



Approximation of a weighted Hilbert transform by using perturbed Laguerre zeros

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Abstract

In the present paper is proposed a numerical method to approximate Hilbert transforms of the type

$$H(fw, t) = \int_0^{+\infty} \frac{f(x)}{x-t} w(x) dx, \quad t > 0,$$

where $w(x) = e^{-x}x^\alpha$, $\alpha > -1$ is a Laguerre weight, by means of a new Lagrange interpolation process essentially based on the midpoints between two consecutive zeros of Laguerre polynomials. Theoretical error estimates are proved in some weighted uniform spaces and some numerical tests which confirm the theoretical estimates are shown.

1 Introduction

Let $H(fw, t)$ be the Hilbert transform of the function f

$$H(fw, t) = \int_0^{+\infty} \frac{f(x)}{x-t} w(x) dx, \quad w(x) = e^{-x}x^\alpha, \quad \alpha > -1, \quad t > 0, \quad (1)$$

provided that the integral exists as a principal value. In the present paper, starting from the decomposition

$$H(fw, t) = \int_0^{+\infty} \frac{f(x) - f(t)}{x-t} w(x) dx + f(t) \int_0^{+\infty} \frac{w(x)}{x-t} dx, \quad (2)$$

we propose a numerical method to approximate the function

$$\mathcal{F}(fw; t) := \int_0^{+\infty} \frac{f(x) - f(t)}{x-t} w(x) dx, \quad t > 0, \quad (3)$$

since the function $H(w; t)$ is computable with the desired accuracy by means of special functions.

By virtue of the known smoothness of $\mathcal{F}(fw)$ when f belongs to some uniform Zygmund spaces [11], we propose to approximate $\mathcal{F}(fw)$ by a suitable Lagrange polynomial interpolating $\mathcal{F}(fw)$. This approach was firstly considered in [6] for Hilbert transforms in $[-1, 1]$ and successively applied in $[0, +\infty)$ in [11], where the function $\mathcal{F}(fw)$ is approximated by the truncated Lagrange polynomial introduced in [3] (see also [4]). One advantage offered by the global approximation of the function $\mathcal{F}(f)$ is the use of the same samples of the function f independent of the point t . For instance, many values of $H(fw, t)$ are required in some projection methods to approximate the solution of singular integral equations. However, the choice of the interpolation nodes that we go to propose here seems to be new. To be more precise, we introduce here the truncated Lagrange polynomial $L_m^*(\mathcal{F}(fw))$ defined as

$$\begin{aligned} L_m^*(\mathcal{F}(fw), z_i) &= \mathcal{F}(fw, z_i), \quad i = 1, 2, \dots, m-1, \quad L_m^*(\mathcal{F}(fw), 4m) = \mathcal{F}(fw, 4m), \\ L_m^*(\mathcal{F}(fw)) &= \sum_{i=1}^j l_{m,i}(x) \mathcal{F}(fw, z_i), \end{aligned} \quad (4)$$

where z_i are the midpoints of the intervals between two consecutive zeros of the m -th Laguerre polynomial $p_m(w, x)$ and the index $j \leq m$ is defined in (7). Since in the general case the samples of the function

$$\mathcal{F}(fw, z_i) \int_0^{+\infty} \frac{f(x) - f(z_i)}{x - z_i} w(x) dx, \quad i = 1, 2, \dots, j,$$

cannot be exactly computed, we will approximate them by the truncated Gauss-Laguerre formula [5] based on the zeros $\{x_k\}_{k=1}^m$ of $p_m(w)$, i.e.

$$\mathcal{F}(fw; z_i) \sim \sum_{k=1}^{\ell} \frac{f(x_k) - f(z_i)}{x_k - z_i} \lambda_k, \quad \{\lambda_k\}_{k=1}^m \text{ Christoffel numbers}, \quad i = 1, 2, \dots, \ell, \quad (5)$$

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where the index $\ell \leq m$ is defined in (9). By this way, with the employment of only one set of Laguerre zeros, the interpolation nodes z_i and the Gaussian knots x_k in (5) are sufficiently far among them, avoiding possible numerical cancellation phenomena. In addition, a reduction of the computational cost is realized w.r.t. the analogous procedure introduced in [11] where two different Laguerre polynomials sequences are involved.

The paper is organized as follows. In Section 2 are collected some basic results about orthogonal polynomials and functional spaces, needed to introduce the main results. Section 3 contains the definition and a result about the rate of convergence of the new Lagrange process defined in (4). In Section 4 the numerical method to approximate $H(f, w, t)$ is described and a result about the rate of convergence is stated. Section 5 contains some tests for showing the behavior of the Lebesgue constants of the process in (4). In Section 6 we present some numerical tests about the approximation of the Hilbert transform, which confirm the theoretical estimates. Finally, in Section 7 the proofs of the main results are stated.

2 Notations and preliminary results

In the sequel C will denote any positive constant which can be different in different formulas. Moreover $C \neq C(a, b, \dots)$ will be used to say that the constant C is independent of a, b, \dots . The notation $A \sim B$, where A and B are positive quantities depending on some parameters, will be used if and only if $(A/B)^{\pm 1} \leq C$, with C positive constant independent of the above parameters.

Throughout the paper θ will denote a fixed real number, with $0 < \theta < 1$, which can be different in different formulas.

Denote by \mathbf{P}_m the space of all algebraic polynomials of degree at most m .

Consider the weight $w(x) = e^{-x}x^\alpha$, $\alpha > -1$, let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients,

$$p_m(w, x) = \gamma_m(w)x^m + \text{terms of lower degree}, \quad \gamma_m(w) > 0 \tag{6}$$

and denote by $\{x_k\}_{k=1}^m$ the zeros of $p_m(w)$ in increasing order, i.e.

$$x_k < x_{k+1}, \quad k = 1, \dots, m-1.$$

We recall that

$$x_k \in \zeta_m := \left(\frac{C}{m}, 4m - Cm^{\frac{1}{3}} \right).$$

For any fixed $0 < \theta < 1$, let

$$x_j = x_{j(m)} = \min \{x_k : x_k \geq 4m\theta, \quad k = 1, 2, \dots, m\}. \tag{7}$$

2.1 Functional spaces and best approximation estimates

Setting $u(x) = e^{-x/2}x^\gamma$, $\gamma \geq 0$, we define the functional space

$$C_u = \left\{ f : fu \in C^0(\mathbf{R}^+), \lim_{x \rightarrow 0^+} f(x)u(x) = 0 = \lim_{x \rightarrow \infty} f(x)u(x) \right\},$$

equipped with the norm $\|f\|_{C_u} = \sup_{x \geq 0} |f(x)u(x)|$ and $Z_\lambda(u)$, with $\lambda > 0$, will denote the following Zygmund-type space

$$Z_\lambda(u) = \left\{ f \in C_u : \|f\|_{Z_\lambda(u)} := \|fu\|_\infty + \sup_{t>0} \frac{\Omega_\varphi^r(f, t)_{u, \infty}}{t^\lambda} < \infty, \quad r > \lambda, \quad \varphi(x) = \sqrt{x} \right\},$$

where

$$\Omega_\varphi^r(f, t)_{u, \infty} = \sup_{0 < h \leq t} \|(\bar{\Delta}_{h\varphi}^r f)u\|_{L^\infty(I_{rh})},$$

$I_{rh} = [8(rh)^2, Ch^{-2}]$, C an arbitrary constant and

$$\bar{\Delta}_{h\varphi}^r = \sum_{i=0}^r (-1)^i \binom{r}{i} f \left(x + \left(\frac{r}{2} - i \right) h\sqrt{x} \right).$$

Moreover, we denote by $W_s(u)$, $s = 1, 2, \dots$, the Sobolev space

$$W_s(u) = \left\{ f \in C_u : f^{(s-1)} \in AC(\mathbf{R}^+) \|f^{(s)}\varphi^s u\|_\infty < \infty \right\},$$

where $AC(\mathbf{R}^+)$ denotes the set of the functions which are absolutely continuous on every closed subset of \mathbf{R}^+ , equipped with the norm

$$\|f\|_{W_s(u)} = \|fu\|_\infty + \|f^{(s)}\varphi^s u\|_\infty.$$

Denoting by

$$E_m(f)_u = \inf_{P \in \mathbf{P}_m} \|(f - P)u\|_\infty$$

the error of the best approximation in C_u , the following weaker version of the Jackson theorem holds [2, Corollary 3.6]

$$E_m(f)_u \leq C \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^k(f, t)_u}{t} dt, \quad C \neq C(m, f). \tag{8}$$

