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Numerical methods for Fredholm integral equations based on Padua points

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Abstract

The numerical solution of two-dimensional Fredholm integral equations on the square by Nyström and collocation methods based on the Padua points is investigated. The convergence, stability and well conditioning of the methods are proved in suitable subspaces of continuous functions of Sobolev type. Some numerical examples illustrate the efficiency of the methods. A comparison with the tensorial approximation methods, of Nyström and collocation type, based on Legendre zeros, is given.

1 Introduction

The paper deals with the numerical approximation of the solution of Fredholm integral equations (FIEs) of the second kind

$$f(x,y)-\mu\int_S k(x,y,s,t)f(s,t)dsdt = g(x,y), \qquad (x,y)\in S, \tag{1}$$

where $S=[-1,1]^2$, $\mu\in\mathbb{R}$, k and g are given continuous functions defined on $S^2 := S\times S$ and S respectively, while f is the unknown function.

Such equations appear in different areas, as computer graphics, aerodynamics, mathematical physics, electromagnetic scattering etc. Indeed some of the problems directly lead to FIEs, as in the case of the rendering equation [10], while some others, for instance the boundary value ones, can be rewritten in this form, see e.g. [2].

In the last years several different numerical approaches for solving FIEs appeared. Some examples are collocation or Nyström methods based on piecewise polynomial approximation [11], Galerkin methods based on wavelets (see [3] and the references therein), discrete or iterated Galerkin methods [9]. Next, global approximation strategies were presented in [12] and [13].

In particular in [12] it was proposed a Nyström method based on the tensor product of two univariate Gaussian rules as well as a collocation one using the product of two Lagrange interpolating polynomials in some Jacobi zeros.

Moreover in that paper the more general case of functions having singularities along the sides of S was considered and the whole study of the methods was carried out in subspaces of weighted continuous functions. Nevertheless it is well known that, if on the one hand the convergence of the proposed methods behaves like the best polynomial approximation of the solution and the approximation functions can be evaluated wherever it is necessary, on the other hand the tensor product strategy is usually expensive and the linear systems arising from the methods and which have full matrices of coefficients, can have a large size.

In [6] new sets of nodes, called Padua points, and constructed by means of suitable sub-sequences of Chebychev polynomials of the second kind, were introduced. Moreover the interpolation and the cubature rules using this set of nodes were studied in [6] and in [14], respectively. One of the most interesting thing from the application point of view is that the proposed interpolating process has optimal Lebesgue constants and the cubature error behaves like the best polynomial approximation of the integrating function.

In addition, with respect to the same order of convergence, the number of interpolation/cubature nodes is about half of that necessary to build concurrent interpolation/cubature processes of the tensor type using Jacobi zeros.

For the above mentioned reasons, in the present paper we propose a Nyström method and a collocation one, both based on Padua points, in order to numerically solve equations of the type (1).

We prove convergence, stability and well conditioning of the methods and show in particular when these methods perform better than those proposed in [12].

The outline of the paper is the following. Section 2 is devoted to notations and preliminary results in particular about the Padua points. In Section 3 the proposed methods for solving equation (1) are presented and discussed. Section 4 is dedicated to the numerical experiments, showing the effectiveness of the methods and discussing the comparison with other concurrent numerical strategies. Finally Section 5 includes the proofs.

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2 Notations and preliminary results

In this section we introduce some notations and preliminary results, useful in the paper. We will denote by C a generic positive constant, having different values in different estimates. Moreover if C depends on some quantities say $a, b, \ldots$ then we will write $C = C(a, b, \ldots)$. On the contrary for saying that C is independent of $a, b, \ldots$, we will use $C \neq C(a, b, \ldots)$.

Next, by $C^0(S)$ we will intend the space of continuous functions on S equipped by the uniform norm $||f||_\infty = \max_{(x,y)\in S} |f(x, y)|$. It will be also handy to write $f_x = f(x, \cdot)$ and $f_y = f(\cdot, y)$ for fixed $x, y \in S$. For smoother functions we introduce the following Sobolev-type space

$$W_r = \left\{ f \in C^0(S) : M_r(f) := \sup \{ ||f_y^{(r)} \varphi^r||_\infty, ||f_x^{(r)} \varphi^r||_\infty \} < +\infty \right\}, \qquad r \in \mathbb{N},$$

where the superscript (r) denotes the rth derivative of the one-dimensional function f_y or f_x, and $\varphi(z) = \sqrt{1 - z^2}$, is computed in the variable with respect to which we are considering the derivative of f (x in the first norm, y in the second one). We equip W_r with the norm

$$||f||_{W_r} = ||f||_\infty + M_r(f).$$

Now, let Π_n^2 be the space of bivariate algebraic polynomials of total degree n and $\mathbb{P}_{n,n}$ be the space of bivariate polynomials of degree n in each variable. Obviously $\mathbb{P}_{n,n} \subset \Pi_{2n}^2$.

We denote by $E_n(f)$ and $E_{n,n}(f)$ the errors of best polynomial approximation for bivariate continuous functions by means of polynomials in Π_n^2 and $\mathbb{P}_{n,n}$ respectively, i.e.

$$E_n(f) := \inf_{P \in \Pi_n^2} ||f - P||_\infty, \qquad E_{n,n}(f) := \inf_{P \in \mathbb{P}_{n,n}} ||f - P||_\infty.$$

From the definitions it follows

$$E_{2n}(f) \le E_{n,n}(f) \le E_n(f). \tag{2}$$

In this framework, we recall that in [12] the following Favard-type inequality was proved

$$\forall f \in W_r \quad E_{n,n}(f) \le \frac{C}{n^r} ||f||_{W_r}, \quad C \neq C(n, f), \tag{3}$$

and consequently by (2) the same estimate holds true also for $E_n(f)$ i.e.

$$\forall f \in W_r \quad E_n(f) \le \frac{C}{n^r} ||f||_{W_r}, \quad C \neq C(n, f). \tag{4}$$

2.1 The Padua points

Let us recall the so-called Padua points, introduced in [6]. The Padua points is a set of nodes defined as the union of two bidimensional Chebychev-like grids

$$Pad_n = C_{n+1}^O \times C_{n+2}^E \cup C_{n+1}^E \times C_{n+2}^O \subset C_{n+1} \times C_{n+2},$$

where $C_{n+1} = \left\{ z_j^n = \cos \frac{(j-1)\pi}{n} \quad j = 1, ..., n+1 \right\}$ denotes the set of Chebychev nodes of second kind(including ± 1) and C_{n+1}^E, C_{n+1}^O are the restrictions to the even and odd indexes respectively. By definition the points Pad_n lie into the square S.
From now we denote in bold a two-dimensional point $\boldsymbol{x} = (x_1, x_2)$.

