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$q-$extensions for the $U-$Bernoulli and $U-$Euler polynomials

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Abstract

This paper defines the $q-U-$Bernoulli, $q-U-$Euler polynomials and numbers and obtains some of their feature relations. We also explore the $q-U-$Bernoulli polynomials $M_s(\gamma,q)$, Rogers-Szegö polynomials $H_s(\gamma,q)$ and the $q-$Bernoulli numbers $B_s(q)$ are related via some connection formulas. Further, we define various properties for the $q-$extensions $U-$Bernoulli and $U-$Euler polynomials like generating functions, $q-$partial derivatives and summation relations.

1 Introduction

Quantum calculus, often known as $q-$calculus is an important study issue in classical mathematical analysis. It aims to generalize differentiation and integration procedures. The use of $q-$calculus in several disciplines such as mathematics, mechanics, and physics is rapidly growing.
Scientists investigating $q-$special functions have invented and analyzed several functions, including their qualities. (see examples [1, 6, 9, 15, 20, 32]

Throughout this paper, we will use the following notation: $\mathbb{N}$ for the set of all natural numbers, $\mathbb{N}_0$ for the set of all non-negative integers, $\mathbb{Z}$ for the set of all integers, $\mathbb{R}$ for the set of all real numbers, and $\mathbb{C}$ for the set of all complex numbers. We briefly review several important definitions and notations from [2, 7, 8, 10] as an introduction to the $q-$calculus ($q \in \mathbb{C}$ with $0 < |q| < 1$).

In [10], the $q-$shifted factorial is defined by

$$(a;q)_0 = 1, \quad (a;q)_s = \prod_{j=0}^{s-1}\left(1-q^j a\right), \quad s \in \mathbb{N},$$

$$(a;q)_\infty = \prod_{j=0}^{\infty}\left(1-q^j a\right), \quad |q| < 1, a \in \mathbb{C}.$$

The $q-$numbers and $q-$factorial are defined by

$$[a]_q = \frac{1-q^a}{1-q}, \qquad q \neq 1, a \in \mathbb{C},$$

and

$$[s]_q! = \frac{(q;q)_s}{(1-q)^s}, \quad q \neq 1, s \in \mathbb{N}, \quad \text{and} \quad [0]_q! = 1,$$

in [14].

The $q-$binomial coefficient is defined by

$$\binom{s}{u}_q = \frac{(q;q)_s}{(q;q)_{s-u}(q;q)_u} = \frac{[s]_q!}{[s-u]_q![u]_q!}.$$

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Some properties of the $q-$binomial coefficient are as follows:

$$\binom{s+1}{u}_q = q^u \binom{s}{u}_q + \binom{s}{u-1}_q$$

and

$$\binom{s+1}{u}_q = \binom{s}{u}_q + q^{s-u} \binom{s}{u-1}_q .$$

In [14], the Gauss's binomial formula is defined by

$$(\gamma+\beta)_q^s := \sum_{u=0}^{s} \binom{s}{u}_q q^{\frac{1}{2}u(u-1)} \gamma^{s-u} \beta^u, \quad s \in \mathbb{N}_0.$$

Specially,

$$(\gamma-1)_q^s = \sum_{u=0}^{s} \binom{s}{u}_q (-1)^{s-u} q^{\frac{1}{2}(s-u)(s-u-1)} \gamma^u.$$

If $\beta\gamma = q\gamma\beta$, where q is a number commuting with both γ and β, then

$$(\gamma+\beta)_q^s = \sum_{u=0}^{s} \binom{s}{u}_q \gamma^u \beta^{s-u}$$

in [14].

In the standard approach to the $q-$calculus, two exponential functions are used

$$e_q(z) = \frac{1}{(1-z(1-q))_q^\infty} = \sum_{s=0}^{\infty} \frac{z^s}{[s]_q!}, \qquad |z| < \frac{1}{|1-q|} \tag{1}$$

$$E_q(z) = (1+z(1-q))_q^\infty = \sum_{s=0}^{\infty} \frac{q^{\frac{1}{2}s(s-1)} z^s}{[s]_q!}, \qquad z \in \mathbb{C}.$$

Moreover, we have [12]

$$D_q e_q(z) = e_q(z), \tag{2}$$

$$D_q E_q(z) = E_q(qz), \tag{3}$$

where the $q-$derivative D_q is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 \neq z \in \mathbb{C}.$$

The above $q-$standard notation can be found in (see, [12]).

In [14], the Jackson integral is as follows

$$\int_a^b f(\gamma) d_q\gamma = \int_0^b f(\gamma) d_q\gamma - \int_0^a f(\gamma) d_q\gamma, \ 0 < a < b$$

where

$$\int_0^a f(\gamma) d_q\gamma = a(1-q) \sum_{i=0}^{\infty} q^i f\left(aq^i\right).$$

Specifically, we see that

$$\int_0^1 \gamma^s d_q\gamma = \frac{1}{[s+1]_q}. \tag{4}$$

The Rogers-Szegö polynomials $H_s(\gamma, q)$ and the Al-Salam Carlitz I polinomials $U_s^{(a)}(\gamma, q)$ [34, p. 1183], [16] are defined by the generating functions

$$e_q(z) e_q(\gamma z) = \sum_{s=0}^{\infty} H_s(\gamma, q) \frac{z^s}{[s]_q!} \tag{5}$$

and

$$\frac{e_q(\gamma z)}{e_q(z) e_q(az)} = \sum_{s=0}^{\infty} U_s^{(a)}(\gamma, q) \frac{z^s}{[s]_q!} \tag{6}$$

respectively.

