, it will be worthless afterward, i.e., the solution remains at the boundary [12].

### 3 Computational domain

Since numerical computations can only be performed on finite domains, the first step is to reduce the PIDE to a bounded domain. The PDE term in equation (5) is defined on  $\Omega_\infty := \mathbb{R}^d$  and we truncate the domain so that to work on  $\Omega_x \subset \Omega_\infty$ . We also localize the integration domain by truncating large jumps. First, we take the linear transformation in the integral term of equation (5)

$$\int_{\Omega_\infty} V(x + y, \tau) g(y) dy = \int_{\Omega_\infty} V(y, \tau) g(y - x) dy, \quad (8)$$

and then divide the integral term (8) into two parts

$$\int_{\Omega_\infty} V(y, \tau) g(y - x) dy = \int_{\Omega_x} V(y, \tau) g(y - x) dy + \int_{\Omega_\infty \setminus \Omega_x} V(y, \tau) g(y - x) dy \quad (9)$$

such that the value of the second integral term in equation (9) be less than a given tolerance, by choosing the appropriate bounded core domain  $\Omega_j \subset \Omega_x$ . It is supposed that the jump magnitude lies into the bounded domain  $\Omega_j$ . We define the residual function  $R(\tau, x, \Omega_x)$  by

$$R(\tau, x, \Omega_x) = \int_{\Omega_\infty \setminus \Omega_x} V(y, \tau) g(y - x) dy. \quad (10)$$

We can expect the enough small value for  $g(y-x)$ , for all  $x \in \Omega_J$  and  $y \in \Omega_\infty \setminus \Omega_X$  if the distance between the core domain  $\Omega_J$  and the infinite domain  $\Omega_\infty \setminus \Omega_X$  is wide enough. The asymptotic behavior of the vanilla option price induces the residual  $R(\tau, x, \Omega_X)$  be ignorable. However, the truncated domain  $\Omega_J$  is possible and has a small size in practice because the probability density function  $g(\cdot)$  decays exponentially [5]. Both localization of the PDE domain and the core domain induce localization error and are exponentially small in terms of truncation parameters and of course have to be estimate and controlled by an appropriate choice of localization. Left figure 1 displays the sample core domain for integral term and computational domain for PDE part when we tackle in two-asset options. We notice that  $x \in \Omega_X$  for the PDE term in equation (5) and  $x \in \Omega_J$  for the integral term.

### 3.1 Estimation of localization error

The decay behavior of residual  $R(x, \Omega_X, \tau)$  when  $\Omega_J, \Omega_X \rightarrow \Omega_\infty$  is investigated on [4, 6]. We focus our attention to evaluate the upper bound of residual with respect to the appropriate choice of  $\Omega_J$ . For  $y \in \Omega_\infty \setminus \Omega_X$  and  $x \in \Omega_J$ , the  $(y-x)$  distance is minimum at  $x \in \partial\Omega_J$ . Furthermore, the density function of the jump magnitudes  $g(\cdot)$  decays exponentially. Then we can write

$$\max_{x \in \Omega_J} g(y-x) \leq \max_{x \in \partial\Omega_J} g(y-x), \quad y \in \Omega_\infty \setminus \Omega_X. \quad (11)$$

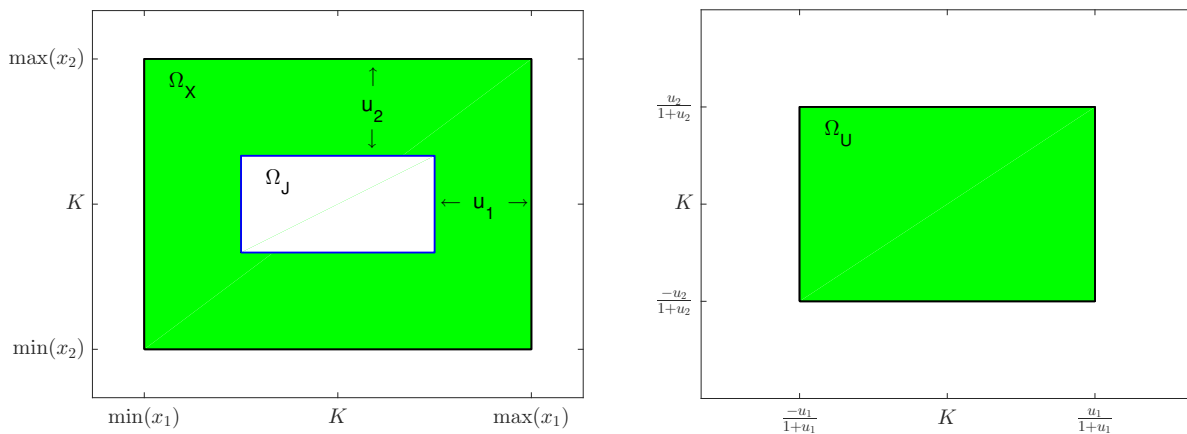
The solution of PIDE (5) is asymptotically linear when the asset price is far from the strike price [16], then we can write

$$\max_{x \in \Omega_J} R(x, \Omega_X, \tau) \leq \int_{\Omega_\infty \setminus \Omega_X} V(y, \tau) g(y-x) dy, \quad x \in \partial\Omega_J. \quad (12)$$

Notice that the value of put and call price is less than the exercise and asset price, respectively. i.e.

$$V(y, \tau) \leq (E + (1-E)\theta) \sum_{i=1}^d w_i e^{\theta y_i}, \quad (13)$$

where  $\theta = 0$  for put option case and  $\theta = 1$  for call option case respectively.



