



On Mellin-type Integral Operators preserving x^a and x^{2a} , $a > 0$

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Dedicated to the memory of Professor P.L. Butzer (1928–2026)

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Abstract

In this paper we study a general class of Mellin-type integral operators that preserve the functions x^a and x^{2a} , with $a > 0$. Weighted convergence with respect to certain weights is studied. Moreover quantitative estimate, using a suitable modulus of continuity, and an asymptotic formula are obtained. Finally the theory is applied to various specific examples.

1 Introduction

Approximation theory by means of sequences of Mellin-type convolution integral operators is a widely studied topic in recent years. We quote here for example [16], [17], [31]. A general approach in this direction was given in the papers [34] and [35], in which a complete theory of approximation is outlined. Note that an abstract approach for functions belonging to abstract function spaces, like e.g., modular spaces, was introduced in [14] in which the Mellin convolution operators are particular cases of a very general class of integral operators (including operators both in linear and nonlinear form). The main aim of the present article is the construction of particular classes of Mellin-type convolution operators that preserve the functions x^a and x^{2a} , for $a > 0$. This represents a suitable and direct approach to the approximation properties of sequences of such operators using weighted versions of Korovkin-type theorems.

The classical Korovkin theorem ([28] see also [29]) is one of the most impressive results in approximation theory. It comes from a generalization of the well-known proof of the Weierstrass approximation theorem that was given by Bernstein using his polynomials. It states that a sequence of positive linear operators T_n , acting on the set of continuous functions over the compact set $[0, 1]$, converges to the identity operator if and only if it converges on the functions $1, x, x^2$.

After the paper of Korovkin several mathematicians started to extend it in several directions and settings: multidimensional case, abstract space and unbounded sets, for example, see [24], [25], [15]. Altomare's survey [6] furnishes a complete picture of the matter (see also [7]).

Thus if a sequence of positive linear operators T_n fixes certain functions, this property can be used to obtain the convergence for every function of the space and sometime provides a better order of approximation. In 2003 King (see [27]) presented a modification of Bernstein operators which preserves 1 and x^2 , obtaining in this way, a better order of approximation in the interval $[0, 1/3]$. Later on many other mathematicians started to construct modifications of well known approximation processes that fix certain functions. We cite here among others [5], [21], [22], [23], [26] and the survey [3]. More recent developments consist in the study of operators preserving exponential functions as testified by the considerable number of paper on this topic in the last years. The first case of positive linear integral operators fixing e^{ax} and e^{2ax} with $a > 0$, is studied in [8] and some other studies can be found in [1], [3], [4], [11], [32] and [33].

Mellin convolution operators represent an important tool in Mellin transform theory widely exposed in [30], [20] and in the recent monograph [13]. These studies generated high research interest throughout the years, we can quote for example [16], [17], [34], [35] (see also [12]).

Starting from [19] (see also [32]), the goal of our paper is not simply to provide convergence theorems for Mellin convolution operators but, following a wide literature on classical integral convolution operators, to construct an entire class of positive linear operators of Mellin type with the property of fixing the functions x^a and x^{2a} , $a > 0$. The most important point is that the method we use, which is based on the properties of fixed points, allows us to use more easily Korovkin's theorem to achieve convergence in various situations.

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More precisely, for every $n \in \mathbb{N}$ and fixed $a > 0$, we define

$$(\mathcal{T}_n f)(x) = \mathcal{A}_{a,n} \int_0^\infty u^{-a} f(\lambda_n(x)u) \mathcal{K}_n(u) \frac{du}{u},$$

where $\{\mathcal{K}_n\}$ is a non-negative kernel over $\mathbb{R}^+$ belonging to an appropriate function space and

$$\mathcal{A}_{a,n} := \int_0^\infty u^a \mathcal{K}_n(u) \frac{du}{u}, \quad \lambda_n(x) := \frac{x}{(\mathcal{A}_{a,n})^{1/a}}.$$

The paper is organized in the following way: In Section 2, we give preliminaries. In Section 3, we introduce the class of modified Mellin-type integral operators. In Section 4, we study weighted convergence results in weighted spaces of continuous functions. Moreover in Section 5, we obtain quantitative estimates of the rate of convergence using suitable moduli of continuity. Related definitions and properties of other kind of moduli of continuity can be found in the recent papers [9], [10]. In Section 6, a Voronovskaya asymptotic formula (see [16], and [17]) is obtained. In Section 7, we apply the theory to various kinds of Mellin-type operators. Finally, in Section 8, we provide some graphical examples.

2 Preliminaries

We denote the set of all positive real numbers by $\mathbb{R}^+$ and the set of all positive integers by $\mathbb{N}$.

The symbols $L^1(\mu)$ and $L^\infty(\mu)$ represent the space of all Lebesgue integrable functions on $\mathbb{R}^+$ with respect to the measure μ with $\mu(S) := \int_S du/u$, where S is a Lebesgue measurable set, and the space of all essentially bounded functions on $\mathbb{R}^+$, respectively. The corresponding norms are denoted as usual by $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

The space of all continuous functions on $\mathbb{R}^+$ is denoted by $C(\mathbb{R}^+)$ and by $C_B(\mathbb{R}^+)$ we denote the space of all the uniformly continuous and bounded functions.

We say that a function $f \in C(\mathbb{R}^+)$ is "log-uniformly continuous" on $\mathbb{R}^+$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(u) - f(v)| < \varepsilon \quad \text{whenever} \quad |\log u - \log v| \leq \delta \quad (u, v \in \mathbb{R}^+).$$

We denote by $C_B(\mathbb{R}^+)$ the space containing the log-uniformly continuous and bounded functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$. Note that in general a log-uniformly continuous function is not necessarily uniformly continuous and conversely a uniformly continuous function is not necessarily log-uniformly continuous. The two notions are equivalent on compact intervals in $\mathbb{R}^+$.

Lastly, we say that a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is locally of class C^r for $r \in \mathbb{N}$ at a point $x \in \mathbb{R}^+$ if there exists a neighborhood $\mathcal{N}_x$ of the point x such that f is $(r-1)$ -fold continuously differentiable in $\mathcal{N}_x$ and $f^{(r)}(x)$ exists.

The pointwise Mellin differential operator Θ , or the pointwise Mellin derivative Θf of a function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is defined by

$$\Theta f(x) := x f'(x), \quad x \in \mathbb{R}^+$$

provided f' exists a.e. on $\mathbb{R}^+$. The Mellin differential operator of order $r \in \mathbb{N}$ is defined iteratively by

$$\Theta^1 := \Theta, \quad \Theta^r := \Theta(\Theta^{r-1}).$$

For convenience we set $\Theta^0 := I$, I denoting the identity. For instance, the first three Mellin derivatives are given by:

$$\begin{aligned} \Theta f(x) &= x f'(x), \\ \Theta^2 f(x) &= x^2 f''(x) + x f'(x), \\ \Theta^3 f(x) &= x^3 f'''(x) + 3x^2 f''(x) + x f'(x). \end{aligned}$$

In general, we have (see [20])

$$\Theta^r f(x) = \sum_{k=0}^r S(r, k) x^k f^{(k)}(x),$$

where $S(r, k)$, $0 \leq k \leq r$, denote the (classical) Stirling numbers of second kind.

We have the following Taylor formula with Mellin derivatives (see [17]).

Proposition 1. For any $f \in C_B(\mathbb{R}^+)$ belonging to C^r locally at a point $x \in \mathbb{R}^+$ we have

$$f(tx) = f(x) + (\Theta f)(x) \log t + \frac{(\Theta^2 f)(x)}{2!} \log^2 t + \cdots + \frac{(\Theta^r f)(x)}{r!} \log^r t + h(t) \log^r t,$$

where $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a bounded function such that $h(t) \rightarrow 0$ for $t \rightarrow 1$.