By (8), [2], [10]

$$E_m(f)_u \leq C \frac{\|f\|_{Z_\lambda(u)}}{(\sqrt{m})^\lambda}, \quad C \neq C(m, f), \quad \forall f \in Z_\lambda(u), \quad \lambda > 0,$$

$$E_m(f)_u \leq \frac{C}{(\sqrt{m})^s} \|f^{(s)} \varphi^s u\|_\infty, \quad C \neq C(m, f), \quad \forall f \in W_s(u), \quad s \geq 1.$$

2.2 Truncated Gaussian rule

The so called "truncated" Gauss-Laguerre rule [5] (see also [8], [4]) is based on the first ℓ zeros of $p_m(w)$, i.e., for a fixed $\theta_1 \in (0, 1)$, denoted by x_ℓ the zero of $p_m(w)$ s.t.

$$x_\ell = \min \{x_i : x_i \geq 4m\theta_1, \quad i = 1, 2, \dots, m\}, \tag{9}$$

the formula is

$$\int_0^{+\infty} f(x)w(x)dx = \sum_{k=1}^{\ell} f(x_k)\lambda_k + R_m(f), \tag{10}$$

where $\{\lambda_k\}_{k=1}^m$ are the Christoffel numbers w.r.t. the weight w and $R_m(f)$ is the remainder term. For all $f \in W_r(u)$, under the assumption $\alpha - \gamma > -1$ [4, Proposition 2.3]

$$|R_m(f)| \leq \frac{C}{(\sqrt{m})^r} \|f^{(r)} \varphi^r u\|_\infty, \quad 0 < C \neq C(m, f).$$

2.3 Lagrange Interpolation at Laguerre zeros

Let $\mathcal{L}_{m+1}(w, g)$ be the Lagrange polynomial interpolating a given function g at the zeros of $p_m(w, x)(4m - x)$, i.e.

$$\mathcal{L}_{m+1}(w, g, x_i) = g(x_i), \quad i = 1, 2, \dots, m, \quad \mathcal{L}_{m+1}(w, g, 4m) = g(4m).$$

In [7] it was proved that for any $g \in C_u$,

$$\|\mathcal{L}_{m+1}(w, g)u\|_\infty \leq C \|gu\|_\infty \log m,$$

if and only if

$$\frac{\alpha}{2} + \frac{1}{4} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4}, \tag{11}$$

where $C \neq C(m, g)$.

Later, a more convenient interpolation process was introduced in [3], namely the "truncated Lagrange polynomial" interpolating f defined as

$$\mathcal{L}_{m+1}^*(w, g, x) := \mathcal{L}_{m+1}(w, g \chi_{m,\theta}, x) = \sum_{k=1}^j g(x_k) \ell_{m+1,k}(w, x), \quad \ell_{m+1,k}(w, x) = \frac{p_m(w, x)(4m - x)}{p'_m(w, x_k)(4m - x_k)(x - x_k)}, \tag{12}$$

where for a fixed $\theta \in (0, 1)$ $\chi_{m,\theta}$ is the characteristic function of the interval $(0, x_j)$, j defined in (7).

About the Lebesgue constants in the space C_u , in [4] the authors proved that for any $g \in C_u$, the assumption (11) is sufficient to prove

$$\|\mathcal{L}_{m+1}^*(w, g)u\|_\infty \leq C \|gu\|_\infty \log m,$$

where $C \neq C(m, g)$.

In such a way, under the same assumptions on the parameters α, γ they obtain a Lagrange process which requires less samples of the function and whose Lebesgue constants are of order $\log m$ again.

3 The Lagrange interpolation process at "midpoints"

Let j be the index defined in (7) and let $\gamma_m(w)$ be the leading coefficient of the polynomial $p_m(w)$, as well as defined in (6). Let $q_{m-1} \in \mathbf{P}_{m-1}$ be the polynomial

$$q_{m-1}(x) := \gamma_m(w) \prod_{i=1}^{m-1} (x - z_i),$$

$$z_i = \frac{x_i + x_{i+1}}{2} = x_i + \frac{\Delta x_i}{2}, \quad i = 1, 2, \dots, m-1, \quad \Delta x_i = x_{i+1} - x_i.$$

Let $L_m(f)$ be the Lagrange polynomial interpolating a given function f at the zeros of $q_{m-1}(x)(4m - x)$, i.e.

$$L_m(f, x) = \sum_{k=1}^{m-1} l_{m,k}(x) f(z_k) + \frac{q_{m-1}(x)}{q_{m-1}(4m)} f(4m), \quad l_{m,k}(x) = \frac{q_{m-1}(x)(4m-x)}{q'_{m-1}(z_k)(4m-z_k)(x-z_k)}.$$

For a fixed $\theta \in (0, 1)$ set $z_j = \frac{x_j+x_{j+1}}{2}$, where j is the index defined in (7) and denote by $\mu_{m,\theta}$ the characteristic function of the interval $(0, z_j)$. Let us introduce the truncated Lagrange polynomial $L_m^*(f)$

$$L_m^*(f, x) := L_m(f \mu_{m,\theta}, x) = \sum_{k=1}^j l_{m,k}(x) f(z_k). \tag{13}$$

We observe that $L_m^*(f, z_k) = 0 = L_m^*(f, 4m)$, $k > j$. The Lagrange operator L_m projects $C^0(\mathbb{R}^+)$ onto \mathbb{P}_{m-1} , while L_m^* does not. However, letting

$$\mathcal{P}_{m-1}^* = \{p \in \mathbb{P}_{m-1} : p(4m) = p(z_i) = 0, \quad i > j\} \subset \mathbb{P}_{m-1},$$

L_m^* projects $C^0(\mathbb{R}^+)$ onto \mathcal{P}_{m-1}^* . Moreover, similarly to Lemma 2.1 in [9], it can be proved that $\cup_m \mathcal{P}_{m-1}^*$ is dense in C_u , since

$$\tilde{E}_{m-1}(f)_u := \inf_{p \in \mathcal{P}_{m-1}^*} \|(f - p)u\|_\infty \rightarrow 0, m \rightarrow \infty.$$

Moreover, $\tilde{E}_{m-1}(f)_u$ can be estimated in terms of the best approximation error of order M , where M is a proper fraction of m , depending on θ , i.e.

$$\tilde{E}_{m-1}(f)_u \leq C(E_M(f)_u + e^{-Am} \|f u\|_\infty) \tag{14}$$

with $M = \lceil (m-1) \frac{\theta}{\theta+1} \rceil$ ($\lceil a \rceil$ denotes the integer part of $a \in \mathbb{R}$) and $A \neq A(m, f)$ is a positive constant.

About the above introduced interpolation process we are able to state the following result

Theorem 3.1. For any function $f \in C_w$, under the assumption

$$\frac{1}{4} \leq \gamma - \frac{\alpha}{2} \leq \frac{5}{4}, \tag{15}$$

we have

$$\|L_m^*(f)u\|_\infty \leq C \|f u\|_\infty m^{\frac{2}{3}} \log m, \tag{16}$$

where $0 < C \neq C(m, f)$.

Remark 1. Recalling that in a more general context [13], [14] P. Vértési proved that for any matrix of interpolation knots in $[0, 4m]$, the Lebesgue constants in C_u are greater than or equal to $\log m$ (as order), the above introduced Lagrange polynomial is not an optimal process, since the corresponding Lebesgue constants algebraically diverge. On the other hand this represents a first result in the study of Lagrange interpolation processes on perturbed zeros of orthogonal polynomials. Moreover, as announced in the Introduction, the polynomial $L_m^*(f)$ will be usefully employed to approximate the Hilbert transform (1).

About the convergence of the Lagrange interpolation on the zeros of $q_{m-1}(x)(4m-x)$, the following result holds

Theorem 3.2. For any function $f \in C_w$, under the assumption (15) the following error estimate holds

$$\|(f - L_m^*(f))u\|_\infty \leq C \left\{ E_M(f)_u m^{\frac{2}{3}} \log m + e^{-Am} \|f u\|_\infty \right\}$$

where $M = \lceil m \left(\frac{\theta}{1+\theta} \right) \rceil \sim m$, $0 < C \neq C(m, f)$, $0 < A \neq A(m, f)$.