The interpolating polynomial on Padua points is defined as

$$\mathcal{L}_n f(x, y) = \sum_{\xi \in Pad_n} \ell_{n,\xi}(x, y) f(\xi_1, \xi_2), \tag{5}$$

where $\ell_{n,\xi}(x, y)$ are the fundamental Lagrange polynomials and satisfy

$$\ell_{n,\xi}(\boldsymbol{\eta}) = \delta_{\xi\,\eta}, \quad \delta_{\xi\,\eta} = \begin{cases} 1 & \text{if } \xi = \boldsymbol{\eta} \\ 0 & \text{if } \xi \neq \boldsymbol{\eta} \end{cases}, \quad \xi, \boldsymbol{\eta} \in Pad_n. \tag{6}$$

The fundamental Lagrange polynomials can be written as

$$\ell_{n,\xi}(x, y) = \omega_\xi (\mathcal{K}_n(\xi_1, \xi_2, x, y) - T_n(\xi_1) T_n(x)), \qquad \xi = (\xi_1, \xi_2),$$

where the quantities $\{\omega_\xi\}$ are weights of a cubature formula for the product Chebychev measure, exact on almost Π_{2n}^2, and they are defined as follows

$$\omega_\xi = \frac{1}{n(n+1)} \begin{cases} \frac{1}{2} & \text{if } \xi \text{ is a vertex} \\ 1 & \text{if } \xi \text{ is an edge point} \\ 2 & \text{if } \xi \text{ is an interior point} \end{cases},$$

$T_n(z) = \cos(k \arccos(z))$ denotes the Chebychev polynomial of first kind and $\mathcal{K}_n$ is the reproducing kernel of Π_n^2 equipped with the inner product

$$\langle f, g \rangle = \int_S f(x_1, x_2) g(x_1, x_2) W(x_1, x_2) dx_1 dx_2 = \int_S f(x_1, x_2) g(x_1, x_2) \frac{dx_1}{\pi\sqrt{1 - x_1^2}} \frac{dx_2}{\pi\sqrt{1 - x_2^2}},$$

that is, with $\hat{T}_n = \sqrt{2}T_n$, $\hat{T}_0 = 1$, denoting the Chebychev first kind orthonormal sequence,

$$\mathcal{K}_n(\boldsymbol{x},\boldsymbol{y}) = \sum_{k=0}^{n}\sum_{j=0}^{k} \hat{T}_j(x_1)\hat{T}_{k-j}(x_2)\hat{T}_j(y_1)\hat{T}_{k-j}(y_2), \qquad \boldsymbol{x} = (x_1, x_2), \quad \boldsymbol{y} = (y_1, y_2),$$

with the reproduction property

$$\int_S \mathcal{K}_n(\boldsymbol{x},\boldsymbol{y})p_n(\boldsymbol{y})W(\boldsymbol{y})d\boldsymbol{y} = p_n(\boldsymbol{x}), \quad \forall p_n \in \Pi_n^2.$$

There is another way of writing the interpolant at Padua points i.e. via its representation in the basis $\{\hat{T}_j(x_1)\hat{T}_{k-j}(x_2)\}, 0 \le j \le k \le n$, which is orthonormal with respect to the product Chebychev measure. In fact we have

$$\mathcal{L}_n f(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{k} c_{j,k-j}\hat{T}_j(x)\hat{T}_{k-j}(y),$$

where

$$c_{j,k-j} = \sum_{\xi\in Pad_n} f(\xi_1,\xi_2)\omega_\xi \hat{T}_j(\xi_1)\hat{T}_{k-j}(\xi_2), \quad (k,j) \neq (n,n),$$

$$c_{n,0} = \frac{1}{2}\sum_{\xi\in Pad_n} f(\xi_1,\xi_2)\omega_\xi \hat{T}_n(\xi_1).$$

The Lebesgue constants of the interpolating operator at the Padua points $\Lambda_n = \max_{(x,y)\in S}\sum_{\xi\in Pad_n}|\ell_{n,\xi}(x,y)|$ have order of growth $\mathcal{O}(\log^2 n)$[4], where n is the degree of the polynomial, which is the optimal order as proved in [8]. Hence by the results in [5, 4] and by (4) it immediately follows

Theorem 2.1. *For any $f \in W_r$, $r \ge 1$, there holds*

$$\|f - \mathcal{L}_n f\|_\infty \le (1+\Lambda_n)E_n(f) \le C\frac{\log^2 n}{n^r}\|f\|_{W_r}, \tag{7}$$

where C is a suitable positive constant, independent on f and n.

In [14] the following nontensorial Cleshaw-Curtis cubature formula was introduced, by integrating the interpolating polynomial at the Padua points

$$\int_S f(x,y)dxdy = \sum_{\xi\in Pad_n} \lambda_\xi f(\xi_1,\xi_2) + \mathcal{E}_n(f), \tag{8}$$

where $\mathcal{E}_n(f)$ denotes the error of the formula and it depends on f and n,

$$\lambda_\xi = \omega_\xi \sum_{|\alpha|\le n} p_\alpha(\xi)m_\alpha, \quad p_\alpha(x,y) = \hat{T}_{\alpha_1}(x)\hat{T}_{\alpha_2}(y), \quad \alpha = \alpha_1 + \alpha_2,$$

and

$$m_\alpha = \int_{-1}^{1}\hat{T}_{\alpha_1}(x)dx\int_{-1}^{1}\hat{T}_{\alpha_2}(y)dy = \mu_{\alpha_1}\mu_{\alpha_2},$$

with

$$\mu_k = \begin{cases} 2 & k = 0 \\ \dfrac{2\sqrt{2}}{1-k^2} & k \text{ even}, k \neq 0 \\ 0 & k \text{ odd} \end{cases}.$$

This formula has weights λ_ξ that are not always positive. About the degree of precision, the convergence and stability of cubature rule the following result was proved in [14].

Theorem 2.2. *Formula (8) is exact on Π_n^2, is convergent, holding*

$$\left|\int_S f(x,y)dxdy - \sum_{\xi\in Pad_n}\lambda_\xi f(\xi_1,\xi_2)\right| \le C\pi^2 E_n(f), \quad C \neq C(n,f)$$

and is stable, since

$$\lim_n \sum_{\xi\in Pad_n}|\lambda_\xi| = 4.$$

From the previous theorem and (4) it immediately follows.

