



Approximation results in Orlicz space using max-min Fractional type neural network operators

Shivani Luhar^a · Shailesh Kumar Srivastava^{b,*}

Communicated by Laura Angeloni

Abstract

In this study, we introduce and study the max-min fractional type neural network operators. The operators are generated by sigmoidal activation functions, which play an important role in the approximation process. These operators provide a constructive tool for the approximation of functions that may not necessarily be continuous. In particular, the main result proved here is a modular convergence theorem in Orlicz spaces. We provide some graphical representations to illustrate the approximation error of the function through activation functions.

Keywords: Orlicz space, Order of approximation, Fractional calculus, Max-min fractional type Neural network operators

2010 Mathematics Subject Classification: 41A05 , 41A25 , 41A30 , 47A58.

1 Introduction

Neural network operators play an important role in approximation theory because they can effectively approximate complex functions. In 1989, using a sigmoidal activation function, Cybenko [15] proved that a single-hidden-layer neural network can approximate any continuous function to any desired level of accuracy, provided enough neurons are used. He established this result using the Hahn-Banach theorem and also showed that such networks can handle classification problems with arbitrarily small error. In 1998, Burton [5] extended the above result to the L^p -class of functions defined on compact sets. Later, many researchers developed a theoretical framework for neural network operators using different types of activation functions, and investigated their properties in various function spaces [11, 12, 18, 19, 20]. NN operators, both in their linear and nonlinear (max-product) forms, have been extensively investigated for their role in approximation processes [1, 2]. In recent years, nonlinear operators (max-product method) are widely studied in approximation theory [8, 10, 11]. These operators can give better performance than classical linear operators. The max-product method, initially proposed by Coroianu and Gal [13, 14] is based on replacing the usual summation symbol $\sum$ used in operators with the supremum symbol $\bigvee$. The main advantage of the max-product approach is that it gives a more accurate approximation than its linear counterparts. This improvement may appear in the order of approximation, computational efficiency, and applications such as image processing, etc. The primary benefit of studying neural network-type operators in Orlicz spaces is that they provide a unified framework for approximating functions belonging to various function spaces, including those that are not necessarily continuous. Costarelli and Vinti [9] developed the theory for a family of neural network operators of the Kantorovich-type in Orlicz spaces. Costarelli and Sambucini [6] proved the convergence results by max-product Kantorovich type NNO in Orlicz space. In 2019, Costarelli et al. [7] extended this work and obtained the approximation results by multivariate max-product Kantorovich type NNO. Further, Costarelli and Vinti [10] approximated the continuous and p -integrable functions using max-product Kantorovich type NNO. They approximated the function in terms of the modulus of continuity and Peetre's K -functionals. Now a days, the study of max-min operators has gained significant attention. Max-min operators are mainly associated with fuzzy theory [22], their good approximation properties and low computational cost have recently motivated the researchers to apply them in approximation theory [16]. Aslan et al. [3] studied the L^p convergence of Kantorovich type max-min neural network operators. In 2020, Gökçe and Duman [17] studied a general approximation theorem for max-min operators with several important applications. This idea was further extended by

*Corresponding author

^aDepartment of Mathematics, Sardar Vallabhbhai National Institute of Technology, Surat, 395007, Gujarat, India, Email: shivani9luhar@gmail.com

^bDepartment of Mathematics, Sardar Vallabhbhai National Institute of Technology, Surat, 395007, Gujarat, India, Email: shaileshiitr2010@gmail.com



Gökçe and Aslan [16], who constructed Kantorovich type max-min operators. They used the modulus of continuity and Hölder continuous functions to estimate the inaccuracy.

In this article, we introduce max-min Fractional type NNOs and study their convergence properties in Orlicz space. For $g \in L^1([a, b], \mathbb{R}^+)$, the Riemann-Liouville fractional integral operator (RLFIO) of order $\alpha \in \mathbb{R}^+$, denoted by I_a^α , is given by

$$I_a^\alpha g(v) = \frac{1}{\Gamma(\alpha)} \int_a^v (v-t)^{\alpha-1} g(t) dt, \quad (1)$$

where, $\Gamma(\cdot)$ denotes the Euler gamma function [19].

The basic properties of RLFIO are given below:

- (i) $I_a^\alpha(I_a^\gamma g(y)) = I_a^{\alpha+\gamma} g(y)$, $\alpha, \gamma > 0$.
- (ii) $I_a^\alpha(c_1 g(y) + c_2 h(y)) = c_1 I_a^\alpha g(y) + c_2 I_a^\alpha h(y)$, where c_1, c_2 are constants.
- (iii) $I_a^\alpha(y^p) = \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} y^{p+\alpha}$, $p > -1$.

Let g be a real-valued function defined on interval $[a, b]$. Then the average value of g over $[a, b]$ can be defined as follows [19]

$$g_{avg} = \frac{1}{b-a} \int_a^b g(v) dv.$$

If $g \in C([a, b])$ (the space of all real valued continuous functions defined on $[a, b]$), then $\exists$ a point $d \in [a, b]$ with $b \geq d \geq a$, such that $g_{avg} = g(d)$.

We now recall the statement of the fractional mean value theorem for a real-valued function g on $[a, b]$, a concept closely associated with the average value derived from RLFIO of order β .

