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Some open problems in the theory of analytic multifunctions

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Abstract

After a brief historical introduction to analytic multifunctions, we discuss open problems relating to multifunction analogues of four classical results of complex analysis: the open mapping theorem, the identity principle, Picard's theorem and Bloch's theorem.

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1 Introduction

Let S and T be Banach-space operators. What can we say about the spectral radius of $S + zT$ as a function of the complex variable z? What about the spectrum of $S + zT$?

In 1968, Vesentini [28] proved that, if D is a plane domain, A is a Banach algebra and $f : D \to A$ is a holomorphic map, then the spectral radius $\rho(f(z))$ is a subharmonic function on D. He subsequently improved this by showing that even $\log \rho(f(z))$ is subharmonic [29]. The proof is simple and elegant. For each $m \geq 1$, the function $z \mapsto \log \|f(z)^m\|$ is subharmonic on D, since it is the log of the norm of a holomorphic function. Further, by the spectral radius formula, the sequence $2^{-n} \log \|f(z)^{2^n}\|$ decreases to $\log \rho(f(z))$. It follows that $\log \rho(f(z))$ is subharmonic.

In the years that followed, Vesentini's theorem and its variants found numerous applications in the theory of Banach algebras, many of them due to Aupetit, who published a book on the subject [1]. At the same time, the search was on for an analogous characterization of the set-valued function $z \mapsto \sigma(f(z))$, where σ denotes the spectrum. Such a characterization was finally discovered by Słodkowski in 1981 in the landmark paper [25].

With the benefit of hindsight, it is very easy to explain Słodkowski's solution. Let Γ denote the graph of $\sigma(f(z))$, namely

$$\Gamma := \{(z, w) \in D \times \mathbb{C} : w \in \sigma(f(z))\}.$$

Then we have

$$(D \times \mathbb{C}) \setminus \Gamma = g^{-1}(\mathrm{Inv}(A)), \tag{1}$$

where $\mathrm{Inv}(A)$ denotes the set of invertible elements of A, and $g : D \times \mathbb{C} \to A$ is the map defined by $g(z, w) := f(z) - w1$. Now, it is well known that $\mathrm{Inv}(A)$ is open in A, and in fact, as observed for example in [32], it is even a pseudoconvex set. (Indeed, the map $a \mapsto \|a^{-1}\|$ is a plurisubharmonic exhaustion function on $\mathrm{Inv}(A)$.) Thus the right-hand side of (1), being the inverse image of a pseudoconvex set by a holomorphic map, is itself pseudoconvex. We deduce that the complement of the graph of $\sigma(f(z))$ is a pseudoconvex set. This is the condition found by Słodkowski. He also proved the converse, showing that every bounded set-valued function satisfying this pseudoconvexity condition arises as the spectrum of a suitably chosen operator-valued holomorphic function.

This characterization was both beautiful and surprising. However, as it turned out, it was not a completely new idea. Many years earlier, Oka and his students had studied set-valued functions whose graphs are the complement of pseudoconvex sets [17, 16, 31], calling them pseudoconcave functions or analytic multivalued functions. Their original motivation was a theorem of Hartogs [11], according to which a complex-valued function is holomorphic if and only if the complement of its graph is pseudoconvex. Thus analytic multifunctions were a natural set-valued generalization of holomorphic functions. However, no one made the link with spectral theory until Słodkowski.

Following Słodkowski's breakthrough, the subject has matured and even earned its own Mathematics Subject Classification number (32A12). There have been developments both in the theory and in the applications. Słodkowski, in the same paper [25], gave an application to uniform algebras that turns out to be useful in the study of analytic structure in polynomial hulls. Further applications include the study of the variation of Julia sets in complex dynamics [8], and a proof of the corona theorem [9, 26].

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On the theory side, the subject resembles a blend of complex analysis and potential theory, and many well-known theorems from these areas have multifunction counterparts. Accounts of this theory can be found in [2, 3, 4, 18, 22]. Yet there still remain interesting unsolved problems, and it is these that form the subject of this article. We shall discuss open problems relating to four classical results: the open mapping theorem, the identity principle, Picard's theorem and Bloch's theorem.

