PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 19 · 2026 · Pages 156–173

Estimates for the Best M–Term Approximations of Function Classes with Bounded Mixed Derivatives in Lorentz Spaces

Aigul Myrzagaliyeva[a]

Communicated by Leonard P. Bos

Abstract

Order-sharp estimates are eatablished for the best M-term approximations of functions from $W^{\bar{r}}_{q,\tau_1}$ with respect to the multiple trigonometric system in the metric of L_{p,τ_2} for a number of relations between the parameters p, q, τ_1, and τ_2. Constructive methods of nonlinear approximation are used in the proofs of upper estimates in most of the main results.

1 Introduction

Let $\mathbb{R}^m$ be an m-dimensional Euclidean space of points $\bar{x} = (x_1, \dots, x_m)$ with real coordinates; $\mathbb{T}^m = [0, 2\pi)^m$ and $\mathbb{I}^m = [0, 1)^m$ be m-dimensional cubes.

By $L_p(\mathbb{T}^m)$ we denote the Lebesgue space equipped with the norm (see [1, p. 500])

$$\|f\|_p = \left[\int_0^1 \dots \int_0^1 |f(2\pi x_1, \dots, 2\pi x_m)|^p dx_1 \dots dx_m\right]^{\frac{1}{p}}, \ 1 \le p < \infty.$$

$L_{p,\tau}(\mathbb{T}^m)$ denotes the Lorentz space of all real-valued Lebesgue measurable functions f, which have a 2π-period in each variable and for which the quantity

$$\|f\|_{p,\tau} = \left\{\frac{\tau}{p}\int_0^1 \left(f^*(t)\right)^{\tau} t^{\frac{\tau}{p}-1} dt\right\}^{\frac{1}{\tau}}, \ 1 < p < \infty, 1 \le \tau < \infty,$$

is finite, where $f^*(t)$ is the nonincreasing rearrangement of the function $|f(2\pi\bar{x})|$, $\bar{x} \in \mathbb{I}^m$ (see [2, p. 213–216]). Note that for $\tau = p$, the Lorentz space $L_{p,\tau}(\mathbb{T}^m)$ coincides with the Lebesgue space $L_p(\mathbb{T}^m)$.

We denote by $\mathring{L}_{p,\tau}(\mathbb{T}^m)$ the set of all functions $f \in L_{p,\tau}(\mathbb{T}^m)$ such that

$$\int_0^1 f(2\pi\bar{x})\, dx_j = 0, \ \ j = 1, \dots, m.$$

Here $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ are sets of natural, integer, real numbers, respectively, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

We begin by introducing some notation that will be used throughout this work. First, we associate a function $f \in L_1(\mathbb{T}^m)$ with its Fourier series $\sum_{\bar{n}\in\mathbb{Z}^m} a_{\bar{n}}(f) e^{i\langle \bar{n}, 2\pi\bar{x}\rangle}$, where $a_{\bar{n}}(f)$ is the Fourier coefficients of a function $f \in L_1(\mathbb{T}^m)$ by the system $\{e^{i\langle\bar{n},2\pi\bar{x}\rangle}\}_{\bar{n}\in\mathbb{Z}^m}$. Here the inner product of $\bar{y}$ and $\bar{x}$ is the number $\langle \bar{y}, \bar{x}\rangle = \sum_{j=1}^m y_j x_j$;

$$\delta_{\bar{s}}(f, \bar{x}) = \sum_{\bar{n}\in\rho(\bar{s})} a_{\bar{n}}(f)\, e^{i\langle\bar{n},\bar{x}\rangle},$$

where

$$\rho(\bar{s}) = \left\{\bar{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m : [2^{s_j - 1}] \le \left|k_j\right| < 2^{s_j}, j = 1, \dots, m\right\},$$

$[a]$ is the integer part of the number a, $\bar{s} = (s_1, \dots, s_m)$, $s_j = 0, 1, 2, \dots$.

[a]Department of Computing and Data Science, Astana IT University, Kazakhstan

For a given vector $\overline{r} = (r_1, \ldots, r_m) > \overline{0} = (0, \ldots, 0)$, we set $\overline{\gamma} = \frac{\overline{r}}{r_1}$ and $Q_{n,\overline{\gamma}} = \cup_{\langle \overline{s}, \overline{\gamma}\rangle < n} \rho(\overline{s})$ and

$$S_{Q_{n,\overline{\gamma}}}(f, \overline{x}) = \sum_{\overline{k} \in Q_{n,\overline{\gamma}}} a_{\overline{k}}(f) e^{i\langle \overline{k}, \overline{x}\rangle},$$

where $S_{Q_{n,\overline{\gamma}}}(f, \overline{x})$ is a partial sum of the Fourier series of a function f (see [1, p. 134]).

Consider the one-dimensional Bernoulli kernel (see, for example, [1, p. 130])

$$F_r(x) = 1 + 2\sum_{k=1}^{\infty} k^{-r} \cos(kx - r\pi/2), \ r > 0.$$

For a vector $\overline{r} = (r_1, \ldots, r_m)$, $r_j > 0$, $j = 1, \ldots, m$, we put

$$F_{\overline{r}}(\overline{x}) = \prod_{j=1}^{m} F_{r_j}(x_j).$$

Consider the Sobolev class (see [1, p. 105])

$$W^{\overline{r}}_{p,\tau} = \{f : f = \varphi * F_{\overline{r}}, \ \|\varphi\|_{p,\tau} \le 1\},$$

where $1 < p < \infty$, $1 \le \tau < \infty$,

$$(\varphi * F_{\overline{r}})(\overline{x}) = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} \varphi(\overline{x} - \overline{u}) F_{\overline{r}}(\overline{u}) d\overline{u}.$$

In the case $\tau = p$, we write $W^{\overline{r}}_p$ instead of $W^{\overline{r}}_{p,p}$ (see [1, p. 105], [3], [4]).

Let $X = X(\mathbb{T}^m)$ be a Banach space. The quantity (see [1, Sec. 8.1])

$$e_M(f)_X = \inf_{\bar{k}^{(j)}, b_j} \Big\| f - \sum_{j=1}^{M} b_j e^{i\langle \bar{k}^{(j)}, \bar{x}\rangle} \Big\|_X$$

is called the best M–term trigonometric approximation of a function $f \in X$, where the infimum is taken over the coefficients b_j and $\bar{k}^{(j)}$. For a class of functions $F \subset X(\mathbb{T}^m)$, we set

$$e_M(F)_X = \sup_{f \in F} e_M(f)_X.$$

The theory of M-term approximations originates from the work of S.B. Stechkin [5], who studied best approximations in orthonormal systems and obtained criteria for absolute convergence of the Fourier series. Classical results of Vallée-Poussin and N.S. Bernstein [1, p. 21] established the exact order of polynomial approximation of functions such as $f(x) = |x|$, $x \in [-1, 1]$.

The advantage of nonlinear M-term approximations over linear methods was first demonstrated by R.S. Ismagilov [6], who proved that

$$e_M(|\sin 2\pi x|)_\infty \le C_\varepsilon M^{-\frac{6}{5}+\varepsilon}, \ \forall \varepsilon > 0,$$

while V.E. Mayorov [7] later obtained the sharp estimate

$$e_M(|\sin 2\pi x|)_\infty \asymp M^{-\frac{3}{2}}.$$

For functions of one variable, sharp estimates of M-term approximations in Sobolev classes were obtained by E.S. Belinsky [8]:

Theorem 1.1 (Belinsky E.S. [8]). *Let $1 \le p \le 2 < q < \infty$. Then*

$$e_n(W^r_p)_q \asymp \begin{cases} n^{-(r-\frac{1}{p}+\frac{1}{2})}, & r > \frac{1}{p}, \\ n^{-\frac{q}{2}(r-\frac{1}{p}+\frac{1}{q})}, & \frac{1}{p} - \frac{1}{q} < r < \frac{1}{p}, \\ n^{-\frac{1}{2}} \log^{1-\frac{1}{p}} n, & r = \frac{1}{p}, p > 1 \\ n^{-\frac{1}{2}} \log n, & r = 1, p = 1. \end{cases}$$

In the multidimensional case, for $1 < q \le p < 2$, $r_1 > 2(\frac{1}{q} - \frac{1}{p})$ order-sharp estimates of M-term approximations in Sobolev classes $W^{\overline{r}}_q$ with respect to the norm of the space $L_p(\mathbb{T}^m)$ were established by V.N. Temlyakov [3, 4] and further for $1 < q \le p < 2$ and $r_1 \le 2(\frac{1}{q} - \frac{1}{p})$ was developed by E.S. Belinsky [9]. A systematic study of such problems for Sobolev, Nikol'skii–Besov and Lizorkin–Triebel classes can be found in [10, 11, 12]. Further results for M-term approximations in Lorentz spaces and related function classes were obtained in [13]–[16].

V.N. Temlyakov [17], using constructive methods, established estimates for M-term approximations of functions from the class $W^{\overline{r}}_q$ in the space $L_p(\mathbb{T}^m)$ for $1 < q \le 2 < p < \infty$ and $(\frac{1}{q} - \frac{1}{p})p' < r_1 < \frac{1}{q}$, $p' = \frac{p}{p-1}$. He also posed the problem of finding a constructive estimation method in the case $\frac{1}{q} - \frac{1}{p} < r_1 \le (\frac{1}{q} - \frac{1}{p})p'$. This problem was solved in [13]–[16]. In particular, order-sharp estimates for the quantities $e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2}$ were established in various parameter ranges in [13, 14, 16]. The case of generalized classes $W^{a,b(\cdot),\bar{r}}_{q,\tau_1,\theta}$ was investigated in [15], where upper bounds and partial sharpness were obtained.

Let us consider these results in more detail. In [13], order-sharp estimates were obtained for the best M-term approximations of functions from the class $W^{\bar{r}}_{q,\tau_1}$ in the space $L_{p,\tau_2}(\mathbb{T}^m)$ for $1 < q < 2 < p < \infty$, $1 < \tau_1 \le \tau_2 < \infty$, and $\frac{1}{q} - \frac{1}{p} < r_1 < \frac{1}{q}$. In the case $\tau_1 = q$ and $\tau_2 = p$, Theorem 4 [13] implies the results of E.S. Belinsky [9] and V.N. Temlyakov [18]. For $q = 2$, upper bounds for the quantity $e_M(W^{\bar{r}}_{2,\tau_1})_{p,\tau_2}$ were obtained in [14].

In [15], the class $W^{a,b(\cdot),\bar{r}}_{q,\tau_1,\theta}$ was introduced, where $a > 0$, $0 < \theta < \infty$ and $b(t)$ is a slowly varying function on $[0,\infty)]$. In the special case $b(t) = (1+\log_2 t)^{(\nu-1)b}$, $t \ge 1$, $b \in \mathbb{R}$ with $\tau = q$ and $\theta = \infty$, the class $W^{a,b(\cdot),\bar{r}}_{q,\tau,\theta} = W^{a,b,\bar{r}}_{q}$ was introduced by V.N. Temlyakov in [17, 18] to study M–term approximations of functions from the Sobolev and the Nikol'skii–Besov classes. In the case $b = 0$, the definitions of the classes $W^{a,b,\bar{r}}_{q,\tau}$ and $W^{\bar{r}}_{q,\tau}$ imply the embedding $W^{\bar{r}}_{q,\tau} \hookrightarrow W^{r_1,0,\bar{r}}_{q,\tau}$ (see [13, 14]). In particular, for $\tau = q$, this embedding was established in [18].

In [15], upper bounds were obtained for the best M-term approximations of functions from the class $W^{a,b(\cdot),\bar{r}}_{q,\tau_1,\theta}$ in the metric of the space $L_{p,\tau_2}(\mathbb{T}^m)$ for $1 < q < 2 < p < \infty$, under certain conditions on the parameters a, τ_1, and τ_2. For some ranges of these parameters, the obtained upper bounds were shown to be order-sharp. Similar results for the class of functions with bounded mixed derivatives $W^{\bar{r}}_{q}$ in the space $L_p(\mathbb{T}^m)$ were proved in [18].

In [16], order-sharp estimates were obtained for the quantity $e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2}$ in the cases $1 < q < 2 < p < \infty$, $r_1 > \frac{1}{q}$ and $r_1 = \frac{1}{q}$, with either $1 < q < 2$ and $1 < \tau_1 < \infty$ or $q = 2$ and $2 < \tau_1 < \infty$. The exact order of this quantity was also established in the diagonal case $p = q$ and $\tau_2 = \tau_1 = \tau$, as well as upper estimates for $1 < p < q < \infty$. Further applications of the constructive method can be found in [18, 19] and [20].

Despite these advances, the problem of obtaining order-sharp estimates for the quantities $e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2}$ with $1 < q < p \le 2$ and $1 < \tau_1 \le \tau_2 \le 2$ remains incomplete.

