



On the backward stability of the second barycentric formula for interpolation

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Abstract

We present a new stability analysis for the second barycentric formula, showing that this formula is backward stable when the relevant Lebesgue constant is small.

1 Introduction

We discuss the numerical stability of the second barycentric formula for interpolation at nodes $x_0 < x_1 < \dots < x_n$. This formula is given by

$$q(x; \mathbf{x}, \mathbf{y}, \mathbf{w}) := \frac{\sum_{k=0}^n \frac{w_k y_k}{x - x_k}}{\sum_{k=0}^n \frac{w_k}{x - x_k}} \quad (1.1)$$

(note that we write the vector $\mathbf{x} \in \mathbb{R}^{n+1}$ in bold face, and x_k is its k th coordinate.) The function q may be a polynomial in x for particular choices of the weights w_k , but we work in the more general context of rational barycentric interpolation discussed in [1],[2], [3], [4] and [7].

In practice, we approximate a function $f : [x^-, x^+] \rightarrow \mathbb{R}$ using the formula q in (1.1) in three steps:

Step I: Abstract approximation theory provides convenient nodes x_k and weights w_k , so that, in exact arithmetic, the error $f(x) - q(x; \mathbf{x}, f(\mathbf{x}), \mathbf{w})$ is small for $x \in [x^-, x^+]$.

Step II: We then obtain floating point approximations \hat{x}_k , y_k and \hat{w}_k for x_k , $f(x_k)$ and w_k .

Step III: Finally, we approximate $f(x)$ evaluating $q(x; \hat{\mathbf{x}}, \mathbf{y}, \hat{\mathbf{w}})$ numerically.

This article presents upper and lower bounds on the backward errors in Steps II and III. We also emphasize the importance of considering the errors in these two steps. We focus on the effects of the errors in the nodes \mathbf{x} and weights \mathbf{w} , and assume that the function values y_k are exact, because perturbations in the function values can be easily handled using Lebesgue constants or by assuming that the perturbed function values are of the form $y_k(1 + \beta_k)$ with β_k small. We discuss both the case in which the end points of the interval $[x^-, x^+]$ are nodes and the case in which x^- and x^+ are not nodes, and allow for the possibility that some nodes lie outside of the interval $[x^-, x^+]$.

The overall conclusion is that formula q in (1.1) is backward stable when the relevant Lebesgue constant is small, in the sense that the values $q(x; \hat{\mathbf{x}}, \mathbf{y}, \hat{\mathbf{w}})$ obtained numerically in Steps II and III are equal to the exact value $q(\tilde{x}; \mathbf{x}, \tilde{\mathbf{y}}, \mathbf{w})$, with \tilde{x} near x and $\tilde{y}_k = y_k(1 + \beta_k)$ for small β_k s. This conclusion is different from the one presented in [6], which states that the second barycentric formula is not backward stable. However, there is no contradiction between our conclusion and Higham's, because we consider the favorable case in which the Lebesgue constant is small and his conclusion applies to the worst possible scenarios.

This article has three more sections. Section 2 presents upper bounds on the backward errors, showing that the second barycentric formula is backward stable under reasonable assumptions. Section 3 gives lower bounds on the backward errors, showing that, for Lagrange polynomials and except for $\log n$ factors, the bounds in Section 2 are sharp. The last section contains a perturbation theory for the barycentric formula, which covers the rational as well as the polynomial case. It also presents a proof of the main theorem, which is stated in Section 2.

2 Upper bounds on the backward error

In this section we present upper bounds on the backward errors in the evaluation of the second barycentric formula (1.1), complementing the bounds presented in [8] and [9]. We look at the second formula in (1.1) as a linear transformation $I_{\mathbf{x}, \mathbf{w}}$

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Email: andreuler@yahoo.com.br, supported by grant 14225012012-0 from Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq)

mapping $\mathbf{y} \in \mathbb{R}^{n+1}$ to the rational function defined by

$$I_{\mathbf{x},\mathbf{w}}[\mathbf{y}](x) := \begin{cases} y_k & \text{when } x = x_k \in \{x_0, x_1, \dots, x_n\}, \\ \frac{\sum_{k=0}^n \frac{w_k y_k}{x-x_k}}{\sum_{k=0}^n \frac{w_k}{x-x_k}} & \text{for } x \in \mathbb{R} \setminus \{x_0, \dots, x_n\}. \end{cases} \quad (2.1)$$

The letter I in $I_{\mathbf{x},\mathbf{w}}$ stems from *Interpolant*, because the function $I_{\mathbf{x},\mathbf{w}}[\mathbf{y}] : \mathbb{R} \rightarrow \mathbb{R}$ defined by (2.1) interpolates the y_k at the x_k .

The linear map $I_{\mathbf{x},\mathbf{w}}$ is an abstract way of looking at the second barycentric formula (1.1), and a practical minded reader can think of $I_{\mathbf{x},\mathbf{w}}[\mathbf{y}]$ as a synonym for the function q in (1.1). However, by considering the linear map $I_{\mathbf{x},\mathbf{w}}$ we can think at a deeper level. When the nodes and weights are such that

$$\sum_{k=0}^n \frac{w_k}{x-x_k} \neq 0 \quad \text{for } x \in [x^-, x^+] \setminus \{x_0, \dots, x_n\}, \quad (2.2)$$

the function q in (1.1) does not have poles in the interval $[x^-, x^+]$, and $I_{\mathbf{x},\mathbf{w}}[\mathbf{y}]$ is an element of the vector space $Q(x^-, x^+)$ of continuous rational functions from $[x^-, x^+]$ to \mathbb{R} , in which we can define the sup norm. It is then natural to study the norm of $I_{\mathbf{x},\mathbf{w}}$ with respect to the sup norm in \mathbb{R}^{n+1} and $Q(x^-, x^+)$. This norm is the Lebesgue constant mentioned in the abstract, and we denote it by $\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}}$. Formally, we define

$$\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}} := \|I_{\mathbf{x},\mathbf{w}}\|_{\infty} := \sup_{x \in [x^-, x^+] \text{ and } y \neq 0} \frac{|q(x; \mathbf{x}, \mathbf{y}, \mathbf{w})|}{\|\mathbf{y}\|_{\infty}}. \quad (2.3)$$

The articles [2], [3] and [4] present bounds on these Lebesgue constants, and their bounds allow us to apply the theory developed in Section 4 to the Floater-Hormann interpolants.

Throughout the article we consider a reference interval $[x^-, x^+]$, nodes \mathbf{x} and weights \mathbf{w} , and perturbed (or rounded) nodes $\hat{\mathbf{x}}$, with a corresponding interval $[\hat{x}^-, \hat{x}^+]$ and weights $\hat{\mathbf{w}}$. Besides the Lebesgue constant, our analysis of the backward stability of the second barycentric formula is based on the relative errors in the length of the intervals $[x_j, x_k]$, which are measured by

$$\delta_{kk} := \delta_{kk}(\mathbf{x}, \hat{\mathbf{x}}) := 0 \quad \text{and} \quad \delta_{jk} := \delta_{jk}(\mathbf{x}, \hat{\mathbf{x}}) := \frac{x_j - x_k}{\hat{x}_j - \hat{x}_k} - 1. \quad (2.4)$$

In order to handle rounding errors in the endpoints x^- and x^+ , and errors in nodes close to them, we also consider

$$\delta_j^- := \delta_j^-(x^-, \mathbf{x}, \hat{x}^-, \hat{\mathbf{x}}) := \frac{x^- - x_j}{\hat{x}^- - \hat{x}_j} - 1, \quad (2.5)$$

$$\delta_j^+ := \delta_j^+(x^+, \mathbf{x}, \hat{x}^+, \hat{\mathbf{x}}) := \frac{x^+ - x_j}{\hat{x}^+ - \hat{x}_j} - 1, \quad (2.6)$$

with $\delta_j^- = 0$ in the particular case in which $\hat{x}^- = \hat{x}_j$, and $\delta_j^+ = 0$ when $\hat{x}^+ = \hat{x}_j$. We combine the δ_j^- , δ_{jk} and δ_j^+ in the δ given by

$$\delta := \max_{0 \leq j, k \leq n} \left\{ \left| \delta_j^- \right|, \left| \delta_{jk} \right|, \left| \delta_j^+ \right| \right\}. \quad (2.7)$$