Theorem 1.1. [19] (Fractional Mean Value Theorem) Consider the interval $[a, b] \subset \mathbb{R}$, and $\beta > 0$. Let $g : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then, there exists a point $c \in [a, b]$ such that

$$g(c) = \frac{\beta}{(b-a)^\beta} \int_a^b (b-v)^{\beta-1} g(v) dv. \quad (2)$$

Structure of the article

The manuscript is organized as follows. Section 2 presents the preliminary concepts, definitions, and auxiliary results that are required throughout the paper. In Section 3, we briefly review the fundamental notions of Orlicz spaces and their associated modular functionals. Section 4 is devoted to the introduction of the proposed max-min fractional type neural network operators and the discussion of their basic properties. In Section 5, several auxiliary lemmas are established, which play a crucial role in proving the main results. The modular convergence theorem for the proposed operators in the framework of Orlicz spaces is proved in Section 6. Finally, Section 7 provides graphical illustrations and numerical examples, including error analysis, to demonstrate the approximation behavior and effectiveness of the proposed operators.

2 Preliminary results

A function $\varrho : \mathbb{R} \rightarrow \mathbb{R}$ is said to be sigmoidal if it is a measurable function and satisfies

$$\lim_{y \rightarrow +\infty} \varrho(y) = 1, \text{ and } \lim_{y \rightarrow -\infty} \varrho(y) = 0.$$

In this paper, we consider a sigmoidal function ϱ , which is non-decreasing and satisfies the following properties:

- (D1) $\varrho(y) - 1/2$ is an odd function and $\varrho(1) < 1$.
- (D2) $\varrho(y) \in C^2(\mathbb{R})$ is concave for $y \geq 0$.
- (D3) $\varrho(y) = O(|y|^{-\beta})$ as $y \rightarrow -\infty$, for some $\beta > 0$.

For any given non-decreasing sigmoidal function, we define the density function

$$\Theta_\varrho(y) = \frac{1}{2} [\varrho(y+1) - \varrho(y-1)], y \in \mathbb{R}.$$

Considering the imposed conditions (D1), (D2), and (D3) on ϱ , the function Θ_ϱ satisfies the following properties:

Lemma 2.1. [8]

- (i) $\Theta_\varrho(y) \geq 0$ for every $y \in \mathbb{R}$, with $\Theta_\varrho(2) > 0$, and further $\lim_{y \rightarrow \pm\infty} \Theta_\varrho(y) = 0$;

- (ii) $\Theta_\varrho(y) = \Theta_\varrho(-y)$, $y \geq 0$;
- (iii) $\Theta_\varrho(y)$ has non-decreasing behavior for $y \in (-\infty, 0)$ and non-increasing behavior for $y \in [0, \infty)$;
- (iv) $\Theta_\varrho(0) \geq \Theta_\varrho(y)$, for all $y \in \mathbb{R}$, and $\Theta_\varrho(0) \leq 1/2$;
- (v) $\Theta_\varrho(y) = O(|y|^{-\beta})$, as $y \rightarrow \pm\infty$, where β is the same constant as in condition (D3);
- (vi) $\sum_{k \in \mathbb{Z}} \Theta_\varrho(y - k) = 1$, for $y \in \mathbb{R}$.

The symbol $\bigvee$ is defined as

$$\bigvee_{k=\lceil ma \rceil}^{\lfloor mb \rfloor - 1} A_k := \max\{A_k, \quad k = \lceil ma \rceil, \dots, \lfloor mb \rfloor - 1\}, \quad (3)$$

for $m \in \mathbb{N}$.

Lemma 2.2. [8, 11]

- (i) For any fixed $y \in [a, b]$, we have $\bigvee_{k=\lceil ma \rceil}^{\lfloor mb \rfloor - 1} \Theta_\varrho(my - k) \geq \Theta_\varrho(2) > 0$, for sufficiently large $m \in \mathbb{N}$.
- (ii) For any fixed $\gamma > 0$, we have $\bigvee_{\substack{k \in \mathbb{Z} \\ |y-k| > m\gamma}} \Theta_\varrho(y - k) = O(m^{-\beta})$, as $m \rightarrow \infty$, uniformly in $y \in \mathbb{R}$, and $\beta (> 0)$ is the same constant as in condition (D3).

We now recall several fundamental properties of the max and min operators.

Lemma 2.3. [4] If $\bigvee_{k \in \mathbb{Z}} p_k < \infty$ or $\bigvee_{k \in \mathbb{Z}} q_k < \infty$, then the following holds

$$\left| \bigvee_{k \in \mathbb{Z}} p_k - \bigvee_{k \in \mathbb{Z}} q_k \right| \leq \bigvee_{k \in \mathbb{Z}} |p_k - q_k|.$$

Lemma 2.4. [3]

- (i) For every $p, q, r \in [0, 1]$, we have

$$|p \wedge q - p \wedge r| \leq p \wedge |q - r|,$$

where $p \wedge q$ denotes $\min\{p, q\}$.

- (ii) For any $q_k \geq 0$ and $p \geq 0$, the following holds

$$\left(\bigvee_{k \in \mathbb{Z}} q_k \right)^p = \bigvee_{k \in \mathbb{Z}} q_k^p \text{ and } \left(\bigwedge_{k \in \mathbb{Z}} q_k \right)^p = \bigwedge_{k \in \mathbb{Z}} q_k^p.$$

3 Orlicz Space

Now, we recall some basic notions concerning Orlicz spaces which can be found in [9].