Taking $a = -1$ in equation (6), we obtain the following relation

$$U_s^{(-1)}(\gamma, q) = h_s(\gamma, q),$$

where $h_s(\gamma, q)$ denotes the discrete $q-$Hermite I polynomials (see [17, Eq. 4.46]).

The inversion formulas for the polynomials $H_s(\gamma, q)$ [3, Eq. 1] and $U_s^{(a)}(\gamma, q)$ [34, p. 1180] are defined by

$$\gamma^u = \sum_{j=0}^{u} (-1)^{u-j} \binom{u}{j}_q q^{\binom{u-j}{2}} H_j(\gamma, q) \tag{7}$$

and

$$\gamma^u = \sum_{j=0}^{u} \binom{u}{j}_q \left(\sum_{s=0}^{u-j} \binom{u-j}{s}_q a^s \right) U_j^{(a)}(\gamma, q). \tag{8}$$

Taking $y = 0$ in [18, Eq. 2.8] gives the $q-$Fubini polynomials with two parameters by means of the following generating function

$$\frac{e_q(\gamma z)}{1 - w\left(e_q(z) - 1\right)} = \sum_{s=0}^{\infty} F_{s,q}(\gamma; w) \frac{z^s}{[s]_q!}. \tag{9}$$

When $w = 1$ and $\gamma = 0$ in the above equation, we obtain the s-th $q-$Fubini number $F_{s,q}$.

For $n \in \mathbb{N}_0$, in [19, Eq. (10)], the U-Bernoulli polynomials $M_s(\gamma)$ are defined by the following generating function:

$$\left(\frac{z}{e^{-z} - 1} \right) e^{-\gamma z} = \sum_{s=0}^{\infty} M_s(\gamma) \frac{z^s}{s!}, \quad |z| < 2\pi. \tag{10}$$

Note that, if $\gamma = 0$ in (10), the $U-$Bernoulli numbers M_s are obtained.

Similarly, the authors of [19, Eq. (19)] defined the U-Euler polynomials $A_s(\gamma)$ by the following generating function:

$$\left(\frac{2}{e^{-\frac{z}{2}} + 1} \right) e^{-\frac{\gamma z}{2}} = \sum_{s=0}^{\infty} A_s(\gamma) \frac{z^s}{s!}, \quad |z| < 2\pi. \tag{11}$$

For $\gamma = 0$ in ((11) the $U-$Euler numbers M_s are obtained.

We motivated by the introducing of $U-$Bernoulli, $U-$Euler polynomials and numbers [19]. Also, motivated from the applications of $q-$special polynomials in different fields of mathematics and sciences. In section 2, we define $q - U-$Bernoulli, $q - U-$Euler polynomials and numbers and obtains some their feature relations. Then we explore the $q - U-$Bernoulli polynomials $M_s(\gamma, q)$, Rogers-Szegö polynomials $H_s(\gamma, q)$ and the $q-$Bernoulli numbers $B_s(q)$ are related via some connection formulas in section 3. Moreover, in section 4, we derive various features for the $q-$extensions $U-$Bernoulli/Euler polynomials like generating functions, $q-$partial derivatives and summation relations.

2 $q - U-$Bernoulli, $q - U-$Euler polynomials and numbers

The $q-$Bernoulli polynomials and numbers, first defined by Carlitz [21], have attracted the attention of many mathematicians, and a variety of their properties have been investigated. Kupershmidt, introduced $q-$Bernoulli polynomials and established their reflection symmetry [22]. After that, Kim first introduced the concept of $q-$Euler numbers and new $q-$extensions of polynomials using Kupershmidt's method, and investigated the symmetry properties of these $q-$Euler polynomials using $q-$differentiation and $q-$integration, and introduced various applications and symmetries [23].

In recent studies, degenerate Bernoulli numbers and polynomials have been investigated from a generalized perspective of $q-$Bernoulli polynomials [25, 31, 33], and numerous papers have been published examining these polynomials by deriving moment values from probability theory and relating them to various identities and values of the zeta function [26, 27, 29].

Moreover, the relationship between these degenerate polynomials and harmonic numbers has been established [28, 30]. The significance of $q-$Bernoulli numbers in the $q-$Laplace or degenerate Laplace transforms for different values of the parameter λ used in these studies has been demonstrated [24].

In this paper, we describe the $q - U-$Bernoulli, $q - U-$Euler polynomials and numbers. We obtain various properties these polynomials and numbers.

The $q-$Bernoulli polynomials $B_s(\gamma, q)$ are defined by means of the following generating function

$$\frac{z e_q(\gamma z)}{e_q(z) - 1} = \sum_{s=0}^{\infty} B_s(\gamma, q) \frac{z^s}{[s]_q!}. \tag{12}$$

Clearly, in case $\gamma = 0$, $B_s(0,q) = B_s(q)$ are $q-$Bernoulli numbers [4, 5, 13, 11].

Moreover, in the studies conducted, we can observe various generalizations of these polynomials [35, 36, 37, 38].

Definition 2.1. Let n be non-negative integer. The $q-U-$Bernoulli polynomials $M_s(\gamma,q)$ are defined by means of the following generating function

$$\sum_{s=0}^{\infty} M_s(\gamma,q)\frac{z^s}{[s]_q!} = \frac{-ze_q(z)e_q(-\gamma z)}{e_q(z)-1}. \tag{13}$$

When written $\gamma = 0$ in (13), the $q-U-$Bernoulli numbers $M_s(q)$ are defined by the generating function

$$\sum_{s=0}^{\infty} M_s(q)\frac{z^s}{[s]_q!} = \frac{-ze_q(z)}{e_q(z)-1}. \tag{14}$$

Here, we see that

$$\lim_{q\to 1} M_s(\gamma,q) = M_s(\gamma)$$

and

$$\lim_{q\to 1} M_s(q) = M_s,$$

where the $M_s(\gamma)$ is s-th $U-$Bernoulli polynomial and M_s is s-th $U-$Bernoulli number.