Another important measure of the perturbations are the relative differences ζ_k between the reference weights \mathbf{w} and the weights $\hat{\mathbf{w}}$ used in computation:

$$\zeta_k := \zeta_k(\mathbf{w}, \hat{\mathbf{w}}) := \frac{w_k - \hat{w}_k}{\hat{w}_k}, \quad (2.8)$$

and to avoid pathological cases we assume that

$$w_k \neq 0 \quad \text{and} \quad \hat{w}_k \neq 0. \quad (2.9)$$

We also make the following definitions and assumptions regarding the nodes and endpoints:

$$x_k < x_{k+1}, \quad (2.10)$$

$$x_k \in (x^-, x^+) \text{ if and only if } \hat{x}_k \in (\hat{x}^-, \hat{x}^+), \quad (2.11)$$

$$x_k = x^- \text{ if and only if } \hat{x}_k = \hat{x}^-, \quad (2.12)$$

$$x_k = x^+ \text{ if and only if } \hat{x}_k = \hat{x}^+, \quad (2.13)$$

$$k^- \text{ is the smallest } k \text{ such that } x_k > x^-, \quad (2.14)$$

$$k^+ \text{ is the largest } k \text{ such that } x_k < x^+, \text{ and } k^+ \geq k^-. \quad (2.15)$$

We can now state our main theorem, which provides an upper bound on the backward errors in steps II and III.

Theorem 2.1. Under the conditions (2.2) and (2.9)–(2.15), let ϵ be the machine precision, assume that $(2n + 5)\epsilon < 1$ and define

$$Z := \frac{\|\zeta(\mathbf{w}, \hat{\mathbf{w}})\|_\infty + (n + 2)\epsilon}{1 - (n + 2)\epsilon}. \tag{2.16}$$

If, for δ in (2.7),

$$(\delta + Z)\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}} + Z < 1, \tag{2.17}$$

and $\hat{x} \in [\hat{x}^-, \hat{x}^+]$ is a floating point number then the computed value $\text{fl}(q(\hat{x}; \hat{\mathbf{x}}, \mathbf{y}, \hat{\mathbf{w}}))$ is equal to $q(x; \mathbf{x}, \tilde{\mathbf{y}}, \mathbf{w})$, for some $x \in [x^-, x^+]$ such that

$$|x - \hat{x}| \leq \max\{\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty, |\hat{x}^- - x^-|, |\hat{x}^+ - x^+|\}, \tag{2.18}$$

$$\tilde{y}_k = y_k(1 + \alpha_k)(1 + \nu_k) \quad \text{with} \quad \|\nu\|_\infty \leq \frac{(2n + 5)\epsilon}{1 - (2n + 5)\epsilon} \tag{2.19}$$

and

$$\|\alpha\|_\infty \leq \frac{(1 + \Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}})(\delta + Z)}{1 - Z - (\delta + Z)\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}}}. \tag{2.20}$$

Theorem 2.1 states that the value obtained by the numerical evaluation of the second barycentric formula using approximate nodes $\hat{\mathbf{x}}$ and approximate weights $\hat{\mathbf{w}}$ is the exact value corresponding to $\tilde{\mathbf{y}}$ “near” \mathbf{y} and x “near” \hat{x} , according to the measures of nearness in (2.18)–(2.20). We emphasize that this theorem takes into account the fact that both the nodes and the weights may have errors in practice, and that by disregarding one of these errors we may underestimate the backward error.

Theorem 2.1 is abstract and general, and we now present examples of its applicability in concrete situations. We state two corollaries regarding polynomial interpolation at the Chebyshev points of the second kind, which are defined as

$$x_k^{(c)} := -\cos(k\pi/n), \tag{2.21}$$

in combination with weights obtained using the traditional formula

$$w_k = \lambda_k(\mathbf{x}) := \prod_{j \neq k} \frac{1}{x_k - x_j}. \tag{2.22}$$

We analyze two scenarios:

- In the first case we consider weights $\hat{\mathbf{w}}$ given by the closed form expressions in [11]. These weights are floating point numbers and we call them *Salzer’s weights*.
- In the second case we consider the weights obtained by evaluating (2.22) numerically, using the rounded nodes $\hat{x}^c := \text{fl}(x^c)$ and $\hat{\mathbf{w}} = \text{fl}(\lambda(\hat{x}^c))$, and we call them *Numerical weights*.

Figure 1 shows that these two cases are quite different for Lagrange polynomials: although Salzer’s weights contain no rounding errors, they lead to much worse results for large n (the data in this plot comes from Tables 1 and 2 in Section 3.) This difference is also present in Corollaries 2.2 and 2.3 below, and by studying their proof the reader will appreciate how Theorem 2.1 can be applied in practice. Note that these corollaries provide upper bounds on the backward error of order $\epsilon n^2 \log n$ for Salzer’s weights and $\epsilon n \log n$ for the Numerical weights, and these numbers are in remarkable agreement with the corresponding lines fitted by the least squares method in Figure 1 (recall that $\epsilon \approx 2.3 \times 10^{-16}$.)

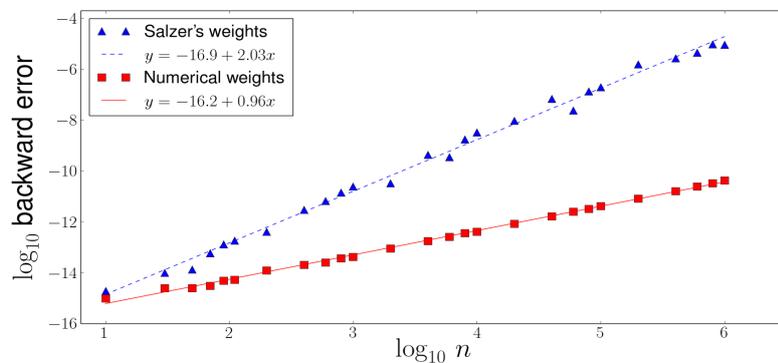


Figure 1: The dependency on the weights of the errors with rounded nodes.

We now present our corollaries and their proofs. In the statement of these corollaries, x^c are the Chebyshev nodes in (2.21), \hat{x}^c are their rounded counterparts and $q(x; \hat{x}^c, \mathbf{y}, \lambda(\hat{x}^c))$ is the n th degree polynomial that interpolates \mathbf{y} at the nodes \hat{x}^c used in practice.

Corollary 2.2. *If $\hat{x} \in [-1, 1]$ is a floating point number, $10 \leq n \leq 2.000.000$, $\epsilon \leq 2.3 \times 10^{-16}$, $\|\hat{\mathbf{x}}^c - \mathbf{x}^c\|_\infty \leq 2\epsilon$, and $\hat{\mathbf{w}}^s$ are Salzer's weights, then there exists $x \in [-1, 1]$ with $|x - \hat{x}| \leq \|\hat{\mathbf{x}}^c - \mathbf{x}^c\|_\infty$ and $\beta \in \mathbb{R}^{n+1}$ such that*

$$\|\beta\|_\infty \leq 3.7 \times (3 + \log n) \epsilon n^2, \tag{2.23}$$

for which the vector $\tilde{\mathbf{y}} \in \mathbb{R}^{n+1}$ with entries $\tilde{y}_k = (1 + \beta_k) y_k$ satisfies

$$\text{fl}(q(\hat{x}; \hat{\mathbf{x}}^c, \mathbf{y}, \hat{\mathbf{w}}^s)) = q(x; \hat{\mathbf{x}}^c, \tilde{\mathbf{y}}, \lambda(\hat{\mathbf{x}}^c)). \tag{2.24}$$