The aim of this paper is to obtain order-sharp estimates for $e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2}$ in the case $1 < q < p \le 2$ and $1 < \tau_1 \le \tau_2 \le 2$ under appropriate conditions on the smoothness parameters r_1, τ_1, and τ_2.

The results of this paper extend and complement earlier works in several directions. Order-sharp estimates are obtained in parameter ranges where only partial results were previously known (see [13, 14, 16]). In particular, the case $1 < q < p \le 2$ is covered, which is essentially different in order from the classical case studied by V.N. Temlyakov and E.S. Belinsky. Furthermore, the dependence on the Lorentz parameters τ_1 and τ_2 is refined. Finally, new upper bounds are provided based on constructive methods of nonlinear approximation.

All results are formulated in Section 2 and proved in Section 4. Section 3 contains auxiliary statements required for the proofs of the main results. Finally, a comparison with previously known results is provided.

Throughout the paper, we state our results in terms of order relations and say that $\mu_1(e)$ and $\mu_2(e)$ are functions of the same order, written $\mu_1 \asymp \mu_2$, if there are constants C_1, C_2 with $0 < C_1 \le C_2$ such that $C_1\mu_2(e) \le \mu_1(e) \le C_2\mu_2(e)$. The constants C_1 and C_2 do not depend on e. The order inequalities $\mu_1 \ll \mu_2$ and $\mu_1 \gg \mu_2$ can be defined likewise. If A is a finite set, then $|A|$ stands for the number of its elements. Here and below $\log M$ is the logarithm with base 2 of the number $M > 1$.

2 Main result

Theorem 2.1. *Let $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \le \ldots \le r_m$, $1 < q < p \le 2$ and $1 < \tau_1 \le \tau_2 \le 2$. If $\frac{1}{q} - \frac{1}{p} < r_1 < \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$, then*

$$e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2} \asymp M^{-(r_1+\frac{1}{p}-\frac{1}{q})}.$$

Theorem 2.2. *Let $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \le \ldots \le r_m$, $1 < q < p \le 2$ and $1 < \tau_1 \le \tau_2 \le 2$. If $r_1 = \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$, then*

$$e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2} \asymp M^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}.$$

Theorem 2.3. *Let $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \le \ldots r_m$, $1 < q \le p \le 2$, $1 < \tau_1 \le \tau_2 \le 2$. If $r_1 > \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$, then*

$$e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2} \asymp M^{-(r_1+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1})},\ M > 1.$$

Theorem 2.4. *Let $0 < r_1 = \ldots = r_\nu < r_{\nu+1} \le \ldots \le r_m$, $1 < q < p \le 2$ and $1 < \tau_2 \le 2 < \tau_1 < \infty$. If $\frac{1}{q} - \frac{1}{p} < r_1 < \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_2} - \frac{1}{\tau_1}$, then*

$$e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2} \le M^{-(r_1+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})+(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}.$$

3 Auxiliary statements

Theorem 3.1 ([21]). *Let $1 < q < \lambda < \infty$, $1 < \tau, \theta < \infty$. If a function $f \in L_{q,\tau}(\mathbb{T}^m)$, then*

$$\|f\|_{q,\tau} \geqslant C\Big(\sum_{\bar{s}\in\mathbb{Z}^m_+}\prod_{l=1}^{m} 2^{s_l(1/\lambda-1/q)\tau}\|\delta_{\bar{s}}(f)\|^{\tau}_{\lambda,\theta}\Big)^{1/\tau}.$$

Theorem 3.2 ([21]). *Let* $1 < p < q < \infty$, $1 < \tau_1, \tau_2 < \infty$. *If a function* $f \in L_{p,\tau_1}(\mathbb{T}^m)$ *satisfies the condition*

$$\sum_{\bar{s}\in\mathbb{Z}_+^m} \prod_{j=1}^m 2^{s_j \tau_2(1/p-1/q)} \|\delta_{\bar{s}}(f)\|_{p,\tau_1}^{\tau_2} < \infty,$$

then $f \in L_{q,\tau_2}(\mathbb{T}^m)$ *and the inequality*

$$\|f\|_{q,\tau_2} \leqslant C\left(\sum_{\bar{s}\in\mathbb{Z}_+^m} \prod_{j=1}^m 2^{s_j \tau_2(1/p-1/q)} \|\delta_{\bar{s}}(f)\|_{p,\tau_1}^{\tau_2}\right)^{1/\tau_2}$$

holds.

Consider the set

$$\Pi(\overline{N}, m) = \{\bar{k} \in \mathbb{Z}^m : |k_j| \le N_j,\ j = 1, \ldots, m\},\ N_j \in \mathbb{Z}_+$$

and the set of trigonometric polynomials

$$`(\overline{N}, m)) = \Big\{T(\bar{x}) : T(\bar{x}) = \sum_{\bar{k}\in\Pi(\overline{N},m)} a_{\bar{k}} e^{i\langle\bar{k},\bar{x}\rangle}\Big\}.$$

For a function $f \in L(\mathbb{T}^m)$, we put

$$\|f\|_A = \sum_{\bar{k}\in\Pi(\overline{N},m)} |a_{\bar{k}}(f)|.$$

In this section, we recall the definition of the fractional derivative of a function and formulate statements that are often used in the proofs of the main results.

For a function $f \in \mathring{L}(\mathbb{T}^m)$ and a vector $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$ with nonnegative coordinates, the fractional differentiation operator is determined by the formula (see [22, Chapter 3, Section 15])

$$f^{(\bar{\alpha})}(\bar{x}) := f^{(\alpha_1,\ldots,\alpha_m)}(\bar{x}) = \sum_{\bar{n}\in\mathring{\mathbb{Z}}^m} \prod_{j=1}^m (in_j)^{\alpha_j} a_{\bar{n}}(f) e^{i\langle\bar{n},\bar{x}\rangle},$$

where $\mathring{\mathbb{Z}}^m = \{\bar{n} \in \mathbb{Z}^m : \prod_{j=1}^m n_j \neq 0\}$ and $(in_j)^{\alpha_j} = |n_j|^{\alpha_j} e^{i\frac{\pi}{2}\alpha_j sign n_j}$, $j = 1, \ldots, m$.

It is known that the following relation [22, Chapter 3, Section 15] is true

$$\|f^{(\bar{\alpha})}\|_p \asymp \Big\|\Big(\sum_{\bar{s}\in\mathbb{Z}_+^m} 2^{2\langle\bar{s},\bar{\alpha}\rangle} |\delta_{\bar{s}}(f)|^2\Big)^{1/2}\Big\|_p, \tag{1}$$

for a function $f \in \mathring{L}_p(\mathbb{T}^m)$, $1 < p < \infty$. Here relation (1) means that the finiteness of any part implies the finiteness of its other part and two-sided inequalities are satisfied.

Using relation (1) and the interpolation theorem in the Lorentz space, it is easy to prove the following statement.

Theorem 3.3 ([22]). *Suppose* $1 < p, \tau < \infty$ *and* $f \in L_{p,\tau}(\mathbb{T}^m)$. *Then the relation*

$$\|f^{(\bar{\alpha})}\|_{p,\tau} \asymp \Big\|\Big(\sum_{\bar{s}\in\mathbb{Z}_+^m} 2^{2\langle\bar{s},\bar{\alpha}\rangle} |\delta_{\bar{s}}(f)|^2\Big)^{1/2}\Big\|_{p,\tau}.$$

holds.

Theorem 3.4 ([21]). *Let* $1 < p, \tau < \infty$. *Then for any function* $f \in L_{p,\tau}(\mathbb{T}^m)$ *the following relation holds*

$$\|f\|_{p,\tau} \asymp \Big\|\Big(\sum_{\bar{s}\in\mathbb{Z}_+^m} |\delta_{\bar{s}}(f)|^2\Big)^{1/2}\Big\|_{p,\tau}.$$

For a function $f \in L_1(\mathbb{T}^m)$, we set

$$f_{l,\bar{r}}(\bar{x}) = \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} \delta_{\bar{s}}(f,\bar{x}),\ l \in \mathbb{Z}_+,$$

where $\bar{\gamma} = (\gamma_1, \ldots, \gamma_m)$, $\gamma_1 = \ldots = \gamma_\nu < \gamma_{\nu+1} \le \ldots \le \gamma_m$, $\gamma_j = \frac{r_j}{r_1}$, $r_j > 0$, $j = 1, \ldots, m$.

Consider the class defined in [5, 6]

$$W_A^{a,b,\bar{r}} = \big\{f \in L_1(\mathbb{T}^m) : \|f_{l,\bar{r}}\|_A \le 2^{-la} \bar{l}^{(\nu-1)b},\ l \in \mathbb{Z}_+\big\},$$

where

$$\|f_{l,\bar{r}}\|_A = \sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1} \sum_{\bar{n}\in\rho(\bar{s})} |a_{\bar{n}}(f)|.$$

In proving the main results, we often use the following statement.

Lemma 3.5. *Let $2 \le p < \infty$, $1 < \tau < \infty$, $a > 0$ and $b \in \mathbb{R}$. Then there is a constructive method $G_M(f, \bar{x})$ based on greedy algorithms. This leads to the estimate*

$$\|f - G_M(f)\|_{p,\tau} \le C(m) M^{-a-\frac{1}{2}} (\log M)^{(\nu-1)(a+b)},$$

where $f \in W_A^{a,b,\bar{r}}$ and $M > 1$.

Remark 1. The formulation of Lemma 3.5 and a brief outline of its proof are given in [13], the complete proof can be found in [23]. For $\tau = p$, Lemma 3.5 was previously proved by V.N. Temlyakov [17].

Let $\Gamma(N, \overline{\gamma}) = \{\overline{k} = (k_1, \dots, k_m) \in \mathbb{N} \colon \prod_{j=1}^m k_j^{\gamma_j} \le N\}$ (see [1, p. 131]). Consider a trigonometric polynomial over the set $\Gamma(N, \overline{\gamma})$

$$T_{N,\bar{\gamma}}(\overline{x}) = \sum_{|\overline{k}| \in \Gamma(N,\overline{\gamma})} a_{\overline{k}} e^{i\langle \bar{k}, \bar{x}\rangle}.$$

Assume that $\mathbb{T}(N, \overline{\gamma}) = \{T_{N,\bar{\gamma}}\}$, where $|\overline{k}| = (|k_1|, \dots, |k_m|)$, $\|T_{N,\bar{\gamma}}\|_A = \sum_{|\overline{k}| \in \Gamma(N,\overline{\gamma})} |a_{\overline{k}}|$.

Theorem 3.6. *Let $1 < p \le 2$, $1 < \tau < \infty$ and $T_{N,\bar{\gamma}} \in \mathbb{T}(N, \overline{\gamma})$.*

1. *If $1 < p < 2$, $1 < \tau < \infty$, then*
$$\|T_{N,\bar{\gamma}}\|_A \le C N^{\frac{1}{p}} (\log(N+1))^{(\nu-1)(1-\frac{1}{\tau})} \|T_{N,\bar{\gamma}}\|_{p,\tau}.$$
2. *If $p = 2$, $2 \le \tau < \infty$, then*
$$\|T_{N,\bar{\gamma}}\|_A \le C N^{\frac{1}{2}} (\log(N+1))^{(\nu-1)(1-\frac{1}{\tau})+\frac{1}{2}-\frac{1}{\tau}} \|T_{N,\bar{\gamma}}\|_{2,\tau}.$$

Proof. Assume that there is $\varepsilon_{\overline{k}}$ such that $|\varepsilon_{\overline{k}}| = 1$ and $\varepsilon_{\overline{k}} a_{\overline{k}} = |a_{\overline{k}}|$. Consider the polynomial

$$T_{n,\varepsilon}(\overline{x}) = \sum_{\overline{k} \in Q_{n,\overline{\gamma}}} \varepsilon_{\overline{k}} e^{i\langle \overline{k}, \overline{x}\rangle},$$

where $n = \log N + \gamma(m)$, $\gamma(m) = \sum_{j=1}^m \gamma_j$. Then, by the orthogonality of the system $\{e^{i\langle \bar{k}, \bar{x}\rangle}\}$ and Hölder's inequality in the Lorentz space, we have

$$\left\|T_{N,\bar{\gamma}}\right\|_A = \left\| \sum_{\bar{s} \in \Gamma(N,\bar{\gamma})} \varepsilon_{\bar{k}} a_{\bar{k}}(f_l) \right\| = \frac{1}{(2\pi)^m} \int_{\mathbb{T}^m} T_{N,\bar{\gamma}}(\overline{x}) T_{n,\varepsilon}(\overline{x}) \le C \|T_{N,\bar{\gamma}}\|_{p,\tau} \|T_{n,\varepsilon}\|_{p',\tau'}, \tag{2}$$

where $\frac{1}{b} + \frac{1}{b'} = 1$.

