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On the Numerical Approximation of Some Non-standard Volterra Integral Equations

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Abstract

In this paper, the numerical solution of non-standard and nonlinear Volterra integral equations is studied and computationally efficient schemes based on quadrature methods are presented. The numerical methods are of Runge-Kutta and barycentric rational quadrature types. A convergence analysis of the barycentric rational quadrature method is discussed and for the Runge-Kutta method, we analyze numerically its convergence properties. The option pricing application of the proposed equation is discussed and finally some open problems in this field are given.

1 Introduction

During the recent decades, integral equations of Volterra type have been employed as important modeling tools in many branches of science and engineering [7, 20]. In some application areas such as financial mathematics, we are usually led to consider Volterra integral equations of the general form

$$u(t) = g(t, u(t)) + \int_a^t k(t, s, u(t), u(s))\mathrm{d}s, \qquad t \in [a, b], \tag{1}$$

which have a significant role in finding the early exercise boundary of an American option contract (see e.g. [24] and also Section 6 of this paper). In the above equation, $g(t, u)$ and $k(t, s, u, v)$ are the forcing and kernel functions respectively and $u(t)$ is an unknown function to be determined.

Based on the fact that these equations could not be included in the classical categorization of integral equations (due to the appearance of $u(t)$ and $u(s)$ terms in the kernel), we adopt the terminology introduced by Brunner in [7] and call them "non-standard" Volterra integral equations. A subclass of Eq. (1) could be considered as

$$u(t) = g(t) + \int_a^t k(t, s, u(t), u(s))\mathrm{d}s, \qquad t \in [a, b], \tag{2}$$

which has some applications in the theory of radiative transfer and astrophysics [8] and also population growth dynamics [15].

The numerical investigation of Eq. (2) has attracted the attention of some researchers in the field: Cahlon in [8] considered a direct quadrature method of higher order to numerically solve Eq. (2) both with smooth and weakly singular kernels. Recently, the convergence and superconvergence properties of the piecewise polynomial collocation scheme applied to Eq. (2) are also considered in [15].

Although, there exist some studies on the regularity properties of the solution to (2) and its numerical approximation, there is still a lack of attention to deal with Eq. (1), both theoretically and numerically. In this paper, we propose two numerical schemes, one based on a higher order quadrature method using barycentric rational interpolation and the other based on Pouzet-type Runge-Kutta theory for both smooth and weakly singular non-standard Volterra integral equations.

With a rich history in computer-aided geometric design and data visualization, barycentric interpolation has found its way into the scientific computing literature with a variety of applications in approximation and interpolation disciplines [14]. Barycentric Lagrange interpolation and the extended version of it, called the barycentric rational interpolation are practical forms of the classical Lagrange polynomials and portray the simplicity of the method in a stable and efficient manner [30]. This method suits for the case of equations with a smooth solution and so it will be employed in this paper when the forcing and kernel are smooth enough functions.

Up to now, there have been some efforts to numerically solve Volterra integral equations via the barycentric quadrature methods [3]. Among the advantages of this numerical approach, we could mention its stability, high polynomial order of convergence and ease of implementation [13].

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On the other hand, we will also present an s-stage Pouzet type Runge-Kutta time marching method for the weakly singular case and report the numerical simulations showing a better performance in comparison with the barycentric rational quadrature method. The presented method could be considered as an extension of the approach developed in [6, 19] for the numerical solution of nonlinear integral equations.

The paper is structured as follows. Sections 3 and 4 are devoted respectively to the barycentric rational quadrature and its implementation to find an approximant for Eq. (1) as well as a complete analysis of the method. In Section 5, we introduce the Pouzet-Runge-Kutta family of methods and apply them to Eq. (1) based on a quasi-uniform mesh distribution. In Section 6, an interesting application of the introduced Volterra integral equation in the American option pricing problem is discussed. In the last section, we report on some numerical experiments illustrating the efficiency of the proposed schemes, compared to the PSOR method applied to the complementarity formulation of the problem. We conclude the paper by pointing out to some possible future research directions.