Proof of Corollary 2.2. In the context of Corollary 2.2, the δ in (2.7) is equal to zero, because we consider the rounded nodes as the interpolation points from the start. When $\mathbf{w} = \lambda(\hat{\mathbf{x}}^c)$ and $\hat{\mathbf{w}} = \lambda(\mathbf{x}^c)$, the ζ_k in definition (17) in [9] is the same as the ζ_k in definition (2.8) here and Table 2 in that article shows that

$$\|\zeta\|_\infty \leq 2.4624 \|\mathbf{x}^c - \hat{\mathbf{x}}^c\|_\infty n^2 \leq 4.9248 \epsilon n^2. \tag{2.25}$$

Using that $10 \leq n \leq 2 \times 10^6$, we conclude that Z in (2.16) satisfies

$$\begin{aligned} Z &\leq \frac{4.9248 \epsilon n^2 + (n+2) \epsilon}{1 - (2 \times 10^6 + 2) \times 2.3 \times 10^{-16}} \leq (4.9249 n^2 + 1.0001 n + 2.0001) \epsilon \\ &\leq \left(4.9249 + \frac{1.0001}{n} + \frac{2.0001}{n^2} \right) \epsilon n^2 \leq 5.0450 \epsilon n^2 \\ &\leq 5.0450 \times 4 \times 10^{12} \times 2.3 \times 10^{-16} \leq 0.0046414. \end{aligned}$$

Table 2 in [9] also shows that

$$\Lambda_{-1,1,\hat{\mathbf{x}}^c} \leq 0.67667 \log n + 1.0236 \quad \text{and} \quad \Lambda_{-1,1,\hat{\mathbf{x}}^c} \leq 10.841,$$

and it follows that the α in (2.19) satisfies

$$\|\alpha\|_\infty \leq \frac{0.67667 \log n + 2.0236}{1 - 0.0046414 \times 11.841} \times 5.0450 \epsilon n^2 \leq 3.6124 (3 + \log n) \epsilon n^2.$$

The assumptions on ϵ and n lead to

$$(2n + 5) \epsilon \leq (4 \times 10^6 + 5) \times 2.3 \times 10^{-16} < 9.2001 \times 10^{-10},$$

and (2.19) leads to $|v|_k \leq 1.0001 (2n + 5) \epsilon < 9.2011 \times 10^{-10}$. It follows that $\beta_k := v_k + (1 + v_k) \alpha_k$ satisfies

$$\begin{aligned} |\beta_k| &\leq 1.0001 (2n + 5) \epsilon + (1 + 9.2011 \times 10^{-10}) \times 3.6124 (3 + \log n) \epsilon n^2 \\ &\leq \left(\frac{1.0001 (2n + 5)}{3.6124 n^2 (3 + \log n)} + 1 \right) \times 3.6124 \times (3 + \log n) \epsilon n^2 \leq 3.6597 (3 + \log n) \epsilon n^2 \end{aligned}$$

Therefore, β_k satisfies (2.23). Theorem 2.1 yields \hat{x} as in (2.24) and we are done. □

Corollary 2.3. *If $\hat{x} \in [-1, 1]$ is a floating point number, $10 \leq n \leq 2.000.000$, $\epsilon \leq 2.3 \times 10^{-16}$, $\|\hat{\mathbf{x}}^c - \mathbf{x}^c\|_\infty \leq 2\epsilon$, and $\hat{\mathbf{w}}^r$ are the Numerical weights $\text{fl}(\lambda(\hat{\mathbf{x}}^c))$, then there exists $x \in [-1, 1]$ with $|x - \hat{x}| \leq \|\hat{\mathbf{x}}^c - \mathbf{x}^c\|_\infty$ and $\beta \in \mathbb{R}^{n+1}$ such that*

$$\|\beta\|_\infty \leq (2.2 \log n + 9.1) \epsilon n. \tag{2.26}$$

for which the the vector $\tilde{\mathbf{y}} \in \mathbb{R}^{n+1}$ with entries $\tilde{y}_k = (1 + \beta_k) y_k$ satisfies

$$\text{fl}(q(\hat{x}; \hat{\mathbf{x}}^c, \mathbf{y}, \hat{\mathbf{w}}^r)) = q(x; \hat{\mathbf{x}}^c, \tilde{\mathbf{y}}, \lambda(\hat{\mathbf{x}}^c)). \tag{2.27}$$

Proof of Corollary 2.3. Lemma 3.1 in [6] states that

$$w_k^{(r)} = w_k \langle 2n \rangle_k,$$

and using Lemma 3.1 in [5] we conclude that the ζ_k in (2.8) satisfy

$$|\zeta_k| = \left| \frac{w_k - w_k \langle 2n \rangle_k}{w_k \langle 2n \rangle_k} \right| = \frac{|1 - \langle 2n \rangle_k|}{|\langle 2n \rangle_k|} = |\langle 2n \rangle_{k'} - 1| \leq \frac{2\epsilon n}{1 - 2\epsilon n} \leq \frac{2\epsilon n}{1 - 2 \times 2 \times 10^6 \times 2.3 \times 10^{-16}} \leq 2.0001 \epsilon n,$$

and the arguments after equation (2.25) lead to (2.26) and (2.27). □

3 Lower bounds on the backward error

This section shows that Theorem 2.1 is sharp, except for $\log n$ factors. These factors are not relevant for n up to one million and we present examples in which the upper bounds provided by Theorem 2.1 are not much larger than the maximum error observed in practice. We use a combination of theory and experiments. We cannot prove that the rounding errors will be **always** above some positive number, because sometimes the value we obtain numerically is exact. For instance, the numerical result is exact when we evaluate the second barycentric formula at the node x_k and y_k is exact, regardless of the errors in the weights. Therefore, we can only obtain meaningful lower bounds under appropriate hypothesis, and experiments help us to show that these hypothesis are fulfilled in practice.

We use Lagrange polynomials as guinea-pigs. Since we consider reference weights $\lambda(\mathbf{x})$, where \mathbf{x} are the nodes used in interpolation, there are no errors in the Step I mentioned in the introduction in this case. Moreover, we have only one y_k to worry about. Formally, Lagrange polynomials can be written in second barycentric form as

$$\ell_k(x; \mathbf{x}) = \frac{\lambda_k(\mathbf{x}) y_k}{x - x_k} \bigg/ \sum_{k=0}^n \frac{\lambda_k(\mathbf{x})}{x - x_k} = q(x; \mathbf{x}, \mathbf{e}^k, \lambda(\mathbf{x})),$$

where $y_k = 1$ and $\mathbf{e}^k \in \mathbb{R}^{n+1}$ is the vector with $e_k^{(k)} = 1$ and $e_j^{(k)} = 0$ for $j \neq k$.

There is a simple expression for the backward error in Step II and Step III for Lagrange polynomials. In fact, when there is no perturbation in the nodes and we measure the backward error in terms of the relative perturbation in the function values, the backward error β_k in Steps II and III for Lagrange polynomials is such that $\text{fl}(q(x; \mathbf{x}, \mathbf{e}^k, \hat{\mathbf{w}})) = q(x; \mathbf{x}, \mathbf{e}^k(1 + \beta_k), \lambda(\mathbf{x}))$, and this condition leads to

$$\beta_k = \frac{\text{fl}(q(x; \mathbf{x}, \mathbf{e}^k, \hat{\mathbf{w}})) - q(x; \mathbf{x}, \mathbf{e}^k, \lambda(\mathbf{x}))}{q(x; \mathbf{x}, \mathbf{e}^k, \lambda(\mathbf{x}))}. \tag{3.1}$$

This expression for β_k allows us to prove the following theorem:

Theorem 3.1. *Assume that the x_i and \hat{w}_i are floating point numbers, let j and k be indexes such that $|\zeta_k| = \|\zeta\|_\infty$ and $\zeta_k \zeta_j \leq 0$ and define*

$$S = \sum_{i \neq j} \frac{|w_i|}{|x_j - x_i|}. \tag{3.2}$$

If $2.5(n+3)\epsilon \leq \|\zeta\|_\infty \leq 0.001$ and x is such that

$$0 < \left| \frac{x - x_j}{w_j} \right| S \leq 0.01 \quad \text{and} \quad \sup_{i \neq j} \left| \frac{x - x_j}{x_i - x_j} \right| < 0.01, \tag{3.3}$$

then the backward error β_k in (3.1) satisfies $|\beta_k| \geq 0.16 \|\zeta\|_\infty$.