Since $1 < p < 2$, then $2 < p^{'} < \infty$. Therefore, by virtue of Theorem 3.1 [21] we obtain

$$\|T_{n,\varepsilon}\|_{p',\tau'} \le C \left\| \left\{ \prod_{j=1}^m 2^{s_j(\frac{1}{2}-\frac{1}{p'})} \|\delta_{\bar{s}}(T_{n,\varepsilon})\|_2 \right\}_{\bar{s} \in X(\log N, \overline{\gamma})} \right\|_{l_{\tau'}}$$

where $X(\log N, \overline{\gamma}) = \{\bar{s} = (s_1, \dots, s_m) \in \mathbb{Z}_+^m \colon s_j > 0, \langle \bar{s}, \overline{\gamma}\rangle \le n + \gamma(m) + 1\}$.

Then, using Parseval's equality, Lemma Γ [4] and choosing $\overline{\delta} = (\delta_1, \dots, \delta_m)$ such that $1 \le \delta_j \le \gamma_j$, $j = 1, \dots, m$, we have

$$\begin{aligned} \|T_{n,\varepsilon}\|_{q',\tau_1'} &\le C \left\| \left\{ \prod_{j=1}^m 2^{\frac{s_j}{p}} \right\}_{\bar{s} \in X(\log N, \overline{\gamma})} \right\|_{l_{\tau'}} \\ \le C \left\| \left\{ \prod_{j=1}^m 2^{\frac{s_j \delta_j}{p}} \right\}_{\bar{s} \in X(\log N, \overline{\gamma})} \right\|_{l_{\tau'}} &\le C 2^{(n+\gamma(m)+1)\frac{1}{p}} (n + \gamma(m) + 1)^{\frac{\nu-1}{\tau_1'}} \\ &\le C 2^{\frac{n}{p}} n^{\frac{\nu-1}{\tau_1'}} \le C N^{\frac{1}{p}} (\log N)^{\frac{\nu-1}{\tau_1'}}. \end{aligned} \tag{3}$$

Now from inequalities (2) and (3) it follows that

$$\|T_{N,\bar{\gamma}}\|_A \le C \|T_{N,\bar{\gamma}}\|_{p,\tau} N^{\frac{1}{p}} (\log N)^{\frac{\nu-1}{\tau_1'}}$$

for $1 < p < 2$ and $1 < \tau < \infty$.

If $p = 2$ and $2 \le \tau < \infty$, then by virtue of Theorem 1.3 [21] we get

$$\|T_{n,\varepsilon}\|_{2,\tau'} \le C\Big\|\Big\{\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{\frac{1}{\tau'}-\frac{1}{2}}\|\delta_{\bar{s}}(T_{n,\varepsilon})\|_2\Big\}_{\bar{s}\in X(\log N,\bar{\gamma})}\Big\|_{l_{\tau'}}$$
$$= C\Big\|\Big\{\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{\frac{1}{\tau'}-\frac{1}{2}}\prod_{j=1}^{m}2^{\frac{s_j}{2}}\Big\}_{\bar{s}\in X(\log N,\bar{\gamma})}\Big\|_{l_{\tau'}}$$
$$= C\Big\{\sum_{l=0}^{n+\gamma(m)+1}\sum_{l<\langle\bar{s},\bar{\gamma}\rangle\le l+1}\Big(\sum_{j=1}^{m}(s_j+1)\Big)^{(\frac{1}{\tau'}-\frac{1}{2})\tau'}\prod_{j=1}^{m}2^{\frac{s_j\tau'}{2}}\Big\}^{\frac{1}{\tau'}}$$
$$\le C\Big\{\sum_{l=0}^{n+\gamma(m)+1}\sum_{l<\langle\bar{s},\bar{\gamma}\rangle\le l+1}\Big(\sum_{j=1}^{m}(s_j\gamma_j)\Big)^{(\frac{1}{\tau'}-\frac{1}{2})\tau'}\prod_{j=1}^{m}2^{\frac{s_j\tau'}{2}}\Big\}^{\frac{1}{\tau'}}$$
$$\le C\Big\{\sum_{l=0}^{n+\gamma(m)+1}(l+1)^{(\frac{1}{\tau'}-\frac{1}{2})\tau'}\sum_{l<\langle\bar{s},\bar{\gamma}\rangle\le l+1}\prod_{j=1}^{m}2^{\frac{s_j\tau'}{2}}\Big\}^{\frac{1}{\tau'}}.$$

Let $\delta_j = \gamma_j$, $j = 1, \ldots, \nu$ and $1 < \delta_j < \gamma_j$, $j = \nu+1, \ldots, m$. Then, taking this fact into account and applying Lemma Γ [4] we obtain

$$\|T_{n,\varepsilon}\|_{2,\tau'} \le C\Big\{\sum_{l=0}^{n+\gamma(m)+1}(l+1)^{(\frac{1}{\tau'}-\frac{1}{2})\tau'}\sum_{l<\langle\bar{s},\bar{\gamma}\rangle\le l+1}2^{\langle\bar{s},\bar{\delta}\rangle\frac{\tau'}{2}}\Big\}^{\frac{1}{\tau'}}$$
$$\le C\Big\{\sum_{l=0}^{n+\gamma(m)+1}(l+1)^{(\frac{1}{\tau'}-\frac{1}{2})\tau'}2^{l\frac{\tau'}{2}}l^{\nu-1}\Big\}^{\frac{1}{\tau'}} \le C2^{\frac{n}{2}}l^{\frac{\nu-1}{\tau'}+\frac{1}{2}-\frac{1}{\tau}}. \tag{4}$$

Then, (2) and (4) imply that

$$\|T_{N,\bar{\gamma}}\|_A \le CN^{\frac{1}{2}}(\log N)^{\frac{\nu-1}{\tau'}+\frac{1}{2}-\frac{1}{\tau}}\|T_{N,\bar{\gamma}}\|_{2,\tau}$$

for $p = 2$ and $2 \le \tau < \infty$. □

Let us give a proof of the inequality of different metrics in the Lorentz space from [24]. The proof is given here with the consent of the author G. Akishev.

Suppose

$$T_{n,\bar{\gamma}}(\bar{x}) = \sum_{\bar{k}\in Q_n^{(\bar{\gamma})}} b_{\bar{k}}e^{i\langle\bar{k},\bar{x}\rangle}.$$

Theorem 3.7. *Let $1 < p < q \le 2$, $1 < \tau_2 < \tau_1$ and $2 < \tau_1 < \infty$. Then*

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C2^{n(\frac{1}{p}-\frac{1}{q})}n^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}\|T_{n,\bar{\gamma}}\|_{p,\tau_1}.$$

Proof. In Theorem 3.1 putting $\lambda = \theta = 2$ and using Parseval's equality, we have

$$\|f\|_{q,\tau} \ge C\Big(\sum_{\bar{s}\in\mathbb{Z}_+^m}2^{|\bar{s}|(\frac{1}{2}-\frac{1}{q})\tau}\Big(\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}(f)|^2\Big)^{\tau/2}\Big)^{\frac{1}{\tau}} \tag{5}$$

for $f \in L_{q,\tau}(\mathbb{T}^m)$, $1 < q < 2$ and $1 < \tau < \infty$, where $|\bar{s}| = \sum_{j=1}^{m}s_j = \langle\bar{s},\bar{1}\rangle$, $\bar{1} = (1,\ldots,1)$.

Let $2 < \tau < \infty$. Then applying Hölder's inequality for $\beta = \tau/2$, $1/\beta + 1/\beta' = 1$, we get

$$\Big(\sum_{\langle\bar{s},\bar{\gamma}\rangle<n}\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}|^2\Big)^{\frac{1}{2}} \le \Big\{\sum_{\langle\bar{s},\bar{\gamma}\rangle<n}2^{|\bar{s}|(\frac{1}{2}-\frac{1}{p})\tau}\Big(\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}|^2\Big)^{\tau/2}\Big\}^{\frac{1}{\tau}}\Big\{\sum_{\langle\bar{s},\bar{\gamma}\rangle<n}2^{-|\bar{s}|(\frac{1}{2}-\frac{1}{p})2\beta'}\Big\}^{\frac{1}{2\beta'}}. \tag{6}$$

If $1 < p < 2$, then according to inequality (5) we have

$$\Big\{\sum_{\langle\bar{s},\bar{\gamma}\rangle<n}2^{|\bar{s}|(\frac{1}{2}-\frac{1}{p})\tau}\Big(\sum_{\bar{k}\in\rho(\bar{s})}|a_{\bar{k}}|^2\Big)^{\tau/2}\Big\}^{\frac{1}{\tau}} \le C\|T_{n,\bar{\gamma}}\|_{p,\tau}. \tag{7}$$

By Parseval's equality, inequalities (6), (7) and Lemma Γ [4], we have

$$\|T_{n,\bar{\gamma}}\|_2 \le C\|T_{n,\bar{\gamma}}\|_{p,\tau}\Big\{\sum_{\langle\bar{s},\bar{\gamma}\rangle<n}2^{-|\bar{s}|(\frac{1}{2}-\frac{1}{p})2\beta'}\Big\}^{\frac{1}{2\beta'}} \le C\|T_{n,\bar{\gamma}}\|_{p,\tau}2^{n(\frac{1}{p}-\frac{1}{2})}n^{(m-1)(\frac{1}{2}-\frac{1}{\tau})} \tag{8}$$

for $1 < p < 2$ and $2 < \tau < \infty$.

If $1<\tau\le 2$, then according to Jensen's inequality, we obtain

$$\Big(\sum_{\langle\bar s,\bar\gamma\rangle<n}\sum_{\bar k\in\rho(\bar s)}|a_{\bar k}|^2\Big)^{\frac12}\le 2^{n(\frac1p-\frac12)}\Big\{\sum_{\langle\bar s,\bar\gamma\rangle<n}2^{-|\bar s|(\frac12-\frac1p)\tau}\Big(\sum_{\bar k\in\rho(\bar s)}|a_{\bar k}|^2\Big)^{\frac{\tau}{2}}\Big\}^{\frac1\tau} \tag{9}$$

in the case $1<p<2$. Inequalities (5) and (9) imply that

$$\Big(\sum_{\langle\bar s,\bar\gamma\rangle<n}\sum_{\bar k\in\rho(\bar s)}|a_{\bar k}|^2\Big)^{\frac12}\le C2^{n(\frac1p-\frac12)}\|T_{n,\bar\gamma}\|_{p,\tau} \tag{10}$$

in the case $1<p<2$ and $1<\tau\le 2$.

According to Parseval's equality from (10) we get

$$\|T_{n,\bar\gamma}\|_2\le C2^{n(\frac1p-\frac12)}\|T_{n,\bar\gamma}\|_{p,\tau} \tag{11}$$

in the case $1<p<2$ and $1<\tau\le 2$.

Let $2<q<\infty$. Then, by virtue of Theorem 3.2, we have

$$\|T_{n,\bar\gamma}\|_{q,\tau}\le C\Big\{\sum_{\langle\bar s,\bar\gamma\rangle<n}2^{|\bar s|(\frac12-\frac1q)\tau}\Big(\sum_{\bar k\in\rho(\bar s)}|a_{\bar k}|^2\Big)^{\tau/2}\Big\}^{\frac1\tau} \tag{12}$$

for $1<\tau<\infty$.

Applying Hölder's inequality we obtain

$$\Big|\int_{\mathbb T^m}T_{n,\bar\gamma}(\bar x)g(\bar x)d\bar x\Big|\le\Big(\sum_{\langle\bar s,\bar\gamma\rangle<n}\sum_{\bar k\in\rho(\bar s)}|a_{\bar k}|^2\Big)^{\frac12}\Big(\sum_{\langle\bar s,\bar\gamma\rangle<n}\sum_{\bar k\in\rho(\bar s)}|b_{\bar k}(g)|^2\Big)^{\frac12}. \tag{13}$$

Since $2<q<\infty$, then $1<q'<2$. Therefore, as in the proof of inequalities (8), (10) we can see that

$$\Big(\sum_{\langle\bar s,\bar\gamma\rangle<n}\sum_{\bar k\in\rho(\bar s)}|b_{\bar k}(g)|^2\Big)^{\frac12}\le C2^{n(\frac1{q'}-\frac12)}n^{(m-1)(\frac12-\frac1{\tau'})}\Big\|\sum_{\langle\bar s,\bar\gamma\rangle<n}\sum_{\bar k\in\rho(\bar s)}b_{\bar k}(g)e^{i\langle\bar k,\bar x\rangle}\Big\|_{q',\tau'}\le C2^{n(\frac12-\frac1q)}n^{(m-1)(\frac1\tau-\frac12)}\|g\|_{q',\tau'} \tag{14}$$

in the case $2<\tau'<\infty$ (i.e. $1<\tau<2$) and

$$\Big(\sum_{\langle\bar s,\bar\gamma\rangle<n}\sum_{\bar k\in\rho(\bar s)}|b_{\bar k}(g)|^2\Big)^{\frac12}\le C2^{n(\frac1{q'}-\frac12)}\Big\|\sum_{\langle\bar s,\bar\gamma\rangle<n}\sum_{\bar k\in\rho(\bar s)}b_{\bar k}(g)e^{i\langle\bar k,\bar x\rangle}\Big\|_{q',\tau'}\le C2^{n(\frac12-\frac1q)}\|g\|_{q',\tau'} \tag{15}$$

in the case $1<\tau'\le 2$ (i.e. $2\le\tau<\infty$) for any $g\in L_{q',\tau'}(\mathbb T^m)$. Now inequalities (13)–(15) imply that

$$\|T_{n,\bar\gamma}\|_{q,\tau}\asymp\sup_{\|g\|_{q',\tau'}\le1}\Big|\int_{\mathbb T^m}T_{n,\bar\gamma}(\bar x)g(\bar x)d\bar x\Big|\le C2^{n(\frac12-\frac1q)}n^{(m-1)(\frac1\tau-\frac12)_+}\|T_{n,\bar\gamma}\|_2 \tag{16}$$

in the case $2<q<\infty$ and $1<\tau<\infty$.

