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On Error Bounds for Milne-Mercer type inequalities through differentiable s-convex functions

Benaissa Bouharket[a] · Azzouz Noureddine[b]

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Abstract

This study presents Milne-Mercer-type inequalities for a specific integral connected to differentiable s-convex functions using conformable fractional operators. Furthermore, we introduce novel findings that extend and generalize specific known inequalities in current literature. We examine additional cases where the derivative functions are both Lipschitzian and bounded.

1 Introduction

Fractional calculus is an extension of classical calculus, which has progressed rapidly, particularly thanks to the fascinating idea of convexity. This sophisticated mathematical framework has discovered numerous applications in various fields such as optimization theory, stochastic theory, and functional analysis. Fractional calculus is now a highly sought-after field of study, generating substantial advances and interest in both theoretical and applied mathematics due to its ability to offer more flexible models and deeper perspectives. The constant discovery of new applications and the creation of increasingly complex instruments further enhance its significance and adaptability in contemporary scientific research. In [8], the author presents a class of functions known as s-convex functions.

Definition 1.1. Let $s \in (0,1]$. The function $\Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is defined as s-convex in the second sense if Φ is non-negative and for any $z_1, z_2 \in I$, $\xi \in [0,1]$, the following condition holds

$$\Phi(\xi z_1 + (1-\xi)z_2) \leq \xi^s \Phi(z_1) + (1-\xi)^s \Phi(z_2). \tag{1}$$

If the inequality (1) is reversed, then Φ is classified as s-concave function in the second sense.

By setting

- $s = 1$, the notion of s-convex function simplifies to convex function [18].
- $s \to 0$, the concept of s-convex function reduces to P-functions [20].

According to [17], the Jensen-Mercer inequality is as follows. If f is a convex function on $[b_1, b_2]$, then

$$f\left(b_1 + b_2 - \sum_{j=1}^{n} \xi_j z_j\right) \leq f(b_1) + f(b_2) - \sum_{j=1}^{n} \xi_j f(z_j),$$

for each $z_j \in [b_1, b_2]$ and $\xi_j \in (0,1)$ $(j = \overline{1;n})$ with $\sum_{j=1}^{n} \xi_j = 1$.

Based on [13], the conformable fractional integral operators with order $\alpha > 0$ and $\rho \in (0,1]$ are represented as follows.

$$^{\rho}\mathfrak{I}^{\alpha}_{b_1^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{b_1}^{x} \left(\frac{(x-b_1)^{\rho} - (t-b_1)^{\rho}}{\rho}\right)^{\alpha-1} (t-b_1)^{\rho-1} f(t)dt, \quad x > b_1,$$

$$^{\rho}\mathfrak{I}^{\alpha}_{b_2^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b_2} \left(\frac{(b_2-x)^{\rho} - (b_2-t)^{\rho}}{\rho}\right)^{\alpha-1} (b_2-t)^{\rho-1} f(t)dt, \quad x < b_2.$$

[a]Faculty of Material Sciences, University of Tiaret, Algeria
[b]Faculty of Sciences, University Center Nour Bachir El Bayadh, Algeria

For $\rho = 1$, the preceding operators are reduced to Riemann-Liouville fractional operators of order $\alpha > 0$, as follows:

$$\Im_{b_1^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{b_1}^{x} (x-t)^{\alpha-1} f(t)dt, \quad x > b_1,$$

$$\Im_{b_2^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b_2} (t-x)^{\alpha-1} f(t)dt, \quad x < b_2.$$

The beta-Euler function $\beta(.,.)$ is defined for any $p, q > 0$ as follows:

$$\beta(q,p) = \int_0^1 (1-y)^{q-1} y^{p-1} dy.$$

One of the most well-known quadrature rules is the Milne rule, which is a successful numerical approach for estimating the integral of a function. It is derived from the larger family of Simpson's rule, which is also known as the principle of quadrature. There have been several studies that investigate the error estimates associated with this formula over a variety of integrals and classes of functions, and these works have been used to conduct significant research on the correctness of Milne's rule. Readers are encouraged to consult [3, 4, 5, 6, 9, 14, 15, 16, 19]for more research on this formula.

In 2022, Djenaoui and Meftah introduced a Milne inequality for convex functions with Riemann integral [9, Corollary 2.4.].

$$\left| \frac{1}{3}\left[2f(b_1) - f\left(\frac{b_1+b_2}{2}\right) + 2f(b_2) \right] - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} f(t)dt \right| \le \frac{5(b_2-b_1)}{24}\left(|f'(b_1)| + |f'(b_2)| \right). \tag{2}$$

In 2023, Budak et al. established a fractional Milne type inequality for convex functions using Riemann-Liouville integral operators [7, Theorem 1.].

$$\begin{aligned} &\left| \frac{1}{3}\left[2f(b_1) - f\left(\frac{b_1+b_2}{2}\right) + 2f(b_2) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b_2-b_1)^{\alpha}}\left[\Im_{\frac{b_1+b_2}{2}^+}^{\alpha} f(b_2) + \Im_{\frac{b_1+b_2}{2}^-}^{\alpha} f(b_1) \right] \right| \\ &\le \frac{4\alpha+1}{12(\alpha+1)}(b_2-b_1)\left(|f'(b_1)| + |f'(b_2)| \right). \end{aligned} \tag{3}$$

In [1], Benaissa and Sarikaya established the required Lemma.