Corollary 2.3. *For any $f \in W_r$, $r \ge 1$, there holds*

$$\left|\int_S f(x,y)dxdy - \sum_{\xi\in Pad_n}\lambda_\xi f(\xi_1,\xi_2)\right| \le \frac{C}{n^r}\|f\|_{W_r}, \quad C \neq C(n,f). \tag{9}$$

3 Numerical methods for FIE

Setting

$$Kf(x,y)=\mu\int_S k(x,y,s,t)f(s,t)dsdt,$$

then (1) can be rewritten in operatorial form as

$$(I-K)f=g, \tag{10}$$

where I is the identity operator on $C^0(S)$.
Let us denote $k_{(s,t)}$(respectively $k_{(x,y)}$) the function k of four variables considered as a function of only (x,y) (respectively (s,t)), i.e.

$$k_{(s,t)}=k(\cdot,\cdot,s,t),\ \text{for fixed } s,t\in S,\qquad k_{(x,y)}=k(x,y,\cdot,\cdot),\ \text{for fixed } x,y\in S.$$

Using standard arguments, it is possible to prove that if $k(x,y,s,t)$ is continuous w.r.t. all the four variables, then K is compact, as a map of $C^0(S)$ into itself, and consequently the Fredholm Alternative holds true for (10) in $C^0(S)$, (see for instance [1]). For this reason we assume that k is at least continuous in each variable. Moreover, if for some $r\in\mathbb{N}$, it is

$$\sup_{(s,t)\in S}\|k_{(s,t)}\|_{w_r}<+\infty \tag{11}$$

then immediately follows that $Kf\in W_r$, for any $f\in C^0(S)$ (see for instance [12]). Therefore if also $g\in W_r$ then by (10) we deduce that $f=g+Kf\in W_r$.
We underline that the proofs of the theorems of the next subsections are shown in Section 5.

3.1 A Nyström method

In this subsection we introduce a Nyström method based on the non-tensorial cubature (8), with nodes $\xi\in Pad_n$ and weights λ_ξ. In order to approximate operator K in (10), we can define the discrete linear operator K_n as follows

$$K_nf(x,y)=\mu\sum_{\xi\in Pad_n}\lambda_\xi k(x,y,\xi_1,\xi_2)f(\xi_1,\xi_2)$$

and consider the equation

$$(I-K_n)f_n=g, \tag{12}$$

where f_n is an unknown function. Therefore the new goal is to determine f_n, that means to solve (12) instead of (10). To do this, taking into account the expression of K_n, we collocate (12) on Pad_n, which has cardinality $N=\frac{(n+1)(n+2)}{2}$, i.e. we write down (12) evaluated on each $\eta\in Pad_n$. In this way we obtain that the quantities $a_\xi:=f_n(\xi_1,\xi_2)$, where $\xi=(\xi_1,\xi_2)\in Pad_n$, are the unknowns of the following linear system

$$a_\eta-\mu\sum_{\xi\in Pad_n}\lambda_\xi k(\eta_1,\eta_2,\xi_1,\xi_2)a_\xi=g(\eta_1,\eta_2),\quad \boldsymbol{\eta}=(\eta_1,\eta_2)\in Pad_n. \tag{13}$$

Setting $A_n=(\delta_{\eta\,\xi}-\mu\lambda_\xi k(\boldsymbol{\eta},\boldsymbol{\xi}))_{\xi,\eta\in Pad_n}$, $b=(g(\xi))^T_{\xi\in Pad_n}$ and $a=(a_\xi)_{\xi\in Pad_n}$, with $\delta_{\xi\,\eta}$ defined in (6), linear system (13) can be rewritten in a matrix form as follows

$$A_na=b. \tag{14}$$

The solution of this system (if it exists) allows us to construct the unknown f_n, also called *the Nyström interpolant* in two variables, directly from (12), as

$$f_n(x,y)=g(x,y)+\mu\sum_{\xi\in Pad_n}\lambda_\xi k(x,y,\xi_1,\xi_2)a_\xi,\qquad \xi=(\xi_1,\xi_2)\in Pad_n, \tag{15}$$

which will approximate the unknown f of equation (10). Therefore the finite dimensional equation (12) and the linear system (14) are equivalent. In addition we remark that since we are assuming k and g to be continuous on S^2 and S respectively, then f_n will be continuous on S too.

Now let $cond(A_n)=\|A_n\|\|A_n^{-1}\|$ denote the condition number in the maximum row sum norm of the coefficient matrix A_n of system (13).

Theorem 3.1. *Let $ker(I-K)=\{0\}$ and assume*

$$\sup_{(s,t)\in S}\|k_{(s,t)}\|_{W_r}<+\infty,\quad \sup_{(x,y)\in S}\|k_{(x,y)}\|_{W_r}<+\infty,\quad g\in W_r.$$

Then the method is stable, i.e. operators $(I-K_n)^{-1}$ are uniformly bounded, the equivalent linear system (14) has a unique solution and it is well conditioned since

$$\sup_n cond(A_n)<+\infty. \tag{16}$$

Moreover the Nyström interpolant f_n converges to unique solution $f^*\in W_r$ and there holds

$$\|f^*-f_n\|_\infty\le\frac{C}{n^r}\|f^*\|_{W_r},\quad C\ne C(n,f^*). \tag{17}$$

3.2 A discrete collocation method

We describe now a discrete collocation method based on the interpolating operator $\mathcal{L}_n f(x,y) = \sum_{\xi\in Pad_n} \ell_{n,\xi}(x,y) f(\xi_1,\xi_2)$, defined in (5).

In (10) we first replace Kf with

$$K^* f(x,y) = \mu \int_S \mathcal{L}_n(k_{(x,y)}, s, t) f(s,t) ds dt,$$

i.e. $K^* f$ is obtained from Kf projecting the kernel k on the space Π_n^2 by means of operator $\mathcal{L}_n$.

As a second step we use $\mathcal{L}_n$ once again, to project the modified equation (i.e. with K^* instead of K) on Π_n^2, searching for a solution in Π_n^2. Hence we have

$$F_n - \mathcal{L}_n(K^* F_n) = \mathcal{L}_n g,$$

where $F_n \in \Pi_n^2$ is an unknown polynomial. Setting $H_n F_n = \mathcal{L}_n(K^* F_n)$ we get the following finite dimensional equation

$$(I - H_n) F_n = \mathcal{L}_n g. \tag{18}$$

We search for the unknown polynomial F_n via its representation in the basis $\{\ell_{n,\gamma}\}_{\gamma\in Pad_n}$ of the Lagrange fundamental polynomials, i.e.

$$F_n(x,y) = \sum_{\gamma\in Pad_n} F_n(\gamma_1,\gamma_2)\ell_{n,\gamma}(x,y), \quad \boldsymbol{\gamma} = (\gamma_1,\gamma_2) \in Pad_n. \tag{19}$$

In order to determine the coefficients $F_n(\gamma_1,\gamma_2)$, $\boldsymbol{\gamma} \in Pad_n$, we collocate (18) on the points $\boldsymbol{\eta} = (\eta_1,\eta_2) \in Pad_n$ and we obtain

$$F_n(\eta_1,\eta_2) - \mathcal{L}_n(K^* F_n, \eta_1, \eta_2) = \mathcal{L}_n(g, \eta_1, \eta_2).$$

As the Padua points are the interpolation knots, we have

$$F_n(\eta_1,\eta_2) - K^* F_n(\eta_1,\eta_2) = g(\eta_1,\eta_2), \qquad (\eta_1,\eta_2) \in Pad_n. \tag{20}$$