Remark 1. The boundedness of the function h in the remainder of the Taylor formula with Mellin derivatives comes from the boundedness of the function f . However, the same holds for functions $f \in C(\mathbb{R}^+)$ which have a growth of type

$$|f(x)| \leq a + b |\log x|^r,$$

for positive constants a, b and $x \in \mathbb{R}^+$. See for example [18].

3 The Class of Mellin-type Integral Operators

For a fixed $a > 0$, we define the space $\mathcal{L}_{m_a}(\mu)$ as follows:

$$\mathcal{L}_{m_a}(\mu) := \left\{ g : \mathbb{R}^+ \rightarrow \mathbb{R} : g(\cdot)(c_0 m_a(\cdot) + d_0 m_{-a}(\cdot)) \in L_1(\mu) \right\},$$

where $m_a(u) := u^a$, $m_{-a}(u) := u^{-a}$ for $u \in \mathbb{R}^+$, and c_0, d_0 are arbitrary positive real numbers.

Assume that the kernel function $\mathcal{K}_n(u)$ ($u \in \mathbb{R}^+$, $n \in \mathbb{N}$) satisfies the following conditions:

i) $\mathcal{K}_n$ is a non-negative function over $\mathbb{R}^+$, that is, $\mathcal{K}_n(u) \geq 0$ for every $u \in \mathbb{R}^+$ and $n \in \mathbb{N}$ and $\mathcal{K}_n \in L^1(\mu)$ with

$$\|\mathcal{K}_n\|_1 = \int_0^\infty \mathcal{K}_n(u) \frac{du}{u} = 1;$$

ii) There exists $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ satisfying $n \geq n_0$, $\mathcal{K}_n \in \mathcal{L}_{m_a}(\mu)$.

According to these conditions, we introduce the following operators:

$$\begin{aligned} (\mathcal{T}_n f)(x) &:= \mathcal{A}_{a,n} \int_0^\infty m_{-a}(u) f(\lambda_n(x)u) \mathcal{K}_n(u) \frac{du}{u} \\ &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} f(\lambda_n(x)u) \mathcal{K}_n(u) \frac{du}{u}, \end{aligned} \quad (1)$$

where

$$\mathcal{A}_{a,n} := \int_0^\infty u^a \mathcal{K}_n(u) \frac{du}{u}, \quad \lambda_n(x) := \frac{x}{(\mathcal{A}_{a,n})^{1/a}}$$

for every function f belonging to the domain $\mathcal{D} := \cap_{n \geq n_0} \text{dom} \mathcal{T}_n$, where $\text{dom} \mathcal{T}_n$ denotes the set of all Lebesgue measurable functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $(\mathcal{T}_n |f|)(x)$ is finite almost everywhere on $\mathbb{R}^+$. Note that, if $f \in L^\infty(\mu)$, then $f \in \mathcal{D}$.

Under the condition $\mathcal{K}_n \in \mathcal{L}_{m_a}(\mu)$ for $n \geq n_0$, we easily see that $\mathcal{A}_{a,n}$ and $\mathcal{A}_{-a,n}$ exist and they are finite.

Proposition 2. Under the established conditions on the kernel $\{\mathcal{K}_n\}$, we have

$$(\mathcal{T}_n m_a)(x) = m_a(x), \quad (\mathcal{T}_n m_{2a})(x) = m_{2a}(x). \quad (2)$$

Proof. We have

$$\begin{aligned} (\mathcal{T}_n m_a)(x) &= \mathcal{A}_{a,n} \int_0^\infty m_{-a}(u) m_a(\lambda_n(x)u) \mathcal{K}_n(u) \frac{du}{u} \\ &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} x^a u^a (\mathcal{A}_{a,n})^{-1} \mathcal{K}_n(u) \frac{du}{u} \\ &= x^a, \end{aligned}$$

and

$$\begin{aligned} (\mathcal{T}_n m_{2a})(x) &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} x^{2a} u^{2a} (\mathcal{A}_{a,n})^{-2} \mathcal{K}_n(u) \frac{du}{u} \\ &= x^{2a} (\mathcal{A}_{a,n})^{-1} \int_0^\infty u^a \mathcal{K}_n(u) \frac{du}{u} \\ &= x^{2a}. \end{aligned}$$

Thus, the claim in (2) follows. $\square$

Remark 2. In view of Lebesgue dominated convergence theorem, the classical Mellin-type integral operators of the form

$$(\mathcal{T}_n^0 f)(x) = \int_0^\infty f(xu) \mathcal{K}_n(u) \frac{du}{u}$$

are connected to $(\mathcal{T}_n f)(x)$ via $\lim_{a \rightarrow 0^+} (\mathcal{T}_n f)(x) = (\mathcal{T}_n^0 f)(x)$ provided that

$$\lim_{a \rightarrow 0^+} (\mathcal{A}_{a,n})^{-1/a} = 1$$

for continuous and bounded functions $f \in \mathcal{D}$. In particular, it is easy to see that $\lim_{a \rightarrow 0^+} \mathcal{A}_{a,n} = 1$ and $\lim_{a \rightarrow 0^+} \lambda_n(x) = x$.

4 Weighted Convergence

We start by introducing the following subspace of $\mathcal{L}_{m_a}(\mu)$, which will be used in the following, that is,

$$\mathcal{L}_{m_a}^*(\mu) := \{g \in \mathcal{L}_{m_a}(\mu) : g(\cdot)(c_1 + c_2|\log(\cdot)|)^2(c_0 m_a(\cdot) + d_0 m_{-a}(\cdot)) \in L^1(\mu)\},$$

where c_0, d_0, c_1 and c_2 are arbitrary positive constants.

To go further, we need the following constants:

$$\mathcal{B}_{-a,n} := \int_0^\infty u^{-a} \log(u) \mathcal{K}_n(u) \frac{du}{u}, \quad \mathcal{C}_{-a,n} := \int_0^\infty u^{-a} \log^2(u) \mathcal{K}_n(u) \frac{du}{u},$$

which are called logarithmic moments of order 1 and 2, respectively. Also, $\mathcal{A}_{-a,n}$ is the logarithmic moment of order 0. If $\mathcal{K}_n \in \mathcal{L}_{m_a}^*(\mu)$ for sufficiently large $n \in \mathbb{N}$, then $\mathcal{B}_{-a,n}$ and $\mathcal{C}_{-a,n}$ exist and they are finite. Now, we define the test functions as $e_0(u) := 1$, $e_1(u) := \log(u)$ and $e_2(u) := \log^2(u)$ on $\mathbb{R}^+$.