Let $1<p<2<q<\infty$ and $1<\tau_2\le 2<\tau_1<\infty$. Then, it follows from inequalities (8) and (16) that

$$\|T_{n,\bar\gamma}\|_{q,\tau_2}\le C2^{n(\frac12-\frac1q)}n^{(m-1)(1/\tau_2-1/2)}\|T_{n,\bar\gamma}\|_2\le C2^{n(\frac1p-\frac1q)}n^{(m-1)(1/\tau_2-1/\tau_1)}\|T_{n,\bar\gamma}\|_{p,\tau_1} \tag{17}$$

in the case $1<p<2<q<\infty$ and $1<\tau_2\le 2<\tau_1<\infty$.

Consider the case $1<p<q\le 2$. Choose a number $q_1>2$ such that $1/q=(1-\theta)/p+\theta/q_1$, $0<\theta<1$. Then, by the well-known inequality (see [25, p. 228, Theorem 2.g.18])

$$\|f\|_{q,\tau_2}\le C\|f\|_{p,\tau_1}^{1-\theta}\|f\|_{q_1,\tau_3}^{\theta}$$

for $1<\tau_1,\tau_2,\tau_3<\infty$, and already proven case for $1<p<2<q_1<\infty$ we have (see (17))

$$\|T_{n,\bar\gamma}\|_{q,\tau_2}\le C\|T_{n,\bar\gamma}\|_{p,\tau_1}^{1-\theta}\|T_{n,\bar\gamma}\|_{q_1,\tau_3}^{\theta}\le C\|T_{n,\bar\gamma}\|_{p,\tau_1}^{1-\theta}\Big(2^{n(\frac1p-\frac1{q_1})}n^{(m-1)(1/\tau_3-1/\tau_1)}\|T_{n,\bar\gamma}\|_{p,\tau_1}\Big)^{\theta}. \tag{18}$$

Now, considering that $(1/p-1/q_1)\theta=1/p-1/q$ and choosing the number $\tau_3\in(1,2)$ so that $1/\tau_2=(1-\theta)/\tau_1+\theta/\tau_3$, from inequality (18) we obtain

$$\|T_{n,\bar\gamma}\|_{q,\tau_2}\le C2^{n(\frac1p-\frac1q)}n^{(m-1)(1/\tau_2-1/\tau_1)}\|T_{n,\bar\gamma}\|_{p,\tau_1}$$

in the case $1<p<q\le 2$ and $1<\tau_2<\tau_1$, $2<\tau_1<\infty$. □

Theorem 3.8. *Let $1<p<q\le 2$ and $1<\tau_1\le\tau_2\le 2$ or $1<p\le 2<q<\infty$ and $1<\tau_1\le 2<\tau_2<\infty$ or $2<p<q<\infty$ and $2<\tau_1\le\tau_2<\infty$, then*

$$\|T_{n,\bar\gamma}\|_{q,\tau_2}\le C2^{n(\frac1p-\frac1q)}\|T_{n,\bar\gamma}\|_{p,\tau_1}.$$

Proof. For $T_{n,\bar{\gamma}} \in \mathbb{T}(Q_n^{\bar{\gamma}})$ by Theorem 2.2 [21], we have

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C\Big\{ \sum_{\langle \bar{s},\bar{\gamma}\rangle<n} 2^{|\bar{s}|(1/p-1/q)\tau_2} \|\delta_{\bar{s}}(T_{n,\bar{\gamma}})\|_{p,\tau_1}^{\tau_2} \Big\}^{1/\tau_2}. \tag{19}$$

for $1 < p < q < \infty$ and $1 < \tau_1, \tau_2 < \infty$.

If $1 < \tau_1 \le \tau_2 < \infty$, considering that $|\bar{s}| = \sum_{j=1}^m s_j \le \langle \bar{s}, \bar{\gamma}\rangle$ according to Jensen's inequality we have

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C2^{n(\frac{1}{p}-\frac{1}{q})}\Big\{ \sum_{\langle \bar{s},\bar{\gamma}\rangle<n} \|\delta_{\bar{s}}(T_{n,\bar{\gamma}})\|_{p,\tau_1}^{\tau_2} \Big\}^{1/\tau_2}. \tag{20}$$

If $2 < p < \infty$, $2 < \tau_1 < \infty$, then according to Lemma 1.3 [21] from formula (20) we get

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C2^{n(\frac{1}{p}-\frac{1}{q})}\|T_{n,\bar{\gamma}}\|_{p,\tau_1}, \tag{21}$$

in the case $\tau_1 \le \tau_2$.

If $1 < p \le 2$ and $1 < \tau_1 \le 2$, then (see (11))

$$\|T_{n,\bar{\gamma}}\|_{2} \le C2^{n(\frac{1}{p}-\frac{1}{2})}\|T_{n,\bar{\gamma}}\|_{p,\tau_1}. \tag{22}$$

If $2 < q < \infty$ and $2 < \tau_2 < \infty$, then from inequality (16) we have

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C2^{n(\frac{1}{2}-\frac{1}{q})}\|T_{n,\bar{\gamma}}\|_{2}. \tag{23}$$

Inequalities (22), (23) imply that

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C2^{n(\frac{1}{p}-\frac{1}{q})}\|T_{n,\bar{\gamma}}\|_{p,\tau_1}, \tag{24}$$

in the case $1 < p \le 2 < q < \infty$, $1 < \tau_1 \le 2 < \tau_2 < \infty$.

Recall the well-known inequality (see [25])

$$\|f\|_{q,\tau_2} \le C\|f\|_{p,\tau_1}^{1-\theta}\|f\|_{q_1,\tau_3}^{\theta} \tag{25}$$

for $1 \le \tau_1, \tau_2 < \infty$, $0 < \theta < 1$.

Let $1 < p < q \le 2$ and $1 < \tau_1 \le \tau_2 < 2$. We choose the numbers $q_1 > 2$ and $\tau_3 > 2$ such that $1/q = (1-\theta)/p + \theta/q_1$ and $1/\tau_2 = (1-\theta)/\tau_1 + \theta/\tau_3$. According to inequalities (24) and (25) we get

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C\|T_{n,\bar{\gamma}}\|_{p,\tau_1}^{1-\theta}\|T_{n,\bar{\gamma}}\|_{q_1,\tau_3}^{\theta} \le C\|T_{n,\bar{\gamma}}\|_{p,\tau_1}^{1-\theta}\Big(2^{n(\frac{1}{p}-\frac{1}{q_1})}\|T_{n,\bar{\gamma}}\|_{p,\tau_1}\Big)^{\theta} = C2^{n(\frac{1}{p}-\frac{1}{q_1})\theta}\|T_{n,\bar{\gamma}}\|_{p,\tau_1}. \tag{26}$$

Since $1/q = (1-\theta)/p + \theta/q_1$, then $\theta(1/p - 1/q_1) = 1/p - 1/q$. Therefore, from (26) we obtain

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C2^{n(\frac{1}{p}-\frac{1}{q_1})}\|T_{n,\bar{\gamma}}\|_{p,\tau_1}, \tag{27}$$

in the case $1 < p < q \le 2$ and $1 < \tau_1 \le \tau_2 < 2$.

Thus, from inequalities (21), (24) and (27) it follows that

$$\|T_{n,\bar{\gamma}}\|_{q,\tau_2} \le C2^{n(\frac{1}{p}-\frac{1}{q})}\|T_{n,\bar{\gamma}}\|_{p,\tau_1},$$

in the cases $1 < p < q \le 2$ and $1 < \tau_1 \le \tau_2 < 2$ or $1 < p \le 2 < q < \infty$, $1 < \tau_1 \le 2 < \tau_2 < \infty$ or $2 < p < q < \infty$, $2 < \tau_1 \le \tau_2 < \infty$. □

4 Proofs of Main results

Proof of Theorem 2.1. Let $\frac{1}{q} - \frac{1}{p} < r_1 < \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$. For a natural number M, there exists a $\mu \in \mathbb{N}$ such that $M \asymp 2^{\mu}\mu^{\nu-1}$. For the given number $\mu \in \mathbb{N}$, we set

$$S_{Q_{\mu,\bar{\gamma}}}(f,\bar{x}) = \sum_{\substack{\langle \bar{s},\bar{\gamma}\rangle \le \mu,\\ \bar{s}\in\mathbb{Z}_+^m}} \delta_{\bar{s}}(f,\bar{x}),$$

where $\bar{\gamma} = (\gamma_1,\dots,\gamma_m)$, $\gamma_j = r_j/r_1$, $j = 1,\dots,m$.

Choose a natural number N such that $2^N \asymp 2^{\mu}\mu^{\nu-1}$. For a natural number $l \in (\mu, N]$, we include m_l blocks $\delta_{\bar{s}}(f,\bar{x})$, $l \le \langle \bar{s},\bar{\gamma}\rangle < l+1$, with the largest $\|\delta_{\bar{s}}(f)\|_{p,\tau_2}$ into the approximating polynomial of $f \in W_{q,\tau_1}^{\bar{r}}$. We denote the set of such indices $\bar{s}$ by G_l.

In Theorem 3.3, setting $f = \delta_{\bar{s}}(f)$, we obtain that

$$2^{\langle \bar{s},\bar{r}\rangle}\|\delta_{\bar{s}}(f)\|_{p,\tau} \asymp \|\delta_{\bar{s}}^{(\bar{r})}(f)\|_{p,\tau} = C\|\delta_{\bar{s}}(f^{(\bar{r})})\|_{p,\tau} = C\|\delta_{\bar{s}}(\varphi)\|_{p,\tau},\ 1 < p,\tau < \infty. \tag{28}$$

Using Theorem 3.1 for $\lambda = p$, $\theta = \tau_2$, $\tau = \tau_1$ and relation (28) we have

$$\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\prod_{j=1}^{m}2^{s_j\tau_1(\frac{1}{p}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\Big)^{1/\tau_1} \le \Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\prod_{j=1}^{m}2^{s_j\tau_1(\frac{1}{p}-\frac{1}{q})}2^{-\langle\bar{s},\bar{r}\rangle\tau_1}\|\delta_{\bar{s}}(\varphi)\|_{p,\tau_2}^{\tau_1}\Big)^{1/\tau_1}$$

$$\le C2^{-lr_1}\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\prod_{j=1}^{m}2^{s_j\tau_1(\frac{1}{p}-\frac{1}{q})}\|\delta_{\bar{s}}(\varphi)\|_{p,\tau_2}^{\tau_1}\Big)^{1/\tau_1} \ll 2^{-lr_1}\Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(\varphi)\Big\|_{q,\tau_1},$$

where $\varphi \in L_{q,\tau_1}(\mathbb{T}^m)$ and $\|\varphi\|_{q,\tau_1} \le 1$, $1 < q < p < \infty$, $1 < \tau_1, \tau_2 < \infty$.

By Theorem 3.4 we find from the above relation that

$$\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\prod_{j=1}^{m}2^{s_j\tau_1(\frac{1}{p}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\Big)^{1/\tau_1} \ll 2^{-lr_1}, \tag{29}$$

for a function $f \in W_{q,\tau_1}^{\bar{r}}$, $1 < q < p < \infty$ and $1 < \tau_1, \tau_2 < \infty$.

Since $1 = \gamma_1 = \ldots = \gamma_\nu < \gamma_{\nu+1} \le \ldots \le \gamma_m$ and $\frac{1}{p} - \frac{1}{q} \le 0$, then

$$\prod_{j=1}^{m}2^{s_j\tau_1(\frac{1}{p}-\frac{1}{q})} \ge 2^{\langle\bar{s},\bar{\gamma}\rangle(\frac{1}{p}-\frac{1}{q})\tau_1} \ge 2^{(l+1)(\frac{1}{p}-\frac{1}{q})\tau_1}.$$

Therefore, from inequality (29) we obtain

$$2^{(l+1)(\frac{1}{p}-\frac{1}{q})}\left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\right)^{1/\tau_1} \ll \left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\prod_{j=1}^{m}2^{s_j\tau_1(\frac{1}{p}-\frac{1}{q})}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\right)^{1/\tau_1} \ll 2^{-lr_1}.$$

Hence,

$$\left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\right)^{1/\tau_1} \ll 2^{-l(r_1+\frac{1}{p}-\frac{1}{q})}, \tag{30}$$

for a function $f \in W_{q,\tau_1}^{\bar{r}}$, $1 < q < p < \infty$ and $1 < \tau_1, \tau_2 < \infty$.