Proof of Theorem 3.1. As in the proof of Theorem 2.1, we use Stewart's relative error counter $\langle n \rangle$. Equations (1.1) and (3.1) and the arguments after equation 4.1 in [6] show that β_k satisfies

$$\frac{\hat{w}_k \langle n+3 \rangle_k}{x - x_k} = \frac{w_k (1 + \beta_k)}{x - x_k} \tag{3.4}$$

$$\sum_{i=0}^n \frac{\hat{w}_i \langle n+2 \rangle_i}{x - x_i} = \sum_{i=0}^n \frac{w_i}{x - x_i}$$

The identities $w_i = \hat{w}_i (1 + \zeta_i)$ and $\theta_i := \langle n+2 \rangle_i - 1$ lead to

$$\hat{w}_i \langle n+2 \rangle_i = w_i \frac{1 + \theta_i}{1 + \zeta_i} = w_i (1 - \zeta_i + \psi_i), \quad \text{where} \quad \psi_i := \frac{1 + \theta_i}{1 + \zeta_i} - 1 + \zeta_i = \frac{\zeta_i^2 + \theta_i}{1 + \zeta_i}.$$

The hypothesis $2.5(n+3)\epsilon \leq \|\zeta\|_\infty \leq 0.001$ and Lemma 3.1 in [5] yield

$$|\theta_i| \leq \frac{1}{1 - 0.001/2.5} (n+2)\epsilon \leq 0.401 \|\zeta\|_\infty$$

and

$$|\psi_i| \leq \frac{0.401 + \|\zeta\|_\infty}{1 - \|\zeta\|_\infty} \|\zeta\|_\infty \leq 0.403 \|\zeta\|_\infty. \tag{3.5}$$

Analogously, $\hat{w}_k \langle n+3 \rangle_k = w_k (1 - \zeta_k + \phi)$ for ϕ such that

$$|\phi| \leq 0.403 \|\zeta\|_\infty. \tag{3.6}$$

We can then rewrite (3.4) as

$$(1 - \zeta_k + \phi) \sum_{i=0}^n \frac{w_i}{x - x_i} = (1 + \beta_k) \sum_{i=0}^n \frac{w_i (1 - \zeta_i + \psi_i)}{x - x_i}$$

and deduce that $\beta_k = N/D$ for

$$\xi := (x - x_j)/w_j, \quad D := \xi \sum_{i=0}^n \frac{w_i (1 - \zeta_i + \psi_i)}{x - x_i} \quad \text{and} \quad N := \xi (1 - \zeta_k + \phi) \sum_{i=0}^n \frac{w_i}{x - x_i} - D. \tag{3.7}$$

It follows that

$$D = 1 - \zeta_j + \psi_j + \xi(A - B + C) \tag{3.8}$$

for

$$A := \sum_{i \neq j} \frac{w_i}{x - x_i}, \quad B := \sum_{i \neq j} \frac{w_i \zeta_i}{x - x_i} \quad \text{and} \quad C := \sum_{i \neq j} \frac{w_i \psi_i}{x - x_i} \tag{3.9}$$

and

$$\begin{aligned} N &= (1 - \zeta_k + \phi)(1 + \xi A) - 1 + \zeta_j - \psi_j - \xi(A - B + C) \\ &= \zeta_j - \zeta_k + \phi - \psi_j + \xi(A\phi - A\zeta_k + B - C). \end{aligned} \tag{3.10}$$

The hypothesis (3.3) yields the bound

$$|x - x_i| = |x_j - x_i| \left| 1 - \frac{x_j - x}{x_j - x_i} \right| \geq |x_j - x_i| (1 - 0.01) \geq 0.99 |x_j - x_i|,$$

and combining this bound with (3.3), (3.7) and (3.9) we obtain

$$|A| \leq \frac{1}{0.99} S, \quad |B| \leq \frac{1}{0.99} \|\zeta\|_\infty S \quad \text{and} \quad |C| \leq \frac{1}{0.99} \|\psi\|_\infty S,$$

for S in (3.2). The hypothesis (3.3) tells us that $\xi S \leq 0.01$ and the last equation yields

$$|\xi A| \leq \frac{0.01}{0.99}, \quad |\xi B| \leq \frac{0.01}{0.99} \|\zeta\|_\infty \quad \text{and} \quad |\xi C| \leq \frac{0.01}{0.99} \|\psi\|_\infty. \tag{3.11}$$

Combining this estimate with the hypothesis that ζ_k and ζ_j have opposite signs and using (3.10) and reminding that $|\zeta_k| = \|\zeta\|_\infty$ we deduce that

$$\begin{aligned} |N| &\geq |\zeta_k| - |\phi| - \|\psi\|_\infty - |\xi| (|A| |\zeta_k| + |A| |\phi| + |B| + |C|) \\ &\geq \|\zeta\|_\infty - |\phi| - \|\psi\|_\infty - \frac{0.01}{0.99} (\|\zeta\|_\infty + |\phi| + \|\zeta\|_\infty + \|\psi\|_\infty) \\ &\geq \left(1 - \frac{0.02}{0.99}\right) \|\zeta\|_\infty - \left(1 + \frac{0.01}{0.99}\right) (|\phi| + \|\psi\|_\infty), \end{aligned}$$

and using the bounds (3.5) and (3.6) we conclude that

$$|N| \geq 0.979 \|\zeta\|_\infty - 0.815 \|\zeta\|_\infty = 0.164 \|\zeta\|_\infty. \tag{3.12}$$

Moreover, (3.5), (3.8) and (3.11) lead to

$$\begin{aligned} |D - 1| &\leq |\zeta_j| + |\psi_j| + |\xi| (|A| + |B| + |C|) \\ &\leq \|\zeta\|_\infty + \|\psi\|_\infty + \frac{0.01}{0.99} (1 + \|\zeta\|_\infty + \|\psi\|_\infty) \\ &\leq 10^{-3} + 0.403 \times 10^{-3} + \frac{0.01}{0.99} (1 + 10^{-3} + 0.403 \times 10^{-3}) \leq 0.012. \end{aligned}$$

Therefore, $0.988 \leq D \leq 1.012$. Combining these bounds on D with (3.12) we obtain

$$|\beta_k| = \left| \frac{N}{D} \right| \geq \frac{0.164 \|\zeta\|_\infty}{1.012} \geq 0.162 \|\zeta\|_\infty$$

and we are done. □

Theorem 3.1 is relevant in Salzer's case in Figure 1 because

- (i) The columns for $\|\zeta\|_\infty$ and $\|\zeta\|_\infty / (\epsilon n)$ in Table 1 below provide strong empirical evidence that $2.5(n + 3)\epsilon \leq \|\zeta\|_\infty \leq 0.001$ for $60 \leq n \leq 1.000.000$ in Salzer's case in practice.
- (ii) Table 2 in [9] shows that $\|\zeta\|_\infty \leq 0.005$ in this case, and Lemma 10 in [9] leads to $S \leq 1.23n^2 \leq 5 \times 10^{12}$, for S in (3.2) and $10 \leq n \leq 2.000.000$. This bound on S , the fact that $|w_j| \geq 1/2$ and equation (3.3) show that we can apply Theorem 3.1 if $|x - x_j| \leq 10^{-15}$. Since $|x_k| \leq 1$ and $\epsilon \leq 2.3 \times 10^{-16}$, the floating point number $x \in [-1, 1] - \{x_j\}$ closest to x_j satisfies this condition on x . Therefore, there exist x_j and a floating point number x that satisfies the hypothesis of Theorem 3.1 when $n \leq 10 \leq 2.000.000$.
- (iii) The column for $\|\zeta\|_\infty / (\epsilon n^2)$ in Table 1 shows that in practice $\|\zeta\|_\infty$ is of order ϵn^2 in Salzer's case. This is not surprising because Lemma 1 in [9] shows that $\zeta_k \approx \sum_{j \neq k} \delta_{jk}$, the shortest intervals $[x_{k-1}, x_k]$ have lengths of order $1/n^2$ and $\hat{x}_k - x_k$ is of order ϵ , and as a result the largest δ_{jk} in (2.4), and $\|\zeta\|_\infty$, are of order ϵn^2 .
- (iv) In summary, Theorem 3.1, in combination with the empirical evidence, shows that in Salzer's case the maximum backward error for the barycentric interpolation of Lagrange polynomials grows at least like ϵn^2 , and Theorem 2.1 shows that this error grows at most like $\epsilon n^2 \log n$. Therefore, in this case Theorem 2.1 is sharp except for a factor of order $\log n$.