Proposition 3. Let $\mathcal{K}_n \in \mathcal{L}_{m_a}^*(\mu)$ for sufficiently large $n \in \mathbb{N}$. Then, there hold:

$$(\mathcal{T}_n e_0)(x) = \mathcal{A}_{a,n} \mathcal{A}_{-a,n},$$

$$(\mathcal{T}_n e_1)(x) = \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log(x) - \frac{1}{a} \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log(\mathcal{A}_{a,n}) + \mathcal{A}_{a,n} \mathcal{B}_{-a,n}$$

and

$$\begin{aligned} (\mathcal{T}_n e_2)(x) &= \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log^2(x) + \left(2\mathcal{A}_{a,n} \mathcal{B}_{-a,n} - \frac{2}{a} \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log(\mathcal{A}_{a,n})\right) \log(x) \\ &\quad + \frac{1}{a^2} \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log^2(\mathcal{A}_{a,n}) - \frac{2}{a} \mathcal{A}_{a,n} \mathcal{B}_{-a,n} \log(\mathcal{A}_{a,n}) + \mathcal{A}_{a,n} \mathcal{C}_{-a,n}. \end{aligned}$$

Proof. We have

$$(\mathcal{T}_n e_0)(x) = \mathcal{A}_{a,n} \int_0^\infty u^{-a} \mathcal{K}_n(u) \frac{du}{u} = \mathcal{A}_{a,n} \mathcal{A}_{-a,n},$$

$$\begin{aligned} (\mathcal{T}_n e_1)(x) &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} \log(\lambda_n(x)u) \mathcal{K}_n(u) \frac{du}{u} \\ &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} \left(\log(u) + \log(x) - \frac{1}{a} \log(\mathcal{A}_{a,n}) \right) \mathcal{K}_n(u) \frac{du}{u} \\ &= \mathcal{A}_{a,n} \mathcal{B}_{-a,n} + \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log(x) - \frac{1}{a} \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log(\mathcal{A}_{a,n}) \end{aligned}$$

and

$$\begin{aligned} (\mathcal{T}_n e_2)(x) &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} \log^2(\lambda_n(x)u) \mathcal{K}_n(u) \frac{du}{u} \\ &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} \left(\log(u) + \log(x) - \frac{1}{a} \log(\mathcal{A}_{a,n}) \right)^2 \mathcal{K}_n(u) \frac{du}{u} \\ &= \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log^2(x) + \left(2\mathcal{A}_{a,n} \mathcal{B}_{-a,n} - \frac{2}{a} \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log(\mathcal{A}_{a,n})\right) \log(x) \\ &\quad + \frac{1}{a^2} \mathcal{A}_{a,n} \mathcal{A}_{-a,n} \log^2(\mathcal{A}_{a,n}) - \frac{2}{a} \mathcal{A}_{a,n} \mathcal{B}_{-a,n} \log(\mathcal{A}_{a,n}) + \mathcal{A}_{a,n} \mathcal{C}_{-a,n}. \end{aligned}$$

□

Corollary 1. Under the assumptions of Proposition 3, if moreover

$$\lim_{n \rightarrow +\infty} \mathcal{A}_{a,n} = \lim_{n \rightarrow +\infty} \mathcal{A}_{-a,n} = 1$$

and

$$\lim_{n \rightarrow +\infty} \mathcal{B}_{-a,n} = \lim_{n \rightarrow +\infty} \mathcal{C}_{-a,n} = 0,$$

then

$$\lim_{n \rightarrow +\infty} (\mathcal{T}_n e_j)(x) = e_j(x), \quad (j = 0, 1, 2).$$

Using Proposition 3 and Corollary 1, we now give a uniform convergence result for the operators $(\mathcal{T}_n f)$ when f belongs to suitable weighted spaces of continuous functions. The theorem which will be used is a Korovkin-type theorem proved in [24] (see also [25]).

Given a continuous, strictly increasing function $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$, we define the function $\rho(x) := 1 + \varphi^2(x)$, and assume that $\lim_{x \rightarrow +\infty} \rho(x) = +\infty$.

We will consider two particular cases: $\varphi_1(x) = \log(x)$, and $\varphi_2(x) = x^a$, $a > 0$ and we set $\rho_1(x) := 1 + \log^2(x)$ and $\rho_2(x) := 1 + x^{2a}$, $a > 0$.

Let us consider the spaces, for $j = 1, 2$ and $x \in \mathbb{R}^+$,

$$C_{\rho_j}^0(\mathbb{R}^+) := \{f \in C(\mathbb{R}^+) : \lim_{x \rightarrow 0^+} \frac{f(x)}{\rho_j} \in \mathbb{R}, \lim_{x \rightarrow +\infty} \frac{f(x)}{\rho_j} \in \mathbb{R}\}.$$

Note that if $f \in C_{\rho_j}^0(\mathbb{R}^+)$ then $|f(x)| \leq M_f \rho_j(x)$, for a suitable constant $M_f > 0$ and $x \in \mathbb{R}^+$. We define a norm on the space $C_{\rho_j}^0(\mathbb{R}^+)$ on setting

$$\|f\|_{\rho_j} := \left\| \frac{f}{\rho_j} \right\|_{\infty}.$$

We are ready to prove the main theorem of this section.

Theorem 1. Let $j = 1, 2$ and let $f \in C_{\rho_j}^0(\mathbb{R}^+)$. Then, under the assumptions of Proposition 3 and Corollary 1, we have

$$\lim_{n \rightarrow +\infty} \|\mathcal{T}_n f - f\|_{\rho_j} = 0.$$

Proof. First, consider the case $j = 1$. The test functions e_j obviously belong to $C_{\rho_1}^0(\mathbb{R}^+)$ and moreover

$$\left\| \frac{\mathcal{T}_n e_0 - e_0}{\rho_1} \right\|_{\infty} = \left\| \frac{\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1}{\rho_1} \right\|_{\infty} \leq |\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1|,$$

and the last term tends to 0 as $n \rightarrow +\infty$. Next,

$$\begin{aligned} \left\| \frac{\mathcal{T}_n e_1 - e_1}{\rho_1} \right\|_{\infty} &\leq |\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1| \sup_{x \in \mathbb{R}^+} \frac{|\log(x)|}{1 + \log^2(x)} \\ &\quad + \frac{1}{a} \mathcal{A}_{a,n} \mathcal{A}_{-a,n} |\log \mathcal{A}_{a,n}| + \mathcal{A}_{a,n} |\mathcal{B}_{-a,n}|, \end{aligned}$$

and so

$$\lim_{n \rightarrow +\infty} \left\| \frac{\mathcal{T}_n e_1 - e_1}{\rho_1} \right\|_{\infty} = 0.$$

Analogously, one can see that

$$\lim_{n \rightarrow +\infty} \left\| \frac{\mathcal{T}_n e_2 - e_2}{\rho_1} \right\|_{\infty} = 0.$$

For $j = 2$, taking into account of (2) and Proposition 2, we obtain again the convergence on the test functions x^{ka} , for $k = 0, 1, 2$. Applying Theorem 2 in [24] we obtain the assertion. $\square$

5 Quantitative Estimate of the Convergence

Here we state a quantitative approximation result for functions $f \in C_{\rho_j}^0(\mathbb{R}^+)$, $j = 1, 2$, in terms of the following modulus of continuity:

$$\omega(f, \delta) := \sup\{|f(u) - f(v)| : |\log u - \log v| \leq \delta\}, \quad \delta > 0.$$

Note that ω satisfies all the classical properties of a modulus of continuity. In particular the following one

$$\omega(f, \lambda \delta) \leq (\lambda + 1) \omega(f, \delta),$$

for every $\delta, \lambda > 0$. Moreover for any log-uniformly continuous function $f \in \mathcal{C}(\mathbb{R}^+)$ one has

$$\lim_{\delta \rightarrow 0} \omega(f, \delta) = 0.$$

For $\mathcal{K}_n \in \mathcal{L}_{m_a}^*(\mu)$ we introduce the absolute logarithmic moment of order 1 by

$$|\mathcal{B}|_{-a,n} := \int_0^\infty u^{-a} |\log(u)| \mathcal{K}_n(u) \frac{du}{u}.$$