We set $\sigma_l = \{\bar{s} \in \mathbb{Z}_+^m : l \le \langle\bar{s},\bar{\gamma}\rangle < l+1\}$. By $|\sigma_l|$ is denoted the number of elements of σ_l.

Let $\{a_j\}$ be the nonincreasing rearrangement of numbers $\{\|\delta_{\bar{s}}\|_{p,\tau_2}\}_{\bar{s}\in\sigma_l}$. Then

$$\left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\right)^{1/\tau_1} = \left(\sum_{j=1}^{|\sigma_l|}a_j^{\tau_1}\right)^{1/\tau_1}.$$

By the definition of the set G_l, the number of elements of the set $\sigma_l \setminus G_l$ is equal to $|\sigma_l| - m_l$. Therefore, from the previous equality it follows that

$$\Big(\sum_{\substack{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\Big)^{1/\tau_1} = \Big(\sum_{j=m_l+1}^{|\sigma_l|}a_j^{\tau_1}\Big)^{1/\tau_1}. \tag{31}$$

Now applying Lemma 2.1 in [18] from equality (31) we obtain

$$\left(\sum_{\substack{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_2}\right)^{1/\tau_2} = \left(\sum_{j=m_l+1}^{|\sigma_l|}a_j^{\tau_1}\right)^{1/\tau_1}$$

$$\le m_l^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}\left(\sum_{j=1}^{|\sigma_l|}a_j^{\tau_1}\right)^{1/\tau_1} = (m_l+1)^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}\left(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\right)^{1/\tau_1}, \tag{32}$$

if $1 \le \tau_1 \le \tau_2 < \infty$.

(30) and (32) imply that

$$\Big(\sum_{\substack{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_2}\Big)^{1/\tau_2} \le C(m_l+1)^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}2^{-l(r_1+\frac{1}{p}-\frac{1}{q})} \tag{33}$$

for $1 \le \tau_1 \le \tau_2 < \infty$.

If $1 < \tau_2 \le 2$ and $1 < p < \infty$, then by Theorem 1.2 [21] the following inequality is true

$$\Big\|\sum_{\substack{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2} \le C\Big(\sum_{\substack{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_2}\Big)^{1/\tau_2}. \tag{34}$$

Now from inequalities (32) and (34) it follows that

$$\Big\|\sum_{\substack{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2}\le C(m_l+1)^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}2^{-l(r_1+\frac{1}{p}-\frac{1}{q})}, \tag{35}$$

for $1\le\tau_1\le\tau_2\le 2$, $1<q<p<\infty$.

Suppose that

$$F_l(\bar{x})=\sum_{\substack{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f,\bar{x}).$$

Let κ be a positive number so that $r_1+\frac{1}{p}-\frac{1}{q}<(\frac{1}{\tau_1}-\frac{1}{\tau_2})\kappa<\frac{1}{\tau_1}-\frac{1}{\tau_2}$. Assume $m_l=[2^{(N-l)\kappa}]$, where $[y]$ is the integer part of y. Then, by the property of the norm and inequality (35), we have

$$\Big\|\sum_{\mu<l\le N}F_l\Big\|_{p,\tau_2}\le\sum_{\mu<l\le N}\|F_l\|_{p,\tau_2}\ll 2^{-N(\frac{1}{\tau_1}-\frac{1}{\tau_2})\kappa}\sum_{\mu<l\le N}2^{-l(r_1+\frac{1}{p}-\frac{1}{q}-(\frac{1}{\tau_1}-\frac{1}{\tau_2})\kappa)}$$
$$\le C2^{-N(\frac{1}{\tau_1}-\frac{1}{\tau_2})\kappa}2^{-N(r_1+\frac{1}{p}-\frac{1}{q}-(\frac{1}{\tau_1}-\frac{1}{\tau_2})\kappa)}.$$

Thus,

$$\Big\|\sum_{\mu<l\le N}F_l\Big\|_{p,\tau_2}\le C2^{-N(r_1+\frac{1}{p}-\frac{1}{q})}, \tag{36}$$

for $\frac{1}{q}-\frac{1}{p}<r_1<\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_1}-\frac{1}{\tau_2}$, $1<\tau_1\le\tau_2\le 2$ and $1<q<p\le\infty$.

Consider the polynomial

$$A_n(f,\bar{x})=S_{Q_{\mu,\bar{\gamma}}}(f,\bar{x})+\sum_{\mu<l\le N}\sum_{\substack{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\in G_l}}\delta_{\bar{s}}(f,\bar{x}).$$

The polynomial has no more then n elements

$$n\le|Q_{\mu,\bar{\gamma}}|+\sum_{\mu<l\le N}2^l m_l\le 2^{\mu}\mu^{\nu-1}+\sum_{\mu<l\le N}2^l2^{(N-l)\kappa}\le C2^N\ll M.$$

By the property of the norm and Theorem 3.8 in the cases when $1<q<p\le 2$ and $1<\tau_1\le\tau_2\le 2$ we have

$$\Big\|\sum_{l>N}f_l\Big\|_{p,\tau_2}\le\sum_{l>N}\|f_l\|_{p,\tau_2}\le C\sum_{l>N}2^{l(\frac{1}{q}-\frac{1}{p})}\|f_l\|_{q,\tau_1}. \tag{37}$$

By Theorems 3.3 and 3.4 we have

$$\Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(\varphi)\Big\|_{q,\tau_1}=\Big\|\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f^{(\bar{r})})\Big)\Big\|_{q,\tau_1}\asymp\Big\|\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{r}\rangle 2}|\delta_{\bar{s}}(f)|^2\Big)^{1/2}\Big\|_{q,\tau_1}$$
$$\asymp 2^{lr_1}\Big\|\Big(\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}|\delta_{\bar{s}}(f)|^2\Big)^{1/2}\Big\|_{q,\tau_1}\asymp 2^{lr_1}\Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f)\Big\|_{q,\tau_1} \tag{38}$$

for $1<q<\infty$ and $1<\tau_1<\infty$.

(37) and (38) imply that

$$\Big\|\sum_{l>N}f_l\Big\|_{p,\tau_2}\ll\sum_{l>N}2^{-l(r_1+\frac{1}{p}-\frac{1}{q})}\Big\|\sum_{l\le\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(\varphi)\Big\|_{q,\tau_1}\ll\|\varphi\|_{q,\tau_1}\sum_{l>N}2^{-l(r_1+\frac{1}{p}-\frac{1}{q})}\ll 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})}\|\varphi\|_{q,\tau_1} \tag{39}$$

for a function $f\in W^{\bar{r}}_{q,\tau_1}$ and $\|\varphi\|\le 1$ in the cases $1<q<p\le 2$, $1<\tau_1\le\tau_2\le 2$ and $r_1>\frac{1}{q}-\frac{1}{p}$.

If $1<q<p\le 2$, $1<\tau_2<\tau_1$ and $2<\tau_1<\infty$, then by Theorem 3.7 we get

$$\Big\|\sum_{l>N}f_l\Big\|_{p,\tau_2}\le\sum_{l>N}\|f_l\|_{p,\tau_2}\le C\sum_{l>N}2^{l(\frac{1}{q}-\frac{1}{p})}l^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}\|f_l\|_{q,\tau_1}.$$

Therefore, taking into account inequality (38) and the definition of the class $W^{\bar{r}}_{q,\tau_1}$, we obtain

$$\Big\|\sum_{l>N}f_l\Big\|_{p,\tau_2}\ll\sum_{l>N}2^{-l(r_1+\frac{1}{p}-\frac{1}{q})}l^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}\|\varphi\|_{q,\tau_1}\ll 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})}N^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}, \tag{40}$$

for a function $f\in W^{\bar{r}}_{q,\tau_1}$ in the cases $1<q<p\le 2$, $1<\tau_2<\tau_1$, $2<\tau_1<\infty$ and $r_1>\frac{1}{q}-\frac{1}{p}$.

For a function $f\in W^{\bar{r}}_{q,\tau_1}$, by the property of the norm and inequalities (36), (39), we obtain

$$\Big\|f-A_n(f)\Big\|_{p,\tau_2}\le\Big\|\sum_{\mu<l\le N}F_l\Big\|_{p,\tau_2}+\Big\|\sum_{l>N}f_l\Big\|_{p,\tau_2}\le C2^{-N(r_1+\frac{1}{p}-\frac{1}{q})},$$

in the cases when $\frac{1}{q}-\frac{1}{p}<r_1<\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_1}-\frac{1}{\tau_2}$, $1<\tau_1<\tau_2\le 2$ and $1<q<p\le 2$.

Thus,

$$e_n(W^{\overline{r}}_{q,\tau_1})_{p,\tau_2}\le CM^{-(r_1+\frac{1}{p}-\frac{1}{q})} \tag{41}$$

for $\frac{1}{q}-\frac{1}{p}<r_1<\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_1}-\frac{1}{\tau_2}$, $1<\tau_1\le\tau_2\le 2$ and $1<q<p\le 2$. This completes the proof of the upper estimate in Theorem 2.1.

Let Ω_M be the set of M m–dimensional vectors $\{\overline{k}^{(1)},\dots,\overline{k}^{(M)}\}$ with integer coordinates, that is

$$\Omega_M=\{\overline{k}^{(l)}\in\mathbb{Z}^m_+: l=1,\dots,M\}.$$

There exists $\overline{s}^0=(s^0_1,\dots,s^0_m)\in\mathbb{Z}^m_+$ such that $|\rho(\overline{s}^0)|=\prod_{j=1}^m 2^{s^0_j-1}\asymp M$ and $|\rho(\overline{s}^0)|\ge 2M$. Then

$$|\rho(\overline{s}^0)\cap\Omega_M|\le\frac{|\rho(\overline{s}^0)|}{2}.$$

Consider the function

$$f_1(\overline{x})=\prod_{j=1}^m 2^{-s^0_j(r_1+1-\frac{1}{q})}\sum_{\overline{k}\in\rho(\overline{s}^0)}e^{i\langle\overline{k},\overline{x}\rangle}.$$

We claim that $f_1\in W^{\overline{r}}_{q,\tau_1}$. Indeed, for any function $\psi_1(\overline{x})=f_1^{(\overline{r})}$ using Bernstein's inequality and Dirichlet kernel estimate (see [21]) we have

$$\|\psi_1\|_{q,\tau_1}=\|f_1^{(\overline{r})}\|_{q,\tau_1}=\prod_{j=1}^m 2^{-s^0_j(r_1+1-\frac{1}{q})}\left\|\left(\sum_{\overline{k}\in\rho(\overline{s}^0)}e^{i\langle\overline{k},\overline{x}\rangle}\right)^{(\overline{r})}\right\|_{q,\tau_1}$$

$$\le C\prod_{j=1}^m 2^{-s^0_j(r_1+1-\frac{1}{q})}\prod_{j=1}^m 2^{s^0_j r_1}\left\|\sum_{\overline{k}\in\rho(\overline{s}^0)}e^{i\langle\overline{k},\overline{x}\rangle}\right\|_{q,\tau_1}\le C\prod_{j=1}^m 2^{-s^0_j(1-\frac{1}{q})}\prod_{j=1}^m 2^{s_j(1-\frac{1}{q})}=C_1.$$

Hence, the function $F_1=C_1^{-1}f_1\in W^{\overline{r}}_{q,\tau_1}$, $1<q,\tau_1<\infty$.