We end this section with two tables presenting the results of experiments with rounded Chebyshev nodes. The backward errors β in these tables are the maximum values found by evaluating the second barycentric formula in double precision and comparing the result with the value obtained in quadruple precision, with $\epsilon \approx 10^{-30}$. For each n , we chose trial points near what we expect to be critical nodes, as described in Subsection 3.1. Table 1 regards the Salzer's weights \hat{w}^s , and Table 2 considers the weights obtained by evaluating numerically $\lambda(\hat{x}^c)$.

Table 1: The maximum backward error β and the relative errors ζ^s in the weights for Lagrange polynomials with Salzer's weights

n	β	$\ \zeta^s\ _\infty$	$\frac{\beta}{\ \zeta^s\ _\infty}$	$\frac{\beta}{\epsilon n}$	$\frac{\ \zeta^s\ _\infty}{\epsilon n}$	$\frac{\beta}{\epsilon n^2}$	$\frac{\ \zeta^s\ _\infty}{\epsilon n^2}$
10	1.9e-15	8.5e-16	2.17	8.3e-01	3.8e-01	0.083	0.038
20	9.7e-15	7.1e-15	1.37	2.2e+00	1.6e+00	0.110	0.080
40	1.3e-14	9.0e-15	1.45	1.5e+00	1.0e+00	0.037	0.025
60	5.9e-14	5.0e-14	1.18	4.4e+00	3.7e+00	0.074	0.062
80	1.3e-13	9.0e-14	1.49	7.5e+00	5.1e+00	0.094	0.063
100	1.8e-13	1.6e-13	1.19	8.3e+00	7.0e+00	0.083	0.070
200	4.1e-13	3.1e-13	1.31	9.2e+00	7.1e+00	0.046	0.035
400	2.9e-12	2.3e-12	1.27	3.2e+01	2.6e+01	0.081	0.064
600	6.4e-12	4.1e-12	1.54	4.8e+01	3.1e+01	0.080	0.052
800	1.4e-11	1.2e-11	1.17	7.7e+01	6.5e+01	0.096	0.082
1.000	2.4e-11	2.2e-11	1.10	1.1e+02	9.8e+01	0.107	0.098
2.000	3.2e-11	2.5e-11	1.28	7.3e+01	5.7e+01	0.036	0.028
4.000	4.3e-10	3.8e-10	1.14	4.8e+02	4.2e+02	0.120	0.106
6.000	3.5e-10	1.7e-10	2.00	2.6e+02	1.3e+02	0.044	0.022
8.000	1.7e-09	1.6e-09	1.09	9.8e+02	9.0e+02	0.122	0.112
10.000	3.2e-09	2.7e-09	1.17	1.4e+03	1.2e+03	0.142	0.122
20.000	9.4e-09	7.9e-09	1.19	2.1e+03	1.8e+03	0.106	0.089
40.000	6.6e-08	6.2e-08	1.06	7.4e+03	6.9e+03	0.185	0.174
60.000	2.3e-08	1.7e-08	1.36	1.7e+03	1.2e+03	0.028	0.021
80.000	1.3e-07	1.1e-07	1.20	7.5e+03	6.2e+03	0.093	0.078
100.000	1.9e-07	1.2e-07	1.63	8.5e+03	5.2e+03	0.085	0.052
200.000	1.5e-06	1.1e-06	1.33	3.3e+04	2.5e+04	0.167	0.125
400.000	2.6e-06	1.9e-06	1.39	2.9e+04	2.1e+04	0.073	0.053
600.000	4.4e-06	3.1e-06	1.40	3.3e+04	2.4e+04	0.055	0.039
800.000	9.3e-06	6.9e-06	1.36	5.3e+04	3.9e+04	0.066	0.048
1.000.000	8.9e-06	7.0e-06	1.26	4.0e+04	3.2e+04	0.040	0.032

Tables 1 and 2, the fact that $\epsilon \approx 2.3 \times 10^{-16}$, and the least squares lines in Figure 1, support the back-of-the-envelope estimates

$$\beta_s \approx 0.1\epsilon n^2 \quad \text{and} \quad \beta_r \approx 0.2\epsilon n$$

for the maximum backward errors in the family of Lagrange polynomials, and Tables 1 and 2 show also that $\|\zeta\|_\infty$ gives a quite good estimate of the order of magnitude of the backward error: the ratio $\beta / \|\zeta\|_\infty$ is in the range $[1, 4]$ for all n considered in both tables (and we obtained similar results in literally more than a thousand other experiments.) Therefore, for large n the ζ for Salzer's weights are considerably larger than the ζ for rounded nodes.

Table 2: The maximum backward error β and the relative errors ζ^r in the weights for Lagrange polynomials with rounded weights

n	β	$\ \zeta^r\ _\infty$	$\frac{\beta}{\ \zeta^r\ _\infty}$	$\frac{\beta}{\epsilon n}$	$\frac{\ \zeta^r\ _\infty}{\epsilon n}$	$\frac{\beta}{\epsilon n^2}$	$\frac{\ \zeta^r\ _\infty}{\epsilon n^2}$
10	9.5e-16	3.5e-16	2.70	0.429	0.159	4.3e-02	1.6e-02
20	2.4e-15	6.9e-16	3.44	0.535	0.156	2.7e-02	7.8e-03
40	2.4e-15	8.3e-16	2.92	0.273	0.094	6.8e-03	2.3e-03
60	3.0e-15	1.2e-15	2.55	0.225	0.088	3.8e-03	1.5e-03
80	4.8e-15	2.1e-15	2.27	0.270	0.119	3.4e-03	1.5e-03
100	5.2e-15	2.2e-15	2.37	0.235	0.099	2.3e-03	9.9e-04
200	1.2e-14	5.0e-15	2.36	0.263	0.111	1.3e-03	5.6e-04
400	2.0e-14	9.6e-15	2.06	0.223	0.108	5.6e-04	2.7e-04
600	2.5e-14	1.2e-14	2.09	0.190	0.091	3.2e-04	1.5e-04
800	3.6e-14	1.6e-14	2.20	0.202	0.092	2.5e-04	1.1e-04
1.000	4.2e-14	2.1e-14	2.02	0.188	0.093	1.9e-04	9.3e-05
2.000	8.7e-14	4.3e-14	2.01	0.196	0.097	9.8e-05	4.9e-05
4.000	1.7e-13	8.3e-14	2.11	0.196	0.093	4.9e-05	2.3e-05
6.000	2.5e-13	1.3e-13	2.02	0.190	0.094	3.2e-05	1.6e-05
8.000	3.5e-13	1.6e-13	2.11	0.194	0.092	2.4e-05	1.1e-05
10.000	4.1e-13	2.0e-13	2.03	0.187	0.092	1.9e-05	9.2e-06
20.000	8.4e-13	4.2e-13	2.00	0.188	0.094	9.4e-06	4.7e-06
40.000	1.6e-12	8.2e-13	2.00	0.183	0.092	4.6e-06	2.3e-06
60.000	2.5e-12	1.2e-12	2.00	0.184	0.092	3.1e-06	1.5e-06
80.000	3.2e-12	1.6e-12	2.00	0.182	0.091	2.3e-06	1.1e-06
100.000	4.1e-12	2.1e-12	2.00	0.186	0.093	1.9e-06	9.3e-07
200.000	8.1e-12	4.1e-12	2.00	0.183	0.092	9.2e-07	4.6e-07
400.000	1.6e-11	8.2e-12	2.00	0.183	0.092	4.6e-07	2.3e-07
600.000	2.4e-11	1.2e-11	2.00	0.183	0.091	3.0e-07	1.5e-07
800.000	3.3e-11	1.6e-11	2.00	0.183	0.092	2.3e-07	1.1e-07
1.000.000	4.1e-11	2.0e-11	2.01	0.184	0.092	1.8e-07	9.2e-08