We have the following quantitative estimate

Theorem 2. Let $\mathcal{K}_n \in \mathcal{L}_{m_a}^*(\mu)$ for sufficiently large values of $n \in \mathbb{N}$. If $f \in C_{\rho_j}^0(\mathbb{R}^+)$ then for every $\delta > 0$ we have

$$\begin{aligned} \|(\mathcal{T}_n f)(x) - f(x)\|_{\rho_j} &\leq \omega(f, \delta) \mathcal{A}_{a,n} \left(\mathcal{A}_{-a,n} + \frac{1}{\delta} |\mathcal{B}|_{-a,n} + \frac{1}{a\delta} \mathcal{A}_{-a,n} |\log \mathcal{A}_{a,n}| \right) \\ &\quad + \|f\|_{\rho_j} |\mathcal{A}_{-a,n} \mathcal{A}_{a,n} - 1|. \end{aligned}$$

Proof. For every $x \in \mathbb{R}^+$ and $\delta > 0$ we have

$$\begin{aligned} |(\mathcal{T}_n f)(x) - f(x)| &= |(\mathcal{T}_n f)(x) - \mathcal{A}_{a,n} \mathcal{A}_{a,n} f(x) + \mathcal{A}_{a,n} \mathcal{A}_{a,n} f(x) - f(x)| \\ &\leq \mathcal{A}_{a,n} \int_0^\infty u^{-a} \omega(f, |\log u - \frac{1}{a} \log \mathcal{A}_{a,n}|) \mathcal{K}_n(u) \frac{du}{u} + |f(x)| |\mathcal{A}_{-a,n} \mathcal{A}_{a,n} - 1| \\ &=: I_1 + I_2. \end{aligned}$$

Using the properties of the modulus ω we obtain

$$\begin{aligned} I_1 &\leq \mathcal{A}_{a,n} \omega(f, \delta) \int_0^\infty u^{-a} \left(1 + \frac{|\log u - \frac{1}{a} \log \mathcal{A}_{a,n}|}{\delta} \right) \mathcal{K}_n(u) \frac{du}{u} \\ &\leq \mathcal{A}_{a,n} \omega(f, \delta) \left(\mathcal{A}_{-a,n} + \frac{1}{\delta} \int_0^\infty u^{-a} (|\log u| + \frac{1}{a} |\log \mathcal{A}_{a,n}|) \mathcal{K}_n(u) \frac{du}{u} \right) \\ &\leq \mathcal{A}_{a,n} \omega(f, \delta) \left(\mathcal{A}_{-a,n} + \frac{1}{\delta} |\mathcal{B}|_{-a,n} + \frac{1}{a\delta} \mathcal{A}_{-a,n} |\log \mathcal{A}_{a,n}| \right). \end{aligned}$$

Taking the norm, we obtain the assertion

$$\begin{aligned} \|(\mathcal{T}_n f)(x) - f(x)\|_{\rho_j} &\leq \omega(f, \delta) \mathcal{A}_{a,n} \left(\mathcal{A}_{-a,n} + \frac{1}{\delta} |\mathcal{B}|_{-a,n} + \frac{1}{a\delta} \mathcal{A}_{-a,n} |\log \mathcal{A}_{a,n}| \right) \\ &\quad + \|f\|_{\rho_j} |\mathcal{A}_{-a,n} \mathcal{A}_{a,n} - 1|. \end{aligned}$$

□

Corollary 2. Under the assumptions of Theorem 2 and Corollary 1, if moreover there is $\beta > 0$ such that $(n^\beta |\mathcal{B}|_{-a,n})_n$ and $(n^\beta |\log \mathcal{A}_{a,n}|)_n$ are bounded sequences, then there exists an absolute constant $M > 0$ such that

$$\|(\mathcal{T}_n f)(x) - f(x)\|_{\rho_j} \leq M \omega(f, \frac{1}{n^\beta}) + \|f\|_{\rho_j} |\mathcal{A}_{-a,n} \mathcal{A}_{a,n} - 1|,$$

for sufficiently large values of $n \in \mathbb{N}$.

Proof. It is sufficient to take $\delta = n^{-\beta}$ in Theorem 2. □

We can obtain a different estimate for the rate of convergence using a weighted modulus of continuity. In what follows we limit ourselves to consider the weight ρ_1 . Similar estimates can be obtained also for ρ_2 .

It will be useful the absolute logarithmic moment of order 5 of $\mathcal{K}_n \in \mathcal{L}_{m_a}^*(\mu)$ defined by

$$|\mathcal{E}|_{-a,n} := \int_0^\infty |\log(u)|^5 u^{-a} \mathcal{K}_n(u) \frac{du}{u}.$$

Let us consider the following weighted logarithmic modulus of continuity $\tilde{\omega}$ which is suitable for the space $C_{\rho_1}^0(\mathbb{R}^+)$. This modulus of continuity was introduced in [2] as follows:

$$\tilde{\omega}(f, \delta) := \sup_{|\log t| \leq \delta, x \in \mathbb{R}^+} \frac{|f(tx) - f(x)|}{(1 + \log^2(x))(1 + \log^2(t))},$$

where $\delta > 0$. For $f \in C_{\rho_1}^0(\mathbb{R}^+)$, $\tilde{\omega}$ satisfies the following properties ([2]; see also [36]):

1. $\lim_{\delta \rightarrow 0} \tilde{\omega}(f, \delta) = 0$.
2. $|f(t) - f(x)| \leq 16(1 + \log^2(x))(1 + \delta^2)^2 \left(1 + \frac{|\log(t) - \log(x)|^5}{\delta^5}\right) \tilde{\omega}(f, \delta), \quad t, x \in \mathbb{R}^+.$

Theorem 3. Let $\mathcal{K}_n \in \mathcal{L}_{m_a}^*(\mu)$ with $|\mathcal{E}|_{-a,n}$ finite. For $f \in C_{\rho_1}^0(\mathbb{R}^+)$, there holds

$$\begin{aligned} \|\mathcal{T}_n f - f\|_{\rho_1} &\leq 2^{11} \mathcal{A}_{a,n} (\mathcal{A}_{-a,n} + 1) \tilde{\omega}(f, (|\mathcal{E}|_{-a,n})^{1/5}) \\ &\quad + \|f\|_{\rho_1} |\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1| \end{aligned}$$

provided that $\left| \log \left(\mathcal{A}_{a,n}^{-\frac{1}{a}} \right) \right| \leq (|\mathcal{E}|_{-a,n})^{1/5} < 1$ for sufficiently large n .

Proof. Let $f \in C_{\rho_1}^0(\mathbb{R}^+)$. We easily get

$$\begin{aligned} |(\mathcal{T}_n f)(x) - f(x)| &\leq \mathcal{A}_{a,n} \int_0^\infty |f(\lambda_n(x)u) - f(x)| u^{-a} \mathcal{K}_n(u) \frac{du}{u} \\ &\quad + |f(x)| |\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1|. \end{aligned}$$

Using property (2) of $\tilde{\omega}$, we get

$$|f(\lambda_n(x)u) - f(x)| \leq 16(1 + \log^2(x))(1 + \delta^2)^2 \tilde{\omega}(f, \delta) \left(1 + \frac{|\log(\lambda_n(x)u) - \log(x)|^5}{\delta^5} \right).$$

In view of last observation, we have

$$\begin{aligned} |(\mathcal{T}_n f)(x) - f(x)| &\leq 16 \mathcal{A}_{a,n} (1 + \log^2(x)) (1 + \delta^2)^2 \tilde{\omega}(f, \delta) \\ &\quad \times \left[\int_0^\infty \left(\frac{|\log(\lambda_n(x)u) - \log(x)|^5}{\delta^5} \right) u^{-a} \mathcal{K}_n(u) \frac{du}{u} + \int_0^\infty u^{-a} \mathcal{K}_n(u) \frac{du}{u} \right] \\ &\quad + |f(x)| |\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1|. \end{aligned}$$