On the other hand $K^* F_n(\boldsymbol{\eta})$, with $\boldsymbol{\eta} \in Pad_n$, can be written as

$$\begin{aligned} K^* F_n(\boldsymbol{\eta}) &= \mu \int_S \mathcal{L}_n(k(\eta_1,\eta_2,s,t) F_n(s,t) ds dt = \\ &= \mu \int_S \sum_{\xi\in Pad_n} k(\eta_1,\eta_2,\xi_1,\xi_2)\ell_{n,\xi}(s,t) F_n(s,t) ds dt = \\ &= \mu \sum_{\xi\in Pad_n} k(\eta_1,\eta_2,\xi_1,\xi_2) \int_S \ell_{n,\xi}(s,t) F_n(s,t) ds dt. \end{aligned} \tag{21}$$

Using (19) we finally get

$$K^* F_n(\boldsymbol{\eta}) = \mu \sum_{\xi\in Pad_n} k(\eta_1,\eta_2,\xi_1,\xi_2) \sum_{\gamma\in Pad_n} F_n(\gamma_1,\gamma_2) \int_S \ell_{n,\xi}(s,t)\ell_{n,\gamma}(s,t) ds dt.$$

Substituting these quantities in (20) we get the linear system

$$F_n(\eta_1,\eta_2) - \mu \sum_{\xi\in Pad_n} k(\eta_1,\eta_2,\xi_1,\xi_2) \sum_{\gamma\in Pad_n} F_n(\gamma_1,\gamma_2) \int_S \ell_{n,\xi}(s,t)\ell_{n,\gamma}(s,t) ds dt = g(\eta_1,\eta_2)$$

and denoting $a_\eta = F_n(\eta_1,\eta_2)$, $\boldsymbol{\eta} = (\eta_1,\eta_2) \in Pad_n$ we get

$$a_\eta - \mu \sum_{\gamma\in Pad_n} \sum_{\xi\in Pad_n} k(\eta_1,\eta_2,\xi_1,\xi_2) M_{\xi,\gamma} a_\gamma = g(\eta_1,\eta_2), \tag{22}$$

where we set $M_{\xi,\gamma} = \int_S \ell_{n,\xi}(s,t)\ell_{n,\gamma}(s,t) ds dt$. By construction, linear system (22) is equivalent to the finite dimensional equation (18) and therefore if this system is solvable it allows to construct the approximating polynomial $F_n \in \Pi_n^2$ defined as in (19).

For calculating the integrals $M_{\xi,\gamma}$ we can approximate them with the cubature formula (8), and we obtain

$$M_{\xi,\gamma} = \int_S \ell_{n,\xi}(s,t)\ell_{n,\gamma}(s,t) ds dt \approx \sum_{\zeta\in Pad_n} \lambda_\zeta \delta_{\xi\zeta}\delta_{\gamma\zeta}$$

where $\delta_{\xi\eta}$ denotes the Kronecher symbol $\delta_{\xi\eta} = \begin{cases} 1 \text{ if } \boldsymbol{\xi} = \boldsymbol{\eta} \\ 0 \text{ if } \boldsymbol{\xi} \neq \boldsymbol{\eta} \end{cases}$ $\quad \boldsymbol{\xi}, \boldsymbol{\eta} \in Pad_n$, and then finally we obtain the linear system

$$a_\eta - \mu \sum_{\gamma\in Pad_n} \lambda_\gamma k(\eta_1,\eta_2,\gamma_1,\gamma_2) a_\gamma = g(\eta_1,\eta_2), \qquad (\eta_1,\eta_2) \in Pad_n. \tag{23}$$

We recognize that we get the same linear system (13) obtained in the Nyström method, that we already said to be well conditioned (see (16)).

About the convergence and stability of the proposed method we have the following result.

Theorem 3.2. *Let $ker(I-K)=\{0\}$ and assume*

$$\sup_{(s,t)\in S}||k_{(s,t)}||_{W_r}<+\infty, \quad \sup_{(x,y)\in S}||k_{(x,y)}||_{W_r}<+\infty, \quad g\in W_r. \tag{24}$$

Then the method is stable, i.e. $(I-H_n)^{-1}$ are uniformly bounded, the equivalent linear system (22) has a unique solution. Moreover the approximating polynomial F_n converges to the unique solution $f^*\in W_r$ and there holds

$$||f^*-F_n||_\infty \le \frac{C\log^2 n}{n^r}||f^*||_{W_r}, \quad C\ne C(n,f^*). \tag{25}$$

Remark 1. From theorems 3.1 and 3.2 immediately follows that, except for the factor $\log^2 n$ in the convergence estimate of the collocation method, the two proposed methods are equivalent and they are reduced to solve the same linear system. The substantial difference is that the Nyström interpolant (15) is not a polynomial, while F_n in (19) is.

4 Numerical tests

We consider some numerical tests to confirm the effectiveness of our methods.

All the numerical tests were performed in Matlab (release R2022a) on a 2.9 GHz Intel Core i9 6 core processor. For the computations we used the Matlab package of Padua points described in [7].

The errors were computed as the relative discrete errors on a grid of equidistant nodes in S

$$\text{errMethodPD}=\frac{||f^*(\boldsymbol{x})-f_n^*(\boldsymbol{x})||_\infty}{||f^*(\boldsymbol{x})||_\infty},$$

where $\boldsymbol{x}=(x_i)_{i=1,\dots,M}$, $x_i\in S$, $M=10000$ and f_n^* is one of the approximating functions of the proposed methods.

When f^* was not known, f^*_{Nmax} was used instead, for $Nmax$ large enough. $N=\frac{(n+1)(n+2)}{2}$ in the tables is the cardinality of Pad_n and denotes the dimension of the solved linear systems. Finally in the table it is reported the condition numbers of the matrices, evaluated in the maximum row sum norm and is denoted by condPD.

First of all we propose three tests on equations for which it is known the exact solution.

Example 4.1. We consider the following equation

$$f(x,y)-2\int_S |s+t|^{\frac{20}{3}}(x^2+y)f(s,t)dsdt = x^2+y^2-2^{\frac{5}{3}}\left(\frac{88704}{8671}(x^2+y)\right).$$

The exact solution is $f^*(x,y)=x^2+y^2$ and the kernel, the known function and the parameter μ are

$$\mu=2, \qquad k(x,y,s,t)=|s+t|^{\frac{20}{3}}(x^2+y), \qquad g(x,y)=x^2+y^2-2^{\frac{5}{3}}\left(\frac{88704}{8671}(x^2+y)\right)$$

As $k_{(x,y)}\in W_6$ and $k_{(s,t)}, g\in W_r, \forall r$, according to the theoretical estimates we expect an error behaving as $O(\frac{1}{n^6})$. The numerical results show an estimated convergence order of 10, while machine accuracy cannot be achieved as the condition number is on the order of 10^2.