2 Existence and Uniqueness Issue

The existence and uniqueness issues for Eq.(2) under some smoothness conditions on $k(t,s,u,v)$ and $g(t)$ have been studied in [8, 15] based on contractive mapping fixed point theorem for the Hölder space, $C^\alpha([a,b])$ for $0 \le \alpha < 1$. It should be noticed that $C^\alpha([a,b])$ for $0 \le \alpha < 1$ is a Banach space with the norm

$$\|f\|_{C^\alpha} = \sup_{x\in[a,b]} |f(x)| + \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

For $\alpha \ge 1$, we could write $\alpha = k+\sigma$, $(k \in \mathbb{N},\ 0 \le \sigma < 1)$ and $C^\alpha([a,b])$ could be equipped with the norm

$$\|f\|_{C^\alpha} = \|f\|_{C^k} + \sup_{x,y\in[a,b]} \frac{|f^{(k)}(x)-f^{(k)}(y)|}{|x-y|^\sigma}.$$

Definition 2.1. The function $k(t,s,u,v)$ is said to satisfy condition A on

$$\Omega = \{(t,s,u,v) \mid a \le s,t \le b, u,v \in \mathbb{R}\},$$

if we have

$$|k(t,s,u_1,v_1)-k(t,s,u_2,v_2)| \le L_1(t,s)|u_1-u_2| + L_2(t,s)|v_1-v_2|, \qquad \forall (t,s,u_i,v_i) \in \Omega, \quad i=1,2, \tag{3}$$

where $0 < L_1(t,s) < M_1 < \infty$ and $0 < L_2(t,s) < M_2 < \infty$ and $\int_a^t L_1(t,s)ds \le 1-\beta$ for some $\beta \in (0,1)$.

We also assume that $g(t,u)$ satisfies the Lipschitz condition

$$|g(t,u_1)-g(t,u_2)| \le M_3|u_1-u_2|, \quad t\in[0,b],\ u_i \in \mathbb{R},\ i=1,2. \tag{4}$$

Now, we are ready to state one of the main results of this section in the following theorem.

Theorem 2.1. *Assume that $k(t,s,u,v) \in C(\Omega)$ and $g(t,u) \in C([a,b]\times\mathbb{R})$ satisfy in Condition A stated in Definition 2.1 and Lipschitz condition (4), respectively. Furthermore, let $M = \max\{M_2, M_3\}$ be such that $M < \beta < 1$. Then there exists a unique solution $u \in C([a,b])$ for Eq. (2).*

Proof. The proof is a straightforward extension of Theorem 2.1.10 in [7] and Theorem 2.3 in [15]. □

3 Barycentric Interpolation and Quadrature

In order to make the paper self-contained, this section is devoted to a brief review of the barycentric form of rational interpolation and the corresponding quadrature formulae. Let

$$\mathbf{X} = \{t_0 = a, t_1, \dots, t_{n-1}, t_n = b\}, \tag{5}$$

be a set of strictly ordered equidistant nodes in $[a,b]$ with the mesh spacing $h_i = t_i - t_{i-1}$, $i = 1,2,\cdots,n$. In the reminder, we first introduce the barycentric interpolation procedure and then the resulting quadrature scheme.

3.1 Barycentric Rational Interpolation

The barycentric form of the Lagrange interpolation for the data $\{t_i, f_i\}_{i=0}^n$ is given by

$$(\mathcal{P}_n f)(t) = \frac{\sum_{i=0}^n \frac{\beta_i}{t-t_i} f_i}{\sum_{i=0}^n \frac{\beta_i}{t-t_i}} = \sum_{i=0}^n \mathcal{L}_i(t) f_i, \tag{6}$$

in which $\mathcal{L}_i(t) = \frac{\frac{\beta_i}{t-t_i}}{\sum_{i=0}^n \frac{\beta_i}{t-t_i}}$, $i = 0,1,\cdots,n$ and the weights β_i are defined by

$$\beta_i = \frac{1}{\Pi_{i\neq j}(x_i - x_j)}, \quad i = 0,1,\cdots,n. \tag{7}$$

The idea of using barycentric interpolation is to utilize the stability properties and computational advantages of the barycentric form which is investigated in detail in [18, 23] (see also [2] for a comprehensive review of this interpolation procedure).

The investigations in this area to obtain some weights which will produce interpolants, $\mathcal{P}_n f$, with no poles and good approximation properties have led to the family of barycentric rational interpolations introduced in [12]. In the following, we give a brief account of the barycentric rational interpolation and quadrature. For a fixed integer $0 \leq d \leq n$, the polynomials $\{p_i(t)\}_{i=0}^{n-d}$ interpolate f at the nodes $\{t_i, \dots, t_{i+d}\}$. Let

$$(\mathcal{P}_n f)(t) = \frac{\sum_{i=0}^{n-d} \lambda_i(t) p_i(t)}{\sum_{i=0}^{n-d} \lambda_i(t)}, \tag{8}$$