3.1 Experimental settings

Our experiments used C++11 code, compiled with g++4.8.1, with usual options for optimization in release builds: `-mavx -O3 -DNDEBUG`. We did not use any tricks to improve performance or accuracy. The experiments were performed in standard processors: an Intel Core i7-2700K and an Intel Xeon E5-2640. The quadruple precision computations were performed with g++'s `_float128` type, which has machine precision of order 10^{-30} (we compiled the code with option `-fext-numeric-literals` and linked the library `quadmath` in order to use these floating point numbers and the constant π with precision of 10^{-30} .) We checked the results by comparing them with the ones obtained using the MPFR library (Fousse et al., 2007). There were differences in the results obtained by the two processors. For instance, some nodes computed by the Core i7 have error of order 10^{-16} while the same nodes computed by the Xeon have error of order 10^{-18} , and vice versa. As a result, some numbers in Tables 1 and 2 computed by these two processors had differences even in their leading digit. However, all entries for Tables 1 and 2 computed by both processors had the same order of magnitude, and these tables present the same overall picture in both cases.

For each n in Tables 1 and 2 we computed the weights \hat{w}^r and \hat{w}^s in quadruple precision and then obtained ζ^r and ζ^s . We then choose a set of pairs of indexes (k, j) for Tables 1 and 2 as follows:

- We formed a vector `indexes` containing the indexes 0, $n/2$ and n , the ten indexes corresponding to the largest z_k^r , the ten indexes corresponding to the largest z_k^s , the ten indexes corresponding to the smallest z_k^r and the ten indexes corresponding to the smallest z_k^s , and removed the repetitions.
- We then formed all the pairs (k, j) with distinct j and k in `indexes`, with $j \neq n/2$, because x very close to $x_{n/2} = 0$ may lead to underflow.
- For each pair (k, j) in which $j > 0$, we considered the 5000 floating point numbers x to the left of x_j . When $j < n$, we considered the 5000 floating point numbers to the right of x_j . For each trial point x we computed the backward error for the k th Lagrange polynomial evaluated at x , for both sets of weights, and Tables 1 and 2 report the maximum backward error found for each n .

4 The stability of the linear map $I_{x,w}$

The main result in this section is Theorem 4.3, which gives general bounds on the effects of the perturbations of the nodes and weights of the second barycentric formula, and leads to the proof of Theorem 2.1 presented at the end of the section. In order to allow for general perturbations in the nodes and in the endpoints of the interpolation interval, this theorem is stated in terms of a generic map $\chi : [\hat{x}^+, \hat{x}^-] \rightarrow [x^+, x^-]$, so that the readers could consider maps that would allow them to use Theorem 4.3 in situations which are not considered here. For instance, it is possible to find a map χ which allows one to

handle cases in which $x_k = x^-$ and $\hat{x}_k < \hat{x}^-$, but we do not consider such cases here for the sake of brevity. Lemma 4.1 and its Corollary 4.2 present a canonical map χ , which will be appropriate in most practical situations. The constant d in Theorem 4.3 for this map is the δ in (2.7), and the readers which are not concerned with utmost generality can go directly to Theorem 4.3, ignore the function χ in the statement of this theorem and take $d = \delta$ in (2.7).

Lemma 4.1. *Given numbers $\hat{x}_0 < \hat{x}_1 < \dots < \hat{x}_n$ and $x_0 < x_1 < \dots < x_n$, the piecewise linear map $\chi : [\hat{x}_0, \hat{x}_n] \rightarrow [x_0, x_n]$ given by $\chi(\hat{x}_0) := x_0$ and*

$$\chi(\hat{x}) := x_k + (\hat{x} - \hat{x}_k) \frac{x_{k+1} - x_k}{\hat{x}_{k+1} - \hat{x}_k} \quad \text{for } \hat{x}_k < \hat{x} \leq \hat{x}_{k+1},$$

is strictly increasing, $\chi(\hat{x}_k) = x_k$, $|\chi(\hat{x}) - \hat{x}| \leq \|\hat{\mathbf{x}} - \mathbf{x}\|_\infty$ for $\hat{x} \in [\hat{x}_0, \hat{x}_n]$ and, for $0 \leq j \leq n$ and

$$\left| \frac{\chi(\hat{x}) - x_j}{\hat{x} - \hat{x}_j} - 1 \right| \leq \max \{ |\delta_{jk}|, |\delta_{j(k+1)}| \}, \tag{4.1}$$

for $\hat{x}_k < \hat{x} < \hat{x}_{k+1}$ and δ_{jk} in (2.7).

Proof of Lemma 4.1. The function $|\chi(\hat{x}) - \hat{x}|$ is convex in $[\hat{x}_k, \hat{x}_{k+1}]$ and, therefore, its maximum value in this interval is reached at \hat{x}_k or \hat{x}_{k+1} , that is,

$$\begin{aligned} \max_{\hat{x} \in [\hat{x}_k, \hat{x}_{k+1}]} |\chi(\hat{x}) - \hat{x}| &= \max \{ |\chi(\hat{x}_k) - \hat{x}_k|, |\chi(\hat{x}_{k+1}) - \hat{x}_{k+1}| \} \\ &= \max \{ |x_k - \hat{x}_k|, |x_{k+1} - \hat{x}_{k+1}| \} \leq \|\hat{\mathbf{x}} - \mathbf{x}\|_\infty. \end{aligned}$$

In order to prove (4.1) it suffices to show that the functions

$$h_{jk}(\hat{x}) := \frac{\chi(\hat{x}) - x_j}{\hat{x} - \hat{x}_j} - 1 = \frac{1}{\hat{x} - \hat{x}_j} \left(x_k - x_j + (\hat{x} - \hat{x}_k) \frac{x_{k+1} - x_k}{\hat{x}_{k+1} - \hat{x}_k} \right) - 1$$

satisfy

$$|h_{jk}(\hat{x})| \leq \max \{ |\delta_{jk}|, |\delta_{j(k+1)}| \} \tag{4.2}$$

for $\hat{x}_k < \hat{x} < \hat{x}_{k+1}$. Note that $h_{jk}(\hat{x}) = A/(\hat{x} - \hat{x}_j) + B$ for constants

$$A := x_k - x_j + (\hat{x}_j - \hat{x}_k) \frac{x_{k+1} - x_k}{\hat{x}_{k+1} - \hat{x}_k} \quad \text{and} \quad B := \frac{x_{k+1} - x_k}{\hat{x}_{k+1} - \hat{x}_k} - 1.$$

When $j \in \{k, k+1\}$ we have that $A = 0$, h_{jk} is constant and equal to $\delta_{k(k+1)}$ in the interval $(\hat{x}_k, \hat{x}_{k+1})$ and (4.2) holds. Otherwise, $\hat{x}_j \notin [\hat{x}_k, \hat{x}_{k+1}]$ and h_{jk} is monotone and continuous in this interval. Therefore, when $j \notin \{k, k+1\}$ we have that

$$\max_{\hat{x}_k \leq \hat{x} \leq \hat{x}_{k+1}} |h_{jk}(\hat{x})| = \max \{ |h_{jk}(\hat{x}_k)|, |h_{jk}(\hat{x}_{k+1})| \} = \max \{ |\delta_{jk}|, |\delta_{j(k+1)}| \}.$$