Rewriting the last inequality, we get

$$\begin{aligned} |(\mathcal{T}_n f)(x) - f(x)| &\leq 16(1 + \log^2(x)) (1 + \delta^2)^2 \tilde{\omega}(f, \delta) \\ &\quad \times \mathcal{A}_{a,n} \left[\int_0^\infty \left(\frac{|\log(\mathcal{A}_{a,n}^{-\frac{1}{a}}) + \log(u)|^5}{\delta^5} \right) u^{-a} \mathcal{K}_n(u) \frac{du}{u} + \mathcal{A}_{-a,n} \right] \\ &\quad + |f(x)| |\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1|. \end{aligned}$$

Applying the well-known inequality $|w_1 - w_2|^p \leq 2^{p-1}(|w_1|^p + |w_2|^p)$ for all $w_1, w_2 \in \mathbb{R}$, $p \geq 1$, we get

$$\begin{aligned} |(\mathcal{T}_n f)(x) - f(x)| &\leq 2^8 (1 + \log^2(x)) (1 + \delta^2)^2 \tilde{\omega}(f, \delta) \\ &\quad \times \mathcal{A}_{a,n} \left[\mathcal{A}_{-a,n} \left(1 + \frac{|\log(\mathcal{A}_{a,n}^{-\frac{1}{a}})|^5}{\delta^5} \right) + \int_0^\infty \frac{|\log(u)|^5}{\delta^5} u^{-a} \mathcal{K}_n(u) \frac{du}{u} \right] \\ &\quad + |f(x)| |\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1|. \end{aligned}$$

Choosing $\delta = (|\mathcal{E}|_{-a,n})^{1/5}$ with $|\log(\mathcal{A}_{a,n}^{-\frac{1}{a}})| \leq \delta < 1$ for sufficiently large n , dividing both sides by $1 + \log^2(x)$ and taking supremum over all x , we obtain the desired result, that is,

$$\begin{aligned} \|\mathcal{T}_n f - f\|_{\rho_1} &\leq 2^{11} \mathcal{A}_{a,n} (\mathcal{A}_{-a,n} + 1) \tilde{\omega}(f, (|\mathcal{E}|_{-a,n})^{1/5}) \\ &\quad + \|f\|_{\rho_1} |\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1|. \end{aligned}$$

□

6 An Asymptotic Formula

For functions $f \in C_{\rho_j}^0(\mathbb{R}^+)$, $j = 1, 2$ which are locally of class C^2 at a point x we can obtain asymptotic formulae of Voronovskaya type. Let us consider the case $j = 1$.

We have the following result.

Theorem 4. Let $f \in C_{\rho_1}^0(\mathbb{R}^+)$ be locally of class C^2 at a point $x \in \mathbb{R}^+$. Let $\mathcal{K}_n \in \mathcal{L}_{m_a}^*(\mu)$ for sufficiently large $n \in \mathbb{N}$ satisfying assumptions of Corollary 1. Assume that there exists $\alpha > 0$ such that the following conditions for the kernel $\mathcal{K}_n$ are satisfied:

(1)

$$\begin{aligned} \lim_{n \rightarrow +\infty} n^\alpha (\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1) &= \ell_0, \quad \lim_{n \rightarrow +\infty} n^\alpha (\mathcal{B}_{-a,n} - \mathcal{A}_{-a,n} \frac{1}{a} \log \mathcal{A}_{a,n}) = \ell_1, \\ \lim_{n \rightarrow +\infty} n^\alpha (\mathcal{C}_{-a,n} + \frac{1}{a^2} \mathcal{A}_{-a,n} \log^2 \mathcal{A}_{a,n} - \frac{2}{a} \mathcal{B}_{-a,n} \log \mathcal{A}_{a,n}) &= \ell_2, \end{aligned}$$

with $\ell_j \in \mathbb{R}$ for $j = 0, 1, 2$.

(2) For every $\eta > 0$

$$\lim_{n \rightarrow +\infty} n^\alpha \int_{|\log u| \geq \eta} u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) \mathcal{K}_n(u) \frac{du}{u} = 0.$$

Then we have

$$\lim_{n \rightarrow +\infty} n^a [(\mathcal{T}_n f)(x) - f(x)] = \ell_0 f(x) + \ell_1(\Theta f)(x) + \frac{\ell_2}{2}(\Theta^2 f)(x).$$

Proof. By the fact that the function f grows like a power of the logarithm and is locally of class C^2 at the point x , we can use a local Taylor formula of the second order, obtaining by Proposition 1

$$\begin{aligned} f(\lambda_n(x)u) &= f\left(\frac{x}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}u\right) = f(x) + (\Theta f)(x) \log\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) + \frac{(\Theta^2 f)(x)}{2!} \log^2\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \\ &\quad + h\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \log^2\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \end{aligned}$$

with h a bounded function such that $\lim_{t \rightarrow 1} h(t) = 0$. Then

$$\begin{aligned} (\mathcal{T}_n f)(x) - f(x) &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} [f(\lambda_n(x)u) - f(x)] \mathcal{K}_n(u) \frac{du}{u} + (\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1) f(x) \\ &:= I_1 + I_2. \end{aligned}$$

For what concerns I_2 we have to evaluate $\lim_{n \rightarrow +\infty} n^a (\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1)$ and by assumptions we obtain that $\lim_{n \rightarrow +\infty} n^a I_2 = \ell_0 f(x)$. Now we consider the term I_1 . We have

$$\begin{aligned} I_1 &= \mathcal{A}_{a,n} \int_0^\infty u^{-a} [f(\lambda_n(x)u) - f(x)] \mathcal{K}_n(u) \frac{du}{u} = (\Theta f)(x) \mathcal{A}_{a,n} \int_0^\infty u^{-a} \log\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \mathcal{K}_n(u) \frac{du}{u} \\ &\quad + \frac{(\Theta^2 f)(x)}{2!} \mathcal{A}_{a,n} \int_0^\infty u^{-a} \log^2\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \mathcal{K}_n(u) \frac{du}{u} \\ &\quad + \mathcal{A}_{a,n} \int_0^\infty u^{-a} h\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \log^2\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \mathcal{K}_n(u) \frac{du}{u} = I_1^1 + I_1^2 + R. \end{aligned}$$

Then

$$\begin{aligned} I_1^1 &= (\Theta f)(x) \mathcal{A}_{a,n} \int_0^\infty u^{-a} \left[\log u - \frac{1}{a} \log \mathcal{A}_{a,n}\right] \mathcal{K}_n(u) \frac{du}{u} \\ &= (\Theta f)(x) \mathcal{A}_{a,n} \left(\mathcal{B}_{-a,n} - \frac{1}{a} \mathcal{A}_{-a,n} \log \mathcal{A}_{a,n}\right) \end{aligned}$$

and

$$\begin{aligned} I_1^2 &= \frac{(\Theta^2 f)(x)}{2} \mathcal{A}_{a,n} \int_0^\infty u^{-a} \left[\log u - \frac{1}{a} \log \mathcal{A}_{a,n}\right]^2 \mathcal{K}_n(u) \frac{du}{u} \\ &= \frac{(\Theta^2 f)(x)}{2} \mathcal{A}_{a,n} \left(\mathcal{C}_{-a,n} + \frac{1}{a^2} \mathcal{A}_{-a,n} \log^2 \mathcal{A}_{a,n} - \frac{2}{a} \mathcal{B}_{-a,n} \log \mathcal{A}_{a,n}\right). \end{aligned}$$

So we have by assumptions that

$$\lim_{n \rightarrow +\infty} n^a I_1^1 = \ell_1(\Theta f)(x), \quad \lim_{n \rightarrow +\infty} n^a I_1^2 = \frac{(\Theta^2 f)(x)}{2} \ell_2.$$