where

$$\lambda_i(t) = \frac{(-1)^i}{(x - x_i) \dots (x - x_{i+d})}.$$

Eq. (8) can be rewritten in the barycentric form (6) with the weights

$$\beta_i = (-1)^{i-d} \sum_{j \in J_i} \binom{d}{i-j}, \tag{9}$$

where J_i is defined as

$$J_i = \{\max(1, i-d) \leq j \leq \min(i, n-d-1)\}.$$

It has been shown that the β_i, $i = 0, \dots, n$ in (7) are not the only possible choice for the weights. Another option for the β_i is the Berrut's weights given by

$$\beta_i = (-1)^i, \quad i = 0, 1, \dots, n, \tag{10}$$

where n is odd and the (6) with the above coefficients has no poles [1]. Indeed, it has been proven that the relation

$$\sum_{i=0}^{n} \beta_i = 0, \tag{11}$$

guarantees Eq. (6) to be a barycentric rational interpolation [13, Proposition 3].

Rational barycentric interpolation with the weights (9) has a superior advantage compared to the other forms of the barycentric interpolation and it attains an order of convergence

$$\|f - \mathcal{P}_n f\|_\infty = \mathcal{O}(h^{d+1}),$$

for a given function $f \in C^{d+2}[a, b]$. This is an improvement with respect to the order $\mathcal{O}(\frac{1}{n})$ as $n \to \infty$ obtainable using the Berrut's weights [2, 12].

3.2 Barycentric Rational Quadrature

Barycentric quadrature and its features have been studied extensively in [22, 30]. The linear interpolant

$$(\mathcal{P}_n f)(t) = \sum_{i=0}^{n} f_i \mathcal{L}_i(t)$$

naturally leads to the following classical quadrature formula

$$Q_n[f] = \sum_{i=0}^{n} \omega_{i,n} f(t_i), \tag{12}$$

where the corresponding quadrature weights, $\omega_{i,n}$, $i = 0, \dots, n$ are defined by

$$\omega_{i,n} = \int_a^b \mathcal{L}_i(t) \mathrm{d}t. \tag{13}$$

The stability condition of the quadrature method is given by (see [17])

$$\sup \Big\{ \sum_{i=0}^{n} |\omega_{i,n}|, n \in \mathbb{N} \Big\} < \infty.$$

It follows from (13) that

$$\sum_{i=0}^{n} |\omega_{i,n}| \leq \int_a^b \sum_{i=0}^{n} \left| \frac{\frac{\beta_i}{t-t_i}}{\sum_{i=0}^{n} \frac{\beta_i}{t-t_i}} \right| \mathrm{d}t = \int_a^b \Lambda_n(t) \mathrm{d}t, \tag{14}$$

where the function

$$\Lambda_n(t) = \sum_{i=0}^{n} |\mathcal{L}_i(t)|,$$

is called the *Lebesgue function* and its supremum over the interval $[a,b]$, given by

$$\Lambda_n = \sup_{t\in[a,b]} \Lambda_n(t),$$

is the *Lebesgue constant* [30]. By this relation, the following upper bound could be obtained for (14)

$$\sum_{i=0}^{n} |\omega_{i,n}| \le (b-a)\Lambda_n, \tag{15}$$

so the stability of the direct quadrature method depends on the stability of the interpolation process.

It is proved in [4] that for equidistant nodes, the barycentric rational interpolation offers much better approximation order than the Lagrange polynomial, because of the behavior of its Lebesgue constant Λ_n. In practice, the weights (13) should be computed numerically by an efficient quadrature rule with rapid convergence, such as Gauss-Legendre or Clenshaw-Curtis method.

Theorem 3.1. *([22, Theorem 4.1]) Suppose n and d with $d \le n$ are positive integers, $f \in C^{d+2}([a,b])$ and $\mathcal{P}_n f$ is the rational interpolant with parameter d given in (8). Let the quadrature weights (9) be approximated by a quadrature rule which convergence at least at the rate $\mathcal{O}(h^{d+1})$ and degree of precision at least $d+1$. Then*

$$\left| \int_a^b f(t)\mathrm{d}t - \sum_{i=0}^{n} \omega_{i,n} f_i \right| \le C h^{d+1}, \tag{16}$$

where C is a constant depending on d, derivatives of f and the length of the interval.

In the following the concept of consistency for quadrature rule is given.

Definition 3.1. ([16]) A quadrature rule $\{Q_n\}_{n\in\mathbb{N}}$ in the interval $[a,b]$ is called consistent of order $\lambda > 0$, if the quadrature error,

$$R_n(f) = \int_a^b f(x)\mathrm{d}x - Q_n(f),$$

satisfies the estimate

$$|R_n(f)| \le C(b-a)^{1+\lambda} n^{-\lambda} \|f\|_{C^\lambda(I)}, \quad \forall f \in C^\lambda([a,b]), \tag{17}$$

with $C > 0$.