Theorem 2.1. *Let $B_s(q)$ be s-th $q-$Bernoulli numbers.*

$$M_s(q) = -\sum_{u=0}^{s}\binom{s}{u}_q B_u(q).$$

Proof. If we use generating function of $M_s(q)$ numbers, we obtain

$$\begin{aligned}\sum_{s=0}^{\infty} M_s(q)\frac{z^s}{[s]_q!} &= \frac{-ze_q(z)}{e_q(z)-1} \\ &= -\sum_{s=0}^{\infty} B_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\frac{z^s}{[s]_q!} \\ &= \left(-\sum_{u=0}^{s}\binom{s}{u}_q B_u(q)\right)\frac{z^s}{[s]_q!}.\end{aligned} \tag{15}$$

□

Example 2.1. The first few $q-U-$Bernoulli numbers $M_s(q)$ are as follows:

$$\begin{aligned} M_0(q) &= -1, \\ M_1(q) &= -\frac{1}{2}, \\ M_2(q) &= -\frac{5[2]_q}{12} - 1, \\ M_3(q) &= -1 + \frac{[3]_q}{2} - \frac{[3]_q[2]_q}{12}, \\ M_4(q) &= -1 + \frac{[4]_q}{2} - \frac{[4]_q[3]_q}{12} + \frac{[4]_q!}{720}, \\ M_5(q) &= -1 + \frac{[5]_q}{2} - \frac{[5]_q[4]_q}{12} + \frac{[5]_q!}{720}. \end{aligned}$$

Theorem 2.2. *Let $M_s(q)$ be s-th $q-U-$Bernoulli numbers. Then*

$$M_s(\gamma,q) = \sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u} M_u(q)\gamma^{s-u}, \tag{16}$$

or

$$M_s(\gamma,q) = (-1)^{s+1}\gamma^s + \sum_{u=1}^{s}\binom{s}{u}_q (-1)^{s-u} M_u(q)\gamma^{s-u}.$$

Proof. Expanding the right hand side of equation (13) via equations (15) and (1), we have

$$\begin{aligned}\sum_{s=0}^{\infty} M_s(\gamma,q)\frac{z^s}{[s]_q!} &= \frac{-ze_q(z)e_q(-\gamma z)}{e_q(z)-1}\\ &= \sum_{s=0}^{\infty} M_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}(-1)^s\gamma^s\frac{z^s}{[s]_q!}\\ &= \sum_{s=0}^{\infty}\left(\sum_{k=0}^{s}\binom{s}{k}_q(-1)^{s-k}M_k(q)\gamma^{s-k}\right)\frac{z^s}{[s]_q!}.\end{aligned}$$

Thus, the proof is complete. □

Example 2.2. The first few $M_s(\gamma,q)$ polynomials are as follows:

$$\begin{aligned}M_0(\gamma,q) &= -1,\\ M_1(\gamma,q) &= \gamma-\frac{1}{2},\\ M_2(\gamma,q) &= -\gamma^2+\frac{[2]_q x}{2}+\frac{5[2]_q}{12}-1,\\ M_3(\gamma,q) &= \gamma^3-\frac{[3]_q\gamma^2}{2}-[3]_q\left(\frac{5[2]_q}{12}-1\right)\gamma,\\ M_4(\gamma,q) &= -\gamma^4+\frac{[4]_q}{2}\gamma^3+\frac{[4]_q[3]_q}{[2]_q}\left(\frac{5[2]_q}{12}-1\right)\gamma^2-[4]_q\left(-1+\frac{[3]_q}{2}-\frac{[3]_q[2]_q}{12}\right)\gamma\\ &\quad+\left(-1+\frac{[4]_q}{2}-\frac{[4]_q[3]_q}{12}+\frac{[4]_q!}{720}\right).\end{aligned}$$

Theorem 2.3. *If $s\in\mathbb{N}_0$ and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$*

$$M_s(\gamma+\beta,q)=\sum_{u=0}^{s}\binom{s}{u}_q(-1)^{s-u}M_u(\gamma,q)\beta^{s-u}$$

for $q-$commuting variables γ and β.

Proof. We have

$$\begin{aligned}\sum_{u=0}^{\infty} M_s(\gamma+\beta,q)\frac{z^s}{[s]_q!} &= \frac{-ze_q(z)e_q(-(\gamma+\beta)z)}{e_q(z)-1}\\ &= \frac{-ze_q(z)}{e_q(z)-1}e_q(-\gamma z)e_q(-\beta z)\\ &= e_q(-\beta z)\sum_{s=0}^{\infty}M_s(\gamma,q)\frac{z^s}{[s]_q!}\end{aligned}$$

and

$$e_q(-\beta z)=\sum_{s=0}^{\infty}(-\beta)^s\frac{z^s}{[s]_q!}. \tag{17}$$

On the other hand,

$$\sum_{s=0}^{\infty}(-\beta)^s\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}M_s(\gamma,q)\frac{z^s}{[s]_q!}=\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q M_u(\gamma,q)(-\beta)^{s-u}\right)\frac{z^s}{[s]_q!}. \tag{18}$$