Lemma 1.1. *Let $\xi \in (0,1)$ and $s \in [0,1]$. The following inequality holds:*

$$\xi^s + (1-\xi)^s \le 2^{1-s}, \tag{4}$$

Theorem 1.2 (Hölder inequality)**.** *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If Ψ and Φ are real functions defined on $[\lambda_1, \lambda_2]$ and if $|\Psi|^p$, $|\Phi|^q$ are integrable functions on $[\lambda_1, \lambda_2]$ then*

$$\int_{\lambda_1}^{\lambda_2} |\Psi(t)\Phi(t)|\, dt \le \left(\int_{\lambda_1}^{\lambda_2} |\Psi(t)|^p\, dt \right)^{\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |\Phi(t)|^q\, dt \right)^{\frac{1}{q}}.$$

The power-mean integral inequality, derived from the Hölder inequality, can be expressed as follows:

Theorem 1.3 (Power mean integral inequality)**.** *Let $p \ge 1$ and W, Φ be two real functions defined on $[\lambda_1, \lambda_2]$. If $|W|$, $|W||\Phi|^p$ are integrable functions on $[b_1, b_2]$ then*

$$\int_{\lambda_1}^{\lambda_2} |W(t)\Phi(t)|\, dt \le \left(\int_{\lambda_1}^{\lambda_2} |W(t)|\, dt \right)^{1-\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |W(t)|\,|\Phi(t)|^p\, dt \right)^{\frac{1}{p}}.$$

For additional details and improvements of the power-mean integral inequality, consult references [12] and [2].

Building on previous research, we have developed an enhanced version of the Milne-Mercer type inequality specifically for s-convex functions. This new formulation utilizes conformable fractional integral operators, offering a more refined and comprehensive approach. Our work expands the theoretical framework, providing deeper insights and potentially broader applications in the study of s-convex functions. By integrating these advanced mathematical tools, we aim to contribute significantly to the ongoing development and understanding of inequalities in the context of conformable fractional calculus.

2 Fundamental Identity

Lemma 2.1. *Let $\alpha > 0$, $\rho \in (0,1]$ and $z, y \in [b_1, b_2] \subseteq \mathbb{R}$ where $z < y$. If $f : [b_1, b_2] \to \mathbb{R}$ is a differentiable mapping such that $f' \in L_1([b_1, b_2])$, then the following identity holds.*

$$\begin{aligned}
&\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\\
&-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\\
&=\frac{y-z}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left\{f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)\right.\\
&\left.-f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)\right\}dt.
\end{aligned} \tag{5}$$

Proof. First, the following integral calculus is required. Let $z, y \in [b_1, b_2]$ where $z < y$ and putting $\xi = t(b_1+b_2-y)+(1-t)(b_1+b_2-\frac{z+y}{2})$, we get

$$\begin{aligned}
&\int_0^1 (1-t^{\rho})^{\alpha-1}t^{\rho-1}f\left(t(b_1+b_2-y)+(1-t)\left(b_1+b_2-\frac{z+y}{2}\right)\right)dt\\
&=\left(\frac{2}{y-z}\right)^{\rho\alpha}\int_{b_1+b_2-y}^{b_1+b_2-\frac{z+y}{2}}\left[\left(\left(b_1+b_2-\frac{z+y}{2}\right)-(b_1+b_2-y)\right)^{\rho}-\left(\left(b_1+b_2-\frac{z+y}{2}\right)-\xi\right)^{\rho}\right]^{\alpha-1}\\
&\times\left(\left(b_1+b_2-\tfrac{z+y}{2}\right)-\xi\right)^{\rho-1}f(\xi)d\xi\\
&=\left(\frac{2}{y-z}\right)^{\rho\alpha}\rho^{\alpha-1}\Gamma(\alpha)\,{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y).
\end{aligned}$$

Taking $\xi = (1-t)(b_1+b_2-\frac{z+y}{2})+t(b_1+b_2-z)$, we get

$$\begin{aligned}
&\int_0^1 (1-t^{\rho})^{\alpha-1}t^{\rho-1}f\left((1-t)(b_1+b_2-\frac{z+y}{2})+t(b_1+b_2-z)\right)dt\\
&=\left(\frac{2}{y-z}\right)^{\rho\alpha}\int_{b_1+b_2-\frac{z+y}{2}}^{b_1+b_2-z}\left[\left((b_1+b_2-z)-\left(b_1+b_2-\frac{z+y}{2}\right)\right)^{\rho}-\left(\xi-\left(b_1+b_2-\frac{z+y}{2}\right)\right)^{\rho}\right]^{\alpha-1}\\
&\times\left(\xi-\left(b_1+b_2-\tfrac{z+y}{2}\right)\right)^{\rho-1}f(\xi)d\xi\\
&=\left(\frac{2}{y-z}\right)^{\rho\alpha}\rho^{\alpha-1}\Gamma(\alpha)\,{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z).
\end{aligned}$$