2 Basic properties of analytic multifunctions

In this section, we give the formal definition of analytic multifunctions, and list some of their basic properties. We omit the proofs, referring the interested reader to [4, 18, 22] for further details. Also, we take for granted standard facts about pseudoconvex sets and plurisubharmonic functions. The background for these can be found in the book of Klimek [12].

To get started, we need some topological preliminaries. Let X and Y be Hausdorff topological spaces. We denote by $\kappa(Y)$ the collection of all non-empty compact subsets of Y. A map $K : X \to \kappa(Y)$ is called a *multifunction*. Its *graph* is the set

$$\operatorname{graph}(K) := \{(x,y) \in X \times Y : y \in K(x)\}.$$

Given $C \subset X$, the *image* of C under K is the set

$$K(C) := \bigcup_{x \in C} K(x).$$

We say that the multifunction $K : X \to \kappa(Y)$ is *upper semicontinuous* if, whenever V is open in Y, the set $\{x \in X : K(x) \subset V\}$ is open in X. If K is upper semicontinuous, then $\operatorname{graph}(K)$ is closed in $X \times Y$, and $K(C)$ is compact whenever C is. The proofs are routine.

We are now ready for the definition of analytic multifunction.

Definition 2.1. Let D be a plane domain. We say that $K : D \to \kappa(\mathbb{C})$ is an *analytic multifunction* if it is upper semicontinuous and $(D \times \mathbb{C}) \setminus \operatorname{graph}(K)$ is pseudoconvex.

At first sight, this definition of analyticity looks rather strange. We hope that our historical introduction makes it more plausible. As explained there, the spectrum of a holomorphic Banach-algebra-valued function is analytic in the sense just defined. In particular, constant multifunctions are analytic.

We now consider some further examples of analytic multifunctions. Our first result is just a restatement of the theorem of Hartogs mentioned in the introduction.

Theorem 2.1. *Let $f : D \to \mathbb{C}$ be a function. Define $K : D \to \kappa(\mathbb{C})$ by*

$$K(z) := \{f(z)\} \quad (z \in D).$$

Then K is an analytic multifunction on D if and only if f is holomorphic on D.

The next result generalizes this.

Theorem 2.2. *Let $n \geq 1$ and let $a_1, a_2, \dots, a_n : D \to \mathbb{C}$ be holomorphic functions. For $z \in D$, define*

$$K(z) := \Big\{ w \in \mathbb{C} : w^n + a_1(z)w^{n-1} + \dots + a_{n-1}(z)w + a_n(z) = 0 \Big\}. \tag{2}$$

Then K is an analytic multifunction on D. Conversely, if K is an analytic multifunction on D such that $K(z)$ is a finite set for all $z \in D$, then K has the form (2) for some choice of n and holomorphic maps $a_1, \dots, a_n : D \to \mathbb{C}$.

Hence, in particular, the classic example $K(z) := \{\pm\sqrt{z}\}$ is an analytic multifunction. The definition also includes examples of a very different kind.

Theorem 2.3. *Let $r : D \to [0, \infty)$ be a positive function. For $z \in D$, define*

$$K(z) := \{w : |w| \leq r(z)\}.$$

Then K is analytic on D if and only if $\log r$ is subharmonic on D.

>From the basic examples of analytic multifunctions given above, it is possible to build new ones using the following results.

Theorem 2.4. *Let K_1, K_2 be analytic multifunctions on D. Define*

$$K(z) := K_1(z) \cup K_2(z) \quad (z \in D).$$

Then K is analytic on D.

Theorem 2.5. *Let $K_1, K_2, K_3, \dots$ be analytic multifunctions on D such that $K_1(z) \supset K_2(z) \supset K_3(z) \supset \dots$ for all $z \in D$. Define*

$$K(z) := \cap_{n \geq 1} K_n(z) \quad (z \in D).$$

Then K is analytic on D.

Theorem 2.6. *Let $K_1, \dots, K_n$ be analytic multifunctions on D, let U be an open subset of $\mathbb{C}^n$ such that $K_1(z) \times \dots \times K_n(z) \subset U$ for all $z \in D$, and let $f : U \to \mathbb{C}$ be a holomorphic function. Define*

$$K(z) := f(K_1(z), \dots, K_n(z)) \quad (z \in D).$$

Then K is analytic on D.