Thus, we achieve the desired result from the equality of (17) and (18). □

Theorem 2.4. *The $q-$derivative of $M_s(\gamma,q)$ polynomials is*

$$D_{q,\gamma}M_s(\gamma,q)=-[s]_q M_{s-1}(\gamma,q).$$

Proof. We obtain

$$\begin{aligned} D_{q,\gamma} M_s(\gamma,q) =& D_{q,\gamma}\left(\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u} M_u(q)\gamma^{s-u}\right) \\ =& \sum_{u=0}^{s-1}\binom{s}{u}_q (-1)^{s-u} M_u(q)[s-u]_q \gamma^{s-1-u} \\ =& -[s]_q \sum_{u=0}^{s-1}\binom{s-1}{u}_q (-1)^{s-1-u} M_u(q)[s-u]_q \gamma^{s-1-u} \\ =& -[s]_q M_{s-1}(\gamma,q). \end{aligned}$$

□

Theorem 2.5. *The $M_s(\gamma,q)$ polynomials have the following integral property.*

$$\int_0^1 M_s(\gamma,q)\, d_q(\gamma) = -\frac{M_{s+1}(1,q) - M_{s+1}(q)}{[s+1]_q}.$$

Proof. If we apply (4) to (16), we obtain

$$\begin{aligned} \int_0^1 M_s(\gamma,q)\, d_q(\gamma) =& \int_0^1 \sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u} M_u(q)\gamma^{s-u} d_q(\gamma) \\ =& \sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u} M_u(q)\int_0^1 \gamma^{s-u} d_q(\gamma) \\ =& \sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u} M_u(q)\frac{1}{[s-u+1]_q} \\ =& \frac{-1}{[s+1]_q}\sum_{u=0}^{s}\binom{s+1}{u}_q (-1)^{s+1-u} M_u(q) \end{aligned}$$

From (16), we have

$$M_s(1,q) = \sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u} M_u(q)$$

and

$$M_s(0,q) = M_s(q).$$

Thus

$$\begin{aligned} \int_0^1 M_s(\gamma,q)\, d_q(\gamma) =& \frac{-1}{[s+1]_q}\left(\sum_{u=0}^{s+1}\binom{s+1}{u}_q (-1)^{s+1-u} M_u(q) - M_{s+1}(0,q)\right) \\ =& \frac{-(M_{s+1}(1,q) - M_{s+1}(q))}{[s+1]_q}. \end{aligned}$$

□

Theorem 2.6. *The $M_s(\gamma,q)$ polynomials have the following property.*

$$M_{s+1}(\gamma,q) + \gamma M_s(\gamma,q) = \sum_{u=0}^{s} q^{s-u-1}\binom{s}{u}_q (-1)^{s-u} M_{u+1}(q)\gamma^{s-u}.$$

Proof. From (16), we have

$$\begin{aligned}
M_{s+1}(\gamma,q) &= \sum_{u=0}^{s+1}\binom{s+1}{u}_q (-1)^{s+1-u} M_u(q)\gamma^{s+1-u} \\
& \sum_{u=0}^{s+1}\left(\binom{s}{u}_q + q^{s-u}\binom{s}{u-1}_q\right)(-1)^{s+1-u} M_u(q)\gamma^{s+1-u} \\
&= \sum_{u=0}^{s+1}\binom{s}{u}_q (-1)^{s+1-u} M_u(q)\gamma^{s+1-u} \\
&\quad + \sum_{u=0}^{s+1} q^{s-u}\binom{s}{u-1}_q (-1)^{s+1-u} M_u(q)\gamma^{s+1-u} \\
&= -\gamma\sum_{u=0}^{s+1}\binom{s}{u}_q (-1)^{s-u} M_u(q)\gamma^{s-u} \\
&\quad + \sum_{u=1}^{s+1} q^{s-u}\binom{s}{u-1}_q (-1)^{s+1-u} M_u(q)\gamma^{s+1-u} \\
&= -\gamma\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u} M_u(q)\gamma^{s-u} \\
&\quad + \sum_{u=0}^{s} q^{s-u-1}\binom{s}{u}_q (-1)^{s-u} M_{u+1}(q)\gamma^{s-u} \\
&= -\gamma M_s(\gamma,q) + \sum_{u=0}^{s} q^{s-u-1}\binom{s}{u}_q (-1)^{s-u} M_{u+1}(q)\gamma^{s-u}.
\end{aligned}$$

□

Now, we define $q-U-$Euler polynomials $A_s(\gamma,q)$ and study some of their feature relations.

Definition 2.2. Let n be non-negative integer. The $q-U-$Euler polynomials $A_s(\gamma,q)$ are defined by means of the following generating function

$$\left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1}\right)e_q(-\gamma z/2) = \sum_{s=0}^{\infty} A_s(\gamma,q)\frac{z^s}{[s]_q!}. \tag{19}$$

For $\gamma = 0$ in (19) the $q-U-$Euler numbers $A_s(q)$ are defined by the generating function

$$\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1} = \sum_{s=0}^{\infty} A_s(q)\frac{z^s}{[s]_q!}. \tag{20}$$

Moreover, we see that

$$\lim_{q\to 1} A_s(\gamma,q) = A_s(\gamma)$$

and

$$\lim_{q\to 1} A_s(q) = A_s,$$

where the $A_s(\gamma)$ is s-th $U-$Euler polynomial and A_s is s-th $U-$Euler number.