Remark 1. Let n and d be two non-negative integers with $d \le n$ and also assume that $f \in C^{d+2}([a,b])$. Then the barycentric rational quadrature method has the consistency order $d+1$ (see [22] for more details).

4 Discretization by the Direct Quadrature Method

The idea is to approximate the integral $\int_a^t k(t,s,u(t),u(s))\mathrm{d}s$ in the fully nonlinear Volterra equation (1) by barycentric rational quadrature method. The function to be integrated is $\psi(s) = k(t,s,u(t),u(s))$ or in short notation $\psi = k(t,.,u(t),u(.))$. Replacing the integral by quadrature gives

$$\tilde{u}(t) = g(t,\tilde{u}(t)) + Q_{[a,t]}(k(t,.,\tilde{u}(t),\tilde{u}(.))), \quad t \in [a,b],$$

where $Q_{[a,t]}$ is the quadrature formula for the interval $[a,t]$ and the function $\tilde{u}(t)$ is approximation of $u(t)$. The quadrature formula (12) on the support abscissae (5) for Eq. (1) yields the following system of equations

$$u_i = g(t_i,u_i) + Q_{[a,t_i]}(k(t_i,.,u_i,u(.)), \quad i = 0,1,\cdots,n, \tag{18}$$

in which $Q_{[a,t_i]} = Q_i$ is defined in (12) and $u_i = \tilde{u}(t_i)$, so Eq. (18) could be written as

$$u_i = g(t_i,u_i) + \sum_{l=0}^{i} \omega_{l,i} k(t_i,t_l,u_i,u_l), \quad i = 0,1,\cdots,n. \tag{19}$$

4.1 Analysis of the Quadrature Method

Before presenting an error estimate for the proposed barycentric quadrature method, we need the following definition.

We now present some estimates on the local discretization error, $e_i = u_i - u(t_i)$ in the sequel. In this respect, we conclude from the integral equation and the discretized form (19) that

$$\begin{aligned} e_i &= \Big[g(t_i,u_i) + \sum_{l=0}^{i} \omega_{l,i} k(t_i,t_l,u_i,u_l) \Big] - \Big[g(t_i,u(t_i)) + \int_0^{t_i} k(t_i,s,u(t_i),u(s))\mathrm{d}s \Big] \\ &= \Big[g(t_i,u_i) - g(t_i,u(t_i)) \Big] + \sum_{l=0}^{i} \omega_{l,i} \Big[k(t_i,t_l,u_i,u_l) - k(t_i,t_l,u(t_i),u(t_l)) \Big] + R(q_i), \end{aligned} \tag{20}$$

where $R(q_i)$ is the quadrature error of the integrand, $k(t_i,s,u(t_i),u(s))$. In the following, an estimation for the upper bound is drived.

Theorem 4.1. *Let $k(t,s,u,v) \in C^{d+2}(\Omega)$ be such that it fulfils the condition A geven in Definition 2.1. Furthermore, let $g(t,u)$ satisfy the relation (4). Let $M = \max\{M_1, M_2, M_3\}$, where M_1 and M_2 are defined in condition A and M_3 given by (4) be such that $M < \frac{1}{1+2(b-a)C_1}$. Then there exist appropriate constants C_1^* and C_2^* such that*

$$|e_i| \le C_1^* \exp(iC_2^* h)h^{d+1}.$$

Proof. It is seen from Definition 3.1 and Theorem 3.1 that the barycentric rational quadrature scheme has the consistency order $d+1$ and we also have

$$|R(q_i)| \le C|t_i - a|h^{d+1}. \tag{21}$$

In the above inequality, the constant C will depend on the smoothness of the kernel $k(t,s,u,v)$. Also, it is proved in [3] that the barycentric rational quadrature weights given by (9) are approximately (equally) distributed in the following sense [16]:

$$|\omega_{l,i}| \le C_1 h, \quad \text{for all } 0 \le l \le i. \tag{22}$$

So by the triangle inequality, we have

$$|e_i| \le M_3|e_i| + \sum_{l=1}^{i} |\omega_{l,i}|(M_1|e_i| + M_2|e_l|) + |R(q_i)|, \tag{23}$$

which using (22) and the fact that the nodes are equidistant, we obtain

$$|e_i| \le (M_3 + M_1 C_1(b-a) + M_2 C_1 h)|e_i| + M_2 \sum_{l=0}^{i-1} |\omega_{l,i}||e_l| + |R(q_i)|. \tag{24}$$