Theorem 2.7. *Let K be an analytic multifunction on D. Let $L : D \to \kappa(\mathbb{C})$ be an upper semicontinuous multifunction such that*

$$\partial L(z) \subset K(z) \subset L(z) \quad (z \in D).$$

Then L is analytic on D. In particular, if we define

$$\widehat{K}(z) := \text{the polynomial hull of } K(z) \quad (z \in D),$$

then $\widehat{K}$ is analytic on D.

The next result, obtained by Słodkowski in [25], is a fundamental tool for analyzing analytic multifunctions.

Theorem 2.8. *Let K be an analytic multifunction on D, let v be a plurisubharmonic function defined on an open neighborhood of* graph(K)*, and define*

$$u(z) := \max_{w \in K(z)} v(z,w) \quad (z \in D).$$

Then u is a subharmonic function on D.

In fact, as shown by Słodkowski, this property actually characterizes the analyticity of upper semicontinuous multifunctions. Thus it can be used as an alternative definition of analytic multifunction, and indeed this is the approach taken in [4, 18].

It also quickly leads to the following result. We write ρ for radius (i.e., $\rho(S) := \max_{w \in S} |w|$) and c for logarithmic capacity.

Theorem 2.9. *If K is an analytic multifunction on D, then both functions $z \mapsto \log \rho(K(z))$ and $z \mapsto \log c(K(z))$ are subharmonic on D.*

Notice that, combining this result with the fact that $z \mapsto \sigma(f(z))$ is an analytic multifunction, we recover Vesentini's theorem that $\log \rho(f(z))$ is subharmonic. The new proof is more complicated than the original, but the tools now made available are considerably more powerful.

3 The open mapping theorem

Let D be a plane domain. According to the classical open mapping theorem, if $f : D \to \mathbb{C}$ is a non-constant holomorphic function, then $f(D)$ is open in $\mathbb{C}$. What about analytic multifunctions?

It is easy to see that there are non-constant analytic multifunctions K such that $K(D)$ is not open in $\mathbb{C}$. For example, one can take $K(z) := \{f(z)\} \cup \{w_0\}$, where $w_0 \notin f(D)$. This example and other similar ones strongly suggest that non-interior points of $K(D)$ should be permanently in $K(z)$, in other words, that

$$K(D) \setminus \operatorname{int}(K(D)) \subset \bigcap_{z \in D} K(z). \tag{3}$$

In particular, if $K(D)$ has empty interior, then K should be constant on D. It turns out that a weak version of this is indeed true.

Theorem 3.1 ([19])**.** *If $K(D)$ has empty fine interior, then K is constant.*

However, maybe a little surprisingly, without the 'fine', the result is false.

Theorem 3.2 ([19])**.** *There exist a compact set F with empty interior and a non-constant multifunction K analytic on a domain D such that $K(D) \subset F$.*

K and F can be constructed as follows. Let u be a subharmonic function on $\mathbb{C}$ such that $\log 2 < u(0) < \log 3$ and $u = -\infty$ on a dense subset E of the unit disk $\mathbb{D}$. Denoting by $\mathbb{T}$ the unit circle, define

$$K(z) := \mathbb{T} \cup \{w \in \mathbb{D} : u(w) \geq \log|z|\} \quad (z \in \mathbb{C} \setminus \{0\}).$$

The complement of graph(K) is the union of the disjoint pseudoconvex sets

$$\Big((\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \overline{\mathbb{D}})\Big) \cup \Big(\Big\{(z,w) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{D} : u(w) - \log|z| < 0\Big\}\Big),$$

so K is analytic on $\mathbb{C} \setminus \{0\}$. Set $D := \{z : 1 < |z| < 4\}$ and $F := K(\overline{D})$. Then F is compact and $K(D) \subset F$. Also F has empty interior since $F \subset \overline{\mathbb{D}} \setminus E$. Finally K is non-constant on D, since $0 \in K(2) \setminus K(3)$.

The picture of $K(z)$ is that of a Swiss cheese with infinitely many (non-circular) holes. The holes grow with $|z|$. This suggests two problems, both of which were proposed by Aupetit in [4, §2].