By utilizing the integration by parts method, we can obtain the following expression:

$$\begin{aligned}
I_1 &= \int_0^1\left((1-t^{\rho})^{\alpha}-\frac{4}{3}\right)f'\left(t(b_1+b_2-y)+(1-t)\left(b_1+b_2-\frac{z+y}{2}\right)\right)dt\\
&=-\left(\frac{2}{y-z}\right)\left((1-t^{\rho})^{\alpha}-\frac{4}{3}\right)f\left(t(b_1+b_2-y)+(1-t)\left(b_1+b_2-\frac{z+y}{2}\right)\right)\Bigg|_0^1\\
&-\left(\frac{2\alpha\rho}{y-z}\right)\int_0^1(1-t^{\rho})^{\alpha-1}t^{\rho-1}f\left(t(b_1+b_2-y)+(1-t)\left(b_1+b_2-\frac{z+y}{2}\right)\right)dt\\
&=\left(\frac{2}{y-z}\right)\left[\frac{4}{3}f(b_1+b_2-y)-\frac{1}{3}f\left(b_1+b_2-\frac{z+y}{2}\right)\right]\\
&-\left(\frac{2}{y-z}\right)^{\rho\alpha+1}\rho^{\alpha}\Gamma(\alpha+1)\,{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y).
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 &= \int_0^1 \left((1-t^\rho)^\alpha - \frac{4}{3}\right) f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)dt \\
&= \left(\frac{2}{y-z}\right)\left((1-t^\rho)^\alpha - \frac{4}{3}\right) f\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)\Big|_0^1 \\
&+\left(\frac{2\alpha\rho}{y-z}\right)\int_0^1 (1-t^\rho)^{\alpha-1}t^{\rho-1} f\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)dt \\
&= -\left(\frac{2}{y-z}\right)\left[\frac{4}{3}f(b_1+b_2-z)-\frac{1}{3}f\left(b_1+b_2-\frac{z+y}{2}\right)\right] \\
&+\left(\frac{2}{y-z}\right)^{\rho\alpha+1}\rho^\alpha\Gamma(\alpha+1)\,{}^\rho\mathfrak{J}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^+} f(b_1+b_2-z).
\end{aligned}$$

As a result, the following equality is valid:

$$\begin{aligned}
\frac{y-z}{4}(I_1-I_2) = &\ \frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right] \\
&-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^\rho\mathfrak{J}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^-} f(b_1+b_2-y)+{}^\rho\mathfrak{J}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^+} f(b_1+b_2-z)\right].
\end{aligned}$$

So the proof is complete. □

3 Milne-Mercer inequality via power-mean integral inequality

To demonstrate the next results, we require the following inequality.

$$\xi^s + 2^{1-s}(1-\xi)^s \le 4^{1-s}, \text{for all } \xi \in (0,1) \text{ and } s \in (0,1]. \tag{6}$$

Proof. For $s \in (0,1]$, it follows that $0 \le 1-s < 1$ and $1 \le 2^{1-s}$. By using (4), we obtain

$$\xi^s + 2^{1-s}(1-\xi)^s \le 2^{1-s}\xi^s + 2^{1-s}(1-\xi)^s = 2^{1-s}[\xi^s + (1-\xi)^s] \le \left(2^{1-s}\right)^2.$$

□

We present the first result for the Milne-Mercer type inequality relying on conformable fractional integral operators.

Theorem 3.1. *Let $p \ge 1$ and $s \in (0,1]$. Assume that the assumptions of Lemma 5 hold. If $|f'|^p$ is an s-convex mapping in the second sense on $[b_1, b_2]$, then the following Milne-Mercer type inequality via conformable fractional integral operators holds.*

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right] \\
&-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^\rho\mathfrak{J}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^-} f(b_1+b_2-y)+{}^\rho\mathfrak{J}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^+} f(b_1+b_2-z)\right]\Bigg| \\
&\le (y-z)\left(\tfrac{1}{2}\right)^{\frac{p+2s-1}{p}}\left(|f'(b_1+b_2-y)|^p+|f'(b_1+b_2-z)|^p\right)^{\frac{1}{p}}\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right).
\end{aligned} \tag{7}$$