Fixing the notation, $M = M_3 + M_1 C_1(b-a) + M_2 C_1 h$, $M^* = (b-a)Ch^{d+1}$ and supposing that $M < 1$, then the above inequality with (21) yields

$$|e_i| \le M|e_i| + M_2 \sum_{l=0}^{i-1} |\omega_{l,i}||e_l| + M^*.$$

Now using the Gronwall Lemma [16, Lemma 2.2.7], we obtain

$$|e_i| \le \frac{M^*}{1-M} \exp\Big(\sum_{l=0}^{i-1} \frac{M_2 C_1 h}{1-M}\Big),$$

which finally leads to

$$|e_i| \le \frac{M^*}{1-M} \exp\Big(\frac{M_2 C_1}{1-M} hi\Big).$$

□

In the following, we present the details of a Runge-Kutta scheme to be applied on the general version (1) of the integral equation.

5 Pouzet type Runge-Kutta Method

In this section, we consider an s-stage Runge-Kutta method represented by the Butcher tableau

$$\begin{array}{c|c} \mathbf{c} = (c_i)_{i=1}^s & \mathbf{A} = (a_{i,j})_{i,j=1}^s \\ \hline & \mathbf{b} = \big((b)_{i=1}^s\big)^T \end{array}$$

Due to the properties of the solutions to Volterra integral equations arising in some applications (see e.g. Theorem 6.1 in Section 6), we consider a special distribution of nodes, the local quasi-uniform mesh for the Runge-Kutta discretization [26].

Definition 5.1. ([26]) Let

$$a = t_0 < t_1 < \ldots < t_N = b, \quad h_i = t_i - t_{i-1}, \quad h = \max_{1 \le i \le N} h_i,$$

A mesh for the interval $[a,b]$ is called *locally quasi-uniform*, if there exists a finite constant, γ, such that

$$\gamma = \frac{1}{2} \max_{2 \le i \le n} \Big(\frac{h_i}{h_{i-1}} + \frac{h_{i-1}}{h_i}\Big).$$

The quasi-uniform mesh has the property that $h \le \gamma(b-a)/n$. This choice of mesh is compatible with the solution of the weakly singular integral equations (see e.g. [7]). Borrowing the idea of Runge-Kutta method for initial value problems, a numerical scheme could be designed for the Volterra integral equation (1) as follows:

$$U_n^{(i)} = \Phi_n(t_n + \theta_i h_n, U_n^{(i)}) + h_n \sum_{j=1}^{s} a_{ij} k(t_n + d_{ij} h_n, t_n + c_j h_n, U_n^{(i)}, U_n^{(j)}), \tag{25}$$

$$u_{n+1} = \Phi_n(t_n + h_n, u_{n+1}) + h_n \sum_{j=1}^{s} b_j k(t_n + e_j h_n, t_n + c_j h_n, u_{n+1}, U_n^{(j)}), \tag{26}$$

in which

$$\Phi_n(t,u) = g(t,u) + F_n^*(t,u),$$

and

$$F_n^*(t,u) = \sum_{l=1}^{n-1} h_l \sum_{j=1}^{s} b_j k(t, t_l + c_j h_l, u, U_j^l).$$

In the above scheme, the parameters a_{ij}, b_j, c_j are obtained from the Butcher Table 1 and the strategy for choosing the other parameters leads us to two families of Runge-Kutta methods, known in the literature as Pouzet and Bel'tyukov [5, 6]. We note that for the Pouzet family, we put $\theta_i = c_i$, $e_i = 1$ and $d_{ij} = c_i$ and for the Bel'tyukov family, we set $\theta_i = c_i$, $d_{ij} = e_i$ for $i, j = 1, \cdots, s$.

It follows from the theory of Pouzet type Runge-Kutta methods for Volterra integral equations (see [19] and also [6, Theorem 4.1.4]) that the method (25) with the coefficients given by Table 1 provides an approximation u_{n+1} to the exact solution $u(t_{n+1})$ of order 5.

Table 1: Coefficients for Runge-Kutta method of order 5.