Finally, we have to estimate the term R . Since $\lim_{t \rightarrow 1} h(t) = 0$, given an arbitrary $\varepsilon > 0$, there is $\delta > 0$ such that $|h(t)| < \varepsilon$ whenever $|\log t| \leq \delta$. We take now an index $\bar{n}$ such that for $n \geq \bar{n}$ one has $|\frac{1}{a} \log(\mathcal{A}_{a,n})| < \frac{\delta}{2}$ and u such that $|\log u| < \frac{\delta}{2}$. In this case we have

$$\left| \log\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \right| = \left| \log u - \frac{1}{a} \log \mathcal{A}_{a,n} \right| < \delta$$

for $n \geq \bar{n}$ and for $|\log u| < \frac{\delta}{2}$. Then

$$R = \mathcal{A}_{a,n} \left\{ \int_{|\log u| < \delta/2} + \int_{|\log u| \geq \delta/2} \right\} u^{-a} h\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \log^2\left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}}\right) \mathcal{K}_n(u) \frac{du}{u} =: R_1 + R_2$$

and we have

$$|R_1| < \mathcal{A}_{a,n} \varepsilon \left(\mathcal{C}_{-a,n} + \frac{\mathcal{A}_{-a,n}}{a^2} \log^2 \mathcal{A}_{a,n} - \frac{2}{a} \mathcal{B}_{-a,n} \log \mathcal{A}_{a,n} \right).$$

By assumption (1) we have that

$$\limsup_{n \rightarrow +\infty} n^a |R_1| \leq \varepsilon \ell_2.$$

For what concerns R_2 , by the boundedness of h we have

$$|R_2| \leq \mathcal{A}_{a,n} \|h\|_\infty \int_{|\log u| \geq \delta/2} u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) \mathcal{K}_n(u) \frac{du}{u}$$

and by assumption (2) we obtain

$$\lim_{n \rightarrow +\infty} n^\alpha |R_2| = 0.$$

This means that $\limsup_{n \rightarrow +\infty} n^\alpha |R| \leq \varepsilon \ell_2$ and by arbitrariness of ε $\lim_{n \rightarrow +\infty} n^\alpha R = 0$. $\square$

7 Some Examples

In this section we apply the previous theory to various specific examples (see e.g. [17]).

Mellin-Gauss-Weierstrass operators. Let us consider the kernel:

$$\mathcal{K}_n(u) = \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u}$$

and easily we have that $\|\mathcal{K}_n\|_1 = 1$ (see [20]). Now we begin with the evaluation of the coefficients $\mathcal{A}_{a,n}$. We obtain by substitution $\frac{n \log u}{2} = z$

$$\mathcal{A}_{a,n} = \int_0^\infty u^a \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty e^{\frac{2}{n} z a - z^2} dz.$$

By differentiation under the integral sign and solving a simple first order linear differential equation, we have

$$\mathcal{A}_{a,n} = \mathcal{A}_{-a,n} = e^{\frac{a^2}{n^2}}$$

and so $\lim_{n \rightarrow +\infty} \mathcal{A}_{a,n} = 1$. Moreover, taking $\alpha = 2$

$$\ell_0 = \lim_{n \rightarrow +\infty} n^2 (\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1) = \lim_{n \rightarrow +\infty} n^2 (e^{\frac{2a^2}{n^2}} - 1) = 2a^2.$$

For what concerns the moment $\mathcal{B}_{-a,n}$ using an integration by parts, we have

$$\begin{aligned} \mathcal{B}_{-a,n} &= \int_0^\infty u^{-a} \log u \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} = -\frac{1}{n\sqrt{\pi}} \int_0^\infty u^{-a} \left(e^{-\frac{n^2}{4} \log^2 u} \right)' du \\ &= -\frac{a}{n\sqrt{\pi}} \int_0^\infty u^{-a} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} = -\frac{2a}{n^2} e^{\frac{a^2}{n^2}} \end{aligned}$$

and so

$$\ell_1 = \lim_{n \rightarrow +\infty} n^2 \left(\mathcal{B}_{-a,n} - \mathcal{A}_{-a,n} \frac{1}{a} \log \mathcal{A}_{a,n} \right) = \lim_{n \rightarrow +\infty} -3a e^{\frac{a^2}{n^2}} = -3a.$$

Finally we consider the moment $\mathcal{C}_{-a,n}$. In a similar way as before we have

$$\begin{aligned} \mathcal{C}_{-a,n} &= \int_0^\infty u^{-a} \log^2 u \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} = -\frac{1}{n\sqrt{\pi}} \int_0^\infty u^{-a} \log u \left(e^{-\frac{n^2}{4} \log^2 u} \right)' du \\ &= -\frac{a}{n\sqrt{\pi}} \int_0^\infty u^{-a} \log u e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} + \frac{1}{n\sqrt{\pi}} \int_0^\infty u^{-a} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} \\ &= -\frac{2a}{n^2} \mathcal{B}_{-a,n} + \frac{2}{n^2} \mathcal{A}_{-a,n} = \frac{2}{n^2} e^{\frac{a^2}{n^2}} \left(\frac{2a^2}{n^2} + 1 \right) \end{aligned}$$

and so we have

$$\begin{aligned} \ell_2 &= \lim_{n \rightarrow +\infty} n^2 \left(\mathcal{C}_{-a,n} + \frac{1}{a^2} \mathcal{A}_{-a,n} \log^2 \mathcal{A}_{a,n} - \frac{2}{a} \mathcal{B}_{-a,n} \log \mathcal{A}_{a,n} \right) \\ &= \lim_{n \rightarrow +\infty} e^{\frac{a^2}{n^2}} \left(\frac{9a^2}{n^2} + 2 \right) = 2. \end{aligned}$$

Hence

$$\ell_0 = 2a^2, \quad \ell_1 = -3a, \quad \ell_2 = 2.$$

Now, we prove that the Gauss-Weierstrass kernel satisfies also assumption (2) of Theorem 4 for $\alpha = 2$. Let $\eta > 0$ be fixed and

$$\begin{aligned} J &= n^2 \int_{|\log u| \geq \eta} u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) \mathcal{K}_n(u) \frac{du}{u} \\ &= n^2 \int_{|\log u| \geq \eta} u^{-a} \left(\log u - \frac{a}{n^2} \right)^2 \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u}. \end{aligned}$$

Then

$$\begin{aligned} J &\leq n^2 \int_{|\log u| \geq \eta} u^{-a} \log^2 u \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} + n^2 \int_{|\log u| \geq \eta} u^{-a} \frac{a^2}{n^4} \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} \\ &+ n^2 \int_{|\log u| \geq \eta} u^{-a} \frac{2a|\log u|}{n^2} \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} =: J_1 + J_2 + J_3. \end{aligned}$$

As to J_1 we can write

$$J_1 = \frac{n^3}{\sqrt{4\pi}} \left(\int_{\eta}^{\infty} e^{-at} t^2 u e^{-\frac{n^2}{4} t^2 u} dt + \int_{\eta}^{\infty} e^{at} t^2 u e^{-\frac{n^2}{4} t^2 u} dt \right) = J_1^1 + J_1^2.$$

For J_1^1 we have

$$J_1^1 \leq \frac{4}{\sqrt{\pi}} e^{-\eta a} \int_{n\eta/2}^{\infty} z^2 e^{-z^2} dz,$$

and so by the absolute continuity of the Lebesgue integral, we obtain $\lim_{n \rightarrow +\infty} J_1^1 = 0$. In the same way we have

$$J_1^2 \leq \frac{4}{\sqrt{\pi}} \int_{n\eta/2}^{\infty} e^{av} v^2 e^{-v^2} dv$$

and again, since the integrand in the right-hand side is Lebesgue integrable, we have $\lim_{n \rightarrow +\infty} J_1^2 = 0$. As to J_2 we can write

$$\lim_{n \rightarrow +\infty} J_2 \leq \lim_{n \rightarrow +\infty} \frac{a^2}{n^2} \mathcal{A}_{-a,n} = 0.$$

Finally for J_3 we have

$$\begin{aligned} J_3 &= 2a \int_{|\log u| > \eta} u^{-a} |\log u| \frac{n}{\sqrt{4\pi}} e^{-\frac{n^2}{4} \log^2 u} \frac{du}{u} \\ &\leq \frac{an}{\sqrt{\pi}} \left(e^{-a\eta} \int_{\eta}^{\infty} t e^{-\frac{n^2}{4} t^2} dt + \int_{\eta}^{\infty} e^{at} t e^{-\frac{n^2}{4} t^2} dt \right) = J_3^1 + J_3^2 \end{aligned}$$

and in a similar way to the estimate of J_1 we obtain $\lim_{n \rightarrow +\infty} J_3^1 = \lim_{n \rightarrow +\infty} J_3^2 = 0$.