If $\varphi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ be a function that satisfies the following properties

- $\varphi 1$: $\varphi(v) > 0$ for every $v > 0$ and $\varphi(0) = 0$;
- $\varphi 2$: $\lim_{v \rightarrow \infty} \varphi(v) = \infty$;
- $\varphi 3$: φ is continuous as well as non-decreasing on $\mathbb{R}_0^+$,

then it is known as φ -function. Let $\mathbb{M}[a, b]$ be the set of all measurable functions $g : [a, b] \rightarrow \mathbb{R}$. For any fixed φ -function, the functional $I^\varphi : \mathbb{M}[a, b] \rightarrow [0, \infty)$ can be defined as follows:

$$I^\varphi[g] := \int_a^b \varphi(|g(y)|) dy, \quad g \in \mathbb{M}[a, b].$$

The space generated by the function φ is known as Orlicz space which is defined as follows:

$$L^\varphi[a, b] := \{g \in \mathbb{M}[a, b] : I^\varphi[\lambda g] < \infty, \text{ for some } \lambda > 0\}.$$

Now, a set of functions $(g_w)_{w>0} \subset L^\varphi[a, b]$ is known as modularly convergent to a function $g \in L^\varphi[a, b]$, if the following condition holds:

$$\lim_{w \rightarrow +\infty} I^\varphi[\lambda(g_w - g)] = 0, \text{ for some } \lambda > 0. \quad (4)$$

Moreover, the set of functions $(g_w)_{w>0}$ is said to be norm convergent to g , if equation (4) holds for every $\lambda > 0$. Hence, modular convergence is weaker than norm convergence. Note that the Δ_2 -condition of the φ -function in Orlicz space is given by

$$\varphi(2v) \leq M\varphi(v), \quad (v \in \mathbb{R}_0^+) \quad (5)$$

for some $M > 0$ [9]. If φ -function satisfies Δ_2 -condition, then the modular convergence is equivalent to norm convergence. This equivalence is a fundamental result in the theory of Orlicz space, and the proof can be observed in [21].

4 Max-min Fractional type NNOs

If we replace a, b and c with $a = c = \frac{k}{m}$ and $b = \frac{k+1}{m}$, for all $k \in \mathbb{N}$ in (2), then there exists $c \in [a, b]$ such that

$$g\left(\frac{k}{m}\right) = \frac{\alpha}{\left(\frac{k+1}{m} - \frac{k}{m}\right)^\alpha} \int_{\frac{k}{m}}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v\right)^{\alpha-1} g(v) dv, \quad (6)$$

where $\alpha > 0$. In view of above lines, now we define the NN operators by substituting the fractional mean value for the sample values $g\left(\frac{k}{m}\right)$ of the approximating function g on $\left[\frac{k}{m}, \frac{k+1}{m}\right]$.

Definition 4.1. Let $g : [a, b] \rightarrow [0, 1]$ be a bounded function and $\alpha \in \mathbb{R}^+$. The max-min fractional type NNOs, denoted by $N_m^\alpha(g; y)$, and activated by a sigmoidal function ϱ , are defined as follows:

$$N_m^\alpha(g, y) := \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor - 1} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v\right)^{\alpha-1} g(v) dv \right] \wedge \frac{\Theta_\varrho(my - k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor - 1} \Theta_\varrho(my - d)} \right\}, \quad (7)$$

$m \in \mathbb{N}$.

Since $\sup\{g(v) : v \in [a, b]\} \leq 1$, we have

$$\begin{aligned} N_m^\alpha(g, y) &\leq \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor - 1} 1 \wedge \frac{\Theta_\varrho(my - k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor - 1} \Theta_\varrho(my - d)} \\ &= \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor - 1} \frac{\Theta_\varrho(my - k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor - 1} \Theta_\varrho(my - d)} = 1 < \infty. \end{aligned}$$

Thus, max-min fractional type NNO is well defined for bounded functions.

5 Lemmas

We need following lemmas, which are essential for establishing the modular convergence theorem in Orlicz space.

Lemma 5.1. Consider real-valued bounded functions $g, h : [a, b] \rightarrow [0, 1]$. Then the following conditions are satisfied for large $m \in \mathbb{N}$:

- (a) If $\varrho(y)$ is continuous on $\mathbb{R}$, then $N_m^\alpha(g, \cdot)$ is continuous on $[a, b]$.
- (b) If $g(y) \leq h(y)$ then $N_m^\alpha(g, y) \leq N_m^\alpha(h, y)$, $y \in [a, b]$.
- (c) N_m^α is sublinear. i.e. $N_m^\alpha(g + h, y) \leq N_m^\alpha(g, y) + N_m^\alpha(h, y)$, $y \in [a, b]$.
- (d) $|N_m^\alpha(g, y) - N_m^\alpha(h, y)| \leq N_m^\alpha(|g - h|, y)$, $y \in [a, b]$.

Using the definition of the operator and the lemmas discussed in preliminary results, the proof can be derived.

Lemma 5.2. Let $g : [a, b] \rightarrow [0, 1]$ be a bounded function which is continuous at $y \in [a, b]$ and $\alpha \in \mathbb{R}^+$. Then, we have

$$\lim_{m \rightarrow \infty} N_m^\alpha(g, y) = g(y).$$

Moreover, if $g \in C([a, b], [0, 1])$ (the space of positive continuous functions defined on $[a, b] \rightarrow [0, 1]$), then we get

$$\lim_{m \rightarrow \infty} \|N_m^\alpha(g, \cdot) - g(\cdot)\|_\infty = 0.$$

Proof. Let $y \in [a, b]$ be a continuity point of g . Consider

$$|N_m^\alpha(g, y) - g(y)| \leq \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor - 1} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v\right)^{\alpha-1} g(v) dv \right] \wedge \frac{\Theta_\varrho(my - k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor - 1} \Theta_\varrho(my - d)} \right\}$$

$$\begin{aligned}
& - \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} g(y) dv \right] \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right\} \\
& + \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} g(y) dv \right] \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right\} - g(y) \Bigg|.
\end{aligned}$$