N	n	errNyströmPD	errCollocationocationPD	condPD
45	8	8.564786984888906e-05	8.564786985372077e-05	2.067606475761332e+02
153	16	2.857144285031600e-09	2.857153275729234e-09	2.079016750954457e+02
561	32	6.724067533925515e-13	1.118669127180640e-12	2.094718270026205e+02
2145	64	4.370579275376739e-14	6.638302972643527e-14	2.100237468954103e+02

Table 1: Numerical results for Example 4.1

Example 4.2. Consider the equation

$$f(v,w)-\int_{[0,1]^2}\frac{v}{(8+w)(1+z+\zeta)}f(z,\zeta)dzd\zeta=\frac{1}{(1+v+w)^2}-\frac{v}{48+6w}, \quad (v,w)\in[0,1]^2,$$

taken from [3] where a discrete Galerkin method based on wavelets approximation was proposed. The exact solution is $f^*(v,w)=\frac{1}{(1+v+w)^2}$.

By linear transformation the integral equation can be considered on the square $[-1,1]^2$ with kernel, known function and parameter μ as

$$\mu=\frac{1}{2}, \qquad k(x,y,s,t)=\frac{1+x}{(17+y)(4+s+t)}, \qquad g(x,y)=\frac{4}{(4+x+y)^2}-\frac{1+x}{2(51+3y)}.$$

As $k_{(x,y)}, k_{(s,t)}, g\in W_r, \forall r$, so also the solution $f_n\in W_r, \forall r$, and according to the theoretical results we expect a very fast convergence. The numerical tests confirm this expectation. We remark that for this test function in [3] the better error shows an estimated order of convergence 4.

N	n	errNyströmPD	errCollocationPD	condPD
45	8	2.109545408864362e-07	6.415945822506977e-05	1.137326923692149e+00
153	16	1.035211849592365e-12	2.429720107599548e-09	1.137134675872489e+00
561	32	2.491699150543592e-17	1.223701138378075e-15	1.137114729433788e+00

Table 2: Numerical results for Example 4.2

Example 4.3. Consider the equation, once again taken from [3]

$$f(v,w)+\int_{[0,1]^2} vwe^{z+\zeta} f(z,\zeta)dzd\zeta = e^{-v-w},$$

and the exact solution of which is $f^*(v,w)=e^{-v-w}-\frac{1}{2}vw$. By linear transformation the integral equation can be considered on the square $[-1,1]^2$ with kernel, known function and parameter μ as

$$\mu=-\frac{1}{16}, \qquad k(x,y,s,t)=(1+x)(1+y)e^{\frac{2+s+t}{2}}, \qquad g(x,y)=e^{-\frac{2+x+y}{2}}.$$

As $k_{(x,y)}, k_{(s,t)}, g \in W_r, \forall r$, according to the theoretical results we expect a very fast convergence and the numerical tests confirm this expectation.

N	n	errNyströmPD	errCollocationPD	condPD
45	8	8.813092808718864e-10	4.256299436234507e-09	9.371767692740676e+00
153	16	1.145068066566441e-16	7.317906264493577e-16	9.674264996711024e+00

Table 3: Numerical results for Example 4.3

Also in this case the better estimated convergence order shown in [3] is 4 and the corresponding better performance in term of absolute error is 10^{-8}, while we catch the machine precision solving a linear system of order 153.

Example 4.4. Consider the equation

$$f(x,y)-\frac{\pi}{2}\int_S (|x+s|^5+t)e^y f(s,t)dsdt = \cos(xy),$$

for which we do not known the exact solution. The kernel, the known function and the parameter μ are

$$\mu=\frac{\pi}{2}, \qquad k(x,y,s,t)=(|x+s|^5+t)e^y, \qquad g(x,y)=\cos(xy).$$

As $k_{(x,y)}, k_{(s,t)} \in W_5$ and $g \in W_r, \forall r$, according to the theoretical results we expect an error $O(\frac{1}{n^5})$ and the numerical tests confirm this expectation.

N	n	errNyströmPD	errCollocationPD	condPD
45	8	3.025017657965084e-03	3.062913257102731e-03	1.490551539782651e+03
153	16	4.248170182393344e-06	4.353132976187091e-06	1.660326660545173e+03
561	32	4.766217466662518e-08	5.332790966304148e-08	1.702893406960435e+03
2145	64	7.633074826073621e-10	7.896547794091552e-10	1.714034174040502e+03
8385	128	1.113975003958422e-11	1.206663215091149e-11	1.716941317381609e+03

Table 4: Numerical results for Example 4.4

Now we compare the Nyström and collocation methods just described to those proposed in [12]. These are a Nyström method based on a cubature formula obtained as the tensor product of two univariate Gaussian rules and a polynomial collocation method using a tensor product of two Lagrange polynomials, both based on Legendre zeros.

Starting from the Gaussian formula

$$\int_S f(x,y)dxdy=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\lambda_i^{n_1}\lambda_j^{n_2} f(x_i^{n_1},x_j^{n_2})+E_{n_1,n_2}(f), \tag{26}$$

where x_k^m and λ_k^m are the zeros and the Christoffel numbers related to the Legendre orthonormal polynomials sequence $\{p_m\}_m$ respectively, then the corresponding Nyström interpolant is defined as

$$f_{n_1,n_2}(x,y)=\mu\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\lambda_i^{n_1}\lambda_j^{n_2}k(x,y,x_i^{n_1},x_j^{n_2})f(x_i^{n_1},x_j^{n_2})+g(x,y). \tag{27}$$

In [12] the convergence and stability of the method were proved for $n_1=n_2$. Here we reformulate the results in a more general framework.

Theorem 4.1. *Assume that k satisfy (11) and* $ker(I-K)=\{0\}$ *in* $C^0(S)$*. Denote by* f^* *the unique solution of (10) in* $C^0(S)$ *for a given* $g\in C^0(S)$*. If in addition, for some* $r\in\mathbb{N}$*,*

$$g\in W_r,\qquad \sup_{(x,y)\in S}||k_{(x,y)}||_{W_r}<+\infty, \tag{28}$$

then there holds

$$||f^*-f_{n_1,n_2}||_\infty\le C\left[\frac{||f_x^{*(r)}\varphi^r||_\infty}{n_2^r}+\frac{||f_y^{*(r)}\varphi^r||_\infty}{n_1^r}\right], \tag{29}$$

where $C\ne C(n_1,n_2,f^*)$*. In the particular case* $n_1=n_2=n$ *the convergence estimate becomes*

$$||f^*-f_{n,n}||_\infty\le C\frac{||f^*||_{W_r}}{n^r},\qquad C\ne C(n,f^*). \tag{30}$$

Proof. The proof can be lead as done in [12], simply considering, when necessary, the estimate of the best approximation error in $\mathbb{P}_{n_1,n_2}$ in terms of the estimates of the univariate best approximation of the functions f_x^* and f_y^*. □