**Figure 1:** Computational domain  $\Omega_X$  and core domain  $\Omega_J$  are in left figure and transformed domain  $\Omega_U$  in right figure for two-asset case.  $K$  is strike price in log scale.

We define the diagonal matrix  $U = \text{diag}(u_1, \dots, u_d)$  where diagonal elements are domain distance criteria  $u_i = \min\{|x_i - y_i| : x = (x_1, \dots, x_d) \in \partial\Omega_X, y = (y_1, \dots, y_d) \in \partial\Omega_J\}$ . Substitution of  $y-x = (U+I)v$  on inequality (12) together (13) leads to

$$\max_{x \in \Omega_J} R(x, \Omega_X, \tau) \leq (E + (1-E)\theta) \int_{\Omega \setminus \Omega_X} \sum_{i=1}^d w_i e^{\theta y_i} g(y-x) dy, \quad x \in \partial\Omega_J \quad (14)$$

$$\leq \max_{x \in \partial\Omega_J} e^{\theta x} (E + (1-E)\theta) \mathbf{J} \int_{\Omega \setminus \Omega_U} g(v+Uv) \sum_{i=1}^d w_i e^{\theta(1+u_i)v_i} dv, \quad (15)$$

where  $\Omega_U = [-\frac{u_1}{1+u_1}, \frac{u_1}{1+u_1}] \times \dots \times [-\frac{u_d}{1+u_d}, \frac{u_d}{1+u_d}]$  and Jacobian  $\mathbf{J}$  is determinant of diagonal matrix  $(U+I)$  in which  $I$  is identity matrix. Since  $\theta$  is zero for put option case, the inequality (15) takes the simple form

$$\max_{x \in \Omega_J} R(x, \Omega_X, \tau) \leq E \mathbf{J} \int_{\Omega \setminus \Omega_U} g(v+Uv) dv. \quad (16)$$

We can see that the truncation error bound depends on payoff function and density function  $g$ . For any given value of  $U$ , the truncation error is easy to evaluate. Asymptotic behavior of the option price enables to scale  $\Omega_X$  sufficiently large such that the effect of solution details outside  $\Omega_X$  can be ignored. Now, we can use the truncation criteria (15) to determine diagonal element of  $U$  and appropriate core domain  $\Omega_J$  such that localization error be less than given tolerance. This choice of core domain allows to control the localization error. It is also enable to reduce the complexity when we approximate the internal term in the PIDE by setting a few number of grid points.

### 3.2 Boundary conditions for localized domain

For the numerical implementation, we can consider  $\Omega_x \subset \Omega_\infty$  as a general computational domain and set  $\mathcal{I}V = 0$ ,  $\forall x \in \Omega_x \setminus \Omega_J$  so that the PIDE (5) is computed with the jump component in  $\Omega_J$  only. Typically, the domain  $\Omega_x$  is sufficiently large if compared to the  $\Omega_J$  which allows to control better the error of boundary truncating [4].

Instead of solving (5) in an unbounded domain  $\Omega_\infty \times (0, T]$ , we will solve the PDE problem on truncated domain  $\Omega_x \times (0, T]$  and integral term on core domain  $\Omega_J \times (0, T]$ . Let us define  $\hat{V}(x, t)$  as a solution of the localized version of equation (5) with truncated domain

$$\frac{\partial \hat{V}}{\partial \tau}(x, \tau) = \mathcal{L}V(x, \tau) + \lambda_J \mathcal{I}V(x, \tau) \quad (x, \tau) \in \Omega_x \times (0, T], \quad (17)$$

where  $\lambda_J = 0$ ,  $x \in \Omega_x \setminus \Omega_J$  localize the domain of the integral term. The payoff function and far-field conditions are unchanged but defined on domain  $\Omega_x$  instead of  $\Omega_\infty$  [2]. And we notice that  $\lambda_J$  is zero for near-field boundary conditions which lead to  $(d-1)$ -dimensional solution of the corresponding PDE in equation (5) where the integral term is dismissed.