Now,

$$\begin{aligned}
& \left| \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} g(y) dv \right] \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right\} - g(y) \right| \\
& = \left| \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \left\{ g(y) \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right\} - g(y) \right| \\
& = \left| g(y) \wedge \left\{ \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right\} - g(y) \right| = 0.
\end{aligned}$$

Also, g is continuous at y , so for each $\epsilon > 0$, $\exists$ a corresponding $\gamma > 0$ such that $|v - y| < \gamma$ leads to $|g(v) - g(y)| < \epsilon$, for all $v \in [a, b]$. Then by Lemma 2.3 and condition (i) of Lemma 2.4, we obtain

$$\begin{aligned}
|N_m^\alpha(g, y) - g(y)| & \leq \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \left| \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} g(v) dv \right] \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right. \\
& \quad \left. - \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} g(y) dv \right] \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right| \\
& \leq \bigvee_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} |g(v) - g(y)| dv \right] \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right\} \\
& \leq \max \left\{ \bigvee_{\left| \frac{k}{m} - y \right| \leq \frac{\gamma}{2}} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} |g(v) - g(y)| dv \right] \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right\}, \right. \\
& \quad \left. \bigvee_{\left| \frac{k}{m} - y \right| > \frac{\gamma}{2}} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} |g(v) - g(y)| dv \right] \wedge \frac{\Theta_\varrho(my-k)}{\bigvee_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right\} \right\}
\end{aligned}$$

$$= \max\{I_1, I_2\}.$$

Let us start with estimation of expression I_1 . For every $v \in [\frac{k}{m}, \frac{k+1}{m}]$ and for any fixed m' such that $\frac{1}{m} < \frac{\gamma}{2}$, $\forall m \geq m'$, we obtain

$$|v - y| \leq |v - \frac{k}{m}| + |\frac{k}{m} - y| < \gamma,$$

whenever, $|\frac{k}{m} - y| < \frac{\gamma}{2}$. Hence for $m \geq m'$, we get

$$\begin{aligned} I_1 &= \bigvee_{|\frac{k}{m} - y| \leq \frac{\gamma}{2}} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} |g(v) - g(y)| dv \right] \wedge \frac{\Theta_\varrho(my - k)}{\bigvee_{d=[ma]}^{[mb]-1} \Theta_\varrho(my - d)} \right\} \\ &\leq \bigvee_{|\frac{k}{m} - y| \leq \frac{\gamma}{2}} \left\{ \epsilon \wedge \frac{\Theta_\varrho(my - k)}{\bigvee_{d=[ma]}^{[mb]-1} \Theta_\varrho(my - d)} \right\} \\ &\leq \epsilon. \end{aligned}$$

Now, for I_2 , we have

$$\begin{aligned} I_2 &= \bigvee_{|\frac{k}{m} - y| > \frac{\gamma}{2}} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} |g(v) - g(y)| dv \right] \wedge \frac{\Theta_\varrho(my - k)}{\bigvee_{d=[ma]}^{[mb]-1} \Theta_\varrho(my - d)} \right\} \\ &\leq 2\|g\|_\infty \bigvee_{|\frac{k}{m} - y| > \frac{\gamma}{2}} \frac{\Theta_\varrho(my - k)}{\Theta_\varrho(2)}. \end{aligned}$$

Using condition (ii) of Lemma 2.2, we get $I_2 = O(m^{-\beta})$, as $m \rightarrow \infty$.

In conclusion $|N_m^\alpha(g, y) - g(y)| < \epsilon$, for sufficiently large $m \in \mathbb{N}$. For $g \in C([a, b])$, the result follows by letting γ to be the parameter of uniform continuity of g . $\square$

Lemma 5.3. Let $g \in C([a, b], [0, 1])$. Then, for every $\lambda > 0$ we have

$$\lim_{m \rightarrow \infty} I^\varphi[\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] = 0.$$

Proof. For a given $\epsilon > 0$, in view of Lemma 5.2 there exists $m_0 \in \mathbb{N}$ such that $\|N_m^\alpha(g, \cdot) - g(\cdot)\|_\infty \leq \epsilon$ whenever $m \geq m_0$. Using the non-decreasing nature of φ , we can write

$$\begin{aligned} I^\varphi[\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] &= \int_a^b \varphi(\lambda |N_m^\alpha(g, y) - g(y)|) dy \\ &\leq \int_a^b \varphi(\lambda \|N_m^\alpha(g, \cdot) - g(\cdot)\|_\infty) dy \\ &\leq \int_a^b \varphi(\lambda \epsilon) dy \\ &\leq \varphi(\epsilon \lambda)(b - a) \leq \epsilon \varphi(\lambda)(b - a). \end{aligned} \tag{8}$$

Since $\epsilon > 0$ is arbitrary, so from equation (8), we can write $\lim_{m \rightarrow \infty} I^\varphi[\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] = 0$, for every $m \geq m_0$, in view of properties of φ function. $\square$

6 Main Result

In this section, we establish the Modular convergence theorem for max-min Fractional type NNO in $L^\varphi[a, b]$. Here, we assume φ -function as a convex function.