Proof. Using the absolute value of identity (5) gets

$$\begin{aligned}
&\Bigg|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right] \\
&-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^\rho\mathfrak{J}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^-} f(b_1+b_2-y)+{}^\rho\mathfrak{J}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^+} f(b_1+b_2-z)\right]\Bigg| \\
&\le \frac{y-z}{4}\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right) \\
&\times\left[\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)\right|+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)\right|\right]dt.
\end{aligned}$$

Let $p \geq 1$, applying the Power-mean integral inequality yields:

$$\int_0^1 \left(\frac{4}{3}-(1-t^\rho)^\alpha\right)$$

$$\times\left[\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-z)\right)\right|+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-y)\right)\right|\right]dt$$

$$\leq\left(\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)dt\right)^{1-\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-z)\right)\right|^p dt\right)^{\frac{1}{p}}\right.$$

$$\left.+\left(\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-y)\right)\right|^p dt\right)^{\frac{1}{p}}\right].$$

Given that $A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, we get

$$\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)\left[\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-z)\right)\right|\right.$$

$$\left.+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-y)\right)\right|\right]dt$$

$$\leq\left(\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)dt\right)^{1-\frac{1}{p}}2^{1-\frac{1}{p}}$$

$$\times\left[\int_0^1\left(\frac{4}{3}-(1-t^\rho)^\alpha\right)\left(\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-z)\right)\right|^p\right.\right.$$

$$\left.\left.+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-y)\right)\right|^p\right)dt\right]^{\frac{1}{p}}.$$

As $|f'|^p$ is an s-convex function, then

$$\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-z)\right)\right|^p\leq(1-t)^s\left|f'\left(b_1+b_2-\frac{z+y}{2}\right)\right|^p+t^s\left|f'(b_1+b_2-z)\right|^p,$$

hence

$$\begin{aligned}&\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-z)\right)\right|^p+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-y)\right)\right|^p\\&\leq 2\,(1-t)^s\left|f'\left(b_1+b_2-\frac{z+y}{2}\right)\right|^p+t^s\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right],\end{aligned}\tag{8}$$

and

$$\begin{aligned}\left|f'\left(b_1+b_2-\frac{z+y}{2}\right)\right|^p&=\left|f'\left(\frac{b_1+b_2-y}{2}+\frac{b_1+b_2-z}{2}\right)\right|^p\\&\leq\left(\frac{1}{2}\right)^s\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right].\end{aligned}\tag{9}$$

Adding (9) to (8) and applying (6), we deduce

$$\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-z)\right)\right|^p+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-y)\right)\right|^p$$

$$\leq\left(2^{1-s}(1-t)^s+t^s\right)\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]$$

$$\leq 4^{1-s}\left[|f'(b_1+b_2-y)|^p+|f'(b_1+b_2-z)|^p\right].$$

Therefore

$$\int_0^1 \left(\frac{4}{3}-(1-t^\rho)^\alpha\right)\left[\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-z)\right)\right|\right.$$

$$\left.+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t\,(b_1+b_2-y)\right)\right|\right]dt$$

$$\leq \left(\int_0^1 \left(\frac{4}{3}-(1-t^\rho)^\alpha\right)dt\right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}}\left[\int_0^1 \left(\frac{4}{3}-(1-t^\rho)^\alpha\right)4^{1-s}\left(\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right)dt\right]^{\frac{1}{p}}$$

$$=2^{1-\frac{1}{p}}\left(\frac{1}{4}\right)^{\frac{s-1}{p}}\left(\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right)^{\frac{1}{p}}\int_0^1 \left(\frac{4}{3}-(1-t^\rho)^\alpha\right)dt.$$

Consequently

$$\left|\frac{1}{3}\left[2f(b_1+b_2+y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^\rho\mathfrak{I}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^\rho\mathfrak{I}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$

$$\leq \frac{y-z}{4}2^{1-\frac{1}{p}}\left(\frac{1}{4}\right)^{\frac{s-1}{p}}\left(\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right)^{\frac{1}{p}}\int_0^1 \left(\frac{4}{3}-(1-t^\rho)^\alpha\right)dt$$

$$=(y-z)\left(\frac{1}{2}\right)^{\frac{p+2s-1}{p}}\left(\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right)^{\frac{1}{p}}\int_0^1 \left(\frac{4}{3}-(1-t^\rho)^\alpha\right)dt.$$

Since

$$\int_0^1 (1-t^\rho)^\alpha\,dt=\frac{1}{\rho}\int_0^1 (1-t)^\alpha t^{\frac{1}{\rho}-1}dt=\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right).$$

This accomplishes the proof of the desired. □

Using $p=1$ in Theorem 3.1 yields the following Corollary.

Corollary 3.2. *Let $s\in(0,1]$ and assume that the assumptions of Lemma 5 hold. If $|f'|$ is an s-convex mapping in the second sense on $[b_1,b_2]$, then the following Milne-Mercer type inequality via conformable fractional integral operators holds.*

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^\rho\mathfrak{I}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^\rho\mathfrak{I}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right| \tag{10}$$

$$\leq \frac{y-z}{4^s}\left[\left|f'(b_1+b_2-y)\right|+\left|f'(b_1+b_2-z)\right|\right]\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right).$$

Remark 1. Theorem 3.1 is a generalization of Theorem 3 in [21], simply by setting $p=1$, $s=1$, $z=b_1$ and $y=b_2$.