0							
$\frac{1}{5}$	$\frac{1}{5}$						
$\frac{3}{10}$	$\frac{3}{40}$	$\frac{9}{40}$					
$\frac{4}{5}$	$\frac{44}{45}$	$-\frac{56}{15}$	$\frac{32}{9}$				
$\frac{8}{9}$	$\frac{19372}{6561}$	$\frac{-25360}{2187}$	$\frac{64448}{6561}$	$\frac{-212}{729}$			
1	$\frac{9017}{3168}$	$\frac{-355}{33}$	$\frac{46732}{5247}$	$\frac{49}{176}$	$\frac{-5103}{18656}$		
1	$\frac{35}{384}$	0	$\frac{500}{1113}$	$\frac{125}{192}$	$\frac{-2187}{6784}$	$\frac{11}{84}$	
u_{n+1}	$\frac{35}{384}$	0	$\frac{500}{1113}$	$\frac{125}{192}$	$\frac{-2187}{6784}$	$\frac{11}{84}$	0

6 Application to Option Pricing

One of the durable concerns in the financial mathematics is the American option pricing problem. Among the well-known numerical methods for pricing American options, we could mention the Black-Scholes approach which is based on the following linear degenerate parabolic free boundary problem [24]

$$\begin{gathered}\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r-\delta) S \frac{\partial P}{\partial S} - rP = 0, \\ \lim_{S\to\mathcal{B}(t)} P(t,S) = K - \mathcal{B}(t), \quad \lim_{S\to\mathcal{B}(t)} \frac{\partial P}{\partial S}(t,S) = -1, \\ \lim_{t\to 0} P(t,S) = \max\{0, K-S\}, \quad \lim_{S\to 0} P(t,S) = 0,\end{gathered} \tag{27}$$

in which the constants r, δ, σ and K are the risk-free interest rate, the continuous dividend yield, the volatility and the exercise (strike) price, respectively. Also, $\mathcal{B}(t)$ denotes the unknown free (early exercise) boundary of the option which should be determined alongside the price.

By employing the risk-neutral valuation method of Cox and Ross [10], Kim [21] showed that the early exercise boundary of an American put option satisfies the following equation

$$\begin{aligned}K - \mathcal{B}(t) =& p(t, \mathcal{B}(t)) + \int_0^t \Big[rKe^{-r(t-s)} \aleph(-d_2(\mathcal{B}(t), t-s, \mathcal{B}(s))) \\ &- \delta \mathcal{B}(t) e^{-\delta(t-s)} \aleph\big(-d_1(\mathcal{B}(t), t-s, \mathcal{B}(s))\big)\Big] \mathrm{d}s,\end{aligned} \tag{28}$$

where $p(t, \mathcal{B}(t))$ represents the price of an European put option written on the same underling asset and given by

$$p(t,S) = Ke^{-rt}\aleph(-d_2(S,t,K)) - Se^{-\delta t}\aleph(-d_1(S,t,K)), \tag{29}$$

and $\aleph(.)$ is the standard cumulative normal distribution function. Furthermore, $d_1(x,t,y)$ and $d_2(x,t,y)$ are defined by

$$d_1(x,t,y) = \frac{\log(\frac{x}{y}) + (r - \delta + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}},$$

$$d_2(x,t,y)=\frac{\log(\frac{x}{y})+(r-\delta-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}.$$

It is obvious that Eq. (28) could be stated as the general form (1). Peskir in [28] proved that the early exercise boundary for the American put option can be characterized as the unique solution of the nonlinear integral equation (28). In the following, some regularity properties of the early exercise boundary, $\mathcal{B}(t)$ are presented.

Theorem 6.1. *Assume that the asset price process, S_t, follows a lognormal diffusion of the form $dS_t = rS_t dt + \sigma S_t dW_t$, in which W_t is the standard Wienner process. Let $\mathcal{B}(t)$ be the solution of the nonlinear integral equation (27). Then $\mathcal{B}(t)$ is a continuously differentiable function on $(0,T]$ and*

$$\begin{aligned} lim_{t\to 0}\mathcal{B}(t)&=\mathcal{B}(0)=K, \quad \delta\le r,\\ lim_{t\to 0}\mathcal{B}(t)&=\mathcal{B}(0)=(\frac{r}{\delta})K, \quad \delta> r. \end{aligned} \tag{30}$$

Proof. See [21]. □

In addition, the following asymptotic equivalence relations describe the behavior of the early exercise boundary near the expiry (see e.g. [11])

$$\begin{aligned} 0\le\delta<r, \quad &\mathcal{B}(t)\sim K-K\sigma\sqrt{t\ln[\frac{\sigma^2}{8\pi(r-\delta)^2t}]},\\ \delta=r, \quad &\mathcal{B}(t)\sim K-K\sigma\sqrt{2t\ln\frac{\delta}{4\sqrt{\pi}t}},\\ \delta>r, \quad &\mathcal{B}(t)\sim \frac{r}{\delta}K(1-\sigma\alpha_0\sqrt{2t}), \end{aligned} \tag{31}$$

where α_0 is a constant determined by the following transcendental equation

$$-\alpha_0^3\exp(\alpha_0^2)\int_{\alpha_0}^{\infty}\exp(-u^2)du=\frac{1}{4}(1-2\alpha_0^2).$$

Remark 2. The relations (31) indicate that the integral term in (28) shows a weakly singular behavior as $t\to 0^+$. This is an important consideration to be made in the numerical experiments.