For instance, assuming $f \in Z_{\lambda+\frac{4}{3}}(u)$, by (8) it follows

$$\|(f - L_m^*(f))u\|_\infty \leq C \log m \frac{\|f\|_{Z_{\lambda+\frac{4}{3}}(u)}}{(\sqrt{m})^\lambda}, \quad C \neq C(m, f).$$

Remark 2. We point out that all the previous results hold true in the more general case, where $w(x)$ is a Freud-Laguerre weight i.e. $w(x) = w_{\alpha,\beta}(x) := e^{-x^\beta} x^\alpha$, $\beta > \frac{1}{2}$. For $\beta = 1$, $w_{\alpha,\beta}$ reduces to a Laguerre weight.

4 Approximation of the function $H(f w)$

Start from the decomposition in (2) with $w(x) = e^{-x} x^\alpha$ and assume $-1 < \alpha < 1$. Indeed, the case $\alpha \geq 1$ can be easily treated setting $f(x)x^{[\alpha]}$ instead of f , with weight $\tilde{w}(x) := e^{-x} x^{\alpha-[\alpha]}$. Since $H(w; t)$ is given in a closed form and it can be "exactly" computed (see Section 6), we focus our attention in approximating the function $\mathcal{F}(f w)$ defined in (3).

Thus, following an idea in [11] (see also [6]), we approximate the function $\mathcal{F}(f w)$ by $L_m^*(\mathcal{F}(f w))$, i.e.

$$L_m^*(\mathcal{F}(f w), t) = \sum_{k=1}^j l_{m,k}(t) \mathcal{F}(f w; z_k) = \sum_{k=1}^j l_{m,k}(t) \int_0^{+\infty} \frac{f(x) - f(z_k)}{x - z_k} w(x) dx.$$

Since in the general case the samples $\{\mathcal{F}(f w; z_k)\}_{k=1}^j$ cannot be exactly computed, we approximate them by the m -th "truncated" Gaussian rule (10)

$$\mathcal{F}(f w; z_k) = \mathcal{F}_m(f w; z_k) + e_m(\mathcal{F}(f w); z_k), \quad k = 1, \dots, j,$$

where

$$\mathcal{F}_m(f w; z_k) := \sum_{i=1}^{\ell} \lambda_i \frac{f(x_i) - f(z_k)}{x_i - z_k}.$$

Therefore we have at least

$$\begin{aligned} H(f w, t) &= H_m(f w; t) + \rho_m(f w, t), \\ H_m(f w; t) &:= L_m^*(\mathcal{F}_m(f w); t) + f(t)H(w; t). \end{aligned} \tag{17}$$

Now, recalling that under the assumptions

$$0 \leq \gamma < \alpha + \frac{1}{4}, \quad f \in Z_{\lambda+1}(u),$$

both the functions $\mathcal{F}(f w)$ and $\mathcal{F}_m(f w)$ belong to $Z_{\lambda}(u)$ [11, Lemmas 5.4, 5.7], we are able to prove the following results about the error estimate:

Theorem 4.1. *Let $f \in Z_{\lambda+2+\frac{1}{3}}(u)$ and assume $t \in (0, 4m\theta)$. Then, under the assumption*

$$\max\left(\frac{\alpha}{2} + \frac{1}{4}, 0\right) \leq \gamma < \alpha + \frac{1}{4} \tag{18}$$

we have

$$|\rho_m(f w, t)|u(t) \leq C \frac{\log m}{(\sqrt{m})^{\lambda}} \|f\|_{Z_{\lambda+2+\frac{1}{3}}(u)} \tag{19}$$

where $0 < C \neq C(m, f)$.

As we can see the method is easy to carry out, since it uses tools like the Gaussian rule and Lagrange interpolation, both of them involving only one sequence of orthogonal polynomials, namely $\{p_m(w)\}_m$.

We remark that interpolation knots and Gaussian nodes involved in the computation of $H_m(f w; t)$ are far enough among them, avoiding in such a way a possible numerical instability in computing $(x_i - z_k)^{-1}$. In the meantime the global process requires only one sequence of orthogonal polynomials. This aspect can represent an advantage w.r.t. the procedure proposed in [11], where two sequences of Laguerre polynomials were involved, namely $\{p_m(w)\}_m, \{p_m(\bar{w})\}_m, \bar{w}(x) = xw(x)$. The above mentioned saving becomes much more interesting in the more general case w let be a Freud-Laguerre weight $w_{\alpha,\beta}$ (see Remark 2). Indeed, in this case, the coefficients of the three term recurrence relation are not always known. Therefore, the computation of zeros and Christoffel numbers can be again performed by using the package "OrthogonalPolynomials" in MATHEMATICA [1], which uses "high" variable precision and requires a considerable computational effort.

5 The numerical behavior of the Lebesgue constants

In this section we show the behavior of the Lebesgue constants related to the Lagrange process at midpoints, by proposing some numerical experiments for different choices of the parameters α, γ under the assumption (15). For the same parameters, we show also the behavior of the Lebesgue constants associated to the Lagrange process at Laguerre zeros. To be more precise, denoted by

$$\Lambda_{m,j}(w, x)_u = \sum_{k=1}^j |\ell_{m+1,k}(w, x)| \frac{u(x)}{u(x_k)}, \quad \tilde{\Lambda}_{m,j}(x)_u = \sum_{k=1}^j |l_{m,k}(x)| \frac{u(x)}{u(z_k)}$$

the Lebesgue functions associated to the Lagrange processes defined in (12) and (13) respectively, we will denote by

$$\Lambda_{m,j}^{(u)}(w) := \max_{x \geq 0} \Lambda_{m,j}(w, x)_u, \quad \tilde{\Lambda}_{m,j}^{(u)} := \max_{x \geq 0} \tilde{\Lambda}_{m,j}(x)_u,$$

the corresponding Lebesgue constants. In the tables we will report the ratios $\left\{ \frac{\Lambda_{m,j}^{(u)}(w)}{\log m} \right\}_m, \left\{ \frac{\tilde{\Lambda}_{m,j}^{(u)}}{m^{\frac{2}{3}} \log m} \right\}_m$ for $\theta = 0.7$. In Table 1 we have considered $\gamma = \frac{\alpha}{2} + \frac{1}{4}$ for $\alpha = \pm \frac{1}{2}$, while in Table 2 the case $\gamma = \frac{\alpha}{2} + \frac{1}{2}$ for $\alpha = \pm \frac{1}{2}$. Moreover, we present some graphs of the Lebesgue functions $\Lambda_{m,j}(w, x)_u$ and $\tilde{\Lambda}_{m,j}(x)_u$ for $m = 100$ and for different choices of θ .

| $\alpha = -0.5, \gamma = 0, \theta = 0.7$ | | | |
|---|-----|---|--|
| m | j | $\frac{\Lambda_{m,j}^{(u)}(w)}{\log m}$ | $\frac{\tilde{\Lambda}_{m,j}^{(u)}}{m^{\frac{2}{3}} \log m}$ |
| 10 | 9 | 1.27 | 0.30 |
| 60 | 56 | 1.02 | 0.11 |
| 110 | 102 | 0.96 | 0.09 |
| 160 | 148 | 0.94 | 0.08 |
| 210 | 194 | 0.93 | 0.07 |
| 260 | 240 | 0.92 | 0.06 |
| 310 | 286 | 0.91 | 0.06 |

| $\alpha = 0.5, \gamma = 0.5, \theta = 0.7$ | | | |
|--|-----|---|--|
| m | j | $\frac{\Lambda_{m,j}^{(u)}(w)}{\log m}$ | $\frac{\tilde{\Lambda}_{m,j}^{(u)}}{m^{\frac{2}{3}} \log m}$ |
| 10 | 9 | 1.08 | 0.29 |
| 60 | 55 | 0.89 | 0.11 |
| 110 | 101 | 0.85 | 0.08 |
| 160 | 148 | 0.85 | 0.08 |
| 210 | 194 | 0.83 | 0.07 |
| 260 | 240 | 0.82 | 0.07 |
| 310 | 286 | 0.82 | 0.06 |