Consider some particular cases of the preceding Theorem 3.1.

- Taking $\rho=1$ in Theorem 3.1 and Corollary 3.2, we get Milne-Mercer inequality via Riemann-Liouville operators for s-convex function.

  Corollary 3.3. *Let $p\geq 1$ and $s\in(0,1]$. Assume that the assumptions of Lemma 5 hold. If $|f'|^p$ is an s-convex mapping in the second sense on $[b_1,b_2]$, then the following Milne-Mercer type inequality holds.*

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$

$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-z)^{\alpha}}\left[\mathfrak{I}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+\mathfrak{I}^\alpha_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right| \tag{11}$$

$$\leq (y-z)\left(\tfrac{1}{2}\right)^{\frac{p+2s-1}{p}}\left(|f'(b_1+b_2-y)|^p+|f'(b_1+b_2-z)|^p\right)^{\frac{1}{p}}\left(\frac{4\alpha+1}{3(\alpha+1)}\right).$$

For $p = 1$, we get

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-z)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right| \tag{12}$$
$$\leq \frac{y-z}{4^s}\left(\frac{4\alpha+1}{3(\alpha+1)}\right)\left[\left|f'(b_1+b_2-y)\right|+\left|f'(b_1+b_2-z)\right|\right].$$

The inequalities (11) and (12) are generalizations of the inequality (3).

- Putting $\alpha = 1$ in the Corollary 3.3 gives the following Milne-Mercer inequalities.

Corollary 3.4. *Let $p \geq 1$ and $s \in (0,1]$. Assume that the assumptions of Lemma 5 hold. If $|f'|^p$ is an s-convex mapping in the second sense on $[b_1, b_2]$, then the following Milne-Mercer type inequality via Riemann integral holds.*

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right| \tag{13}$$
$$\leq \frac{5(y-z)}{6\cdot 4^s}\left(\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right)^{\frac{1}{p}}.$$

For $p = 1$, we get

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right|$$
$$\leq \frac{5(y-z)}{6\cdot 4^s}\left[\left|f'(b_1+b_2-y)\right|+\left|f'(b_1+b_2-z)\right|\right].$$

The above inequality (13) generalized the inequality (2), simply take $p = 1$ and $s = 1$.

3.1 Milne-Mercer inequality via class P-functions

Setting $s \to 0$ in Theorem 3.1 and Corollary 3.2 gives the following new results involving the class P-functions.

Corollary 3.5. *Let $p \geq 1$ and $s \in (0,1]$. Assume that the assumptions of Lemma 5 hold. If $|f'|^p$ is a P-functions on $[b_1, b_2]$, then the following Milne-Mercer type inequality via conformable fractional integral operators holds.*

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right| \tag{14}$$
$$\leq (y-z)\left(\tfrac{1}{2}\right)^{\frac{p-1}{p}}\left(|f'(b_1+b_2-y)|^p+|f'(b_1+b_2-z)|^p\right)^{\frac{1}{p}}\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right).$$

For $p = 1$, we get

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right| \tag{15}$$
$$\leq (y-z)\left[\left|f'(b_1+b_2-y)\right|+\left|f'(b_1+b_2-z)\right|\right]\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right).$$

Remark 2. 1. Taking $\rho = 1$ in the inequalities (14) and (15) gives the following Milne-Mercer inequality via Riemann-Liouville operators.

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-z)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right| \tag{16}$$
$$\leq (y-z)\left(\tfrac{1}{2}\right)^{\frac{p-1}{p}}\left(|f'(b_1+b_2-y)|^p+|f'(b_1+b_2-z)|^p\right)^{\frac{1}{p}}\left(\frac{4\alpha+1}{3(\alpha+1)}\right).$$

For $p = 1$, we get

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-z)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right| \tag{17}$$
$$\leq (y-z)\left(\frac{4\alpha+1}{3(\alpha+1)}\right)\left[|f'(b_1+b_2-y)|+|f'(b_1+b_2-z)|\right].$$

2. By setting $\alpha = 1$ in the inequalities (16) and (17), we use the Riemann integral to derive the following Milne-Mercer inequality.

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right|$$
$$\leq \frac{5(y-z)}{6}\left(\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right)^{\frac{1}{p}}.$$

For $p = 1$, we get

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right|$$
$$\leq \frac{5(y-z)}{6}\left[\left|f'(b_1+b_2-y)\right|+\left|f'(b_1+b_2-z)\right|\right].$$

Remark 3. Putting $z = b_1$ and $y = b_2$ in the previous inequalities gives the Milne inequalities version via conformable fractional integral operators, Riemann-Liouville operators and Riemann integral (respectively).