Theorem 6.1. For every $g \in L^\varphi[a, b]$, there exists $\lambda > 0$, such that

$$\lim_{m \rightarrow \infty} I^\varphi[\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] = 0.$$

Proof. The set of continuous functions $C([a, b])$ is modularly dense in $L^\varphi[a, b]$. This means that for every $\epsilon > 0$, and every function $g \in L^\varphi[a, b]$, there exists a function $h \in C([a, b])$ and a positive constant $\bar{\lambda}$ such that $I^\varphi[\bar{\lambda}(g - h)] < \epsilon$.

Now, take a fixed $\lambda > 0$ such that $3\lambda \leq \bar{\lambda}$. Then, using the properties of the modular functional I^φ , and convexity of φ , the following inequality holds for all $m \in \mathbb{N}$.

$$\begin{aligned} I^\varphi[\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] &\leq \frac{1}{3} (I^\varphi[3\lambda(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] + I^\varphi[3\lambda(N_m^\alpha(h, \cdot) - h(\cdot))] + I^\varphi[3\lambda(h(\cdot) - g(\cdot))]) \\ &\leq \frac{1}{3} (I^\varphi[\bar{\lambda}(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] + I^\varphi[\bar{\lambda}(N_m^\alpha(h, \cdot) - h(\cdot))] + I^\varphi[\bar{\lambda}(h(\cdot) - g(\cdot))]). \end{aligned}$$

Using Lemma 5.3, there exists $m_0 \in \mathbb{N}$ such that $I^\varphi[\bar{\lambda}(N_m^\alpha(h, \cdot) - h(\cdot))] < \epsilon$, $\forall m \geq m_0$, we can write

$$I^\varphi[\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] \leq \frac{1}{3} (I^\varphi[\bar{\lambda}(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] + 2\epsilon), \forall m \geq m_0. \quad (9)$$

Now, using the condition (d) of Lemma 5.1 of the operator N_m^α , we get

$$\begin{aligned} &I^\varphi[\bar{\lambda}(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] \\ &= \int_a^b \varphi(\bar{\lambda}|N_m^\alpha(g, y) - N_m^\alpha(h, y)|) dy \\ &\leq \int_a^b \varphi(\bar{\lambda}N_m^\alpha(|g - h|, y)) dy \\ &\leq \int_a^b \varphi \left(\bigvee_{k=[ma]}^{[mb]-1} \left\{ \left[\bar{\lambda} \alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} |g(v) - h(v)| dv \right] \wedge \frac{\bar{\lambda} \Theta_\varrho(my - k)}{\bigvee_{d=[ma]}^{[mb]-1} \Theta_\varrho(my - d)} \right\} \right) dy. \end{aligned}$$

Since φ is a non-decreasing function, we have $\varphi\left(\bigvee_{k=[ma]}^{[mb]-1} p_k\right) = \bigvee_{k=[ma]}^{[mb]-1} \varphi(p_k)$ and $\varphi\left(\bigwedge_{k=[ma]}^{[mb]-1} q_k\right) = \bigwedge_{k=[ma]}^{[mb]-1} \varphi(q_k)$. Using Jensen's inequality, we get

$$\begin{aligned} &I^\varphi[\bar{\lambda}(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] \\ &= \int_a^b \bigvee_{k=[ma]}^{[mb]-1} \left\{ \left[\varphi \left(\bar{\lambda} \alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} |g(v) - h(v)| dv \right) \right] \wedge \left[\varphi \left(\frac{\bar{\lambda} \Theta_\varrho(my - k)}{\bigvee_{d=[ma]}^{[mb]-1} \Theta_\varrho(my - d)} \right) \right] \right\} dy \\ &\leq \int_a^b \bigvee_{k=[ma]}^{[mb]-1} \left\{ \left[\alpha m^\alpha \int_{k/m}^{\frac{k+1}{m}} \left(\frac{k+1}{m} - v \right)^{\alpha-1} \varphi(\bar{\lambda}|g(v) - h(v)|) dv \right] \wedge \left[\varphi \left(\frac{\bar{\lambda} \Theta_\varrho(my - k)}{\bigvee_{d=[ma]}^{[mb]-1} \Theta_\varrho(my - d)} \right) \right] \right\} dy \\ &\leq \int_a^b \bigvee_{k=[ma]}^{[mb]-1} \left\{ \left[\alpha m \int_{k/m}^{\frac{k+1}{m}} \varphi(\bar{\lambda}|g(v) - h(v)|) dv \right] \wedge \left[\varphi \left(\frac{\bar{\lambda} \Theta_\varrho(my - k)}{\bigvee_{d=[ma]}^{[mb]-1} \Theta_\varrho(my - d)} \right) \right] \right\} dy. \end{aligned}$$

Now, considering $\frac{\Theta_\varrho(my-k)}{\sum_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \leq 1$, and φ is a convex function we have,

$$\begin{aligned} I^\varphi [\bar{\lambda}(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] &\leq \int_{\mathbb{R}} \sum_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \left\{ \left[am \int_{k/m}^{\frac{k+1}{m}} \varphi(\bar{\lambda}|g(v)-h(v)|) dv \right] \wedge \left[\varphi(\bar{\lambda}) \left(\frac{\Theta_\varrho(my-k)}{\sum_{d=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \Theta_\varrho(my-d)} \right) \right] \right\} dy \\ &\leq \int_{\mathbb{R}} \sum_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \left\{ \left[am \int_{k/m}^{\frac{k+1}{m}} \varphi(\bar{\lambda}|g(v)-h(v)|) dv \right] \wedge \left[\varphi(\bar{\lambda}) \left(\frac{\Theta_\varrho(my-k)}{\Theta_\varrho(2)} \right) \right] \right\} dy, \end{aligned}$$