Problem A. *Let K be an analytic multifunction on a domain D. Suppose that there exists a compact set F with empty interior such that $K(D) \subset F$. Must we have $\cap_{z \in D} K(z) \neq \emptyset$?*

Problem B. *Let K be an analytic multifunction on a domain D whose values $K(z)$ are polynomially convex sets. Must* (3) *hold?*

Baribeau and Harbottle [7] gave an affirmative answer to Problem B in the special case when $K(z)$ takes convex values. Słodkowski [27] extended their result to the case when the values of $K(z)$ are connected and polynomially convex. It is also easy to see that (3) holds when K takes finite values [19]. The general case is still an open problem (no pun intended!).

4 The identity principle

Again, let D be a plane domain. The standard identity principle says that, if f_1 and f_2 are holomorphic functions on D and if $f_1 = f_2$ on a set with a limit point in D, then $f_1 \equiv f_2$. What about analytic multifunctions?

We saw in Theorem 2.3 that, if $K(z)$ is the disk with centre 0 and radius $r(z)$, then K is analytic if and only if $\log r$ is subharmonic. It is easy to find subharmonic functions $\log r_1$ and $\log r_2$ that agree on an open subset of D without being equal everywhere on D. Thus the identity principle, in the form stated above, does not extend to arbitrary analytic multifunctions.

However, all is not lost. There is a general principle that the 'thinner' the values taken by an analytic multifunction, the more it behaves like a holomorphic function: compare Theorems 2.1 and 2.3, for example. It therefore makes sense to seek identity principles for certain classes of analytic multifunctions taking 'thin' values. We shall do this for polar-valued functions.

We recall that a plane set P is *polar* if there exists a domain D containing P and a subharmonic function $u \not\equiv -\infty$ on D such that $u \equiv -\infty$ on P. It is well known that a compact set P is polar if and only if $c(P) = 0$ (where c denotes logarithmic capacity).

The following two results are prototypical.

Theorem 4.1. *Let K be analytic on D. If $K(z)$ is polar for z in a non-polar subset of D, then $K(z)$ is polar for all $z \in D$.*

Proof. Set $u(z) := \log c(K(z))$. By Theorem 2.9, u is subharmonic on D. Whenever $K(z)$ is polar, we have $u(z) = -\infty$. If this happens for z in a non-polar subset of D, then it follows that $u \equiv -\infty$ on D. This implies that $K(z)$ is polar for all $z \in D$. □

Theorem 4.2 ([6]). *Let K be analytic on D and let P be a compact polar set. If $K(z) = P$ for z in a non-polar subset of D, then $K(z) = P$ for all $z \in D$.*

Proof. By Evans' theorem (see e.g. [21, Theorem 5.5.6]), as P is a compact polar set, there exists a probability measure μ on P such that the logarithmic potential $v(w) := \int \log|w - \zeta|\, d\mu(\zeta)$ is identically $-\infty$ on P. Then v is a subharmonic function on $\mathbb{C}$ and $P = \{w \in \mathbb{C} : v(w) = -\infty\}$. Define

$$u(z) := \max_{w \in K(z)} v(w) \quad (z \in D).$$

By Theorem 2.8, u is subharmonic on D. Also, whenever $K(z) = P$, we have $u(z) = -\infty$. If this happens for z in a non-polar subset of D, then $u \equiv -\infty$ on D. This implies that $K(z) \subset P$ for all $z \in D$. Finally, as P has empty fine interior, Theorem 3.1 shows that K is constant on D. □

These results strongly suggest seeking an identity principle for polar-valued analytic multifunctions.

Problem C. *Let K_1 and K_2 be polar-valued analytic multifunctions on D. If $K_1 = K_2$ on a non-polar subset of D, must we have $K_1 \equiv K_2$ on D?*

A natural approach to this problem is to try to show that graph(K_1) is a *complete pluripolar set*, i.e., there exists a plurisubharmonic function v on $D \times \mathbb{C}$ such that the set where $v = -\infty$ is exactly the graph of K_1. If we could do this, then the proof above would show that $K_2(z) \subset K_1(z)$ for all $z \in D$, and reversing the roles of K_1 and K_2, we could deduce that $K_1 \equiv K_2$.

Problem D. *Let K be a polar-valued analytic multifunction on D. Is* graph(K) *a complete pluripolar set?*

Sadullaev [24] proved that graph(K) is *pluripolar*, in other words, there exists a plurisubharmonic function $v \not\equiv -\infty$ on $D \times \mathbb{C}$ such that $v \equiv -\infty$ on graph(K). This is already a difficult result. However, it does not quite solve the problem, since we need v to be finite everywhere outside graph(K).