Theorem 2.7. *Let $B_s(q)$ be s-th q−Bernoulli numbers and $A_s(q)$ be s-th $q-U-$Euler numbers,*

$$A_s(q) = \sum_{u=0}^{s}\left(\frac{1}{2}\right)^u \binom{s}{u}_q C_u(q) B_{s-u}(q), \tag{21}$$

where

$$C_u(q) = \sum_{i=0}^{u}\frac{1}{[i+1]_q}\binom{u}{i}_q.$$

Proof. Let's prove this using (20)

$$\begin{aligned}
\sum_{s=0}^{\infty} A_s(q)\frac{z^s}{[s]_q!} &= \frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1} \\
&= \frac{2}{z}\sum_{s=0}^{\infty} B_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(\frac{1}{2}\right)^s\frac{z^s}{[s]_q!}\sum_{s=1}^{\infty}\left(\frac{1}{2}\right)^s\frac{z^s}{[s]_q!} \\
&= \sum_{s=0}^{\infty} B_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(\frac{1}{2}\right)^s\frac{z^s}{[s]_q!}\sum_{s=1}^{\infty}\left(\frac{1}{2}\right)^{s-1}\frac{z^{s-1}}{[s]_q!} \\
&= \sum_{s=0}^{\infty} B_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(\frac{1}{2}\right)^s\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(\frac{1}{2}\right)^s\frac{z^s}{[s+1]_q!} \\
&= \sum_{s=0}^{\infty} B_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\frac{1}{[u+1]_q}\binom{s}{u}_q\right)\left(\frac{1}{2}\right)^s\frac{z^s}{[s]_q!} \\
&= \sum_{s=0}^{\infty} B_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty} C_s(q)\left(\frac{1}{2}\right)^s\frac{z^s}{[s]_q!} \\
&= \sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\left(\frac{1}{2}\right)^u\binom{s}{u}_q C_u(q)B_{s-u}(q)\right)\frac{z^s}{[s]_q!}.
\end{aligned}$$

Thus the proof is completed. □

Example 2.3. First few the $A_s(q)$ numbers are as follows:

$$\begin{aligned}
A_s(q) &= 1, \\
A_1(q) &= \frac{1}{2[2]_q}, \\
A_2(q) &= -\frac{[2]_q}{6}+\frac{1}{4[3]_q}+\frac{1}{4} \\
A_3(q) &= \frac{[3]_q\left([2]_q-5\right)}{24}+\frac{[5]_q+q^2+1}{8[4]_q}+\frac{1}{8}, \\
A_4(q) &= -\frac{[4]_q!}{720}+\frac{[4]_q\left(2[3]_q-5\right)}{48}-\frac{[4]_q\left([3]_q-2\right)}{16[2]_q}+\frac{1}{16[5]_q}+\frac{1}{16}.
\end{aligned}$$

Theorem 2.8. *For every $s,k\in\mathbb{N}$,*

$$A_s(\gamma,q)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(q)\gamma^{s-u}. \tag{22}$$

Proof. By virtue of (20), we get

$$\begin{aligned}
\sum_{s=0}^{\infty} A_s(\gamma,q)\frac{z^s}{[s]_q!} &= \left(\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\right)e_q(-\gamma z/2) \\
&= \sum_{s=0}^{\infty} A_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^s\gamma^s\frac{z^s}{[s]_q!} \\
&= \sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(q)\gamma^{s-u}\right)\frac{z^s}{[s]_q!}.
\end{aligned}$$

□

Some of the $A_s(\gamma,q)$ polynomials are as follows:

$$\begin{aligned}
A_0(\gamma,q) =& 1,\\
A_1(\gamma,q) =& -\frac{1}{2}\gamma + \frac{1}{2[2]_q},\\
A_2(\gamma,q) =& \frac{1}{4}\gamma^2 - \frac{1}{4}\gamma - \frac{[2]_q}{6} + \frac{1}{4[3]_q} + \frac{1}{4},\\
A_3(\gamma,q) =& -\frac{1}{8}\gamma^3 + \frac{[3]_q}{8[2]_q}\gamma^2 - \left(-\frac{[3]_q[2]_q}{12} + \frac{[3]_q+1}{8}\right)\gamma\\
&+ \frac{[3]_q([2]_q-5)}{24} + \frac{[5]_q+q^2+1}{8[4]_q} + \frac{1}{8},\\
A_4(\gamma,q) =& \frac{1}{16}\gamma^4 - \frac{[4]_q}{16[2]_q}\gamma^3 + \left(-\frac{[4]_q[3]_q}{24} + \frac{[4]_q[3]_q+[4]_q}{16[2]_q}\right)\gamma^2\\
&- \left(\frac{[4]_q[3]_q([2]_q-5)}{48} + \frac{[5]_q+[4]_q+q^2+1}{16}\right)\gamma\\
&- \frac{[4]_q!}{720} + \frac{[4]_q(2[3]_q-5)}{48} - \frac{[4]_q([3]_q-2)}{16[2]_q} + \frac{1}{16[5]_q} + \frac{1}{16}.
\end{aligned}$$

Theorem 2.9. *The q−derivative of $A_s(\gamma,q)$ polynomials is*

$$D_{q,\gamma}A_s(\gamma,q) = -\frac{1}{2}[s]_q A_{s-1}(\gamma,q).$$

Proof. We obtain

$$\begin{aligned}
D_{q,\gamma}A_s(\gamma,q) =& D_{q,\gamma}\left(\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(q)\gamma^{s-u}\right)\\
=& \sum_{u=0}^{s-1}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(q)[s-u]_q\gamma^{s-1-u}\\
=& -\frac{1}{2}[s]_q\sum_{u=0}^{s-1}\binom{s-1}{u}_q\left(-\frac{1}{2}\right)^{s-1-u}A_u(q)[s-u]_q\gamma^{s-1-u}\\
=& -\frac{1}{2}[s]_q A_{s-1}(\gamma,q).
\end{aligned}$$