Therefore, all the assumptions of Theorem 4 are satisfied with $\alpha = 2$ and so we obtain the following asymptotic formula

$$\lim_{n \rightarrow +\infty} n^2 ((\mathcal{T}_n f)(x) - f(x)) = 2a^2 f(x) - 3a(\Theta f)(x) + (\Theta^2 f)(x).$$

For related results, see [12].

Mellin-Abel-Poisson operators. Let us consider the kernel

$$\mathcal{K}_n(u) = \begin{cases} \frac{n}{2} u^n & 0 \leq u \leq 1 \\ \frac{n}{2} u^{-n} & u > 1 \end{cases}$$

and easily we have that $\|\mathcal{K}_n\|_1 = 1$ (see [20]). As to the coefficients $\mathcal{A}_{a,n}$ we have

$$\mathcal{A}_{a,n} = \mathcal{A}_{-a,n} = \frac{n^2}{n^2 - a^2}$$

for sufficiently large $n > a$, and so, taking $\alpha = 2$, we have

$$\ell_0 = \lim_{n \rightarrow +\infty} n^2 (\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1) = \lim_{n \rightarrow +\infty} n^2 \frac{2a^2 n^2 - a^4}{(n^2 - a^2)^2} = 2a^2.$$

For what concerns the moments $\mathcal{B}_{-a,n}$ and $\mathcal{C}_{-a,n}$ we can obtain

$$\mathcal{B}_{-a,n} = \frac{n}{2} \int_0^1 u^n u^{-a} \log u \frac{du}{u} + \frac{n}{2} \int_1^{\infty} u^{-n} u^{-a} \log u \frac{du}{u} = -\frac{2an^2}{(n-a)^2(n+a)^2}$$

and

$$C_{-a,n} = \frac{n}{2} \int_0^1 u^n u^{-a} \log^2 u \frac{du}{u} + \frac{n}{2} \int_1^\infty u^{-n} u^{-a} \log^2 u \frac{du}{u} = \frac{2n^2(n^2 + 3a^2)}{(n+a)^3(n-a)^3}$$

so that

$$\ell_1 = \lim_{n \rightarrow +\infty} n^2 \left(\mathcal{B}_{-a,n} - \mathcal{A}_{-a,n} \frac{1}{a} \log \mathcal{A}_{a,n} \right) = -3a$$

and

$$\ell_2 = \lim_{n \rightarrow +\infty} n^2 \left(C_{-a,n} + \frac{1}{a^2} \mathcal{A}_{-a,n} \log^2 \mathcal{A}_{a,n} - \frac{2}{a} \mathcal{B}_{-a,n} \log \mathcal{A}_{a,n} \right) = 2.$$

Then we obtain

$$\ell_0 = 2a^2 \quad \ell_1 = -3a \quad \ell_2 = 2.$$

Finally we prove that the kernel satisfies assumption (2) of Theorem 4 for $\alpha = 2$. Let $\eta > 0$. Since $\log u \geq \eta \Leftrightarrow u \geq e^\eta > 1$ we have

$$\begin{aligned} J &= n^2 \int_{|\log u| \geq \eta} u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) \mathcal{K}_n(u) \frac{du}{u} \\ &= n^2 \int_{e^\eta}^\infty u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) \frac{n}{2} u^{-n} \frac{du}{u} + n^2 \int_0^{e^{-\eta}} u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) \frac{n}{2} u^n \frac{du}{u} \\ &= J_1 + J_2. \end{aligned}$$

As to J_1 we obtain

$$\begin{aligned} J_1 &= \frac{n^3}{2} \int_{e^\eta}^\infty u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) u^{-n} \frac{du}{u} \\ &= \frac{n^3}{2} \left[\int_{e^\eta}^\infty u^{-a-n-1} \frac{1}{a^2} \log^2 \mathcal{A}_{a,n} du - \int_{e^\eta}^\infty u^{-a-n-1} \frac{2}{a} \log \mathcal{A}_{a,n} \log u du \right. \\ &\quad \left. + \int_{e^\eta}^\infty u^{-a-n-1} \log^2 u du \right] = J_1^1 - J_1^2 + J_1^3. \end{aligned}$$

Then

$$\begin{aligned} J_1^1 &= \frac{n^3}{2} \int_{e^\eta}^\infty u^{-a-n-1} \frac{1}{a^2} \log^2 \mathcal{A}_{a,n} du = \frac{n^3}{2} \frac{1}{a^2} \log^2 \mathcal{A}_{a,n} \left[\frac{u^{-a-n}}{-a-n} \right]_{e^\eta}^\infty \\ &= \frac{n^3}{2} \frac{1}{a^2} \log^2 \mathcal{A}_{a,n} \frac{e^{-\eta(n+a)}}{a+n} \end{aligned}$$

and so

$$\lim_{n \rightarrow +\infty} J_1^1 = 0.$$

Moreover, using integration by parts, we have

$$\begin{aligned} J_1^2 &= \frac{n^3}{a} \log \mathcal{A}_{a,n} \int_{e^\eta}^\infty u^{-a-n-1} \log u du = \frac{n^3}{a} \log \mathcal{A}_{a,n} \left(\frac{e^{-\eta(a+n)}}{n+a} \eta + \frac{1}{a+n} \int_{e^\eta}^\infty u^{-a-n-1} du \right) \\ &= \frac{n^3}{a} \log \mathcal{A}_{a,n} \left(\frac{e^{-\eta(a+n)}}{n+a} \eta + \frac{e^{-\eta(a+n)}}{(a+n)^2} \right) \end{aligned}$$

and so

$$\lim_{n \rightarrow +\infty} J_1^2 = 0.$$

In a similar way one can prove that also J_1^3 and J_2 tend to zero. Therefore, all the assumptions of Theorem 4 are satisfied with $\alpha = 2$ and so we obtain the following asymptotic formula:

$$\lim_{n \rightarrow +\infty} n^2 ((\mathcal{T}_n f)(x) - f(x)) = 2a^2 f(x) - 3a(\Theta f)(x) + (\Theta^2 f)(x).$$

The moment operators. Let us consider the kernel

$$\mathcal{K}_n(u) = n u^n \chi_{[0,1]}(u)$$

and easily we have that $\|\mathcal{K}_n\|_1 = 1$. For what concerns the coefficients $\mathcal{A}_{a,n}$ we have

$$\mathcal{A}_{a,n} = \int_0^1 n u^{a+n-1} du = \frac{n}{n+a}, \quad \mathcal{A}_{-a,n} = \int_0^1 n u^{-a+n-1} du = \frac{n}{n-a}$$

for sufficiently large $n > a$. Therefore, we can obtain for $\alpha = 2$

$$\ell_0 = \lim_{n \rightarrow +\infty} n^2 (\mathcal{A}_{a,n} \mathcal{A}_{-a,n} - 1) = \lim_{n \rightarrow +\infty} \frac{n^2 a^2}{n^2 - a^2} = a^2.$$