In [12] it was also proposed a collocation method obtained by projecting (10) on the space $\mathbb{P}_{n_1,n_2}$ by means of the Lagrange operator

$$\mathcal{L}_{n_1,n_2}(f,x,y)=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\ell_{n_1,i}(x)\ell_{n_2,j}(y)f(x_i^{n_1},x_j^{n_2}) \tag{31}$$

where $\ell_{m,k}(z)$ denote the Lagrange fundamental polynomials w.r.t. the sequence $\{p_m\}_m$, i.e. $\ell_{m,k}(z)=\frac{p_m(z)}{p_m'(x_k^m)(z-x_k^m)}$. The method searches for a polynomial solution $F_{n_1,n_2}\in\mathbb{P}_{n_1,n_2}$, in the form

$$F_{n_1,n_2}(x,y)=\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\ell_{n_1,i}(x)\ell_{n_2,j}(y)f(x_i^{n_1},x_j^{n_2}).$$

In order to obtain optimal Lebesgue constants for the Lagrange operator (31) it is necessary to consider the approximation of the equation in a weighted spaces of continuous function $C_u(S)$, where $u(x,y)=v^{\gamma_1,\delta_1}(x)v^{\gamma_2,\delta_2}(y)$ and $v^{\gamma,\delta}(z)=(1-z)^\gamma(1+z)^\delta$ denotes a suitable Jacobi weight, with nonnegative exponents. $C_u(S)$ can be equipped with the weighted norm $||fu||_\infty$ and mutatis mutandis Sobolev type subspaces $W_r(u)$ can be introduced (for more details see [12]). As done for the above mentioned Nyström method we reformulate the convergence and stability results for the tensorial collocation method, stated in [12] for $n_1=n_2=n$, in the more general case.

Theorem 4.2. *Assume the exponents of u satisfy* $\frac14\le\gamma_i,\delta_i\le\frac54$*,* $i=1,2$*, and let* $ker(I-K)=\{0\}$ *in* $C_u(S)$*. Denote by* f^* *the unique solution of (10) in* $C_u(S)$ *for a given* $g\in C_u(S)$*. Moreover let*

$$g\in W_r(u),\qquad \sup_{(s,t)\in S}||k_{(s,t)}||_{W_r(u)}<+\infty,\qquad \sup_{(x,y)\in S}u(x,y)||k_{(x,y)}||_{W_r}<+\infty,$$

Then

$$||[f^*-F_{n_1,n_2}]u||_\infty\le C\log n_1\log n_2\left[\frac{||f_x^{*(r)}\varphi^r u||_\infty}{n_2^r}+\frac{||f_y^{*(r)}\varphi^r u||_\infty}{n_1^r}\right] \tag{32}$$

where $C\ne C(n_1,n_2,f^*)$*. In the particular case* $n_1=n_2=n$ *the convergence estimate becomes*

$$||[f^*-F_{n,n}]u||_\infty\le C\log^2 n\frac{||f^*||_{W_r(u)}}{n^r},\qquad C\ne C(n,f^*). \tag{33}$$

Proof. The proof can be lead as done in [12], simply considering, when necessary, the estimates of the univariate Lagrange interpolation error for f_x^* and f_y^*. □

If we compare (17) with (30) we recognize that the two Nyström methods have exactly the same rate of convergence. Analogously if we compare (25) with (33) we see that, except for the difference of the spaces in which one can consider the approximation processes, the two collocation methods are equivalent, from the convergence point of view.

Nevertheless for the same n we have that the methods based on the Padua points need to solve linear systems having a number of equations which is the half of the number of linear equations that are necessary to be solved in the case of the “tensor product" methods.

However if the known functions, and therefore the solution of the equation, have different degree of smoothness with respect to the two variables, the tensorial strategies, according to (29) and (32), allow to use a different number of nodes on the two directions and this can reduce the global computational cost, since the number of linear equations (and therefore the dimension of the involved matrices) can be reduced much more then the half.

What we remarked is confirmed by the following examples. In the tables now we will call errMethodGauss and condGauss the errors of the methods based on the tensorial strategy and the corresponding condition numbers.

Example 4.5. In this example we consider the following integral equation

$$f(x,y)-\frac{\pi}{4}\int_S(\sin(|x-t|^{\frac{5}{2}}|s-y|^{\frac{5}{2}})f(s,t)dsdt=|x-y|^{\frac{5}{2}},$$

of which we do not known the exact solution. The kernel, the known function and the parameter μ are

$$\mu=\frac{\pi}{4},\qquad k(x,y,s,t)=\sin(|x-t|^{\frac{5}{2}}|s-y|^{\frac{5}{2}}),\qquad g(x,y)=|x-y|^{\frac{5}{2}}.$$

As $k_{(x,y)}, k_{(s,t)}, g \in W_2$, according to the theoretical results we expect an error $O(\frac{1}{n^2})$ for the methods based on the Padua points and also for the "tensorial methods". The numerical tests reported in tables 5 and 6 show that effectively the methods are equivalent and that the Padua points methods have to solve linear systems of a dimension which is about the half of the dimension of the corresponding linear systems in the tensorial methods.

N	n	errNyströmPD	errCollocationPD	condPD
45	8	7.576066165065090e-02	9.765082736392187e-02	6.554074700569336e+00
153	16	6.765154198776951e-03	1.029131358515190e-02	6.096416670061345e+00
561	32	6.635519936473319e-06	1.906965645556420e-04	6.000601495523336e+00
2145	64	4.104113301422593e-07	6.906771132743972e-06	5.915043491951402e+00
8385	128	3.504260471892602e-08	8.136562389557896e-07	5.912600645033277e+00

Table 5: Numerical results for Example 4.5: Nyström and collocation methods on Padua points

N	n	errNyströmGauss	errCollocationGauss	condGauss
64	8	1.351963105416440e-02	4.053569579806666e-02	6.212071736299888e+00
256	16	7.388574541307164e-05	3.744924312131188e-03	5.680912444375486e+00
1024	32	6.960662232350263e-06	3.718298215793385e-05	5.893856535088040e+00
4096	64	6.144553341974727e-07	4.613484395560557e-06	5.902255857929990e+00
16384	128	5.747849982383078e-08	5.754961454472095e-07	5.908542935713527e+00

Table 6: Numerical results for Example 4.5: Nyström and collocation methods based on tensorial formulas

Example 4.6. Consider the equation

$$f(x,y)-\frac{\pi}{2}\int_S(|x-0.5|^{\frac{5}{2}}+|s-0.5|^{\frac{5}{2}})^3(|y|^{\frac{5}{2}}+|t|^{\frac{5}{2}})^3f(s,t)dsdt=|x-y|^{\frac{5}{2}},$$

where also in this case we do not known the exact solution. The kernel, the known function and the parameter μ are

$$\mu=\frac{\pi}{2},\qquad k(x,y,s,t)=(|x-0.5|^{\frac{5}{2}}+|s-0.5|^{\frac{5}{2}})^3(|y|^{\frac{5}{2}}+|t|^{\frac{5}{2}})^3,\qquad g(x,y)=|x-y|^{\frac{5}{2}}.$$