By analogy with Evans' theorem in one dimension, we might even hope to construct a plurisubharmonic function v on $D \times \mathbb{C}$ such that $v \equiv -\infty$ on graph(K) and v is pluriharmonic on $(D \times \mathbb{C}) \setminus \text{graph}(K)$. However, Levenberg and Słodkowski [14] showed that this is not always possible.

Thus far we have concentrated on polar-valued analytic multifunctions. There is an interesting subclass for which a little more is known, namely analytic multifunctions K for which $K(z)$ is always a countable set.

Here are two results relating to this class. The first of these, proved by Nishino [16], played an interesting role in the history of the subject (see e.g. [2, §3]). A simpler proof of Nishino's result, deducing it from Theorem 4.1, was given by Levenberg, Ransford, Rostand and Słodkowski in [15].

Theorem 4.3. *Let K be analytic on D. If $K(z)$ is countable for all z in a non-polar subset of D, then $K(z)$ is countable for all $z \in D$.*

The second result is due to Aupetit and Zemánek [5].

Theorem 4.4. *Let K be a countable-valued analytic multifunction on D. If $0 \in K(z)$ for an uncountable set of $z \in D$, then $0 \in K(z)$ for all $z \in D$.*

One might wonder whether the hypothesis can be weakened to supposing merely that $0 \in K(z)$ for z in a set having a limit point in D. This is not the case. For example, consider the multifunction $K(z) := \{z\} + C$, where C is a countable compact set. This shows that the hypothesis in Theorem 4.4 is sharp, and also suggests the following problem.

Problem E. *Let K be a polar-valued analytic multifunction on D such that $0 \in K(z)$ for z in a non-polar subset of D. Does it follow that $0 \in K(z)$ for all $z \in D$?*

Notice that an affirmative answer to Problem C would immediately solve Problem E, just by applying it to the pair of multifunctions $K_1(z) := K(z)$ and $K_2(z) := K(z) \cup \{0\}$. On the other hand, there is no obvious reverse implication, so maybe Problem E is easier.

We end this section with an identity principle of a slightly different type. It is not hard to see that the standard identity principle for holomorphic functions is valid also for analytic multifunctions taking values that are finite sets (so they are of the form (2)). In [30], White generalized this observation as follows.

Theorem 4.5 ([30, Theorem 5.1])**.** *Let $1 \le p < \infty$, and let K_1 and K_2 be analytic multifunctions on D whose values are ℓ^p-sequences whose ℓ^p-norms are locally bounded. If $K_1 = K_2$ on a set with a limit point in D, then $K_1 \equiv K_2$ on D.*

In the same paper, White raised the question as to whether the analogous result holds for c_0. In this case, there is no need to assume local boundedness of the c_0-norm, since it is automatic. Thus his question reads as follows.

Problem F. *Let K_1 and K_2 be analytic multifunctions on D whose values are sequences tending to zero. If $K_1 = K_2$ on a set with a limit point in D, must we have $K_1 \equiv K_2$ on D?*

As far as we know, this problem is still open.

5 Picard's theorem

The little Picard theorem says that, if f is a non-constant entire function, then $\mathbb{C} \setminus f(\mathbb{C})$ contains at most one point. In his Ph.D. thesis in 1905, Rémoundos obtained the following generalization, later published in [23].

Theorem 5.1. *Let $n \ge 1$ and let $a_1, \dots, a_n$ be entire functions. For $z \in \mathbb{C}$, define*

$$K(z) := \{w \in \mathbb{C} : w^n + a_1(z)w^{n-1} + \cdots + a_{n-1}(z)w + a_n(z) = 0\}.$$

If $\mathbb{C} \setminus K(\mathbb{C})$ contains at least $2n$ points then K is constant.

In view of Theorem 2.2, this can be regarded as a Picard theorem for analytic multifunctions taking values that are finite sets. It naturally suggests seeking Picard-type theorems for more general analytic multifunctions. Here are two such results. We recall that $\widehat{K}$ denotes the polynomial hull of K, as defined in Theorem 2.7.