Table 1

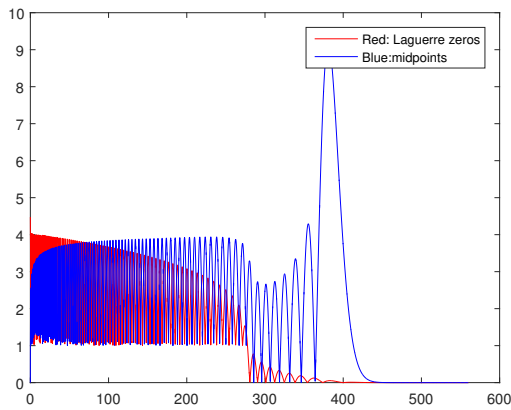


Figure 1: Graphs of the functions $\Lambda_{m,j}(w, x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, \theta = 0.7, \alpha = -0.5, \gamma = 0$, for $x \in [0, 600]$

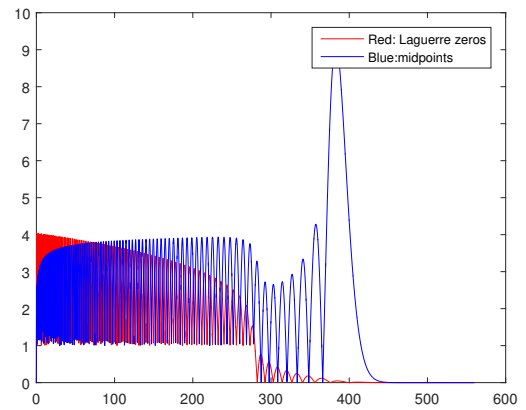


Figure 2: Graphs of the functions $\Lambda_{m,j}(w, x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, \theta = 0.7, \alpha = 0.5, \gamma = 0.5$ for $x \in [0, 600]$

In all the proposed graphs of the Lebesgue functions we have considered ranges such that the attained maxima values are included. Only the Figures 5 – 6 contain zooms of the complete ranges, in order to evidence the comparable behaviors of the Lebesgue functions $\tilde{\Lambda}_{m,j}(x)_u, \Lambda_{m,j}(w, x)_u$ in the ranges $[0, 4m\theta]$, for different choices of α, γ . As we can see by inspecting the plots in Figures 1 – 12, in all the cases the growths of the functions $\tilde{\Lambda}_{m,j}(x)_u, \Lambda_{m,j}(w, x)_u$ are comparable inside the truncation interval $[0, 4m\theta]$, for different admissible choices of the parameters α and γ and for different values of the truncation parameter θ . Moreover, for any m, j the function $\tilde{\Lambda}_{m,j}(x)_u$ takes the maximum outside the truncation range $[0, 4m\theta]$, as the Fig. 1-4 and 7-12 attest. This allow us to hypothesize that in the interval $[0, 4m\theta]$ it is possible to achieve results better than the estimate in (16) and therefore to prove that the new Lagrange process L_m^* at the midpoints can be optimal in some sense. Finally, by the numerical results given in the Tables 1 – 2, it seems that the estimate of the Lebesgue constants in (16) is too pessimistic, since the sequence $\tilde{\Lambda}_{m,j}^{(u)}$ diverges slower than $m^{\frac{2}{3}} \log m$. The conjecture is that in a restricted range of $\gamma - \frac{\alpha}{2}$ it should be possible to improve the estimate in (16). This argument will be elaborated in future research studies.

6 Numerical tests

In this section we propose some numerical tests obtained by using formula (17). Since the exact values of the integrals are not known, we will compare our numerical approximations with the corresponding ones obtained by using $m = 700$.

About the computation of the Hilbert transform of the Laguerre weight we recall [12, p.325, n. 16]

$$\int_0^{+\infty} \frac{e^{-x}}{x-t} dx = -e^{-t} Ei(t),$$

$$\int_0^{+\infty} \frac{e^{-x} x^\alpha}{x-t} dx = -\pi t^\alpha e^{-t} \cot((1+\alpha)\pi) + \Gamma(\alpha) {}_1F_1(1, 1-\alpha, -t), \quad \alpha \neq 0,$$

where Ei is the Exponential Integral function, ${}_1F_1$ is the Confluent Hypergeometric function and Γ is the complete gamma function. Before, we want to give just an idea of the percentage of interpolation knots involved in truncated processes, depending on the choice of the parameter θ .

Defined

$$N_m(a, b) = \text{Number of zeros of } q_{m-1} \text{ in } (a, b)$$

| $\alpha = -0.5, \gamma = 0.25, \theta = 0.7$ | | | |
|--|-----|---|--|
| m | j | $\frac{\Lambda_{m,j}^{(u)}(w)}{\log m}$ | $\frac{\tilde{\Lambda}_{m,j}^{(u)}}{m^{\frac{2}{3}} \log m}$ |
| 10 | 9 | 1.02 | 0.77 |
| 60 | 56 | 0.86 | 0.54 |
| 110 | 102 | 0.83 | 0.49 |
| 160 | 148 | 0.81 | 0.45 |
| 210 | 194 | 0.80 | 0.43 |
| 260 | 240 | 0.79 | 0.42 |
| 310 | 286 | 0.78 | 0.41 |

| $\alpha = 0.5, \gamma = 0.75, \theta = 0.7$ | | | |
|---|-----|---|--|
| m | j | $\frac{\Lambda_{m,j}^{(u)}(w)}{\log m}$ | $\frac{\tilde{\Lambda}_{m,j}^{(u)}}{m^{\frac{2}{3}} \log m}$ |
| 10 | 9 | 1.0 | 0.67 |
| 60 | 55 | 0.86 | 0.48 |
| 110 | 101 | 0.82 | 0.43 |
| 160 | 148 | 0.81 | 0.41 |
| 210 | 194 | 0.81 | 0.39 |
| 260 | 240 | 0.79 | 0.39 |
| 310 | 286 | 0.79 | 0.38 |

Table 2

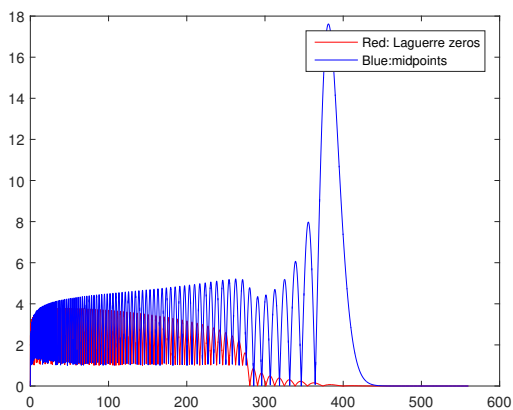


Figure 3: Graphs of the functions $\Lambda_{m,j}(w, x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, j = 92, \theta = 0.7, \alpha = -0.5, \gamma = 0.25$ for $x \in [0, 600]$

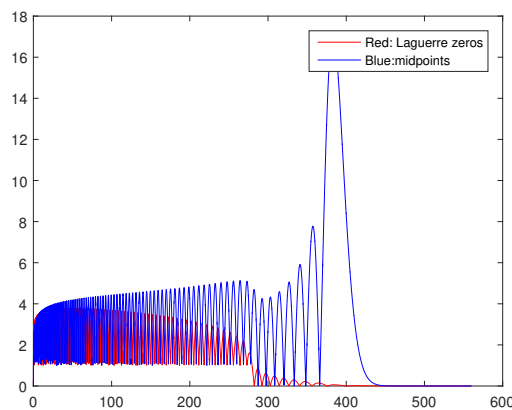


Figure 4: Graphs of the functions $\Lambda_{m,j}(w, x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, j = 92, \theta = 0.7, \alpha = 0.5, \gamma = 0.75$ for $x \in [0, 600]$

for any $\theta \in (0, 1)$ let

$$v_m(\theta) = N_m(0, 4m\theta)\%, \quad \tilde{v}_m(\theta) = \frac{1}{901} \sum_{m=100}^{1000} v_m(\theta).$$

Next Table 3 contains the value of $\tilde{v}_m(\theta)$ for two choices of α .

| $\alpha = -0.5$ | |
|-----------------|---------------------|
| θ | $\tilde{v}(\theta)$ |
| 0.1 | 39.62 |
| 0.2 | 55.00 |
| 0.3 | 66.06 |
| 0.4 | 74.82 |
| 0.5 | 81.88 |
| 0.6 | 87.66 |
| 0.7 | 92.29 |
| 0.8 | 96.00 |
| 0.9 | 98.68 |

| $\alpha = 0.5$ | |
|----------------|---------------------|
| θ | $\tilde{v}(\theta)$ |
| 0.1 | 39.5 |
| 0.2 | 55.00 |
| 0.3 | 66.10 |
| 0.4 | 74.74 |
| 0.5 | 81.83 |
| 0.6 | 87.62 |
| 0.7 | 92.26 |
| 0.8 | 96.00 |
| 0.9 | 98.66 |

Table 3: Mean percentages of interpolation points belonging to $[0, 4m\theta]$, for $\alpha = \pm \frac{1}{2}$.