As $k_{(x,y)}, k_{(s,t)}, g \in W_2$, according to the theoretical results we expect an error $O(\frac{1}{n^2})$ for the methods based on the Padua points and also for the "tensorial methods". The numerical behavior is very similar to that of the previous example as shown in tables 7 and 8

N	n	errNyströmPD	errCollocationPD	CondPD
45	8	6.074070070080574e-02	7.810654934396465e-02	3.050883302886094e+04
153	16	1.449262344141590e-04	7.925157955796673e-04	1.936755708514138e+04
561	32	7.176170195323578e-06	1.002757247756092e-04	1.935826750224334e+04
2145	64	7.026962161203207e-07	1.230733903318241e-05	1.936199543151200e+04
8385	128	6.636672561146159e-08	1.433674390569745e-06	1.936053340255196e+04

Table 7: Numerical results for Example 4.6: Nyström and collocation methods on Padua points

Example 4.7. Consider the equation

$$f(x,y)-\frac{\pi}{2}\int_S(|x+s|^{\frac{7}{2}}+t)e^yf(s,t)dsdt=\cos(xy),$$

of which we do not known the exact solution. The kernel, the known function and the parameter μ are

$$\mu=\frac{\pi}{2},\qquad k(x,y,s,t)=(|x+s|^{\frac{7}{2}}+t)e^y,\qquad g(x,y)=\cos(xy).$$

N	n	errNyströmGauss	errCollocationGauss	condGauss
64	8	1.366384160119497e-03	8.522827814901306e-03	4.779742814002967e+03
256	16	1.243414308289405e-04	5.002503747339286e-04	1.387298027043163e+04
1024	32	1.154655634130743e-05	6.646169208004555e-05	1.782578361506734e+04
4096	64	1.085348771045084e-06	8.741139745367303e-06	1.896533515223295e+04
16384	128	1.098704522358498e-07	1.036503234390004e-06	1.926061334527154e+04

Table 8: Numerical results for Example 4.6: Nyström and collocation methods based on tensorial formulas

As $k_{(x,y)}, k_{(s,t)} \in W_3, g \in W_r, \forall r$, according to the theoretical results we expect the errors of the order $O(\frac{1}{n^3})$. Nevertheless in this case it is clear that $k_{(x,y)}$ is a function with a continuous third partial derivative w.r.t. s, but is a polynomial w.r.t. t. Analogously $k_{(s,t)}$ has a continuous third partial derivative w.r.t. x but is analytic w.r.t. y. Therefore this is a case in which it is possible to choose operators of different order in the tensorial methods. The numerical results confirm this possibility and the computational saving is evident (see Tables 9-10).

N	n	errNyströmPD	errCollocationPD	CondPD
45	8	2.465271482100009e-03	2.515280367465270e-03	3.339068142772825e+02
153	16	2.501026750147732e-05	2.625470787426196e-05	3.636435705114400e+02
561	32	7.715958142138392e-07	9.661530734480123e-07	3.720671883637921e+02
2145	64	3.914863303638266e-08	4.092132825140609e-08	3.744151261351798e+02
8385	128	1.645201417613668e-09	1.782039892581032e-09	3.750206337212849e+02

Table 9: Numerical results for Example 4.7: Nyström and collocation methods on Padua points

N	n_1	n_2	errNyströmGauss	errCollocationGauss	condGauss
128	8	16	4.109901467877955e-04	7.224670933831467e-04	3.128149502945657e+02
256	16	16	1.843816637403101e-05	2.006854630734481e-05	3.513579853458843e+02
512	32	16	8.561272768254221e-07	8.767511886440799e-07	3.636004779596027e+02
1024	64	16	3.592369324561472e-08	3.929492645715875e-08	3.668471166557514e+02
2048	128	16	1.478022896153951e-09	1.750110504860680e-09	3.676840753176024e+02

Table 10: Numerical results for Example 4.7: Nyström and collocation methods based on tensorial formulas

Example 4.8. Consider the equation

$$f(x,y) - \frac{1}{2}\int_S (|x-s|^{\frac{7}{2}} + t)(t^2+y^2)f(s,t)dsdt = x^2y^2,$$

of which we do not known the exact solution. The kernel, the known function and the parameter μ are

$$\mu = \frac{1}{2}, \qquad k(x,y,s,t) = (|x-s|^{\frac{7}{2}} + t)(t^2+y^2), \qquad g(x,y) = x^2y^2.$$

As $k_{(x,y)}, k_{(s,t)} \in W_3, g \in W_r, \forall r$, according to the theoretical results we expect an error $O(\frac{1}{n^3})$. In this case the considerations about the smoothness of the function kernel k and the consequent choice of different n_1 and n_2 in the tensorial methods can be made similarly to what said in the previous example.

N	n	errNyströmPD	errCollocationPD	CondPD
45	8	3.556836420275436e-05	2.621425021666921e-05	4.801963960960753e+01
153	16	1.282345425995348e-06	1.282900279600414e-06	4.766586978932889e+01
561	32	5.629688695654519e-08	5.629868751831099e-08	4.763253425317059e+01
2145	64	2.480710813116984e-09	2.480686129408371e-09	4.762498479136794e+01
8385	128	1.067939388941909e-10	1.057311418982930e-10	4.762384070546259e+01

Table 11: Numerical results for Example 4.8: Nyström and collocation methods on Padua points

5 The proofs

In this section we prove the convergence, stability and well conditioning of the two proposed numerical methods.

N	n_1	n_2	errNyströmGauss	errCollocationGauss	condGauss
128	8	16	2.306513742770721e-05	2.185690392173100e-05	3.999732725700071e+01
256	16	16	1.111282148178759e-06	1.089501576228607e-06	4.450823322173192e+01
512	32	16	5.245033393921879e-08	5.241369126284725e-08	4.583066745452771e+01
1024	64	16	2.394564074974570e-09	2.394490834768147e-09	4.617718486838073e+01
2048	128	16	1.037988288528741e-10	1.037986608106382e-10	4.626773459681050e+01

Table 12: Numerical results for Example 4.8: Nyström and collocation methods based on tensorial formulas

Proof of Theorem 3.1. We want to show that the Nyström method is convergent and stable in $C^0(S)$. Using standard arguments (see [1], Theorem 4.1.1, p. 106 and the related remarks) these claims follow if the sequence $\{K_n\}_n$ is collectively compact. The collectively compactness of the sequence can be obtained by showing that

1. $\lim_n \|Kf - K_n f\|_\infty = 0, \qquad$ as f $\in C^0(S)$,

2. $\sup_n \lim_{m\to\infty} \sup_{\|f\|_\infty=1} E_m(K_n f) = 0.$

Moreover from the definition of f^* and f_n it follows that

$$\|f^* - f_n\|_\infty \sim \|Kf^* - K_n f^*\|_\infty.$$

Therefore proceeding as in [12] (see the proof of Theorem 3.1, p. 2328) and [13] (see the proof of Theorem 5.1, p.162), using (9) we can prove 1. and 2. and we get (17).