7 Numerical Experiments

In this section, the theoretical results of the previous sections are applied on some test examples from the field to confirm their validity. The experiments are implemented in *Mathematica*® software platform. The programs are executed on a PC with 2.60 GHz Intel(R) Core(TM) i7-6700HQ processor.

The first test problem concerns with an option pricing application in which the early exercise boundary is approximated by a Pouzet type Runge-Kutta scheme and also a barycentric rational quadrature. The other two examples which are the special cases of (2) are selected from [8, 15].

For the Runge-Kutta method, we choose a quadratically graded mesh with the nodes

$$t_i=\left(\frac{i}{n}\right)^{\alpha}, \quad i=0,1,\dots,n,$$

and $\alpha=2$. For the uniform mesh, we set $\alpha=1$. In this section, the figures and tables report the value of $\max_{0\le i\le n}|u_i-u(t_i)|$ or $(\sum_{i=0}^{n}(u_i-u(t_i))^2)^{\frac{1}{2}}$. We should notice that the exact solution of (28) is not available, due to this fact we use the PSOR method as a benchmark

Example 7.1. ([25]) Consider the Black-Scholes equation with the parameters $r=0.1$, $K=100$, $\sigma=0.3$ and $\delta=0$. We have reported the results of applying both methods in Table 7.3. In order to compare the obtained results with some existing numerical approaches, alongside the two columns titled BRQ and PRK which correspond respectively to the barycentric quadrature and the Pouzet-Runge-Kutta method, we have also reported the results obtained by the PSOR method (based on the complementarity formulation presented in [24, 25]), EKK method (based on the asymptotic expression (31) presented in [11]), Zhu method (based on an analytical expression to the value of American put options and their optimal exercise boundary presented in [31]) and also the SSCH method (based on the solution of the integral equation obtained in [29] for the early exercise boundary using an iteration method).

As we expect from the discussion at the previous section (especially the comments made at the Remark 2), the Pouzet-Runge-Kutta method has produced approximations which agree with the benchmark solution, in comparison with the barycentic rational quadrature method.

Table 2: The PRK and BRQ results for the Example 7.1

t	PSOR	BRQ	PRK	EKK	Zhu	SSCH
0.00001	99.70	99.68	99.71	99.69	99.51	99.69
0.00005	99.40	99.43	99.37	99.37	99.03	99.36
0.0001	99.20	99.11	99.18	99.14	98.72	99.11
0.0005	98.31	98.29	98.30	98.28	97.57	98.27
0.001	97.73	97.63	97.70	97.10	96.83	97.66
0.01	94.18	94.43	94.21	94.33	92.73	94.07
0.04	90.30	91.02	90.25	91.12	88.66	91.31
0.1	86.94	85.25	86.88	86.29	85.25	86.76

Table 3: The absolute error of BRQ results for the Example 7.2

n	BRQ2	BRQ4
10	3.13E−02	6.06E−04
20	8.26E−03	1.17E−04
30	3.74E−04	8.67E−05
40	2.13E−05	5.09E−07
50	8.17E−06	3.37E−07
60	8.57E−07	1.99E−08
70	1.12E−07	2.71E−09
80	3.24E−08	1.11E−09

Example 7.2. ([15]) We also consider the following nonstandard Volterra integral equation

$$u(t) = \sqrt{2+t} - \frac{1}{2}\Big(\ln(5+3t) - \ln(5+2t)\Big) + \int_0^t \frac{5+t+s}{10+4t+2s}\,\frac{1}{1+u^2(t)+u^2(s)}\mathrm{d}s,$$

with the exact solution $u(t) = \sqrt{2+t}$ for $t \in [0,1]$. It must be noted that the forcing and kernel functions are smooth in the interval $[0,1]$. The barycentric rational quadrature (BRQ) solution is reported for $d = 2$ and $d = 4$ and denoted in the figures by BRQ2 and BRQ4, respectively. It is expected from the Theorem 4.1 that the order of convergence behaves like $\exp(h)h^{d+1}$ and if we compare the absolute error of the solution reported in Table 3 with the order expected in the Theorem 4.1, we see that the numerical results confirm the theoretical one.