Now, about the numerical experiments presented below, we point out that we have empirically detected the truncation intervals. More precisely, in each table the indexes ℓ and j defined in (9) and (7), respectively, have been chosen under the following criteria

$$\ell := \max_{1 \leq k \leq m} \ell_k, \quad \ell_k = \max_{i=1, \dots, m} \left\{ i : \lambda_i \left| \frac{f(x_i) - f(z_k)}{x_i - z_k} \right| \geq eps \right\}$$

and

$$j := \max_{k=1, \dots, m} \{k : |\mathcal{F}_m(fw, z_k)|u(z_k) \geq eps\}.$$

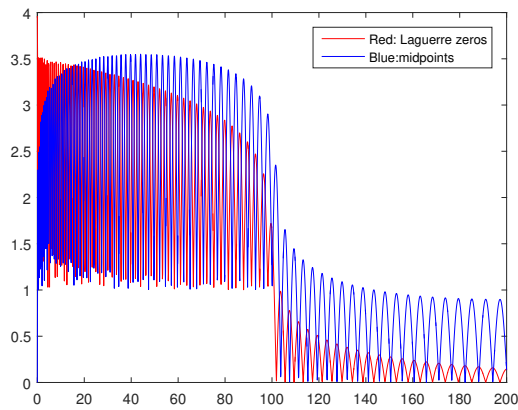


Figure 5: Graphs of the functions $\Lambda_{m,j}(w, x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, j = 61, \theta = 0.25, \alpha = -0.5, \gamma = 0$ for $x \in [0, 200]$

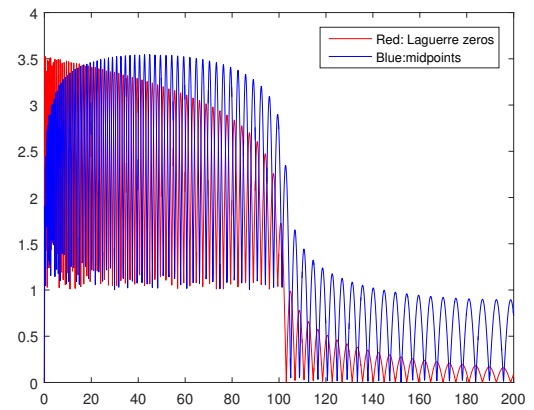


Figure 6: Graphs of the functions $\Lambda_{m,j}(w, x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, j = 61, \theta = 0.25, \alpha = 0.5, \gamma = 0.5$ for $x \in [0, 200]$

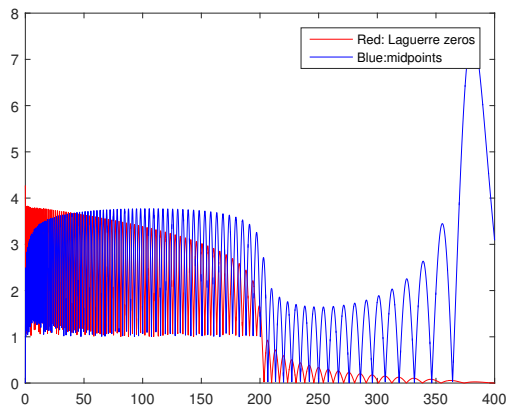


Figure 7: Graphs of the functions $\Lambda_{m,j}(w, x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, j = 82, \theta = 0.5, \alpha = -0.5, \gamma = 0$ for $x \in [0, 400]$

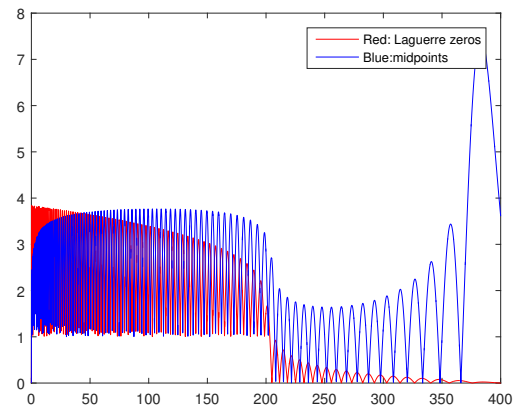


Figure 8: Graphs of the functions $\Lambda_{m,j}(w, x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, j = 82, \theta = 0.5, \alpha = 0.5, \gamma = 0.5$ for $x \in [0, 400]$

In any case, for m and j given, the corresponding value of θ "a posteriori" can easily be deduced.

For each test we produce also the plots of the functions $H_m(fw)$ realized for $m = 200$. Finally, we point out that all the computations have been performed in double-machine precision ($eps \approx 2.22044e - 16$).

Example 1

$$H(fw, t) = \int_0^{+\infty} \frac{f(x)}{x-t} e^{-x} \sqrt{x} dx, \quad f(x) = \frac{\sin x}{(x^2 + 5)^3}, \quad \alpha = 0.5, \quad \gamma = 0.5.$$

We compute $H(fw, t)$ for three values of t , namely $t = 0.2, 2, 10$. The function f is very smooth and we expect for a fast convergence. We remark that even if m takes "large" values, the actual number of samples of the integrand function f is substantially lower. For instance, with $m = 512$ for which 13 exact digits are taken, only 179 evaluations of f are required, for any value of t .

Example 2

$$H(fw, t) = \int_0^{+\infty} \frac{f(x)}{x-t} e^{-x} \sqrt[4]{x} dx, \quad f(x) = \sinh\left(\frac{x}{8}\right) |x-1|^{\frac{13}{2}}, \quad \alpha = 0.25, \quad \gamma = 0.5.$$

We compute $H(fw, t)$ for three values of t , namely $t = 0.1, 1, 10$. In this case the function $f \in Z_{6.5}(u)$, $\gamma = 0.5$ and therefore, according to (19), the error behaves like $(\sqrt{m})^{-3.83} \log m$. Thus, for $m = 512$ we can expect at most 6 exact digits, while 11 digits appear correct. We observe also a lower truncation than the previous test, depending on the exponential growth of f . Indeed almost twice of samples of f are required as compared with Example 1, where f decays to 0 at $x \rightarrow \infty$.

Example 3

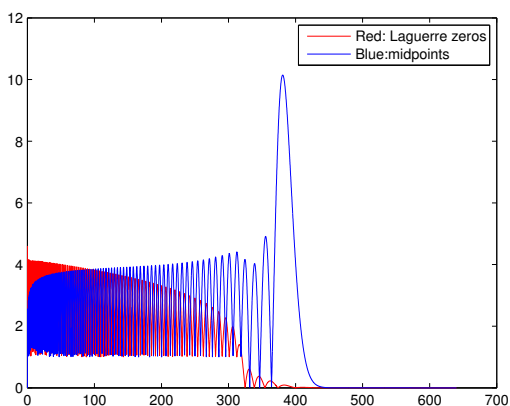


Figure 9: Graphs of the functions $\Lambda_{m,j}(w,x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, j = 96, \theta = 0.8, \alpha = -0.5, \gamma = 0$ for $x \in [0, 700]$

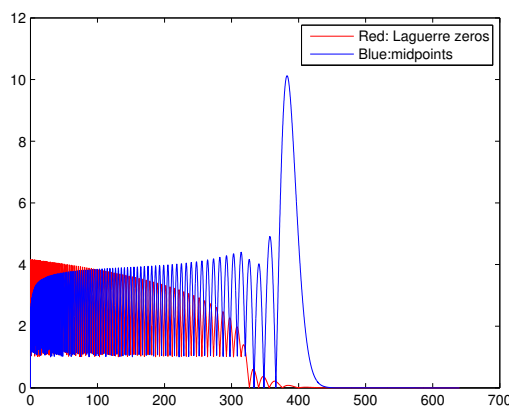


Figure 10: Graphs of the functions $\Lambda_{m,j}(w,x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 100, j = 96, \theta = 0.8, \alpha = 0.5, \gamma = 0.5$ for $x \in [0, 700]$