after substituting $my-k=t$, we get

$$\begin{aligned} I^\varphi [\bar{\lambda}(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] &\leq \int_{\mathbb{R}} \left\{ \left[\sum_{k=\lfloor ma \rfloor}^{\lfloor mb \rfloor-1} \alpha \int_{k/m}^{\frac{k+1}{m}} \varphi(\bar{\lambda}|g(v)-h(v)|) dv \right] \wedge \left[\varphi(\bar{\lambda}) \left(\frac{\Theta_\varrho(t)}{\Theta_\varrho(2)} \right) \right] \right\} dt \\ &\leq \int_{\mathbb{R}} \left\{ \left[\alpha \int_a^b \varphi(\bar{\lambda}|g(v)-h(v)|) dv \right] \wedge \left[\varphi(\bar{\lambda}) \left(\frac{\Theta_\varrho(t)}{\Theta_\varrho(2)} \right) \right] \right\} dt \\ &\leq \int_{\mathbb{R}} \left\{ [\alpha I^\varphi [\bar{\lambda}(g-h)]] \wedge \left[\varphi(\bar{\lambda}) \left(\frac{\Theta_\varrho(t)}{\Theta_\varrho(2)} \right) \right] \right\} dt. \end{aligned}$$

Since $\Theta_\varrho \in L^1(\mathbb{R})$ and $\Theta_\varrho(t) = O(|t|^{-(1+\beta)})$, for some $\beta > 0$ and $|t| \rightarrow +\infty$, $\exists M, N > 0$ such that $\Theta_\varrho(t) \leq \frac{M}{|t|^{(1+\beta)}}$, for $|t| \geq N$. Assume that $I^\varphi(\bar{\lambda}(g-h))$ is sufficiently small such that $\hat{N} = \frac{1}{(I^\varphi[\bar{\lambda}(g-h)])^{\frac{1}{1+\beta}}} \geq N$, then by the following separation of the integral, we get

$$\begin{aligned} &I^\varphi [\bar{\lambda}(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] \\ &\leq \int_{|t|>\hat{N}} \left\{ [\alpha I^\varphi [\bar{\lambda}(g-h)]] \wedge \left[\varphi(\bar{\lambda}) \left(\frac{\Theta_\varrho(t)}{\Theta_\varrho(2)} \right) \right] \right\} dt + \int_{|t|<\hat{N}} \left\{ [\alpha I^\varphi [\bar{\lambda}(g-h)]] \wedge \left[\varphi(\bar{\lambda}) \left(\frac{\Theta_\varrho(t)}{\Theta_\varrho(2)} \right) \right] \right\} dt \\ &\leq \int_{|t|>\hat{N}} \left[\varphi(\bar{\lambda}) \left(\frac{\Theta_\varrho(t)}{\Theta_\varrho(2)} \right) \right] dt + \int_{|t|<\hat{N}} \alpha I^\varphi [\bar{\lambda}(g-h)] dt \\ &\leq \frac{M\varphi(\bar{\lambda})}{\Theta_\varrho(2)} \int_{|t|>\hat{N}} \frac{1}{|t|^{(1+\beta)}} dt + 2\alpha \cdot \hat{N} I^\varphi [\bar{\lambda}(g-h)], \end{aligned}$$

since Θ_ϱ is even function so,

$$\begin{aligned} I^\varphi [\bar{\lambda}(N_m^\alpha(g, \cdot) - N_m^\alpha(h, \cdot))] &\leq \frac{2M\varphi(\bar{\lambda})}{\Theta_\varrho(2)} \int_{\hat{N}}^{\infty} \frac{1}{t^{(1+\beta)}} dt + 2\alpha \frac{1}{(I^\varphi[\bar{\lambda}(g-h)])^{\frac{1}{1+\beta}}} I^\varphi [\bar{\lambda}(g-h)] \\ &\leq \frac{2M\varphi(\bar{\lambda})}{\beta\Theta_\varrho(2)} (I^\varphi [\bar{\lambda}(g-h)])^{\frac{\beta}{1+\beta}} + 2\alpha (I^\varphi [\bar{\lambda}(g-h)])^{\frac{\beta}{1+\beta}} \\ &\leq \left(\frac{2M\varphi(\bar{\lambda})}{\beta\Theta_\varrho(2)} + 2\alpha \right) (I^\varphi [\bar{\lambda}(g-h)])^{\frac{\beta}{1+\beta}}. \end{aligned} \quad (10)$$

Now, combining equation (9) and (10) we get

$$I^\varphi [\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] \leq \left(\frac{2M\varphi(\bar{\lambda})}{\beta\Theta_\varrho(2)} + 2\alpha \right) (\epsilon)^{\frac{\beta}{1+\beta}} + 2\epsilon, \forall m \geq m_0.$$

Since, ϵ is arbitrary, it follows that $\lim_{m \rightarrow \infty} I^\varphi [\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] = 0$. □

7 Graphical representation

The max-min fractional type NNOs are studied in the setting of Orlicz spaces, providing a unified approach for their analysis in different function spaces. Orlicz spaces form a broad class of function spaces that include not only the classical L^p -spaces but also other important spaces such as interpolation and exponential spaces. Now we can deduce the following corollaries from Theorem 6.1.

A well-known example of an Orlicz space is $L^p[a, b]$ spaces, which is generated by the convex φ -function $\varphi(v) = v^p$, $v \geq 0$ and $1 \leq p < \infty$. Further, we have $I^\varphi[g] = \|g\|_p^p$, which is the usual L^p -norm. Here, the Δ_2 -condition is satisfied, hence the norm convergence and the modular convergence are the same.