Theorem 5.2 ([6])**.** *Let K be an analytic multifunction on $\mathbb{C}$. If $\mathbb{C} \setminus K(\mathbb{C})$ is a non-polar set, then $\widehat{K}$ is constant.*

Theorem 5.3 ([20])**.** *Let K be an analytic multifunction on $\mathbb{C}$ such that $K(z)$ is connected for all $z \in \mathbb{C}$. If $\mathbb{C} \setminus K(\mathbb{C})$ contains more than one point, then $\widehat{K}$ is constant.*

Why does $\widehat{K}$ appear here? It is needed in order to exclude the following rather artificial kind of example. Define

$$K(z) := \begin{cases} \overline{\mathbb{D}}, & z \in \overline{\mathbb{D}}, \\ \mathbb{T}, & z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \end{cases}$$

By Theorem 2.7, K is analytic and bounded on $\mathbb{C}$, yet K is non-constant. On the other hand $\widehat{K}$ is constant, as predicted by Theorems 5.2 and 5.3.

It is known that the conditions on $\mathbb{C} \setminus K(\mathbb{C})$ in all three theorems above are sharp. In particular, given a compact polar set P, there exists an analytic multifunction K on $\mathbb{C}$ with $\widehat{K}$ non-constant such that $\mathbb{C} \setminus K(\mathbb{C}) = P$, thereby demonstrating that Theorem 5.2 is sharp. However, the example of such a K constructed in [6] has the defect that the values $K(z)$ are not polynomially convex, and in fact $\widehat{K}(\mathbb{C}) := \cup_{z \in \mathbb{C}} \widehat{K}(z)$ is the whole of $\mathbb{C}$. This suggests the following problem.

Problem G. *Let K be a non-constant analytic multifunction on $\mathbb{C}$. How large can $\mathbb{C} \setminus \widehat{K}(\mathbb{C})$ be?*

It can happen that $\mathbb{C} \setminus \widehat{K}(\mathbb{C})$ is infinite. For example, if

$$K(z) := \{0\} \cup \{w \in \mathbb{C} : \sin(\pi/w) = \exp(z)\} \quad (z \in \mathbb{C}),$$

then one can show that K is analytic on $\mathbb{C}$, and since K takes countable values we have $\widehat{K}(z) = K(z)$ for all z, and $\mathbb{C} \setminus \widehat{K}(\mathbb{C}) = \{1/n : n \ge 1\}$. However, we know of no example where $\mathbb{C} \setminus \widehat{K}(\mathbb{C})$ is uncountable.

6 Bloch's theorem

Bloch's theorem, in its simplest form, states that there is a constant $r > 0$ such that, if f is holomorphic on $\mathbb{D}$ and $f'(0) = 1$, then $f(\mathbb{D})$ contains an open disk of radius r. The supremum of all such r is called *Landau's constant*, denoted L. The exact value of L is not known, but it is known that $0.5 < L < 0.55$.

In [10] Guillot and Ransford generalized Bloch's theorem to finite-valued analytic multifunctions K on $\mathbb{D}$. Recall from Theorem 2.2 that such a K has the form

$$K(z) := \{w \in \mathbb{C} : w^n + a_1(z)w^{n-1} + \cdots + a_{n-1}(z)w + a_n(z) = 0\}, \tag{4}$$

where $n \ge 1$ and $a_1, \dots, a_n$ are holomorphic functions on $\mathbb{D}$. We shall say that K is of *degree* n. As discussed in [10], the appropriate normalization is to suppose that $a_1'(0) = 1$.

Theorem 6.1. *Let $n \ge 1$ and let $\mathcal{F}_n$ be the family of all analytic multifunctions K on $\mathbb{D}$ of degree n such that $a_1'(0) = 1$. Define*

$$L_n := \inf_{K \in \mathcal{F}_n} \Big(\sup\{r : K(\mathbb{D}) \text{ contains a disk of radius } r\} \Big).$$

Then $L_n > 0$.

Thus $L_1 = L$, Landau's constant. The proof of Theorem 6.1 is by contradiction, and does not yield concrete estimates of L_n for $n \ge 2$. However, it is shown in [10] that

$$\frac{1}{L_{m+n}} \ge \frac{1}{L_m} + \frac{1}{L_n} \quad (m, n \ge 1),$$

from which it follows that $nL_n \le L$ for all n and that $\lim_{n \to \infty} nL_n$ exists. This leads to our final problem.

Problem H. *Identify $\lim_{n \to \infty} nL_n$. In particular, is it positive or zero?*

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