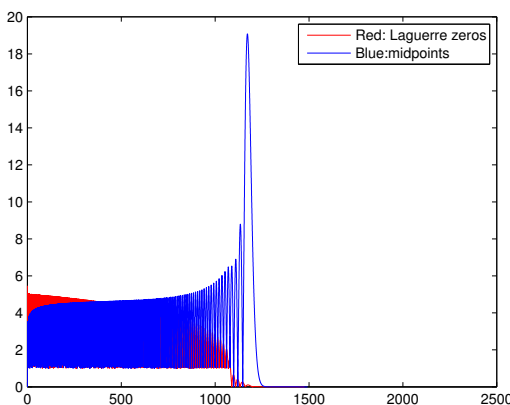


Figure 11: Graphs of the functions $\Lambda_{m,j}(w,x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 300, j = 296, \theta = 0.9, \alpha = -0.5, \gamma = 0$ for $x \in [0, 2500]$

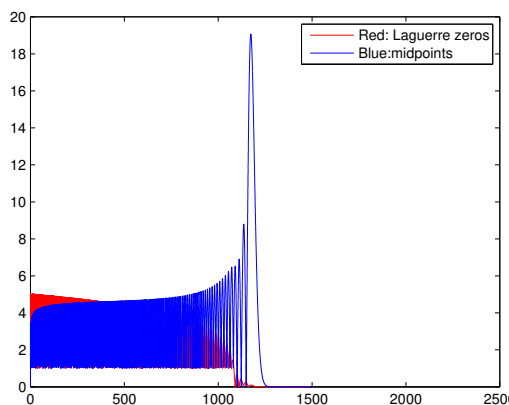


Figure 12: Graphs of the functions $\Lambda_{m,j}(w,x)_u, \tilde{\Lambda}_{m,j}(x)_u$ for $m = 300, j = 296, \theta = 0.9, \alpha = 0.5, \gamma = 0.5$ for $x \in [0, 2500]$

$$H(fw, t) = \int_0^{+\infty} \frac{f(x) e^{-x}}{x-t} \frac{1}{\sqrt[4]{x}} dx, \quad f(x) = \frac{\cos(\log(\pi+x))}{(e+x)^5}, \alpha = -0.25.$$

We compute $H(fw, t)$ for three values of t , namely $t = 0.1, 1, 10$. Since the function f is very smooth we expect for a fast convergence.

We remark that even if m takes "large" values, the actual number of samples of the integrand function f is drastically lower. For instance, with $m = 512$ for which 13 exact digits are taken, only 175 evaluations of f are required.

7 The proofs

Now we collect some polynomial inequalities proved in [7].

Let $x \in [x_1, x_m]$ and $d = d(x) \in \{1, \dots, m\}$ be an index of a zero of $p_m(w)$ closest to x . Then, for some positive constant $C \neq C(m, x, d)$, we have

$$\frac{1}{C} \left(\frac{x-x_d}{x_d-x_{d\pm 1}} \right)^2 \leq p_m^2(w, x) e^{-x} \left(x + \frac{1}{m} \right)^{\alpha+\frac{1}{2}} \sqrt{|4m-x|+m^{\frac{1}{3}}} \leq C \left(\frac{x-x_d}{x_d-x_{d\pm 1}} \right)^2. \tag{20}$$

and for a fixed real number $0 < \delta < 1$,

$$|p_m(w, x)| \sqrt{w(x)} \leq C \frac{1}{\sqrt[4]{x} \sqrt{|4m-x|+m^{\frac{1}{3}}}}, \quad \frac{C}{m} \leq x \leq 4m(1+\delta). \tag{21}$$

| m | j | ℓ | $H_m(fw, 0.2)$ | $H_m(fw, 2)$ | $H_m(fw, 10)$ |
|-----|-----|--------|---------------------|----------------------|----------------------|
| 16 | 16 | 13 | $4.e-3$ | $-2.1e-3$ | $-2.e-4$ |
| 32 | 24 | 19 | $4.12e-3$ | $-2.17e-3$ | $-2.1e-4$ |
| 64 | 36 | 27 | $4.1223e-3$ | $-2.1745e-3$ | $-2.1885e-4$ |
| 128 | 51 | 38 | $4.122234e-3$ | $-2.1745710e-3$ | $-2.18851e-4$ |
| 256 | 73 | 54 | $4.12223453576e-3$ | $-2.17457109172e-3$ | $-2.188510510e-4$ |
| 512 | 103 | 76 | $4.122234535761e-3$ | $-2.174571091722e-3$ | $-2.188510510010e-4$ |

Table 4: Example 1: $H_m(fw, t)$ with $t = 0.2, 2, 10$

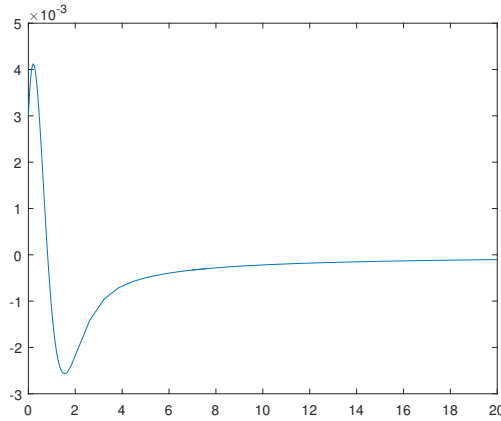


Figure 13: Example 1: $H_m(fw, x)$ for $m = 200, \alpha = 0.5, \gamma = 0.5$ and $t \in (0, 20]$

Moreover, for $k = 1, 2, \dots, m$

$$\frac{1}{|p'_m(w, x_k)|\sqrt{w(x_k)}} \sim \Delta x_k \sqrt[4]{x_k(4m - x_k)}, \quad \Delta x_k = x_{k+1} - x_k, \tag{22}$$

and

$$\Delta x_k \sim \sqrt{\frac{x_k}{4m - x_k}}, \tag{23}$$

uniformly in $m \in \mathbb{N}$.

In particular for $k \leq j$, with j given in (7), by (23), it follows

$$\Delta x_k = x_{k+1} - x_k \sim \sqrt{\frac{x_k}{m}}, \quad k = 1, 2, \dots, j. \tag{24}$$

Lemma 7.1. Let $\{x_k\}_{k=1}^m$ the zeros of $p_m(w)$ and denote with x_d a zero closest to x , $\Delta x_k = x_{k+1} - x_k$. Assuming $0 \leq \tau, \sigma \leq 1$, for $x \in (0, 4m)$ and for m sufficiently large, we have

$$\sum_{k=1, k \neq d}^m \frac{\Delta x_k}{|x - x_k|} \frac{(4m - x)^\tau}{(4m - x_k)^\tau} \frac{x^\sigma}{x_k^\sigma} \leq C \log m,$$

where $C \neq C(m, x)$.

Now we prove some Lemmas useful in the subsequent proofs.

Lemma 7.2. For $k \leq j$, where j has been defined in (7),

$$\frac{|p'_m(w, x_k)|}{|q'_{m-1}(z_k)|} \leq C(4m - x_k) \quad C \neq C(m).$$

Proof. Consider the ratio

$$\begin{aligned} \left| \frac{p'_m(w, x_k)}{q'_{m-1}(z_k)} \right| &= (x_m - x_k) \prod_{i=1}^{k-1} \frac{x_k - x_i}{x_k - x_i + \frac{\Delta x_k - \Delta x_i}{2}} \prod_{i=k+1}^{m-1} \frac{x_i - x_k}{x_i - x_k + \frac{\Delta x_i - \Delta x_k}{2}} \\ &\leq (4m - x_k), \end{aligned}$$

since $\Delta x_r - \Delta x_s \geq 0, \forall r > s$. □

| m | j | ℓ | $H_m(fw, 0.1)$ | $H_m(fw, 1)$ | $H_m(fw, 10)$ |
|-----|-----|--------|---------------------|---------------------|-----------------------|
| 16 | 16 | 16 | $1.8e + 2$ | $2.061e + 2$ | $-1.1e + 02$ |
| 32 | 32 | 30 | $1.8267e + 2$ | $2.06147e + 2$ | $-1.11978e + 02$ |
| 64 | 58 | 45 | $1.82677e + 2$ | $2.061477e + 2$ | $-1.119782e + 02$ |
| 128 | 89 | 66 | $1.8267753e + 2$ | $2.0614775e + 2$ | $-1.1197822e + 02$ |
| 256 | 129 | 94 | $1.82677536e + 2$ | $2.06147754e + 2$ | $-1.11978222e + 02$ |
| 512 | 185 | 133 | $1.8267753669e + 2$ | $2.0614775426e + 2$ | $-1.1197822201e + 02$ |