Let $T(\overline{x})$ denote an arbitrary trigonometric polynomial with "numbers" of harmonics in Ω_M, i.e. $T(\overline{x})=\sum_{\overline{k}\in\Omega_M}b_{\overline{k}}e^{i\langle\overline{k},\overline{x}\rangle}$. Then by Theorem 3.1 with $\lambda=\theta=2$ and Parseval's equality we get

$$\|F_1-T\|_{p,\tau_2}\ge C\prod_{j=1}^m 2^{s^0_j(\frac{1}{2}-\frac{1}{p})}\|\delta_{\overline{s}^0}(F_1-T)\|_2=C\prod_{j=1}^m 2^{-s^0_j(r_1+\frac{1}{p}-\frac{1}{q})}\prod_{j=1}^m 2^{-s^0_j/2}\left(|\rho(\overline{s}^0)|-M\right)^{1/2}. \tag{42}$$

Since $|\rho(\overline{s}^0)|\ge 2M$, then

$$|\rho(\overline{s}^0)|-M\ge\frac{|\rho(\overline{s}^0)|}{2}=\frac{1}{2}\prod_{j=1}^m 2^{s^0_j-1}.$$

Therefore (42) leads to

$$\|F_1-T\|_{p,\tau_2}\ge C\prod_{j=1}^m 2^{-s^0_j(r_1+\frac{1}{p}-\frac{1}{q})}\ge CM^{-(r_1+\frac{1}{p}-\frac{1}{q})}.$$

Hence,

$$e_M(F_1)_{p,\tau_2}\ge CM^{-(r_1+\frac{1}{p}-\frac{1}{q})}$$

means

$$e_M(W^{\overline{r}}_{q,\tau_1})_{p,\tau_2}\ge CM^{-(r_1+\frac{1}{p}-\frac{1}{q})} \tag{43}$$

for $1<q<p<\infty$, $1<\tau_1,\tau_2<\infty$ and $r_1>0$. The proof of the theorem is complete. □

Proof of Theorem 2.2. Let us consider the case $r_1=\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_1}-\frac{1}{\tau_2}$ for $1<\tau_1<\tau_2\le 2$. For a natural number M choose $n\in\mathbb{N}$ such that $M\asymp 2^n n^{\nu-1}$. Suppose that $n_0=n+(\nu-1)\log_2 n$. It is well-known that

$$\sup_{f\in W^{\overline{r}}_{q,\tau_1}}\left\|f-\sum_{\langle\overline{s},\overline{r}\rangle<n_0}\delta_{\overline{s}}(f)\right\|_{p,\tau_2}\le 2^{-n_0(r_1+\frac{1}{p}-\frac{1}{q})} \tag{44}$$

for $1<q<p<\infty$, $1<\tau_1,\tau_2<\infty$, $\overline{\gamma}=(\gamma_1,\dots,\gamma_m)$, $\gamma_j=\frac{r_j}{r_1}$, $j=1,\dots,m$.

Consider the sum

$$\left(\sum_{l\le\langle\overline{s},\overline{\gamma}\rangle<l+1}\|\delta_{\overline{s}}(\psi)\|^{\tau_1}_{p,\tau_2}2^{\langle\overline{s},\overline{1}\rangle(\frac{1}{p}-\frac{1}{q})\tau_1}\right)^{1/\tau_1}=S_l(\psi), \tag{45}$$

where $\psi = f^{(r)}$.

$\{R_k(\psi, p, \tau_2)\}$ denotes the nonincreasing rearrangement (see [26, P. 175–176]) of the sequence $\{\|\delta_{\bar{s}}(\psi)\|_{p,\tau_2}\}$. Then by the property of rearrangement of numbers we have

$$S_l(\psi) \geq 2^{(l+1)(\frac{1}{p}-\frac{1}{q})}\left(\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(\psi)\|_{p,\tau_2}^{\tau_1}\right)^{1/\tau_1} = 2^{(l+1)(\frac{1}{p}-\frac{1}{q})}\left(\sum_{\nu=1}^{|\Omega_l|}R_\nu^{\tau_1}(\psi,p,\tau_2)\right)^{1/\tau_1}$$
$$\geq 2^{(l+1)(\frac{1}{p}-\frac{1}{q})}\left(\sum_{\nu=1}^{k}R_\nu^{\tau_1}(\psi,p,\tau_2)\right)^{1/\tau_1} \geq 2^{(l+1)(\frac{1}{p}-\frac{1}{q})}k^{\frac{1}{\tau_1}}R_k(\psi,p,\tau_2).$$

Here $\Omega_l = \{\bar{s} : l \leq \langle\bar{s},\bar{\gamma}\rangle < l+1\}$, and $|\Omega_l|$ is the number of its elements. So,

$$R_k(\psi,p,\tau_2) \leq 2^{(l+1)(\frac{1}{q}-\frac{1}{p})}k^{-\frac{1}{\tau_1}}S_l(\psi). \tag{46}$$

In the approximating M-term polynomial $T_M(\bar{x})$ of a function $f \in W_{q,\tau_1}^{\bar{r}}$ we include the partial sum

$$S_{Q_{n,\bar{\gamma}}}(f,\bar{x}) = \sum_{l=1}^{n}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f,\bar{x})$$

and from the polynomial

$$\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\delta_{\bar{s}}(f,\bar{x}),\ \ l = n+1,\dots,n+(\nu-1)\log n,$$

we take only the first m_l "blocks" $\delta_{\bar{s}}(f,\bar{x})$ corresponding to the numbers $R_k(\psi,p,\tau_2)$. We denote the set of such $\bar{s}$ by $G(l)$. Then, by the property of the norm, we have

$$\|f-T_M\|_{p,\tau_2} = \left\|f-\left(S_{Q_{n,\bar{\gamma}}}(f)+\sum_{l=n}^{n+(\nu-1)\log n}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\in G_l}}\delta_{\bar{s}}(f,\bar{x})\right)\right\|_{p,\tau_2}$$
$$\leq \left\|\sum_{l=n}^{n_0}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\right\|_{p,\tau_2} + \left\|\sum_{\langle\bar{s},\bar{\gamma}\rangle\geq n_0}\delta_{\bar{s}}(f)\right\|_{p,\tau_2} \tag{47}$$

Using (44) from (47) we obtain

$$\|f-T_M\|_{p,\tau_2} \leq \left\|\sum_{l=n}^{n_0}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\right\|_{p,\tau_2} + C2^{-n_0(r_1+\frac{1}{p}-\frac{1}{q})}. \tag{48}$$

Further, according to Theorem 1.2 [21] and (46), we get

$$\left\|\sum_{l=n}^{n_0}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\right\|_{p,\tau_2} \leq C\left(\sum_{l=n}^{n_0}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\notin G_l}}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_2}\right)^{1/\tau_2} = C\left(\sum_{l=n}^{n_0}\sum_{k=|G_l|}^{|\Omega_l|}R_k^{\tau_2}(f,p,\tau_2)\right)^{1/\tau_2}$$
$$\leq C\left(\sum_{l=n}^{n_0}R_{m_l}^{\tau_2-\tau_1}(f,p,\tau_2)\sum_{k=m_l}^{|\Omega_l|}R_k^{\tau_1}(f,p,\tau_2)\right)^{1/\tau_2} \leq \left(\sum_{l=n}^{n_0}m_l^{-\frac{\tau_2-\tau_1}{\tau_1}}2^{-l(r_1+\frac{1}{p}-\frac{1}{q})(\tau_2-\tau_1)}S_l^{\tau_2-\tau_1}(\psi)\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\right)^{1/\tau_2}$$
$$\leq C\left(\sum_{l=n}^{n_0}m_l^{-\frac{\tau_2-\tau_1}{\tau_1}}2^{-l\tau_2(r_1+\frac{1}{p}-\frac{1}{q})}S_l^{\tau_2}(\psi)\right)^{1/\tau_2}.$$

Hence,

$$\left\|\sum_{l=n}^{n_0}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\right\|_{p,\tau_2} \leq C\left(\sum_{l=n}^{n_0}m_l^{-\frac{\tau_2-\tau_1}{\tau_1}}2^{-l\tau_2(r_1+\frac{1}{p}-\frac{1}{q})}S_l^{\tau_2}(\psi)\right)^{1/\tau_2}. \tag{49}$$

As is known, the number of harmonics of the approximating polynomial does not exceed $\sum_{l=n}^{n_0}2^l m_l$. Assume that

$$m_l = [2^n n^{\nu-1}2^{-l}S_l^{\tau_1}(\psi)]+1.$$

Then

$$\sum_{l=n}^{n_0} 2^l m_l \leq 2^n n^{\nu-1} \sum_{n\leq\langle\bar{s},\bar{\gamma}\rangle<n_0+1} 2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{p}-\frac{1}{q})\tau_2}\|\delta_s(\psi)\|_{p,\tau_2}^{\tau_2} + 2\cdot 2^n n^{\nu-1}.$$

Applying Theorem 3.1 we get

$$\sum_{l=n}^{n_0} 2^l m_l \leq C2^n n^{\nu-1}(1+\|\psi\|_{q,\tau_1}^{\tau_1}) \leq CM.$$

Now, substituting the values of m_l into (49), we obtain

$$\left\|\sum_{l=n}^{n_0}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\right\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}\left(\sum_{l=n}^{n_0}2^{-l(r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1})}S_l^{\tau_1}(\psi)\right)^{1/\tau_2}. \tag{50}$$

If $r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}=0$, then

$$\left\|\sum_{l=n}^{n_0}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\right\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}\left(\sum_{l=n}^{n_0}S_l^{\tau_1}(\psi)\right)^{1/\tau_2}$$
$$= C(2^n n^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}\left(\sum_{l=n}^{n_0}\sum_{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1}2^{\langle\bar{s},\bar{1}\rangle(\frac{1}{p}-\frac{1}{q})\tau_1}\|\delta_s(\psi)\|_{p,\tau_2}^{\tau_1}\right)^{1/\tau_2}. \tag{51}$$

Next, using Theorem 3.1 [21] from (51) we obtain

$$\left\|\sum_{l=n}^{n_0}\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\right\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}\|\psi\|_{q,\tau_1} \leq C(2^n n^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}. \tag{52}$$

Now, equations (48) and (52) lead to

$$\|f-T_M\|_{p,\tau_2} \leq C\left((2^n n^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}+2^{-n_0(r_1+\frac{1}{p}-\frac{1}{q})}\right)$$

for a function $f\in W^{\bar{r}}_{q,\tau_1}$, $1<q<p\leq 2$, $1<\tau_1<\tau_2\leq 2$ in the case when $r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}=0$.

Considering that $n_0=n+[(\nu-1)\log n]$, from here we get

$$\|f-T_M\|_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}$$

for a function $f\in W^{\bar{r}}_{q,\tau_1}$, $1<q<p\leq 2$, $1<\tau_1<\tau_2\leq 2$ in the case when $r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}=0$. Hence,

$$e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2} \leq C(2^n n^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})} \asymp M^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}.$$

The lower estimate in the theorem follows from inequality (43) in the proof of Theorem 2.1 for $r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}>0$. □

Remark 2. The lower estimate in the cases when $p=2$ and $\tau_2\neq 2$ remains open.

Proof of Theorem 2.3. Consider the case $r_1>\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_1}-\frac{1}{\tau_2}$. Assume that $m_l=2^{N-l}$. Then, taking into account that $2^N\asymp 2^\mu\mu^{\nu-1}$ from formula (35) we obtain

$$\left\|\sum_{\mu<l\leq N}F_l\right\|_{p,\tau_2} \leq C\sum_{\mu<l\leq N}\|F_l\|_{p,\tau_2}$$
$$\leq C\sum_{\mu<l\leq N}(m_l+1)^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}2^{-l(r_1+\frac{1}{p}-\frac{1}{q})} \leq C2^{-N(\frac{1}{\tau_1}-\frac{1}{\tau_2})}\sum_{\mu<l\leq N}2^{-l(r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_1}-\frac{1}{\tau_2})}$$
$$\leq C2^{-N(\frac{1}{\tau_1}-\frac{1}{\tau_2})}2^{-\mu(r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_1}-\frac{1}{\tau_2})} \leq C(2^\mu\mu^{\nu-1})^{-(\frac{1}{\tau_1}-\frac{1}{\tau_2})}2^{-\mu(r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_1}-\frac{1}{\tau_2})}$$
$$= C2^{-\mu(r_1+\frac{1}{p}-\frac{1}{q})}\mu^{-(\nu-1)(\frac{1}{\tau_1}-\frac{1}{\tau_2})} \leq C2^{-N(r_1+\frac{1}{p}-\frac{1}{q})}N^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1})}$$
$$\asymp n^{-(r_1+\frac{1}{p}-\frac{1}{q})}(\log^{\nu-1}n)^{r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}}.$$

So, if $r_1>\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_1}-\frac{1}{\tau_2}$, then

$$\left\|\sum_{\mu<l\leq N}F_l\right\|_{p,\tau_2} \leq Cn^{-(r_1+\frac{1}{p}-\frac{1}{q})}(\log^{\nu-1}n)^{r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}} \tag{53}$$

for any function $f \in W^{\overline{r}}_{q,\tau_1}$ in the cases $1 < q < p \le 2$ and $1 < \tau_1 \le \tau_2 \le 2$.

Since $r_1 > \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2} \ge \frac{1}{q} - \frac{1}{p}$ for $\tau_1 \le \tau_2$, (39) implies that

$$\Big\|\sum_{l>N} f_l\Big\|_{p,\tau_2} \le C2^{-N(r_1+\frac{1}{p}-\frac{1}{q})} \le 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})}N^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1})} \le Cn^{-(r_1+\frac{1}{p}-\frac{1}{q})}(\log^{\nu-1} n)^{r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}} \tag{54}$$

for any function $f \in W^{\overline{r}}_{q,\tau_1}$ in the cases when $1 < q < p \le 2$ and $1 < \tau_1 \le \tau_2 \le 2$.