Finally acting as in [13] it immediately follows that

$$cond(A_n) \le \|(I - K_n)\|\|(I - K_n)^{-1}\| < +\infty$$

and the proof is complete. □

In order to prove convergence and stability of the collocation method we introduce a preliminary result. Let $W(x, y) = \frac{1}{\pi^2}\frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-y^2}}$ and denote by $L^p_W(S)$ the space of integrable functions f such that

$$\|f\|_{W,p} = \left(\int_S |f(x, y)|^p W(x, y)dx\,dy\right)^{\frac{1}{p}} < +\infty.$$

In [5] the following result was proved.

Lemma 5.1. *For all $f \in C^0(S)$ it results*

$$\|f - \mathcal{L}_n f\|_{W,1} \le C\, E_n(f), \qquad C \ne C(n, f) \tag{34}$$

where $E_n(f)$ is the error of best polynomial approximation in Π^2_n in uniform norm.

This result is an important tool in proving Theorem 3.2.

Proof of Theorem 3.2. The proof is conducted in a classical way. Indeed the claim is to prove (see for instance [1], Theorem 3.1.1, p. 55)

$$\lim_n ||K - H_n|| = 0. \tag{35}$$

We note that

$$||Kf - H_n f||_\infty \le ||Kf - K^* f||_\infty + ||K^* f - H_n f||_\infty. \tag{36}$$

By Lemma 5.1 applied to $k_{(x,y)} \in W_r$, taking into account that W^{-1} is bounded and in view of (4), we get

$$\begin{aligned}||Kf - K^* f||_\infty &\le C \max_{(x,y)\in S} ||f||_\infty \int_S |k(x, y, s, t) - \mathcal{L}_n(k_{(x,y)}, s, t)|W(s, t)dsdt \\ &\le C||f||_\infty \max_{(x,y)\in S} E_n(k_{(x,y)}) \\ &\le C||f||_\infty \max_{(x,y)\in S} \frac{||k_{(x,y)}||_{W_r}}{n^r} \le C\frac{||f||_\infty}{n^r}, \qquad C \ne C(n, f).\end{aligned} \tag{37}$$

On the other hand, in view of (7) it follows.

$$||K^* f - \mathcal{L}_n(K^* f)||_\infty \le C \log^2 nE_n(K^* f) \tag{38}$$

Now for simplicity let n be an even number and Q_n be a polynomial of degree $\frac{n}{2}$ in each of its four variables. Define a polynomial on $\mathbb{P}_{\frac{n}{2},\frac{n}{2}}$ as follows

$$K^{Q_n} f(x,y) = \int_S Q_n(x,y,s,t) f(s,t) ds dt$$

Since $K^{Q_n} \in \mathbb{P}_{\frac{n}{2},\frac{n}{2}}$, necessarily $K^{Q_n} \in \Pi_n^2$, and the interpolation on the Padua points Pad_n is exact on Π_n^2, then, again by (34),

$$\begin{aligned} \max_{(x,y)\in S} |K^* f(x,y) - K^{Q_n} f(x,y)| &\leq \max_{(x,y)\in S} \|f\|_\infty \int_S |\mathcal{L}_n(k_{(x,y)} - Q_n(x,y,.,.),s,t)| W(s,t) ds dt \\ &\leq C \|f\|_\infty \max_{(x,y)\in S} \|k_{(x,y)} - Q_n(x,y,.,.)\|_\infty . \end{aligned} \tag{39}$$

As $Q_{n(x,y)}$ is a generic polynomial on Π_n^2 we obtain

$$E_n(K^* f) \leq C\|f\|_\infty \max_{(x,y)\in S} E_n(k_{(x,y)}) \leq C \frac{\|f\|_\infty}{n^r}, \qquad C \neq C(n,f). \tag{40}$$

Therefore, since k satisfy (24) and (4) holds true, then from (37) and (38) the claim follows. □

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References

[1] K. E. Atkinson, The Numerical Solution of Integral Equation of the Second Kind. *Cambridge Monographs on Applied and Computational Mathematics*, 1997.

[2] K. E. Atkinson, D. Chien, J. Seol, Numerical analysis of the radiosity equation using the collocation method. *Electron. Trans. Numer. Anal.*, 11 : 94–120, 2000

[3] H. Bin Jebreen, A Novel and Efficient Numerical Algorithm for Solving 2D Fredholm Integral Equations. *J. Math.*, Volume 2020, Article ID 6662721, 9 pages https://doi.org/10.1155/2020/6662721

[4] L. Bos L., S. De Marchi, M. Vianello, Y. Xu, Bivariate Lagrange interpolation at the Padua points: The generating curve approach. *J. Approx. Theory* 143: 15–25, 2006

[5] L. Bos L., S. De Marchi, M. Vianello, Y. Xu, Bivariate Lagrange interpolation at the Padua points: the ideal theory approach. *Numer. Math.* , 108: 43–57, 2007

[6] M. Caliari , S. De Marchi, M. Vianello, Bivariate polynomial interpolation on the square at new nodal sets. *Appl. Math. Comput.*, 165(2): 261–274, 2005

[7] M. Caliari, S. De Marchi, A. Sommariva, M. Vianello,Padua 2DM: fast interpolation and cubature at the Padua points in Matlab/Octave. *Numer. Algorithms* . 56: 45–60, 2011

[8] B. Della Vecchia, G. Mastroianni, P. Vertesi, Exact order of the Lebesgue constants for bivariate Lagrange interpolation at certain node-systems. *Studia Sci. Math. Hungar.* , 46(1): 97-102, 2009

[9] G. Han, R. Wang, Richardson extrapolation of iterated discrete Galerkin solution for two-dimensional Fredholm integral equations. *J. Comput. Appl. Math.* 139 (1): 49–63, 2002

[10] J.T. Kajiya, The rendering equation. *SIGGRAPH Comput. Graph*, vol. 20, ACM, New York, NY, USA, 1986.

[11] F. Liang, F.R. Lin, A fast numerical solution method for two dimensional Fredholm integral equations of the second kind based on piecewise polynomial interpolation. *Appl. Math. Comput.* 216 (10): 3073–3088, 2010

[12] D. Occorsio, M.G. Russo, Numerical methods for Fredholm integral equations on the square. *Appl. Math. Comput.* 218: 2318–2333, 2011

[13] D. Occorsio, M. G. Russo, Bivariate generalized Bernstein operators and their application to Fredholm integral equations. *Publ. Inst. Math. (Beograd) (N.S.)* 100(114): 141–162, 2016

[14] A. Sommariva, M. Vianello, R. Zanovello, Nontensorial Clenshaw-Curtis cubature. *Numer. Algorithms*, 49: 409–427, 2008

[15] Y. Xu, Lagrange Interpolation on Chebychev Points of Two Variables. *J. Approx. Theory* 87 (2): 220–238, 1996