Example 7.3. The third example is chosen from [8] as follows

$$u(t) = \sqrt{t} + \frac{1}{4}t^{\frac{3}{2}}(\ln 2 - 1) + \frac{1}{4}\int_0^t \frac{u^2(s)u(t)}{s+t}\mathrm{d}s. \tag{32}$$

The exact solution is $u(t) = \sqrt{t}$ for $t \in [0,1]$. The results obtained from the proposed Pouzet-type Runge-Kutta and barycentric rational quadrature methods for $d = 2$ and $d = 4$ are reported in Figure 7.3. Eq. (32) is weakly singular, so we expect a better results from Pouzet Runge-Kutta method and the results in Figure 7.3 indicate this. The theoretical order of convergence obtained in the Theorem 4.1 is not expected in this example because the exact solution is not smooth enough. Although, a comparison between Table 1 and Table 2 of [8] shows that the Runge-Kutta method proposed in this paper gives better results for this test problem.

Table 4: The results of PRK(5) for the Example 7.2

n	$\max_{0\le i\le n} \lvert u_i - u(t_i)\rvert$	$(\sum_{i=0}^{n}(u_i - u(t_i))^2)^{\frac{1}{2}}$
5	2.6480E–04	1.8466E–04
10	1.3862E–04	9.7888E–05
20	7.0632E–05	5.0294E–05
30	4.7361E–05	3.3826E–05
40	3.5620E–05	2.5480E–05
50	2.8543E–06	2.0437E–06
60	9.3721E–07	1.7660E–07
70	6.9039E–07	1.4641E–07
80	3.3991E–08	1.0271E–08

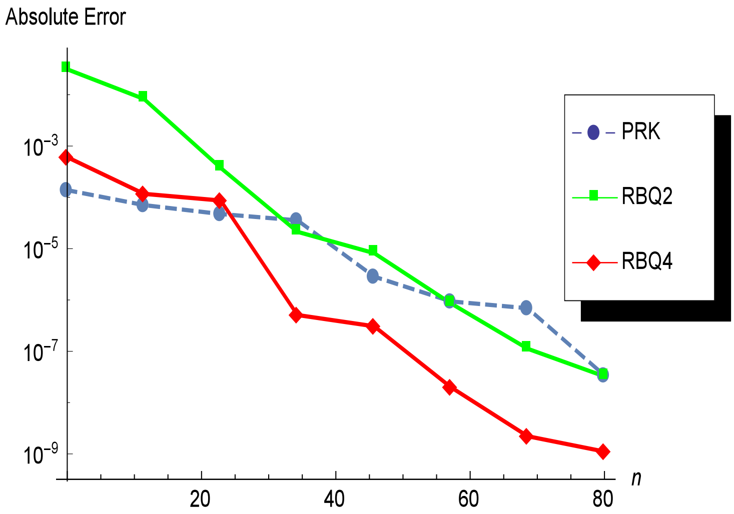

Figure 1: The absolute error of PRK and BRQ solutions for the Example 7.2

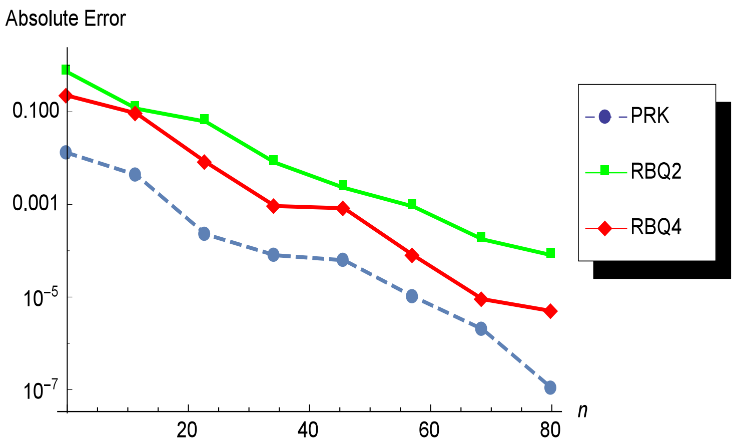

Figure 2: The absolute error of PRK and BRQ solutions for the Example 7.3

8 Conclusion and Some Problems in Progress

In this paper, we have considered the numerical solution of a class of nonlinear and non-standard Volterra integral equations arising mainly from the literature of financial mathematics. A barycentric rational quadrature method and also a Pouzet type Runge-Kutta scheme are presented and the local error of the quadrature solution is analyzed. We have also tested the efficiency of the proposed methods on some test problems from the literature which show their superior performance in comparison with some competing approaches. Among possible future research directions, we could point out to the stability analysis of the proposed schemes as well as their extensions to other integral equations proposed in the American option pricing field. We could also consider the variable step-size counterpart of the Runge-Kutta scheme which will improve the efficiency of the proposed methodology.

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