Corollary 7.1. *For every $g \in L^p[a, b]$, such that*

$$\lim_{m \rightarrow \infty} \|N_m^\alpha(g, \cdot) - g(\cdot)\|_p = 0.$$

The same theory applies to the Zygmund space $L^\alpha \log^\beta L[a, b]$, also known as an interpolation space, which is another example of an Orlicz space. These spaces are important in studying partial differential equations and are generated by the convex φ -functions $\varphi_{\alpha, \beta} := (\log^\beta(v + e)) \cdot v^\alpha$, for $0 \leq v$, $1 \leq \alpha$, $0 < \beta$. The corresponding convex modular functionals are given by [9]:

$$I^{\varphi_{\alpha, \beta}}[g] := \int_a^b |g(y)|^\alpha \log^\beta(e + |g(y)|) dy, \quad (g \in \mathbb{M}[a, b]).$$

Here, we can observe that $\varphi_{\alpha, \beta}$ satisfies the Δ_2 -condition.

Corollary 7.2. *For every $g \in L^{\varphi_{\alpha, \beta}}[a, b]$, for every $\lambda > 0$, such that*

$$\lim_{m \rightarrow \infty} I^{\varphi_{\alpha, \beta}}[\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] = 0.$$

Another important example is the exponential space, which is generated by the convex φ -function $\varphi_\gamma(v) = e^{v^\gamma} - 1$, for $v \geq 0$, $\gamma > 0$. Here, φ_γ does not satisfy the Δ_2 -condition, which means that norm convergence and modular convergence are not equivalent. The corresponding modular functional is given by [9]:

$$I^{\varphi_\gamma}[g] := \int_a^b (e^{|g(y)|^\gamma} - 1) dy, \quad (g \in \mathbb{M}[a, b]).$$

Corollary 7.3. *For every $g \in L^{\varphi_\gamma}[a, b]$, there exists $\lambda > 0$ such that*

$$\lim_{m \rightarrow \infty} I^{\varphi_\gamma}[\lambda(N_m^\alpha(g, \cdot) - g(\cdot))] = 0.$$

The behavior of the max-min fractional type neural network operator is now demonstrated with the help of following example. This example helps us to understand its convergence, efficiency in computation, and stability.

The logistic function $\varrho_l(y) := \frac{1}{1+e^{-y}}$, $y \in \mathbb{R}$ [12] and hyperbolic tangent function $\varrho_h(y) := \frac{\tanh y + 1}{2}$, $y \in \mathbb{R}$ [10] are well-known examples of smooth sigmoidal functions which satisfy all the assumptions in the theory developed in [2] and references therein. In particular, ϱ_l and ϱ_h fulfill condition (D3) of section 2, for all $\beta > 0$.

We now provide numerical example illustrating the approximation of a function, which may not be continuous, using the aforementioned max-min fractional type neural network operator in order to analyze the theory developed in this paper.

Consider following discontinuous function $g : [-1, 1] \rightarrow \mathbb{R}$, which is defined as follows:

$$g(y) = \begin{cases} e^{2y}, & -1 \leq y < 0, \\ y^2, & 0 \leq y \leq 1. \end{cases}$$

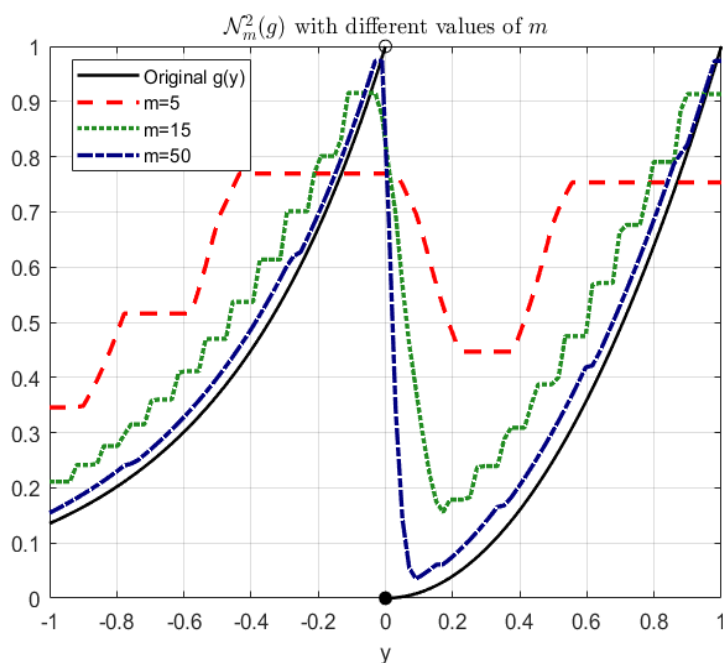
In the following Table 1, we find the absolute error of approximation to the functions g by the operators N_m^α , activated by ϱ_l for different values of α and m .