Now (53) and (54) imply that

$$\|f - A_n(f)\|_{p,\tau_2} \le \Big\|\sum_{\mu<l\le N} F_l\Big\|_{p,\tau_2} + \Big\|\sum_{l>N} f_l\Big\|_{p,\tau_2} \le Cn^{-(r_1+\frac{1}{p}-\frac{1}{q})}(\log^{\nu-1} n)^{r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}}$$

for any function $f \in W^{\overline{r}}_{q,\tau_1}$ in the cases when $r_1 > \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$, $1 < q < p \le 2$ and $1 < \tau_1 \le \tau_2 \le 2$. Hence,

$$e_n(W^{\overline{r}}_{q,\tau_1})_{p,\tau_2} \le CM^{-(r_1+\frac{1}{p}-\frac{1}{q})}(\log^{\nu-1} M)^{r_1+\frac{1}{p}-\frac{1}{q}+\frac{1}{\tau_2}-\frac{1}{\tau_1}}$$

for $r_1 > \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$, $1 < q < p \le 2$, $1 < \tau_1 \le \tau_2 \le 2$. This proves the upper estimate in Theorem 2.3.

The lower estimate in the cases $r_1 > \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$ and $1 < q \le p \le 2$. Let M be a sufficiently large natural number. Choose a natural number n such that $M \asymp 2^n n^{\nu-1}$. Assume that $F^{\overline{\gamma}}_n = \cup_{\langle\overline{s},\overline{\gamma}\rangle=n}\rho(\overline{s})$ is a set such that its number of points is

$$|F^{\overline{\gamma}}_n| > 4M. \tag{55}$$

This is always possible since $|F^{\overline{\gamma}}_n| \asymp 2^n n^{\nu-1} \asymp M$ (see, for example, [4], [18]).

Consider the function

$$f_0(\overline{x}) = 2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}}\sum_{\overline{k}\in F^{\overline{\gamma}}_n} e^{i\langle\overline{k},\overline{x}\rangle}.$$

Since $r_1 > 0$, then $r_1 + \frac{1}{p} - \frac{1}{q} > 0$. Hence, $2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}} \downarrow 0$ as $n \to \infty$. For f_0 we have

$$\|f_{l,\overline{r}}\|_{q,\tau_1} = 2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}}\Big\|\sum_{\langle\overline{s},\overline{\gamma}\rangle=l}\sum_{\overline{k}\in\rho(\overline{s})} e^{i\langle\overline{k},\overline{k}\rangle}\Big\|_{q,\tau_1} \tag{56}$$

for $l = 1, \dots, n$. If $l > n$, then $\|f_{l,\overline{r}}\|_{q,\tau_1} = 0$.

By virtue of the estimate for the norm of the Dirichlet kernel [21] and by Theorem 3.2, we have

$$\|D_{n,\overline{\gamma}}\|_{q,\tau_1} \ll \Big(\sum_{\langle\overline{s},\overline{\gamma}\rangle\le n}\prod_{j=1}^{m} 2^{s_j(\frac{1}{p_0}-\frac{1}{q})}\|\delta_{\overline{s}}(D_{n,\overline{\gamma}})\|^{\tau_1}_{p_0}\Big)^{1/\tau_1} \ll \Big(\sum_{\langle\overline{s},\overline{\gamma}\rangle\le n}\prod_{j=1}^{m} 2^{s_j(1-\frac{1}{q})\tau_1}\Big)^{1/\tau_1} \tag{57}$$

for some $p_0 \in (1, q)$, $1 < \tau_1 < \infty$. Choose numbers δ_j, $j = 1, \dots, m$ such that $\delta_j = \gamma_j = 1$ for $j = 1, \dots, \nu$ and $1 < \delta_j < \gamma_j$ for $j = \nu+1, \dots, m$. Then (57) implies that

$$\|D_{n,\overline{\gamma}}\|_{q,\tau_1} \ll \Big(\sum_{\langle\overline{s},\overline{\gamma}\rangle\le n} 2^{\langle\overline{s},\overline{\delta}\rangle(1-\frac{1}{q})\tau_1}\Big)^{1/\tau_1},$$

where $\overline{\delta} = (\delta_1, \dots \delta_m)$.

Hence, by Lemma Г [4] we have

$$\|D_{n,\overline{\gamma}}\|_{q,\tau_1} \ll 2^{n(1-\frac{1}{q})}n^{(\nu-1)/\tau_1}, \tag{58}$$

whenever $1 < q < \infty$ and $1 < \tau_1 < \infty$. Now from inequalities (56) and (58) it follows that

$$\|f_{l,\overline{r}}\|_{q,\tau_1} \ll 2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}}\Big(\|D_{l,\overline{\gamma}}\|_{q,\tau_1} + \|D_{l-1,\overline{\gamma}}\|_{q,\tau_1}\Big) \ll 2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}}2^{l(1-\frac{1}{q})}l^{\frac{\nu-1}{\tau_1}}$$

for $l = 1, \dots, n$.

Thus,

$$\|f_0\|_{q,\tau_1} \ll 2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}}\sum_{l=1}^{n}\|f_{l,\overline{r}}\|_{q,\tau_1} \ll 2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}}\sum_{l=1}^{n}2^{l(1-\frac{1}{q})}l^{\frac{\nu-1}{\tau_1}} \ll 2^{-nr_1}.$$

Consequently, by Theorem 3.3 and Theorem 3.4 we get

$$\|\varphi_0\|_{q,\tau_1} = \|f_0^{(\overline{r})}\|_{q,\tau_1} \ll \Big\|\Big(\sum_{\langle\overline{s},\overline{\gamma}\rangle=n} 2^{\langle\overline{s},\overline{r}\rangle 2}|\delta_{\overline{s}}(f_0)|^2\Big)^{1/2}\Big\|_{q,\tau_1} \ll 2^{nr_1}\Big\|\Big(\sum_{\langle\overline{s},\overline{\gamma}\rangle=n}|\delta_{\overline{s}}(f_0)|^2\Big)^{1/2}\Big\|_{q,\tau_1} \ll 2^{nr_1}\|f_0\|_{q,\tau_1} \le C_0.$$

This means that the function $F_0 = C_0^{-1}f_0 \in W^{\overline{r}}_{q,\tau_1}$ and $\|\varphi_0 C_0^{-1}\|_{q,\tau_1} \le 1$ for $1 < q, \tau_1 < \infty$.

Let us evaluate $e_M(F_0)_{p,\tau_2}$. Let Ω_M be a set of M integer vectors $\overline{k}^{(j)} = (k_1^{(j)}, \dots, k_m^{(j)})$, $j = 1, \dots, M$. Consider the set $\Omega_M \cap \rho(\bar{s})$ for indices $\bar{s}$ such that $\langle \bar{s}, \bar{\gamma} \rangle = n$. Then, by virtue of (55), the set P of vectors $\bar{s}$ such that $\langle \bar{s}, \bar{\gamma} \rangle = n$ and

$$|\Omega_M \cap \rho(\bar{s})| < \frac{|\rho(\bar{s})|}{2} \tag{59}$$

will contain at least half of all $\bar{s}$ such that $\langle \bar{s}, \bar{\gamma} \rangle = n$, and therefore

$$|P| \asymp n^{\nu-1}. \tag{60}$$

Suppose $T_M(\bar{x})$ is a polynomial with harmonics from Ω_M. Then by Theorem 3.1 we have

$$\|F_0 - T_M\|_{p,\tau_2} \geq C\Big(\sum_{\bar{s}\in\mathbb{Z}_+^m}\prod_{j=1}^m 2^{s_j(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(F_0-T_M)\|_2^{\tau_2}\Big)^{\frac{1}{\tau_2}}$$

$$\geq C\Big(\sum_{\langle\bar{s},\bar{\gamma}\rangle=n}\prod_{j=1}^m 2^{s_j(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(F_0-T_M)\|_2^{\tau_2}\Big)^{\frac{1}{\tau_2}} \geq C\Big(\sum_{\bar{s}\in P}\prod_{j=1}^m 2^{s_j(\frac{1}{2}-\frac{1}{p})\tau_2}\|\delta_{\bar{s}}(F_0-T_M)\|_2^{\tau_2}\Big)^{\frac{1}{\tau_2}} \tag{61}$$

for $1 < p < 2$ and $1 < \tau_2 < \infty$. Applying Hölder's inequality for $\frac{1}{\tau_2} + \frac{1}{\tau_2'} = 1$ and Galeev's lemma we obtain

$$|P| = \sum_{\bar{s}\in P} 1 \leq \Big(\sum_{\bar{s}\in P}\prod_{j=1}^m 2^{s_j(1-\frac{1}{p})\tau_2}\Big)^{\frac{1}{\tau_2}}\Big(\sum_{\bar{s}\in P}\prod_{j=1}^m 2^{-s_j(1-\frac{1}{p})\tau_2'}\Big)^{\frac{1}{\tau_2'}}$$

$$\leq \Big(\sum_{\bar{s}\in P}\prod_{j=1}^m 2^{s_j(1-\frac{1}{p})\tau_2}\Big)^{\frac{1}{\tau_2}}\Big(\sum_{\langle\bar{s},\bar{\gamma}\rangle\geq n}\prod_{j=1}^m 2^{-s_j(1-\frac{1}{p})\tau_2'}\Big)^{\frac{1}{\tau_2'}} \leq C2^{-n(1-\frac{1}{p})}n^{\frac{\nu-1}{\tau_2'}}\Big(\sum_{\bar{s}\in P}\prod_{j=1}^m 2^{s_j(1-\frac{1}{p})\tau_2}\Big)^{\frac{1}{\tau_2}}.$$

Since by (60) it follows that

$$\Big(\sum_{\bar{s}\in P}\prod_{j=1}^m 2^{s_j(1-\frac{1}{p})\tau_2}\Big)^{\frac{1}{\tau_2}} \geq C2^{n(1-\frac{1}{p})}n^{-\frac{\nu-1}{\tau_2'}}|P| \geq C2^{n(1-\frac{1}{p})}n^{\frac{\nu-1}{\tau_2}}.$$

Therefore, using Parseval's equality in the space $L_2(\mathbb{T}^m)$ from (61) we obtain

$$\|F_0 - T_M\|_{p,\tau_2} \gg 2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}}\Big(\sum_{\bar{s}\in P}\prod_{j=1}^m 2^{s_j(1-\frac{1}{p})\tau_2}\Big)^{\frac{1}{\tau_2}}$$

$$\geq C2^{-n(r_1+1-\frac{1}{q})}n^{-\frac{\nu-1}{\tau_1}}2^{n(1-\frac{1}{p})}n^{\frac{\nu-1}{\tau_2}} = C2^{-n(r_1+\frac{1}{p}-\frac{1}{q})}n^{(\nu-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}$$

for $r_1 > \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$, $1 < q < p < 2$ and $1 < \tau_1, \tau_2 < \infty$.

Thus, for the function $F_0 \in W_{q,\tau_1}^{\bar{r}}$ it is proved that

$$\|F_0 - T\|_{p,\tau_2} \gg 2^{-n(r_1+\frac{1}{p}-\frac{1}{q})}n^{(\nu-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \tag{62}$$

for $1 < q < p < 2$ and $1 < \tau_1, \tau_2 < \infty$.

Therefore, taking into account that $M \asymp 2^n n^{\nu-1}$ we have

$$e_M(W_{q,\tau_1}^{\bar{r}})_{p,\tau_2} \gg 2^{-n(r_1+\frac{1}{p}-\frac{1}{q})}n^{(\nu-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \asymp M^{-(r_1+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})}(\log M)^{(\nu-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}, \tag{63}$$

for $r_1 > \frac{1}{q} - \frac{1}{p} + \frac{1}{\tau_1} - \frac{1}{\tau_2}$, $1 < q < p < 2$, $1 < \tau_1, \tau_2 < \infty$. The inequality shows the accuracy of the estimate in Theorem 2.3. □

Proof of Theorem 2.4. Let us prove the upper estimate in Theorem 2.4.

Consider the case $1 < \tau_2 < \tau_1 < \infty$. If $1 < \tau_2 \leq 2$ and $1 < p < \infty$, then by (34) we get

$$\Big\|\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2} \leq C\left(\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_2}\right)^{1/\tau_2}.$$

Since $1 < \tau_2 < \tau_1 < \infty$, applying Hölder's inequality to the sum on the right side of the above inequality we obtain

$$\Big\|\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\delta_{\bar{s}}(f)\Big\|_{p,\tau_2} \leq C\Big(\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\Big)^{1/\tau_1}\Big(\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}1\Big)^{\frac{1}{\tau_2\theta'}}$$

$$\leq Cl^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}\Big(\sum_{\substack{l\leq\langle\bar{s},\bar{\gamma}\rangle<l+1,\\ \bar{s}\notin G_l}}\|\delta_{\bar{s}}(f)\|_{p,\tau_2}^{\tau_1}\Big)^{1/\tau_1}. \tag{64}$$

For a function $f \in W^{\bar{r}}_{q,\tau_1}$, by (30) and (64) we get

$$\Big\| \sum_{\substack{l \le \langle \bar{s},\bar{\gamma}\rangle < l+1, \\ \bar{s}\notin G_l}} \delta_{\bar{s}}(f) \Big\|_{p,\tau_2} \le C l^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} 2^{-l(r_1+\frac{1}{p}-\frac{1}{q})} \tag{65}$$

in the cases $1<\tau_2\le 2$, $\tau_2<\tau_1<\infty$ and $1<q<p<\infty$.