4 Milne-Mercer inequality via Hölder inequality

Theorem 4.1. *Let $p > 1$, $\frac{1}{p'}+\frac{1}{p} = 1$ and assume that α, ρ, f are defined as in Lemma 2.1. If $|f'|^p$ is an s-convex mapping on $[b_1, b_2]$, the following Milne-Mercer type inequality holds*

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right| \tag{18}$$
$$\leq \frac{y-z}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}4^{\frac{1-s}{p}}\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]^{\frac{1}{p}}.$$

Proof. Using the absolute value of identity (5), we get

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$

$$\leq\frac{y-z}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)\right|dt$$

$$+\frac{y-z}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)\right|dt.$$

Since $A^{\frac{1}{p}}+B^{\frac{1}{p}}\leq 2^{1-\frac{1}{p}}(A+B)^{\frac{1}{p}}$, by using Hölder inequality we deduce

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$

$$\leq\frac{b_2-b_1}{4}\left(\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$+\frac{y-z}{4}\left(\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)\right|^p dt\right)^{\frac{1}{p}}$$

$$\leq\frac{y-z}{4}\left(\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}2^{1-\frac{1}{p}}\left\{\int_0^1\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)\right|^p dt\right.$$

$$\left.+\int_0^1\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)\right|^p dt\right\}^{\frac{1}{p}}.$$

Given that $|f'|^p$ is an s-convex function, we result

$$\begin{aligned}\left|f'\left(b_1+b_2-\frac{z+y}{2}\right)\right|^p &=\left|f'\left(\frac{b_1+b_2-y}{2}+\frac{b_1+b_2-z}{2}\right)\right|^p\\ &\leq\left(\frac{1}{2}\right)^s\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right],\end{aligned}$$

hence

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$

$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$

$$\leq\frac{y-z}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left[\int_0^1\left((1-t)^s\left|f'\left(b_1+b_2-\frac{z+y}{2}\right)\right|^p+t^s\left|f'(b_1+b_2-z)\right|^p\right)dt\right.$$

$$\left.+\int_0^1\left((1-t)^s\left|f'\left(b_1+b_2-\frac{z+y}{2}\right)\right|^p+t^s\left|f'(b_1+b_2-y)\right|^p\right)dt\right]^{\frac{1}{p}}$$

$$=\frac{y-z}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left(\int_0^1\left[t^s+2^{1-s}(1-t)^s\right]dt\right)^{\frac{1}{p}}\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]^{\frac{1}{p}}.$$

□

Remark 4. With $s=1$, $z=b_1$ and $y=b_2$, Theorem 4.1 improves Theorem 4 from [21].

We suggest specific Milne-Mercer's type inequalities for s-convex functions.

- By putting $\rho=1$ in inequality (18), we obtain a Milne-Mercer type inequality using Riemann-Liouville operators for s-convex functions.

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-z)^{\alpha}}\left[\mathfrak{J}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+\mathfrak{J}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|\\&\leq\frac{y-z}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t)^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}4^{\frac{1-s}{p}}\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]^{\frac{1}{p}}.\end{aligned}\tag{19}$$

- By setting $\alpha=1$ in inequality (19), we derive a Milne-Mercer type inequality through the Riemann integral for s-convex functions.

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right|\\&\leq\frac{y-z}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}4^{\frac{1-s}{p}}\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]^{\frac{1}{p}}.\end{aligned}\tag{20}$$

- Putting $s=1$ in inequality (20), we get Milne-Mercer inequality via Riemann integral for convex function.

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right|\\&\leq\frac{y-z}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]^{\frac{1}{p}}.\end{aligned}$$

By setting $s\to 0$ in Theorem 4.1, we identify an interesting new results about the class P-functions.

Corollary 4.2. *Assume α, ρ and f are defined according to Theorem 4.1. If $|f'|^p$ is a P-function on $[b_1,b_2]$, then*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.\\&\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{J}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{J}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|\\&\leq\frac{y-z}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left\{4\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]\right\}^{\frac{1}{p}}.\end{aligned}$$

Setting $\rho=1$ yields next Milne-Mercer type inequality through Riemann-Liouville operators.

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-z)^{\alpha}}\left[\mathfrak{J}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+\mathfrak{J}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|\\&\leq\frac{y-z}{4}\left(2\int_0^1\left(\frac{4}{3}-(1-t)^{\alpha}\right)^{p'}dt\right)^{\frac{1}{p'}}\left\{4\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]\right\}^{\frac{1}{p}}.\end{aligned}$$

By setting $\rho=1$ and $\alpha=1$, one obtains the next Milne-Mercer type inequality.