For $\mathcal{B}_{-a,n}$, we have

$$\begin{aligned} \mathcal{B}_{-a,n} &= \int_0^\infty u^{-a} \log u n u^n \chi_{[0,1]}(u) \frac{du}{u} = n \int_0^1 u^{-a+n-1} \log u du \\ &= \frac{n}{a-n} \int_0^1 u^{-a+n-1} du = -\frac{n}{(n-a)^2}. \end{aligned}$$

Therefore we get, considering analogously L'Hospital rule twice,

$$\begin{aligned} \ell_1 &= \lim_{n \rightarrow +\infty} n^2 \left(-\frac{n}{(n-a)^2} - \frac{n}{n-a} \frac{1}{a} \log \left(\frac{n}{n+a} \right) \right) \\ &= \lim_{n \rightarrow +\infty} -n \left(1 + \frac{n-a}{a} \log \left(\frac{n}{n+a} \right) \right) = -\frac{3a}{2}. \end{aligned}$$

Finally, we have for $\mathcal{C}_{-a,n}$

$$\begin{aligned} \mathcal{C}_{-a,n} &= \int_0^\infty u^{-a} \log^2 u n u^n \chi_{[0,1]}(u) \frac{du}{u} = n \int_0^1 u^{-a+n-1} \log^2 u du \\ &= \frac{2n}{a-n} \int_0^1 u^{-a+n-1} \log u du = \frac{2}{a-n} \mathcal{B}_{-a,n} = \frac{2n}{(n-a)^3} \end{aligned}$$

and

$$\ell_2 = \lim_{n \rightarrow +\infty} n^2 \left(\frac{2n}{(n-a)^3} + \frac{1}{a^2} \frac{n}{n-a} \log^2 \left(\frac{n}{n+a} \right) + \frac{2}{a} \frac{n}{(n-a)^2} \log \left(\frac{n}{n+a} \right) \right) = 1.$$

Now we prove that the moment kernel satisfies also assumption (2) of Theorem 4 for $\alpha = 2$. Let $\eta > 0$ be fixed. Since $\log u \geq \eta \Leftrightarrow u \geq e^\eta > 1$

$$\begin{aligned} J &= n^2 \int_{|\log u| \geq \eta} u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) n u^n \chi_{[0,1]}(u) \frac{du}{u} = n^2 \int_0^{e^{-\eta}} u^{-a} \log^2 \left(\frac{u}{(\mathcal{A}_{a,n})^{\frac{1}{a}}} \right) n u^n \frac{du}{u} \\ &= n^3 \int_0^{e^{-\eta}} u^{-a+n-1} \log^2 u du + n^3 \int_0^{e^{-\eta}} u^{-a+n-1} \frac{1}{a^2} \log^2 \mathcal{A}_{a,n} du \\ &\quad - n^3 \int_0^{e^{-\eta}} u^{-a+n-1} \frac{2}{a} \log u \log \mathcal{A}_{a,n} du = J_1 + J_2 - J_3. \end{aligned}$$

Easily we obtain that $\lim_{n \rightarrow +\infty} J_1 = \lim_{n \rightarrow +\infty} J_2 = \lim_{n \rightarrow +\infty} J_3 = 0$ and so also assumption (2) is satisfied. Then we have that

$$\ell_0 = a^2, \ell_1 = -\frac{3a}{2}, \ell_2 = 1$$

and the asymptotic formula with $\alpha = 2$ is given by

$$\lim_{n \rightarrow +\infty} n^2 ((\mathcal{T}_n f)(x) - f(x)) = a^2 f(x) - \frac{3}{2} a (\Theta f)(x) + \frac{1}{2} (\Theta^2 f)(x).$$

8 Graphical Examples

Using (CAS) Mathematica, we provide some graphical examples for modifications of the moment operators. Let M_n and M_n^* denote the classical moment and modified moment operators, respectively, as follows:

$$\begin{aligned} (M_n f)(x) &= n \int_0^\infty f(ux) u^n \chi_{[0,1]}(u) \frac{du}{u}, \\ (M_n^* f)(x) &= \frac{n^2}{n+a} \int_0^\infty u^{-a} f \left(xu \left(\frac{n}{n+a} \right)^{-1/a} \right) u^n \chi_{[0,1]}(u) \frac{du}{u}, \end{aligned}$$

for sufficiently large n . In Figure 1, we compare M_4 (orange), M_4^* (blue) and original function f (green) for $f(x) = e^{-x}$, $x \in [1, 4]$ and $a = 1/2$. In Figure 2, we compare M_5^* (orange), M_{10}^* (blue) and original function f (green) for $f(x) = e^{-x}$, $x \in [1, 4]$ and $a = 3/2$.

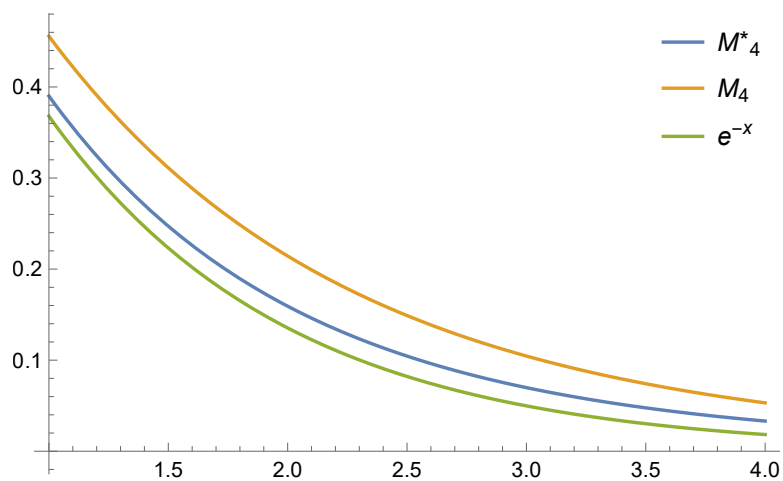


Figure 1: Comparison of the moment and modified moment operators

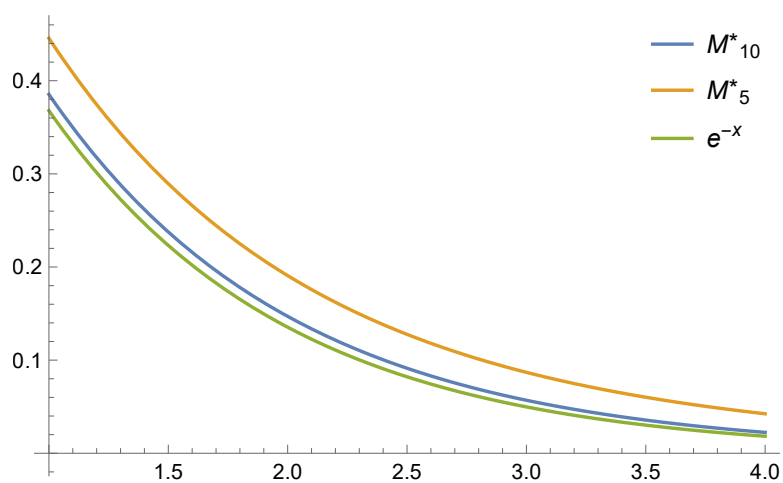


Figure 2: Comparison of the modified moment operators



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