Table 5: Example 2: $H_m(fw, t)$ with $t = 0.1, 1, 10$

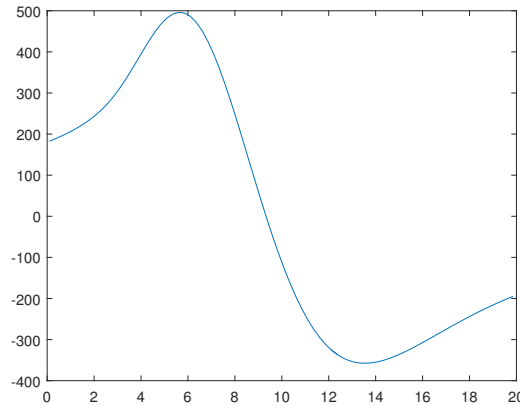


Figure 14: Example 2: $H_m(fw, x)$ for $m = 200, \alpha = 0.25, \gamma = 0.5$ and $t \in (0, 20]$

By Lemma 7.2, taking into account (22), it follows

$$\frac{1}{|q'_m(z_k)|u(x_k)} \leq C(4m - x_k)^{\frac{5}{4}} x_k^{\frac{\alpha}{2} - \gamma + \frac{1}{4}} \Delta x_k. \tag{25}$$

Lemma 7.3. Let be $\alpha > -1$ and $u(x) = e^{-x/2}x^\gamma$. Then for any $x \in \zeta_m = (\frac{c}{m}, 4m - Cm^{\frac{1}{3}})$ it is

$$|q_{m-1}(x)|u(x) \leq \frac{C}{m^{\frac{1}{3}}} \frac{x^{\gamma - \frac{\alpha}{2} - \frac{1}{4}}}{\sqrt[4]{4m - x}}, \tag{26}$$

$0 < C \neq (m, x)$.

Proof. Assume at first $x \neq x_k, k = 1, 2, \dots, m$. Denote by d the index of a zero closest to x , and let, for instance, $x_{d-1} < x < x_d$ and consider at first $x \geq \frac{m}{2}$. By

$$\frac{|q_{m-1}(x)|}{|p_m(w, x)|} = \frac{1}{x - x_1} \prod_{i=1}^{d-3} \frac{x - z_i}{x - x_{i+1}} \prod_{i=d-2}^d \frac{|x - z_i|}{|x - x_{i+1}|} \prod_{i=d+1}^{m-1} \frac{z_i - x}{x_{i+1} - x}$$

since for $i \geq d$ it is $z_i - x < x_{i+1} - x$, and taking into account $x_1 \sim \frac{1}{m}$ and the assumption $x \geq \frac{m}{2}$, we have

$$\frac{|q_{m-1}(x)|}{|p_m(w, x)|} \leq \frac{C}{m} \prod_{i=1}^{d-3} \frac{x - z_i}{x - x_{i+1}} \left(1 + \frac{\Delta x_{d-2}}{x - x_{d-1}}\right) \left(\frac{x_d - z_{d-1}}{x_d - x}\right).$$

Since $\Delta x_{d-2} \sim \Delta x_{d-1} \sim x - x_{d-1}$ and $\frac{x_d - z_{d-1}}{x_d - x} \sim 1$

$$\frac{|q_{m-1}(x)|}{|p_m(w, x)|} \leq C \frac{C}{m} \prod_{i=1}^{d-3} \left(1 + \frac{\Delta x_i}{2(x - x_{i+1})}\right) \leq \frac{C}{m} e^S \tag{27}$$

where

$$\begin{aligned} S &= \sum_{i=1}^{d-3} \frac{\Delta x_i}{2(x - x_{i+1})} \leq \frac{1}{2} \sum_{i=1}^{d-3} \frac{\int_{x_i}^{x_{i+1}} dt}{x - x_i - \Delta x_i} \leq \frac{1}{2} \sum_{i=1}^{d-3} \int_{x_i}^{x_{i+1}} \frac{dt}{x - t - \Delta x_i} \\ &\leq \frac{1}{2} \sum_{i=1}^{d-3} \log \left(\frac{x - x_i - \Delta x_i}{x - x_{i+1} - \Delta x_i} \right) \leq \frac{1}{2} \log \left(\frac{x - x_1 - \Delta x_1}{x - x_{d-2} - \Delta x_{d-3}} \right) \leq \frac{1}{2} \log(Cm) \end{aligned}$$

| m | j | ℓ | $H_m(fw, 0.1)$ | $H_m(fw, 1)$ | $H_m(fw, 10)$ |
|-----|-----|--------|----------------------|----------------------|----------------------|
| 16 | 16 | 13 | $-3.2e-3$ | $-1.8e-3$ | $-1.e-4$ |
| 32 | 24 | 19 | $-3.227e-3$ | $-1.82e-3$ | $-1.34e-4$ |
| 64 | 36 | 26 | $-3.22783e-3$ | $-1.821903e-3$ | $-1.3446e-4$ |
| 128 | 51 | 37 | $-3.2278395029e-3$ | $-1.8219038131e-3$ | $-1.34460808e-4$ |
| 256 | 73 | 51 | $-3.227839502975e-3$ | $-1.82190381314e-3$ | $-1.34460808900e-4$ |
| 512 | 103 | 72 | $-3.227839502975e-3$ | $-1.821903813148e-3$ | $-1.344608089008e-4$ |

Table 6: Example 3: $H_m(fw, t)$ with $t = 0.1, 1, 10$

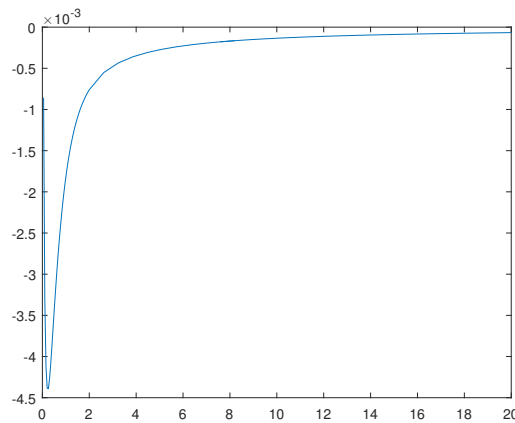


Figure 15: Example 3: $H_m(fw, x)$ for $m = 200, \alpha = -0.25, \gamma = 0$ and $t \in (0, 20]$

and by (27) it follows

$$\frac{|q_{m-1}(x)|}{|p_m(w, x)|} \leq \frac{C}{\sqrt{m}}. \tag{28}$$

Assume now $\frac{1}{2} \leq x \leq \frac{m}{2}$. By similar arguments to those used in the previous case, we have

$$\frac{|q_{m-1}(x)|}{|p_m(w, x)|} = \frac{1}{x_m - x} \prod_{i=1}^{d-1} \frac{x - z_i}{x - x_i} \frac{|x - z_d|}{|x - x_d|} \prod_{i=d+1}^{m-1} \frac{z_i - x}{x_i - x} \leq \frac{C}{m} \prod_{i=d+1}^{m-1} \frac{z_i - x}{x_i - x}$$

and also

$$\frac{|q_{m-1}(x)|}{|p_m(w, x)|} \leq C \frac{C}{m} \prod_{i=d+1}^{m-1} \left(1 + \frac{\Delta x_i}{2(x_i - x)} \right) \leq \frac{C}{m} e^S, \tag{29}$$

where

$$S = \sum_{i=d+1}^{m-1} \frac{\Delta x_i}{2(x_i - x)} \leq \frac{1}{2} \left(\sum_{i=d+1}^{j-1} \int_{x_i}^{x_{i+1}} \frac{dt}{t - x - a} + \sum_{i=j}^{m-1} \int_{x_i}^{x_{i+1}} \frac{dt}{t - x - b} \right), \quad a = \max \Delta x_i < C, \quad i \leq j, \quad b \leq Cm^{\frac{1}{3}}.$$