From Table 1, it is clear that the error of approximation to the function by the max-min fractional type neural network operators depends on α and m both. For a fixed value of α , the error decreases as we increase m and vice-versa. These numerical findings demonstrate that these operators exhibit good convergence behavior and provide improved approximations, depending on the values of both α and m . Further, the convergence performance of the proposed operators much improves when we raise the values of α , order of the Riemann-Liouville fractional integral operator (RLFIO). This characteristic allows the proposed operator

Table 1: Estimation of $|N_m^\alpha(g, y) - g(y)|$, activated by ϱ_l for different values of α and m

y -values	α -values	$ N_{15}^\alpha(g, y) - g(y) $	$ N_{40}^\alpha(g, y) - g(y) $	$ N_{50}^\alpha(g, y) - g(y) $	$ N_{70}^\alpha(g, y) - g(y) $
-0.596	3.5	0.101543	0.032966	0.025563	0.017304
	5	0.098511	0.032027	0.024830	0.016793
	10.5	0.094211	0.030686	0.023780	0.016063
	30.2	0.091290	0.029767	0.023061	0.015561
-0.111	3.5	0.101009	0.027174	0.025324	0.023219
	5	0.094261	0.024865	0.023483	0.021908
	10.5	0.084692	0.021567	0.020850	0.020032
	30.2	0.078192	0.019307	0.019044	0.018744
0.475	3.5	0.152752	0.056124	0.049669	0.042386
	5	0.148160	0.054645	0.048501	0.041563
	10.5	0.141633	0.052531	0.046829	0.040385
	30.2	0.137188	0.051082	0.045682	0.039576
0.798	3.5	0.140378	0.053070	0.042949	0.031476
	5	0.133810	0.050758	0.041114	0.030177
	10.5	0.124449	0.047449	0.038486	0.028315
	30.2	0.118052	0.045177	0.036680	0.027036

to provide enhanced approximation results in applications such as signal and image processing as well as artificial neural networks.

**Figure 1:** Approximation to the original discontinuous function g by the max-min fractional type neural network operator, $N_m^2(g)$, activated by ϱ_l for different values of m .



In Figure 1, we approximate the original discontinuous function $g(y)$ using the max-min fractional type neural network operator, activated by ϱ_l for different values of m . Figure 1 shows that the max-min fractional type neural network operator converges to the original function g as m increases.

References

- [1] G. A. Anastassiou, Approximation by a nonlinear Cardaliaguet-Euvrard neural network operator of max-product kind, *Int. Math.: Comput. Anal.* 261–271, 2011.
- [2] G. A. Anastassiou, Intelligent systems: approximation by artificial neural networks, Springer, 2011.
- [3] İ. Aslan, S. D. Marchi, W. Erb, L^p -convergence of Kantorovich-type max-min neural network operators, *arXiv preprint arXiv:2407.03329*, 2020.
- [4] B. Bede, L. Coroianu, S. G. Gal, Approximation by max-product type operators, *Heidelberg: Springer*, 2016.
- [5] R. M. Burton, H. G. Dehling, Universal approximation in p-mean by neural networks. *Neural Netw.* 11 (04), 661–667, 1998.
- [6] D. Costarelli, A. R. Sambucini, Approximation results in Orlicz spaces for sequences of Kantorovich max-product neural network operators, *Results Math.* 70 (01), 01–15, 2018.
- [7] D. Costarelli, A. R. Sambucini, G. Vinti, Convergence in Orlicz spaces by means of the multivariate max-product neural network operators of the Kantorovich type and applications, *Neural Comput. Appl.* 31 (09), 5069–5078, 2019.
- [8] D. Costarelli, G. Vinti, Approximation by max-product neural network operators of Kantorovich type, *Results Math.*, 69 (03), 505–519, 2016.
- [9] D. Costarelli, G. Vinti, Convergence for a family of neural network operators in Orlicz spaces, *Math. Nachr.* 290 (2-3), 226–235 2017.
- [10] D. Costarelli, G. Vinti, Estimates for the neural network operators of the max-product type with continuous and p -integrable functions, *Results Math.* 73 (12), 01–10, 2018.
- [11] D. Costarelli, G. Vinti, Max-product neural network and quasi-interpolation operators activated by sigmoidal functions, *J. Approx. Theory* 209, 1–22, 2016.
- [12] D. Costarelli, R. Spigler, Solving volterra integral equations of the second kind by sigmoidal functions approximation, *J. Integral Equ. Appl.* 25 (2), 193–222 2013.
- [13] L. Coroianu, S. G. Gal, Approximation by max-product sampling operators based on sinc-type kernels, *Sampl. Theory Signal Image Process.* 10 (03) 211–230, 2011.
- [14] L. Coroianu, S. G. Gal, Approximation by nonlinear generalized sampling operators of max-product kind, *Sampl. Theory Signal Image Process.* 9 (1-3) 59–75, 2010.
- [15] G. Cybenko, Approximation by superpositions of a sigmoidal function. *Math. Control Signals Syst.* 02 (04), 303–314, 1989.
- [16] T. Y. Gökçe, İ. Aslan, Approximation by Kantorovich-type max-min operators and its applications, *Appl. Math. Comput.* 423, 127011, 2022.
- [17] T. Y. Gökçe, O. Duman, Approximation by max-min operators: A general theory and its applications, *Fuzzy Sets Syst.* 394, 146–161, 2020.
- [18] N. J. Guliyev, V. E. Ismailov, On the approximation by single hidden layer Feedforward neural networks with fixed weights, *Neural Netw.* 98, 296–304, 2018.
- [19] U. Kadak, Fractional type multivariate neural network operators, *Math. Methods Appl. Sci.* 46 (03), 3045–3065, 2021.
- [20] U. Kadak, Multivariate neural network interpolation operators, *J. Comput. Appl. Math.* 414, 114426, 2022.
- [21] J. Musielak, W. Orlicz, On modular spaces, *Studia Math.* 18 (01), 49–65, 1959.
- [22] H. Tahayori, A. G. B. Tettamanzi, G. D. Antoni, A. Visconti, On the calculation of extended max and min operations between convex fuzzy sets of the real line, *Fuzzy Sets Syst.* 160, 3103–3114, 2009.