Consider the sum

$$F_l(\bar{x}) = \sum_{\substack{l \le \langle \bar{s},\bar{\gamma}\rangle < l+1, \\ \bar{s}\notin G_l}} \delta_{\bar{s}}(f,\bar{x}).$$

Let $\frac{1}{q}-\frac{1}{p}<r_1<\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_2}-\frac{1}{\tau_1}$ and κ be a positive number such that $0<r_1+\frac{1}{p}-\frac{1}{q}<(\frac{1}{\tau_2}-\frac{1}{\tau_1})\kappa<\frac{1}{\tau_2}-\frac{1}{\tau_1}$. Suppose $m_l=[2^{(N-l)\kappa}]$, where $[y]$ is an integer part of y. Then, by the property of the norm and inequality (65) taking into account that $r_1>\frac{1}{q}-\frac{1}{p}$ we have

$$\Big\| \sum_{\mu<l\le N} F_l \Big\|_{p,\tau_2} \le C \sum_{\mu<l\le N} \|F_l\|_{p,\tau_2} \le \sum_{\mu<l\le N} 2^{-l(r_1+\frac{1}{p}-\frac{1}{q})} l^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \le C 2^{-\mu(r_1+\frac{1}{p}-\frac{1}{q})} \mu^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \tag{66}$$

for $1<\tau_2\le 2$, $\tau_2<\tau_1<\infty$ and $1<q<p\le 2$.

Consider the polynomial

$$A_n(f,\bar{x}) = S_{Q_{\mu,\bar{\gamma}}}(\bar{x}) + \sum_{\mu<l\le N} \sum_{\substack{l \le \langle \bar{s},\bar{\gamma}\rangle < l+1, \\ \bar{s}\in G_l}} \delta_{\bar{s}}(f,\bar{x}).$$

This polynomial has at most n elements

$$n \le |Q_{\mu,\bar{\gamma}}| + \sum_{\mu<l\le N} 2^l m_l \le C 2^N.$$

If $1<q<p\le 2$, $2<\tau_1<\infty$ and $1<\tau_2<\tau_1<\infty$, then by Theorem 5 [27] we have

$$\Big\| \sum_{l>N} f_l \Big\|_{p,\tau_2} \le C \sum_{l>N} \|f_l\|_{p,\tau_2} \le C \sum_{l>N} 2^{l(\frac{1}{q}-\frac{1}{p})} l^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \|f_l\|_{q,\tau_1}.$$

Next, using inequality (38) and the definition of $W^{\bar{r}}_{q,\tau_1}$, we obtain

$$\Big\| \sum_{l>N} f_l \Big\|_{p,\tau_2} \le C \sum_{l>N} 2^{-l(r_1+\frac{1}{p}-\frac{1}{q})} l^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \le C 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})} N^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}. \tag{67}$$

Now, by the property of the norm and inequalities (66), (67), we obtain

$$\begin{aligned} \Big\| f - A_n(f) \Big\|_{p,\tau_2} &\le \Big\| \sum_{\mu<l\le N} F_l \Big\|_{p,\tau_2} + \Big\| \sum_{l>N} f_l \Big\|_{p,\tau_2} \\ &\le C \Big\{ 2^{-\mu(r_1+\frac{1}{p}-\frac{1}{q})} \mu^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} + 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})} N^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \Big\} \end{aligned} \tag{68}$$

for a function $f \in W^{\bar{r}}_{q,\tau_1}$, where $1<\tau_2\le 2<\tau_1<\infty$, $1<q<p\le 2$ and $\frac{1}{q}-\frac{1}{p}<r_1<\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_2}-\frac{1}{\tau_1}$.

Since $2^\mu \mu^{\nu-1} \asymp 2^N$, we obtain

$$\begin{aligned} 2^{-\mu(r_1+\frac{1}{p}-\frac{1}{q})} \mu^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} &= (2^\mu \mu^{\nu-1})^{-(r_1+\frac{1}{p}-\frac{1}{q})} \mu^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})} \mu^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \\ \le C 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})} \mu^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})} \mu^{(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} &\le C 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})} \mu^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})+(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \\ &\le C 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})} N^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})+(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}. \end{aligned}$$

Therefore, (68) implies that

$$\begin{aligned} \Big\| f - A_n(f) \Big\|_{p,\tau_2} &\le C 2^{-N(r_1+\frac{1}{p}-\frac{1}{q})} N^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})+(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \\ &\asymp n^{-(r_1+\frac{1}{p}-\frac{1}{q})} (\log n)^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})+(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})} \end{aligned}$$

for a function $f \in W^{\bar{r}}_{q,\tau_1}$ in the cases when $1<\tau_2\le 2<\tau_1<\infty$, $1<q<p\le 2$ and $\frac{1}{q}-\frac{1}{p}<r_1<\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_2}-\frac{1}{\tau_1}$.

Thus,

$$e_n(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2} \le C n^{-(r_1+\frac{1}{p}-\frac{1}{q})} (\log n)^{(\nu-1)(r_1+\frac{1}{p}-\frac{1}{q})+(m-1)(\frac{1}{\tau_2}-\frac{1}{\tau_1})}$$

for $1<q<p\le 2$, $1<\tau_2\le 2<\tau_1<\infty$ and $\frac{1}{q}-\frac{1}{p}<r_1<\frac{1}{q}-\frac{1}{p}+\frac{1}{\tau_2}-\frac{1}{\tau_1}$. This completes the proof of the theorem. □

5 Conclusion

In [13], order-sharp estimates for the best M–term approximations of functions from the Sobolev class in the norm of the Lorentz space are established for $1 < q < 2 < p < \infty$, $1 < \tau_1 \le \tau_2 < \infty$, and $\frac{1}{q} - \frac{1}{p} < r_1 < \frac{1}{q}$. And for $q = 2$ the upper bounds of the quantity $e_M(W^{\bar{r}}_{2,\tau_1})_{p,\tau_2}$ are established in [14]. In this work, order-sharp estimates of the quantity $e_M(W^{\bar{r}}_{q,\tau_1})_{p,\tau_2}$ are obtained for $1 < q < p \le 2$, $1 < \tau_1 \le \tau_2 \le 2$ and $\frac{1}{q} - \frac{1}{p} < r_1 < \infty$, and the upper bounds of this quantity are established in the case $1 < \tau_2 \le 2 < \tau_1 < \infty$.

Remark 3. For $\tau_1 = q$ and $\tau_2 = p$, Theorem 2.4 implies the results of E.S. Belinsky [9, Theorem 2.1] and V.N. Temlyakov [4, Theorem 2.1], [17, Theorem 2.7, Theorem 2.8], [18, Theorem 1.3, Theorem 1.4].

Funding

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP22683029).

References

[1] Temlyakov V.N. *Multivariate approximation*. Cambridge University Press, Cambridge, 2018.

[2] Stein E.M., Weiss G. *Introduction to Fourier analysis on Euclidean spaces*. Princeton: Princeton Univ. Press, 1971, 312 p.

[3] Temlyakov V.N. *On the approximation of periodic functions of several variables*, Dokl. Akad. Nauk SSSR **279** (1984), P. 301–305; English transl., Soviet Math. Dokl. 30 (1984), P. 659–662.

[4] Temlyakov V.N. *Approximation of functions with bounded mixed derivative*, Tr. Mat. Inst. Steklov., **178** (1986), P. 3–112.

[5] Stechkin S.B. *On the absolute convergence of orthogonal series*, Doklad. Akadem. Nauk SSSR , **102**:2 (1955), P. 37–40.

[6] Ismagilov R.S. *Widths of sets in linear normed spaces and the approximation of functions by trigonometric polynomials*, Uspehi mathem. nauk, **29**:3 (1974), P. 161–178.

[7] Maiorov V.E. *Trigonometric diametrs of the Sobolev classes W^r_p in the space L_q*, Math. Notes, **40**:2 (1986), P. 161–173

[8] Belinsky E.S. *Approximation of periodic functions by a 'floating' system of exponents and trigonometric diametrs*, Issledovanija po teorii funkcij mnogih veshhestvennyh peremennyh, Jaroslavl' – Research on the theory of functions of many real variables, Yaroslavl' State University (1984), 10-24 [in Russian].

[9] Belinsky E.S. *Approximation by a 'floating' system of exponentials on the classes of smooth periodic functions with bounded mixed derivative*, Research on the theory of functions of many real variables Yaroslavl' State University (1988), P. 16-33 .

[10] DeVore R.A. *Nonlinear approximation*, Acta Numerica, *7* (1998), P. 51–150.

[11] Dinh Dũng, Temlyakov V.N., Ullrich T. Hyperbolic Cross Approximation. Advanced Courses in Mathematics. CRM Barcelona. Birkh*ä*user/Springer, Basel/Berlin (2018), 222 p.

[12] Romanyuk A.S. *Best trigonometric approximations for some classes of periodic functions of several variables in the uniform metric*, Math. Notes, V. 82, Issue 2 (2007), P. 216–228.

[13] Akishev G., Myrzagaliyeva A. *On estimates of M–term approximations on classes of functions with bounded mixed derivative in Lorentz spaces*, *Journal Math. Sci.* (2023), P. 1–16. https://doi.org/10.1007/s10958-022-06146-7

[14] Akishev G., Myrzagaliyeva A. *On estimates of M–term approximations of the Sobolev class in the Lorentz space*, Bulletin of the Karaganda University. Mathematics series, **114**:2 (2024), P. 4–21, DOI: https://doi.org/10.31489/2024m2/4-21

[15] Myrzagaliyeva A., Akishev G. *Estimates for the best M–term approximations of functions of class $W^{a,b(\cdot),\bar{r}}_{q,\tau,\theta}$ in the Lorentz space*, Journal Math. Sci., **291**:2 (2025), P. 282–301, DOI: https://doi.org/10.1007/s10958-025-07807-z

[16] Myrzagaliyeva A. Kh., Akishev G. *On estimates of the best M–term approximations of functions of the class $W^{\bar{r}}_{q,\tau}$ in the Lorentz space*, CJM., **42**:1 (2026), P. 47–63, DOI: https://doi.org/10.37193/CJM.2026.01.04

[17] Temlyakov V.N. *Constructive sparse trigonometric approximation and other problems for functions with mixed smoothness*, Sbornic Mathematics, **206**:11 (2015), P. 131–160.

[18] Temlyakov V.N. *Constructive sparse trigonometric approximation for functions with small mixed smoothness*, Constr. Approx., *45* (3), P. 467–495 (2017).

[19] Bazarkhanov D.B., Temlyakov V.N. *Nonlinear tensor product approximation of functions*, ArXiv: *1409.1403v1 [stat ML])* (2014), P. 1–23.

[20] Bazarkhanov D.B. *Nonlinear trigonometric approximation of multivariate function classes*, Proceedings Steklov Inst. Math., **293** (2016), P. 2–36.

[21] Akishev G. *Estimation of the best approximations of the functions Nikol'skii - Besov class in the space of Lorentz by trigonometric polynomials*, Trudy Instituta Matematiki i Mekhaniki UrO RAN, **26**:2 (2020), P. 5–27 .

[22] Besov O.V., Il'in V.P., Nikol'skii S.M. Integral representations of functions and imbedding theorems, v. I, II, V. H. Winston and Sons, Washington, D.C.; Halsted Press [John Wiley and Sons], New York-Toronto, Ont.-London, 1978, 1979.

[23] Akishev G. Otsenki M–chlennykh priblizheniy funktsiy klassa $W^{a,b,\bar{r}}_{q,\tau}$ v prostranstve Lorentsa [Estimates for M-term approximations of functions of the class $W^{a,b,\bar{r}}_{q,\tau}$ in the Lorentz space]. *Materialy mezhdunarodnoy nauchnoy konferentsii, posvyashchennoy 70-letiyu akademika NAN Tadzhikistana M.Sh. Shabozova* (2022), P. 22–25 (in Russian).

[24] Akishev G. *The inequality for trigonometric polynomials by hyperbolic crosses in Lorentz space*, Materials of the 20th international Saratov winter school. – Saratov: Saratov University (2020), P. 19–22.

[25] Lindenstrauss J., Tzafriri L. *Classical Banach spaces II. Function Spaces*. Springer-Verlag, Berlin. Heidelberg, New York, 1979.

[26] Lindenstrauss J., Tzafriri L. *Classical Banach spaces I. Sequence Spaces*. Springer-Verlag, Berlin. Heidelberg, New York, 1977.

[27] Akishev G. *Estimates for best approximations of functions from the logarthmic smoothness class in the Lorentz space*, Trudy Instituta Matematiki i Mekhaniki UrO RAN, **23**:3 (2018), P. 3–21.