□

Theorem 2.10. *For q−commuting variables γ and β, $s\in\mathbb{N}_0$ and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$*

$$A_s(\gamma+\beta,q) = \sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(\gamma,q)\beta^{s-u}.$$

Proof. We have

$$\begin{aligned}
\sum_{u=0}^{\infty}A_s(\gamma+\beta,q)\frac{z^s}{[s]_q!} =& \frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}e_q(-z(\gamma+\beta)/2)\\
=& \frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}e_q(-\gamma z/2)e_q(-\beta z/2)\\
=& e_q(-\beta z/2)\sum_{s=0}^{\infty}A_s(\gamma,q)\frac{z^s}{[s]_q!}
\end{aligned}$$

and from (1) we write

$$e_q(-\beta z/2) = \sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^s\beta^s\frac{z^s}{[s]_q!}.$$

On the other hand,

$$\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^s\beta^s\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}A_s(\gamma,q)\frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty}\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(\gamma,q)\beta^{s-u}\frac{z^s}{[s]_q!}. \tag{23}$$

Thus, the proof is obtained from the equality of (22) and (23). □

Theorem 2.11. *The $A_s(\gamma,q)$ polynomials have the following integral property.*

$$\int_0^1 A_s(\gamma,q)\,d_q(\gamma) = -\frac{2\left(A_{s+1}(1,q)-A_{s+1}(q)\right)}{[s+1]_q}.$$

Proof. If we apply (4) to (21), we obtain

$$\begin{aligned}\int_0^1 A_s(\gamma,q)\,d_q(\gamma) &= \int_0^1 \sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(q)\gamma^{s-u}d_q(\gamma)\\ &=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(q)\int_0^1\gamma^{s-u}d_q(\gamma)\\ &=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(q)\frac{1}{[s-u+1]_q}\\ &=\frac{-1}{[s+1]_q}\sum_{u=0}^{s}\binom{s+1}{u}_q\left(-\frac{1}{2}\right)^{s+1-u}A_u(q)\\ &=\frac{-1}{[s+1]_q}\left(\sum_{u=0}^{s+1}\binom{s+1}{u}_q\left(-\frac{1}{2}\right)^{s+1-u}A_u(q)-A_{s+1}(q)\right).\end{aligned}$$

From (22), we have

$$A_s(1,q)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}A_u(q)$$

and

$$A_s(0,q)=A_s(q).$$

Thus

$$\begin{aligned}\int_0^1 A_s(\gamma,q)\,d_q(\gamma) &=\frac{-1}{[s+1]_q}\left(\sum_{u=0}^{s+1}\binom{s+1}{u}_q\left(-\frac{1}{2}\right)^{s+1-u}A_u(q)-A_{s+1}(q)\right)\\ &=-\frac{2\left(A_{s+1}(1,q)-A_{s+1}(q)\right)}{[s+1]_q}.\end{aligned}$$

The proof is complete. □

3 Some connection formulas for the polynomials $M_s(\gamma,q)$

Next, we explore the $q-U-$Bernoulli polynomials $M_n(\gamma,q)$, the Rogers-Szegö polynomials $H_n(\gamma,q)$, and the q-Bernoulli numbers $B_n(q)$, examining how they are related through various connection formulas.

Theorem 3.1. *The $q-U-$Bernoulli polynomials $M_s(\gamma,q)$ are related with the Rogers-Szegö polynomials $H_s(\gamma,q)$ and the $q-$Bernoulli numbers $B_s(q)$ by the means of the following identities*

$$M_s(\gamma,q)=-\sum_{u=0}^{s}\binom{s}{u}_q B_{s-u}(q)H_u(-\gamma,q). \tag{24}$$

$$H_s(-\gamma,q)=-\sum_{u=0}^{s}\frac{1}{[s+1]_q}\binom{s+1}{u}M_u(\gamma,q). \tag{25}$$

Proof. (24) Using Definition (13), (5) and the $q-$Bernoulli numbers (12) with $\gamma=0$, we have

$$\begin{aligned}\sum_{s=0}^{\infty}M_s(\gamma,q)\frac{z^s}{[s]_q!} &=\frac{-ze_q(z)e_q(-\gamma z)}{e_q(z)-1}\\ &=-\sum_{s=0}^{\infty}B_s(q)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}H_s(-\gamma,q)\frac{z^s}{[s]_q!}\\ &=-\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q B_{s-u}(q)H_u(-\gamma,q)\right)\frac{z^s}{[s]_q!}.\end{aligned}$$

□

Proof. (25) We have

$$
\begin{aligned}
-\frac{e_q(z)-1}{z}\sum_{s=0}^{\infty} M_s(\gamma,q)\frac{z^s}{[s]_q!} &= e_q(z)e_q(-\gamma z)\\
-\sum_{s=0}^{\infty}\frac{z^s}{[s+1]_q!}\sum_{s=0}^{\infty} M_s(\gamma,q)\frac{z^s}{[s]_q!} &= \sum_{s=0}^{\infty} H_s(-\gamma,q)\frac{z^s}{[s]_q!}\\
-\sum_{s=0}^{\infty}\sum_{u=0}^{s}\frac{1}{[s+1]_q}\binom{s+1}{u}_q M_u(\gamma,q)\frac{z^s}{[s]_q!} &= \sum_{s=0}^{\infty} H_s(-\gamma,q)\frac{z^s}{[s]_q!}.
\end{aligned}
$$