This proves (4.2) and this proof is complete. □

Corollary 4.2. *Under the conditions (2.9)–(2.15), if δ in (2.7) is smaller than one then $\hat{x}_k < \hat{x}_{k+1}$ for $0 \leq k \leq n$ and there exists a bijection $\chi : [\hat{x}^-, \hat{x}^+] \rightarrow [x^-, x^+]$ such that $\chi(\hat{x}^-) = x^-$, $\chi(\hat{x}_k) = x_k$, $\chi(\hat{x}^+) = x^+$,*

$$|\chi(\hat{x}) - \hat{x}| \leq \max \{ \|\mathbf{x} - \hat{\mathbf{x}}\|_\infty, |x^- - \hat{x}^-|, |x^+ - \hat{x}^+| \},$$

and, for $0 \leq j \leq n$,

- If $\hat{x}^- < \hat{x} < \hat{x}_{k^-}$ then

$$\left| \frac{\chi(\hat{x}) - x_j}{\hat{x} - \hat{x}_j} - 1 \right| \leq \max \{ |\delta_j^-|, |\delta_{jk^-}| \}.$$

- If $k^- \leq k < k^+$ and $\hat{x}_k < \hat{x} < \hat{x}_{k+1}$ then

$$\left| \frac{\chi(\hat{x}) - x_j}{\hat{x} - \hat{x}_j} - 1 \right| \leq \max \{ |\delta_{jk}|, |\delta_{j(k+1)}| \}.$$

- If $\hat{x}_{k^+} < \hat{x} < \hat{x}^+$ then

$$\left| \frac{\chi(\hat{x}) - x_j}{\hat{x} - \hat{x}_j} - 1 \right| \leq \max \{ |\delta_{jk^+}|, |\delta_j^+| \}.$$

Proof of Corollary 4.2. Corollary 4.2 follows from Lemma 4.1 applied to the vectors obtained by inserting \hat{x}^- , \hat{x}^+ , x^- , x^+ in the appropriate positions of $\hat{\mathbf{x}}$ and \mathbf{x} . In fact, consider the vectors

$$\begin{aligned} \hat{\mathbf{x}}' &:= (\hat{x}_0, \dots, \hat{x}_{k-1}, \{\hat{x}^-, \hat{x}_{k^-}\}, \hat{x}_{k^+}, \dots, \hat{x}_{k^+}, \{\hat{x}_{k^+}, \hat{x}^+\}, \dots, \hat{x}_n)^t, \\ \mathbf{x}' &:= (x_0, \dots, x_{k-1}, \{x^-, x_{k^-}\}, x_{k^+}, \dots, x_{k^+}, \{x_{k^+}, x^+\}, \dots, x_n)^t, \end{aligned}$$

where $\{x^-, x_{k^-}\}$ represents x_{k^-} when $x^- = x_{k^-}$ and the pair x^-, x_{k^-} when $x^- \neq x_{k^-}$, and the other braces are analogous. The hypothesis $\hat{x}^- = \hat{x}_k \Leftrightarrow x^- = x_k$ and $\hat{x}^+ = \hat{x}_k \Leftrightarrow x^+ = x_k$ ensures that $\hat{\mathbf{x}}'$ and \mathbf{x}' have the same dimension, and the definitions of k^- and k^+ guarantee that $x'_k < x'_{k+1}$ for the relevant k . Finally, the vector $\hat{\mathbf{x}}'$ is strictly sorted because, for instance for $j > k^-$,

$$\hat{x}_j - \hat{x}^- = \frac{x_j - x^-}{1 + \delta_j^-} \geq \frac{x_j - x^-}{1 - |\delta_j^-|} > 0,$$

and we can indeed derive Corollary 4.2 from Lemma 4.1. □

Theorem 4.3. Under the conditions (2.2) and (2.9)–(2.15), if $\hat{x} \in [\hat{x}^-, \hat{x}^+] \setminus \{\hat{x}_0, \dots, \hat{x}_n\}$, $d \in \mathbb{R}$ and the function

$$\chi : [\hat{x}^-, \hat{x}^+] \setminus \{\hat{x}_0, \dots, \hat{x}_n\} \rightarrow [x^-, x^+] \setminus \{x_0, \dots, x_n\},$$

are such that

$$\max_{0 \leq k \leq n} \left| \frac{\chi(\hat{x}) - x_k}{\hat{x} - \hat{x}_k} - 1 \right| \leq d < \frac{1 - \|\zeta(\mathbf{w}, \hat{\mathbf{w}})\|_\infty}{\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}}} - \|\zeta(\mathbf{w}, \hat{\mathbf{w}})\|_\infty \tag{4.3}$$

then

$$\sum_{k=0}^n \frac{\hat{w}_k}{\hat{x} - \hat{x}_k} \neq 0 \tag{4.4}$$

and there exists $\beta \in \mathbb{R}^{n+1}$ such that

$$\|\beta\|_\infty \leq \frac{(d + \|\zeta(\mathbf{w}, \hat{\mathbf{w}})\|_\infty)(1 + \Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}})}{1 - \|\zeta(\mathbf{w}, \hat{\mathbf{w}})\|_\infty - (d + \|\zeta(\mathbf{w}, \hat{\mathbf{w}})\|_\infty)\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}}}, \tag{4.5}$$

and

$$I_{\hat{\mathbf{x}}, \hat{\mathbf{w}}}[\mathbf{y}](\hat{x}) = I_{\mathbf{x}, \mathbf{w}}[\tilde{\mathbf{y}}](\chi(\hat{x})) \quad \text{for} \quad \tilde{y}_k = y_k(1 + \beta_k). \tag{4.6}$$

Moreover, if (4.3) holds for all $\hat{x} \in [\hat{x}^-, \hat{x}^+]$ then

$$\Lambda_{\hat{x}^-, \hat{x}^+, \hat{\mathbf{x}}, \hat{\mathbf{w}}} \leq \frac{(1 + d)\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}}}{1 - \|\zeta(\mathbf{w}, \hat{\mathbf{w}})\|_\infty - (d + \|\zeta(\mathbf{w}, \hat{\mathbf{w}})\|_\infty)\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}}}. \tag{4.7}$$

Proof of Theorem 4.3. Equation (4.3) shows that

$$\nu_k := \frac{\chi(\hat{x}) - x_k}{\hat{x} - \hat{x}_k} - 1$$

satisfies $|\nu_k| \leq d$, and also that $\|\zeta\|_\infty < 1$. Therefore $1 + \zeta_k \neq 0$ and we can write $\hat{w}_k = w_k / (1 + \zeta_k)$ and deduce that

$$\begin{aligned} \sum_{k=0}^n \frac{\hat{w}_k}{\hat{x} - \hat{x}_k} &= \sum_{k=0}^n \frac{w_k}{\chi(\hat{x}) - x_k} \frac{1}{1 + \zeta_k} \frac{\chi(\hat{x}) - x_k}{\hat{x} - \hat{x}_k} = \sum_{k=0}^n \frac{w_k}{\chi(\hat{x}) - x_k} \frac{1 + \nu_k}{1 + \zeta_k} \\ &= D \left(1 + \frac{1}{D} \sum_{k=0}^n \frac{w_k}{\chi(\hat{x}) - x_k} \frac{\nu_k - \zeta_k}{1 + \zeta_k} \right) = D(1 + E), \end{aligned} \tag{4.8}$$

for

$$D := \sum_{k=0}^n \frac{w_k}{\chi(\hat{x}) - x_k}, \quad \sigma_k := \frac{\nu_k - \zeta_k}{1 + \zeta_k} \quad \text{and} \quad E := \frac{1}{D} \sum_{k=0}^n \frac{w_k \sigma_k}{\chi(\hat{x}) - x_k}$$