## 4 Numerical study

The collocation method for computing the solution of the PIDE gives an approximated solution of a basket option pricing problem under jump diffusion. In order to approximate without effecting by localization error, we choose the enough small tolerance  $\omega$  so that

$$\max_{x \in \partial \Omega_J} e^{\theta x} (E + (1-E)\theta) \mathcal{J} \int_{\Omega \setminus \Omega_J} g(v + Uv) \sum_{i=1}^d w_i e^{\theta(1+u_i)v_i} dv \leq \omega. \quad (18)$$

A possible and easy choice for the transformed domain is considering the neighborhood centered at the strike price in log scale, and evaluating the widths factor  $u_i$  so that  $(\log(E) \pm u_1, \dots, \log(E) \pm u_d) = \Omega_U$ . The core computational domain  $\Omega_J$  could be constructed by distance criteria  $u_i$ . A sample transformed domain  $\Omega_U$  and finite computational domain  $\Omega_J \subset \Omega_x$  is illustrated in figure 1. For the numerical test, the density distribution parameters considered are  $\tilde{\sigma}_i = \tilde{\sigma}$ ,  $\tilde{\mu} = 0$ ,  $\tilde{\rho} = 0$  and equal distance criteria  $u_i = u$ . In general, the domains need not be square, but for ease of exposition we define the finite domains to be square. Figure 2 presents the numerical investigation of localization error for two-asset and three-asset put option. The transformed computational domain  $\Omega_U = [\log(E) - u, \log(E) + u]^d$  proportional to the given tolerance  $\omega$  is obtainable by numerical integration scheme on (16). Therefore it is possible to choose  $u$  as well as the core computational domain  $\Omega_J$  for any given truncated tolerance  $\omega$ .

Notice that the localization error bound in equation (18) depends on the jump magnitude function  $g(\cdot)$ . In general, the similar numerical results are obtainable for specific formulations of  $g(\cdot)$  with Gaussian and exponentially distributed jumps [9] which follow the exponential decay property. The exponentially distributed jumps is not as straightforward as with the Gaussian case.

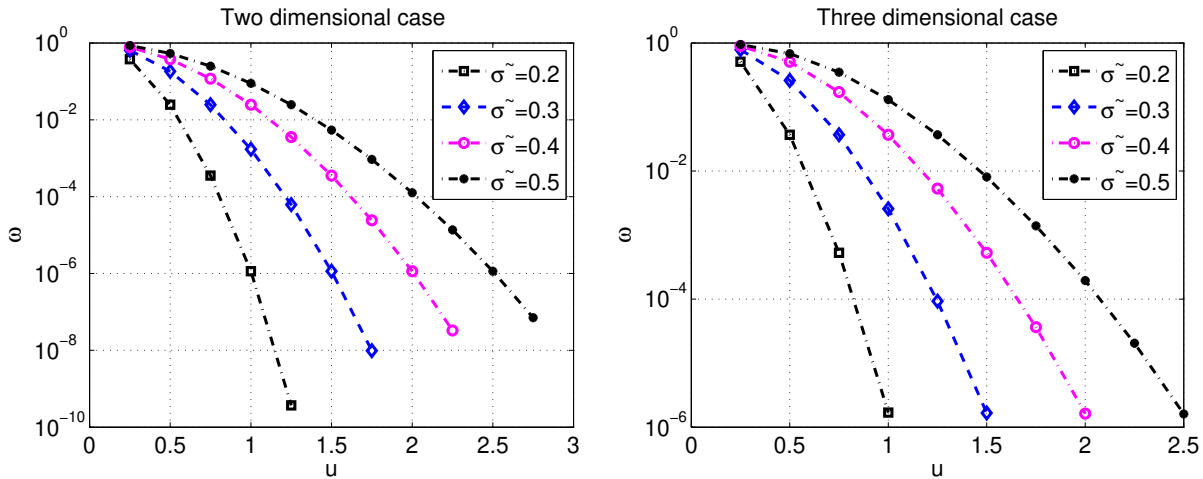


Figure 2: The truncated tolerance  $\omega$  as function of distance criteria  $u$  for different value of  $\tilde{\sigma}$ .

The computational time required to solve the PIDE (17) for two asset cases is displayed in figure 3. The finite difference implementation for the same PIDE was taken in [12] that uses  $n$  equidistance node points in each direction  $x_i$ . The tolerance error is considered  $\omega = 10^{-4}$  and computational core domain is the case analyzed in 2 and other basket option parameters are the same as reported in [12]. The experiments are carried out with Matlab software on an Intel Core i7 processor, 2.7 GHz and 16 GB RAM.

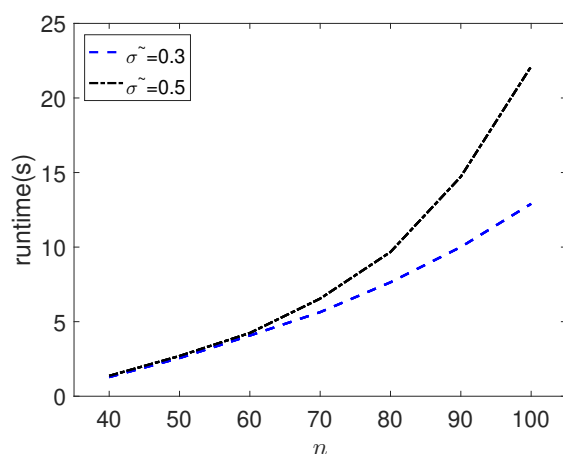


Figure 3: Computational time as function of  $n$  for two asset cases with truncated tolerance  $\omega = 10^{-4}$ .

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