Finally, equating the coefficients of the powers of z on both sides, we get (25). □

Theorem 3.2. *The $q-U-$Bernoulli polynomials $M_s(\gamma,q)$ are related with the $q-$Fubini polynomials $F_{s,q}(\gamma;w)$ by the means of the following identity*

$$
M_s(\gamma,q)=\sum_{u=0}^{s}\binom{s}{u}_q M_{s-u}(q)\left[(1+\omega)F_{u,q}(-\gamma;\omega)-\omega\sum_{k=0}^{u}\binom{u}{k}_q F_{k,q}(-\gamma;\omega)\right]. \tag{26}
$$

Proof. We have

$$
\begin{aligned}
\sum_{s=0}^{\infty} M_s(\gamma,q)\frac{z^s}{[s]_q!} &= \frac{-ze_q(z)\cdot e_q(-\gamma z)}{e_q(z)-1}\cdot\frac{1-\omega\left(e_q(z)-1\right)}{1-\omega\left(e_q(z)-1\right)}\\
&= \frac{-ze_q(z)}{e_q(z)-1}\cdot\left[\frac{e_q(-\gamma z)}{1-\omega\left(e_q(z)-1\right)}-\frac{e_q(-\gamma z)\cdot e_q(z)\cdot\omega}{1-\omega\left(e_q(z)-1\right)}+\frac{\omega e_q(-\gamma z)}{1-\omega\left(e_q(z)-1\right)}\right]
\end{aligned}
$$

using the equations (15) and (9), we obtain

$$
\begin{aligned}
\sum_{s=0}^{\infty} M_s(\gamma,q)\frac{z^s}{[s]_q!} &= \sum_{s=0}^{\infty} M_s(q)\frac{z^s}{[s]!}\\
&\quad\times\left[(1+\omega)\sum_{s=0}^{\infty}F_{s,q}(-\gamma;\omega)\frac{z^s}{[s]_q!}-\omega\sum_{s=0}^{\infty}F_{s,q}(-\gamma;\omega)\frac{z^s}{[s]_q!}\cdot\sum_{s=0}^{\infty}\frac{z^s}{[s]_q!}\right]\\
&= \sum_{s=0}^{\infty} M_s(q)\frac{z^s}{[s]!}\sum_{s=0}^{\infty}\left[(1+\omega)F_{s,q}(-\gamma;\omega)-\omega\sum_{k=0}^{s}\binom{s}{k}_q F_{k,q}(-\gamma;\omega)\right]\frac{z^s}{[s]!}\\
&= \sum_{s=0}^{\infty}\sum_{u=0}^{s}\binom{s}{u}_q M_{s-u}(q)\\
&\quad\times\left[(1+\omega)F_{u,q}(-\gamma;\omega)-\omega\sum_{k=0}^{u}\binom{u}{k}_q F_{k,q}(-\gamma;\omega)\right]\frac{z^s}{[s]!}.
\end{aligned}
$$

Equating the coefficients of $\frac{z^s}{[s]!}$, we derive asserted result. □

Theorem 3.3. *Each of the following identities holds true*

$$
M_s(\gamma,q)=\sum_{i=0}^{s}\left(\sum_{u=i}^{s}\binom{s}{u}_q\binom{u}{i}_q(-1)^{2u-i}q^{\binom{u-i}{2}}M_{s-u}(q)\right)H_i(\gamma,q).
$$

$$
M_s(\gamma;q)=\sum_{j=0}^{s}\left(\sum_{k=j}^{s}\binom{s}{k}_q\binom{k}{j}_q(-1)^k M_{s-k}(q)\sum_{u=0}^{k-1}\binom{k-j}{u}_q a^u\right)U_j^{(a)}(\gamma;q)
$$

Proof. The proof of this theorem uses (16), (7), (8) and the summation formula

$$
\sum_{u=0}^{s}A_u\sum_{i=0}^{u}B_u=\sum_{i=0}^{s}\left(\sum_{u=i}^{s}A_u\right)B_i.
$$

□

Now, we combine of the Rogers-Szegö polynomials $H_s(\gamma,q)$ and the Al-Salam Carlitz polynomials $U_s^{(a)}(\gamma,q)$ with q—Bernoulli polynomials $B_s(\gamma,q)$ to considerto the q—Rogers-Szegö-Bernoulli polynomials ${}_HB_s(\gamma,q)$ and q—Al-Salam Carlitz—Bernoulli polynomials ${}_UB_s^{(a)}(\gamma,q)$ and establishes their series representation.

By employing expansion (1) in equation (12) and then replacing powers of $\gamma^0,\gamma^1,\gamma^2,\ldots,\gamma^s$ of γ with the correlating polynomials $H_0(\gamma,q),H_1(\gamma,q),\ldots,H_s(\gamma,q)$ of Rogers-Szegö polynomials $H_s(\gamma,q)$, and summing up the terms in the resulting equation and denoting the resulting on the right aspect as ${}_HB_s(\gamma,q)$, the following generating equation of q—Rogers-Szegö-Bernoulli polynomials is given:

$$\sum_{s=0}^{\infty} {}_HB_s(\gamma,q)\frac{z^s}{[s]_q!}=\frac{ze_q(\gamma z)}{e_q(z)-1}e_q(z),\ \gamma\in\mathbb{R}. \tag{27}$$