(Note that (2.2) implies that $D \neq 0$.) The bound $|\nu_k| \leq d$ yields

$$|\sigma_k| \leq \frac{d + \|\zeta\|_\infty}{1 - \|\zeta\|_\infty},$$

and (4.3) and the definition of the Lebesgue constant (2.3) lead to

$$|E| = \left| \frac{1}{D} \sum_{k=0}^n \frac{w_k \sigma_k}{\chi(\hat{x}) - x_k} \right| = |\mathbb{I}_{x,w}[\sigma](\chi(\hat{x}))|$$

$$\leq \Lambda_{x^-,x^+,x,w} \|\sigma\|_\infty \leq \frac{d + \|\zeta\|_\infty}{1 - \|\zeta\|_\infty} \Lambda_{x^-,x^+,x,w} < 1, \tag{4.9}$$

and this bound, in combination with (4.8), proves (4.4). Therefore, $\mathbb{I}_{\hat{x},\hat{w}}[y](\hat{x})$ is well defined and $\mathbb{I}_{\hat{x},\hat{w}}[y](\hat{x}) = N/(D(1+E))$, for D and E as above and

$$N := \sum_{k=0}^n \frac{\hat{w}_k y_k}{\hat{x} - \hat{x}_k} = \sum_{k=0}^n \frac{w_k}{\chi(\hat{x}) - x_k} y_k \frac{1}{1 + \zeta_k} \frac{\chi(\hat{x}) - x_k}{\hat{x} - \hat{x}_k} = \sum_{k=0}^n \frac{w_k}{\chi(\hat{x}) - x_k} \theta_k,$$

for

$$\theta_k := \frac{1 + \nu_k}{1 + \zeta_k} y_k.$$

It follows that $\mathbb{I}_{\hat{x},\hat{w}}[y](\hat{x}) = \mathbb{I}_{x,w}[\tilde{y}](\chi(\hat{x}))$, with

$$\tilde{y}_k = \frac{\theta_k}{1 + E} = y_k (1 + \beta_k) \quad \text{and} \quad \beta_k := \frac{\frac{1 + \nu_k}{1 + \zeta_k} - 1 - E}{1 + E} = \frac{\sigma_k - E}{1 + E},$$

and (4.9) leads to

$$|\beta_k| \leq \frac{|\sigma_k| + |E|}{1 - |E|} \leq \frac{(d + \|\zeta\|_\infty)(1 + \Lambda_{x^-,x^+,x,w})}{1 - \|\zeta\|_\infty - (d + \|\zeta\|_\infty)\Lambda_{x^-,x^+,x,w}},$$

and we have verified (4.5) and (4.6). Let us now prove (4.7). For each $\hat{x} \in [\hat{x}^-, \hat{x}^+]$, the hypothesis about d and equation (4.4) guarantee that $\mathbb{I}_{\hat{x},\hat{w}}[y](\hat{x})$ is well defined and

$$|\mathbb{I}_{\hat{x},\hat{w}}[y](\hat{x})| = |\mathbb{I}_{x,w}[\tilde{y}](\chi(\hat{x}))| \leq \Lambda_{x^-,x^+,x,w} \|\tilde{y}\|_\infty \leq \Lambda_{x^-,x^+,x,w} \|y\|_\infty (1 + \|\beta\|_\infty),$$

and the bound (4.5) yields

$$|\mathbb{I}_{\hat{x},\hat{w}}[y](\hat{x})| \leq \frac{1 + d}{1 - \|\zeta\|_\infty - (d + \|\zeta\|_\infty)\Lambda_{x^-,x^+,x,w}} \Lambda_{x^-,x^+,x,w} \|y\|_\infty.$$

Taking the sup in $\hat{x} \in [\hat{x}^-, \hat{x}^+]$ of this expression we deduce (4.7). □

Proof of Theorem 2.1. We use Stewart’s error counter (Higham, 2002)

$$\langle k \rangle = \prod_{i=1}^k (1 + \xi_i)^{\sigma_i}, \quad \text{for } \sigma_i = \pm 1, \text{ and } |\xi_i| \leq \epsilon,$$

and write $\langle k \rangle_\ell$ to give a label ℓ to the specific k rounding errors we are concerned with. We note that [5]’s Lemma 3.1 implies that

$$\text{if } k\epsilon < 1 \text{ then } |\langle k \rangle - 1| \leq \frac{k\epsilon}{1 - k\epsilon} \quad \text{and} \quad 0 < \langle k \rangle \leq \frac{1}{1 - k\epsilon}.$$

If $\hat{x} = \hat{x}_k$ for some k then we can simply take $x = x_k$ and $\tilde{y} = y$. Let us then consider the case $\hat{x} \in [\hat{x}^-, \hat{x}^+] \setminus \{\hat{x}_0, \dots, \hat{x}_n\}$. [6] shows that

$$\text{fl}(\mathbb{I}_{\hat{x},\hat{w}}[y](\hat{x})) = \frac{\sum_{k=0}^n \frac{\hat{w}_k y_k \langle n+3 \rangle_k}{\hat{x} - \hat{x}_k}}{\sum_{k=0}^n \frac{\hat{w}_k \langle n+2 \rangle_k}{\hat{x} - \hat{x}_k}} = \frac{\sum_{k=0}^n \frac{w'_k y'_k}{\hat{x} - \hat{x}_k}}{\sum_{k=0}^n \frac{w'_k}{\hat{x} - \hat{x}_k}} = \mathbb{I}_{\hat{x},w'}[y'](\hat{x}), \tag{4.10}$$

for

$$w'_k := \hat{w}_k \langle n+2 \rangle_k \quad \text{and} \quad y'_k := y_k \frac{\langle n+3 \rangle_k}{\langle n+2 \rangle_k} = y_k \langle 2n+5 \rangle_k.$$

Recalling that $w_k = \hat{w}_k (1 + \zeta_k)$ and using Higham’s Lemma 3.1 we obtain

$$|\zeta'_k| = \left| \frac{w_k - w'_k}{w'_k} \right| = \left| \frac{1 + \zeta_k - \langle n+2 \rangle_k}{\langle n+2 \rangle_k} \right| = |(1 + \zeta_k) \langle n+2 \rangle_{k'} - 1|$$

$$\leq |\zeta_k \langle n+2 \rangle_{k'}| + |\langle n+2 \rangle_{k'} - 1| \leq \frac{|\zeta_k| + (n+2)\epsilon}{1 - (n+2)\epsilon} \leq Z.$$

Corollary 4.1 implies that there exists a function χ as required by the hypothesis of Theorem 4.3 with $d = \delta$ in (2.7), and this theorem applied to $\hat{\mathbf{x}} = \hat{\mathbf{x}}$, $\hat{\mathbf{w}} = \mathbf{w}'$, and $\mathbf{y} = \mathbf{y}'$ and equation (4.10) show that $\text{fl}(\mathbf{I}_{\hat{\mathbf{x}}, \hat{\mathbf{w}}}[\mathbf{y}](\hat{x})) = \mathbf{I}_{\mathbf{x}, \mathbf{w}}[\tilde{\mathbf{y}}](\chi(\hat{x}))$, with

$$\tilde{y}_k := y'_k (1 + \alpha_k) = y_k \langle 2n+5 \rangle_k (1 + \alpha_k),$$

and

$$|\alpha_k| \leq \frac{(\delta + Z)(1 + \Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}})}{1 - Z - (\delta + Z)\Lambda_{x^-, x^+, \mathbf{x}, \mathbf{w}}}.$$

Higham's Lemma shows that $\nu_k := \langle 2n+5 \rangle_k - 1$ satisfies

$$|\nu_k| \leq (2n+5)\epsilon / (1 - (2n+5)\epsilon),$$

and this completes the proof of Theorem 2.1. \square

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