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right|\\&\leq\frac{y-z}{4}\left[\frac{2}{1+p'}\left(\left(\frac{4}{3}\right)^{1+p'}-\left(\frac{1}{3}\right)^{1+p'}\right)\right]^{\frac{1}{p'}}\left\{4\left[\left|f'(b_1+b_2-y)\right|^p+\left|f'(b_1+b_2-z)\right|^p\right]\right\}^{\frac{1}{p}}.\end{aligned}$$

5 Supplementary studies regarding the Milne inequality

Theorem 5.1. *Let $f:[b_1,b_2]\to\mathbb{R}$ be a differentiable function on (b_1,b_2) such that $f'\in L_1([b_1,b_2])$. If there exist constants $-\infty<m<M<+\infty$ such that $m\le f'(x)\le M$ for all $x\in[b_1,b_2]$, then the following inequality holds.*

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.\\&\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|\\&\le\frac{(y-z)(M-m)}{4}\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right).\end{aligned}$$

Proof. Through the Lemma 2.1, we have

$$\begin{aligned}&\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\\&-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\\&=\frac{(y-z)}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left\{\left(f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)-\frac{M+m}{2}\right)\right.\\&\left.-\left(f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)-\frac{M+m}{2}\right)\right\}dt.\end{aligned}$$

Applying the absolute value to the previously equality, we obtain

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.\\&\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|\\&\le\frac{(y-z)}{4}\int_0^1\left|\frac{4}{3}-(1-t^{\rho})^{\alpha}\right|\left[\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)-\frac{M+m}{2}\right|\right.\\&\left.+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)-\frac{M+m}{2}\right|\right]dt.\end{aligned}\tag{21}$$

Given that $m\le f'(x)\le M$ for all $x\in[b_1,b_2]$,

$$\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)-\frac{M+m}{2}\right|\le\frac{M-m}{2},\tag{22}$$

and

$$\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)-\frac{M+m}{2}\right|\le\frac{M-m}{2},\tag{23}$$

adding (22) and (23) to (21) yields

$$\begin{aligned}&\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.\\&\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|\\&\le\frac{(y-z)(M-m)}{4}\int_0^1\left|\frac{4}{3}-(1-t^{\rho})^{\alpha}\right|dt\\&=\frac{(y-z)(M-m)}{4}\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right).\end{aligned}$$

□

- In Theorem 5.1, we can determine the following result using Riemann-Liouville operators by assuming $\rho = 1$.

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-z)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$
$$\leq \frac{(y-z)(M-m)}{4}\left(\frac{4\alpha+1}{3(\alpha+1)}\right).$$

- By setting $\rho = 1$ and $\alpha = 1$ in the above Theorem 5.1, we can derive an interesting result using the Riemann integral.

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right|$$
$$\leq \frac{5(y-z)(M-m)}{24}.$$

Theorem 5.2. *Let $f:[b_1,b_2]\to\mathbb{R}$ be a differentiable function on (b_1,b_2) such that $f'\in L_1([b_1,b_2])$. If f' is a L-Lipschitzian function on $[b_1,b_2]$, then*

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$
$$\leq \frac{L(y-z)^2}{4}\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right).$$

Proof. According to Lemma 2.1, we have

$$\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]$$
$$-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]$$
$$=\frac{(y-z)}{4}\int_0^1\left(\frac{4}{3}-(1-t^{\rho})^{\alpha}\right)\left[\left(f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)-f'(b_1+b_2-z)\right)\right.$$
$$\left.-\left(f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)-f'(b_1+b_2-z)\right)\right]dt.$$

By applying the absolute value to the previous equality, we can derive

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$
$$\leq\frac{(y-z)}{4}\int_0^1\left|\frac{4}{3}-(1-t^{\rho})^{\alpha}\right|\left[\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-z)\right)-f'(b_1+b_2-z)\right|\right.$$
$$\left.+\left|f'\left((1-t)\left(b_1+b_2-\frac{z+y}{2}\right)+t(b_1+b_2-y)\right)-f'(b_1+b_2-z)\right|\right]dt.$$

Given that f' is a L-Lipschitzian function on $[b_1, b_2]$, so

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\rho\alpha-1}\rho^{\alpha}\Gamma(\alpha+1)}{(y-z)^{\rho\alpha}}\left[{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+{}^{\rho}\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$
$$\leq \frac{(y-z)}{4}\int_0^1\left|\frac{4}{3}-(1-t^{\rho})^{\alpha}\right|\left[L(1-t)\left(\frac{y-z}{2}\right)+L(1+t)\left(\frac{y-z}{2}\right)\right]dt$$
$$=\frac{L(y-z)^2}{4}\int_0^1\left|\frac{4}{3}-(1-t^{\rho})^{\alpha}\right|dt$$
$$=\frac{L(y-z)^2}{4}\left(\frac{4}{3}-\frac{1}{\rho}\beta\left(\alpha+1,\frac{1}{\rho}\right)\right).$$

□

- In Theorem 5.2, we can ascertain another result via Riemann-Liouville operators with the supposition that $\rho=1$.