Therefore

$$S \leq \frac{1}{2} \left(\int_{x_{d+1}}^{x_j} \frac{dt}{t - x - a} + \int_{x_j}^{x_m} \frac{dt}{t - x - b} \right) \leq \frac{1}{2} \left(\log \frac{x_j - x - a}{x_{d+1} - x - a} + \log \frac{x_m - x - b}{x_j - x - b} \right) \leq \frac{1}{2} \log(Cm)$$

and by (29) it follows

$$\frac{|q_{m-1}(x)|}{|p_m(w, x)|} \leq \frac{C}{\sqrt{m}}.$$

By (21), we have for $x \in \zeta_m \setminus \{x_1, \dots, x_m\}$

$$|q_{m-1}(x)|u(x) \leq \frac{C}{\sqrt{m}} \frac{x^{\gamma - \frac{\alpha}{2} - \frac{1}{4}}}{\sqrt[4]{4m - x}}.$$

For $x = x_k, k = 1, 2, \dots, j$, with j defined in (7), by following arguments similar to those used in the proof for $x \neq x_k$, it is no hard to prove

$$\frac{|q_{m-1}(x_k)|}{|p_m(w, z_k)|} \leq \frac{C}{\sqrt{m}}, \quad k \leq j \tag{30}$$

and therefore, by (20)

$$|q_{m-1}(x_k)|u(x_k) \leq \frac{C}{\sqrt{m}} \frac{z_k^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}}{\sqrt[4]{4m-z_k}}.$$

Finally, for $x = x_k$, $k > j$,

$$|q_{m-1}(x_k)| = \gamma_m(w) \frac{\Delta x_k}{2} \prod_{i=1}^{k-1} (x_k - z_i) \prod_{i=k+1}^{m-1} (z_i - x_k) \leq \gamma_m(w) \frac{\Delta x_k}{2} \prod_{i=1}^{k-1} (x_k - x_i) \prod_{i=k+1}^{m-1} (x_{i+1} - x_k) = \frac{1}{2} |p'_m(w, x_k)|$$

and therefore by (22), taking into account $\Delta x_m \sim m^{\frac{1}{3}}$

$$|q_{m-1}(x_k)|u(x_k) \leq \frac{C}{\Delta x_k} \frac{x_k^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}}{\sqrt[4]{4m-x_k}} \leq \frac{C}{m^{\frac{1}{3}}} \frac{x_k^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}}{\sqrt[4]{4m-x_k}}$$

and the Lemma is completely proved. □

Lemma 7.4. For $x \in (z_1, z_j)$, j defined in (7) and denoted by z_d a zero of q_{m-1} closest to x , we have

$$\frac{|q_{m-1}(x)|}{|q'_{m-1}(z_d)(x-z_d)|} \frac{u(x)}{u(z_d)} \leq \frac{C}{\sqrt{m}} (4m-x), \quad C \neq C(m, x).$$

Proof. For $x \neq x_d$, by (28) and (20) we have

$$\frac{|q_{m-1}(x)|u(x)}{|x-x_d|} \leq \frac{C}{\sqrt{m}} \frac{x^{\gamma-\frac{\alpha}{2}-\frac{1}{4}}}{\sqrt[4]{4m-x}} \Delta x_d.$$

By taking into account (25), $w(x) \sim w(z_d)$ and that by (24) $\Delta x_d \leq C$, $d \leq j$ we get

$$\frac{|q_{m-1}(x)|}{|q'_{m-1}(z_d)(x-z_d)|} \frac{u(x)}{u(z_d)} \leq \frac{C}{\sqrt{m}} (4m-x) \left(\frac{4m-x_d}{4m-x}\right)^{\frac{1}{4}} \left(\frac{x}{x_d}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}} (\Delta x_d)^2 \leq \frac{C}{\sqrt{m}} (4m-x).$$

Finally, the case $x = x_d$ follows from (26), (25) and (30). □

To prove the next two Theorems, we recall that for any polynomial $P \in \mathbf{P}_m$ the following Remez-type inequality holds [10, Theorem 2.1]

$$\max_{x \geq 0} |P_m(x)u(x)| \leq C \max_{x \in \zeta_m} |P_m(x)u(x)|, \quad \zeta_m = \left(\frac{C}{m}, 4m - Cm^{\frac{1}{3}}\right) \tag{31}$$

where $C \neq C(m)$.

Proof of Theorem 3.1. To prove (16), by (31) and denoting by d the index of an interpolation knot closest to x , we have

$$\begin{aligned} \|L_{m+1}^*(f)u\|_\infty &\leq \max_{x \in \zeta_m} |L_{m+1}^*(f, x)|u(x) \leq \|f u\|_\infty \max_{x \in \zeta_m} \sum_{k=1}^j \frac{|q_{m-1}(x)|(4m-x)u(x)}{|q'_{m-1}(z_k)|(4m-z_k)u(z_k)|x-z_k|} \\ &= \|f u\|_\infty \max_{x \in \zeta_m} \left[\sum_{d \neq k=1}^j \frac{|q_{m-1}(x)|(4m-x)u(x)}{|q'_{m-1}(z_k)|(4m-z_k)u(z_k)|x-z_k|} + \frac{|q_{m-1}(x)|u(x)(4m-x)}{|q'_{m-1}(z_d)|(4m-z_d)u(z_d)|x-z_d|} \right] \\ &=: \|f u\|_\infty \max_{x \in \zeta_m} [\Sigma_2(x) + C_d(x)]. \end{aligned} \tag{32}$$

For $x \in \zeta_m$ and from (26) and (25) it follows

$$\Sigma_2(x) \leq \frac{C}{m^{\frac{1}{3}}} \sum_{d \neq k=1}^j (4m-x)^{\frac{3}{4}} (4m-z_k)^{\frac{1}{4}} \left(\frac{x}{x_k}\right)^{\gamma-\frac{\alpha}{2}-\frac{1}{4}} \frac{e^{\Delta x_k} \Delta z_k}{|x-z_k|}.$$

Moreover, recalling $\Delta x_k \sim \frac{\sqrt{x_k}}{\sqrt{m}}$, $k \leq j$, and by Lemma 7.1, under the assumption $0 \leq \gamma - \frac{\alpha}{2} - \frac{1}{4} \leq 1$,

$$\Sigma_2(x) \leq Cm^{\frac{2}{3}} \log m. \tag{33}$$

Finally, from Lemma 7.4 it easily follows

$$C_d(x) \leq Cm^{\frac{2}{3}}. \tag{34}$$

Theorem 3.1 is then proved by combining (33) and (34) with (32). □

To prove Theorem 4.1 we need the following result proved in [11].

Lemma 7.5. [11] For any $f \in Z_{\lambda+1}(u)$, under the assumption $0 \leq \gamma < \alpha + \frac{1}{4}$,

$$E_m(\mathcal{F}(fw))_u \leq \frac{C}{(\sqrt{m})^\lambda} \|f\|_{Z_{\lambda+1}(u)}, \tag{35}$$

$$E_m(\mathcal{F}_m(fw))_u \leq C \frac{C}{(\sqrt{m})^\lambda} \|f\|_{Z_{\lambda+1}(u)}, \tag{36}$$

where C is independent on f, m .

Proof of Theorem 4.1. Let $P_{m-1} \in \mathcal{P}_{m-1}^*$.

$$|\rho_m(fw, t)|u(t) = |H(fw; t) - H_m(fw; t)|u(t) \leq |\mathcal{F}(fw; t) - P_m(t)|u(t) + |L_m^*(\mathcal{F}_m(fw) - P_m; t)|u(t).$$

Under the assumption (18), which implies (15),

$$|\rho_m(fw; t)|u(t) \leq C|\mathcal{F}(fw; t) - P_{m-1}(t)|u(t) + |\mathcal{F}_m(fw; t) - P_{m-1}(t)|u(t)m^{\frac{2}{3}} \log m$$

and taking the infimum on \mathcal{P}_m^* and by (14) we get

$$|\rho_m(fw; t)|u(t) \leq C \left\{ E_M(\mathcal{F}(fw))_u + E_M(\mathcal{F}_m(fw))_u m^{\frac{2}{3}} \log m \right\},$$

where $M = \lceil m \left(\frac{\theta}{1+\theta} \right) \rceil \sim m$. Finally, from (35) and (36) the thesis follows. □

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