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]\right.$$
$$\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(y-z)^{\alpha}}\left[\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^-}f(b_1+b_2-y)+\mathfrak{I}^{\alpha}_{\left(b_1+b_2-\frac{z+y}{2}\right)^+}f(b_1+b_2-z)\right]\right|$$
$$\leq\frac{L(y-z)^2}{4}\left(\frac{4\alpha+1}{3(\alpha+1)}\right).$$

- Setting $\rho=1$ and $\alpha=1$ in the preceding Theorem 5.2 yields an intriguing conclusion with the Riemann integral.

$$\left|\frac{1}{3}\left[2f(b_1+b_2-y)-f\left(b_1+b_2-\frac{z+y}{2}\right)+2f(b_1+b_2-z)\right]-\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z}f(t)dt\right|\leq\frac{5L(y-z)^2}{24}.$$

6 Applications

for any positive values $b_1, b_2>0$, we consider the following means:

- The arithmetic mean:
$$A(b_1,b_2)=\frac{b_1+b_2}{2}.$$
- The harmonic mean:
$$H(b_1,b_2)=\frac{2b_1b_2}{b_1+b_2}.$$
- The n-logarithmic mean:
$$L_n(b_1,b_2)=\left(\frac{b_2^{n+1}-b_1^{n+1}}{(b_2-b_1)(n+1)}\right)^{\frac{1}{n}},\quad n\in\mathbf{R}-\{-1,0\},\ b_2>b_1.$$
- The logarithmic mean:
$$L(b_1,b_2)=\left(\frac{b_2-b_1}{\ln b_2-\ln b_1}\right),\quad b_2>b_1.$$

In [11], the following example is given: Let $s\in(0,1)$ and $d,k,c\in\mathbf{R}$. We define a function $f:[0,+\infty)\to R$, as

$$f(t)=\begin{cases} d & ,\ t=0;\\ kt^s+c & ,\ t>0.\end{cases}$$

If $k\geq 0$ and $0\leq c\leq d$, then f is an s-convex function.

Example 6.1. Let $t > 0$, $p \geq 1$, $0 < s < 1$ and consider the function $f(t) = t^{\left(\frac{s}{p}+1\right)}$, then

$$f'(t) = \left(\frac{s}{p} + 1\right) t^{\frac{s}{p}}.$$

In reference to

$$|f'(t)|^p = \left(\frac{s}{p} + 1\right)^p t^s,$$

thus, for $d = c = 0$, $k = \left(\frac{s}{p} + 1\right)^p$, we have $|f'|^p$ is an s-convex function.

The next results are attained by employing the previous example to inequality (13).

Proposition 6.1. *Let $b_2 > b_1 > 0$, $p \geq 1$, $0 < s < 1$ and $n = \frac{s}{p} + 1$. Then the following inequality holds:*

$$\left|\frac{4}{3}A((b_1+b_2-y)^n,(b_1+b_2-z)^n) - \frac{1}{3}A^n((b_1+b_2-y),(b_1+b_2-z)) - L_n^n((b_1+b_2-y),(b_1+b_2-z))\right|$$

$$\leq \frac{5(y-z)n}{6}\left(\frac{1}{2}\right)^{2s-\frac{1}{p}} A^{\frac{1}{p}}\left((b_1+b_2-y)^s,(b_1+b_2-z)^s\right).$$

Proof. Let $n = \frac{s}{p} + 1$ and $f(t) = t^n$. We get

$$2\left[f(b_1+b_2-y)+f(b_1+b_2-y)\right] = 4\left[\frac{f(b_1+b_2-y)+f(b_1+b_2-y)}{2}\right] = 4A((b_1+b_2-y)^n,(b_1+b_2-z)^n),$$

$$f\left(b_1+b_2-\frac{y+z}{2}\right) = f\left(\frac{(b_1+b_2-y)+(b_1+b_2-z)}{2}\right) = A^n\left((b_1+b_2-y),(b_1+b_2-z)\right),$$

$$\frac{1}{y-z}\int_{b_1+b_2-y}^{b_1+b_2-z} f(t)dt = \left(\frac{(b_1+b_2-z)^{n+1}-(b_1+b_2-y)^{n+1}}{(y-z)(n+1)}\right) = L_n^n\left((b_1+b_2-y),(b_1+b_2-z)\right),$$

and

$$\left(\left|f'(b_1+b_2-y)\right|^p + \left|f'(b_1+b_2-z)\right|^p\right)^{\frac{1}{p}} = n\left(\frac{1}{2}\right)^{-\frac{1}{p}}\left(\frac{(b_1+b_2-y)^s+(b_1+b_2-z)^s}{2}\right)^{\frac{1}{p}}$$

$$= n\left(\frac{1}{2}\right)^{-\frac{1}{p}} A^{\frac{1}{p}}\left((b_1+b_2-y)^s,(b_1+b_2-z)^s\right).$$

□

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