Theorem 3.4. *For the ${}_HB_s(\gamma,q)$, the following series representation holds true:*

$$ {}_HB_s(\gamma,q)=\sum_{u=0}^{s}\binom{s}{u}_q B_{s-u}(\gamma,q). \tag{28}$$

Proof. Using generating function (12) and expansion (1) in equation (27), we have

$$\sum_{s=0}^{\infty} {}_HB_s(\gamma,q)\frac{z^s}{[s]_q!}=\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}B_s(\gamma,q)\frac{z^{s+u}}{[s]_q!\,[u]_q!},$$

which on simplification becomes

$$\sum_{s=0}^{\infty} {}_HB_s(\gamma,q)\frac{z^s}{[s]_q!}=\sum_{s=0}^{\infty}\sum_{u=0}^{s}B_s(\gamma,q)\frac{z^{s}}{[s-u]_q!\,[u]_q!}.$$

Equating the coefficients of like powers of z on both sides of the previous formula, we obtain assertion (28). □

Similarly, employing expansion (1) in equation (12) and then replacing powers of $\gamma^0,\gamma^1,\gamma^2,\ldots,\gamma^s$ of γ with the correlating polynomials $U_0^{(a)}(\gamma,q),U_1^{(a)}(\gamma,q),\ldots,U_s^{(a)}(\gamma,q)$ of q—Al-Salam Carlitz polinomials $U_s^{(a)}(\gamma,q)$, and summing up the terms in the resulting equation and denoting the resultant on the r.h.s. as ${}_UB_s^{(a)}(\gamma,q)$, the following generating equation of q—Al-Salam Carlitz—Bernoulli polynomials is obtained:

$$\frac{z}{e_q(z)-1}\frac{e_q(\gamma z)}{e_q(z)e_q(az)}=\sum_{s=0}^{\infty} {}_UB_s^{(a)}(\gamma,q)\frac{z^s}{[s]_q!},\ \gamma\in\mathbb{R}. \tag{29}$$

Theorem 3.5. *For the polynomials ${}_UB_s^{(a)}(\gamma,q)$, the following series representation is valid:*

$$ {}_UB_s^{(a)}(\gamma,q)=\sum_{u=0}^{s}\binom{s}{u}_q B_u(q)U_{s-u}^{(a)}(\gamma,q).$$

Proof. Using generating function (6) and expansion for —Bernoulli numbers in equation (29), we have

$$\sum_{s=0}^{\infty} {}_UB_s^{(a)}(\gamma,q)\frac{z^s}{[s]_q!}=\sum_{s=0}^{\infty}\sum_{u=0}^{\infty}B_u(q)U_s^{(a)}(\gamma,q)\frac{z^{s+u}}{[s]_q!\,[u]_q!},$$

which on simplification becomes

$$\sum_{s=0}^{\infty} {}_HB_s(\gamma,q)\frac{z^s}{[s]_q!}=\sum_{s=0}^{\infty}\sum_{u=0}^{s}B_u(q)U_{s-u}^{(a)}(\gamma,q)\frac{z^{s}}{[s-u]_q!\,[u]_q!}.$$

Equating the coefficients of same powers of z on each side of the previous formula, we obtain assertion (28). □

4 The Generalized $q-U-$Bernoulli/Euler Polynomials and their properties

In this section, we derive some properties for the $q-$extensions $U-$Bernoulli and $U-$Euler polynomials like generating functions, $q-$partial derivatives and summation relations.

Definition 4.1. Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$. The generalized $q-U-$Bernoulli numbers $\mathfrak{M}_{s,q}^{(\alpha)}$ and the generalized $q-U-$Bernoulli polynomials $\mathfrak{M}_{s,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:

$$\left(\frac{-ze_q(z)}{e_q(z)-1}\right)^r = \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!}$$

and

$$\sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(\gamma,\beta) \frac{z^s}{[s]_q!} = \left(\frac{-ze_q(z)}{e_q(z)-1}\right)^r e_q(-\gamma z)E_q(-\beta z). \tag{30}$$

It is obvious that $\mathfrak{M}_{s,q}^{(r)}(0,0)$=$\mathfrak{M}_{s,q}^{(r)}$.

Definition 4.2. Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$. The generalized $q-U-$Bernoulli numbers $\mathfrak{A}_{s,q}^{(\alpha)}$ and the generalized $q-U-$Bernoulli polynomials $\mathfrak{A}_{s,q}^{(\alpha)}(x,y)$ in x, y of order α are defined by means of the generating functions:

$$\left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1}\right)^r = \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)} \frac{z^s}{[s]_q!},$$

$$\sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)}(\gamma,\beta) \frac{z^s}{[s]_q!} = \left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1}\right)^r e_q(-\gamma z/2)E_q(-\beta z/2), \tag{31}$$

It is obvious that $\mathfrak{A}_{s,q}^{(r)}(0,0)$=$\mathfrak{A}_{s,q}^{(r)}$.

Theorem 4.1. *The* $\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)$ *polynomials satisfy the following property*

$$\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta) = \sum_{u=0}^{s} \binom{s}{u}_q \mathfrak{M}_{s-u,q} \mathfrak{M}_{u,q}^{(r-1)}(\gamma,\beta).$$

Proof. If we use (30), we obtain

$$\begin{aligned}
\sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(\gamma,\beta) \frac{z^s}{[s]_q!} &= \left(\frac{-ze_q(z)}{e_q(z)-1}\right)^r e_q(-\gamma z)E_q(-\beta z) \\
&= \frac{-ze_q(z)}{e_q(z)-1}\left(\frac{-ze_q(z)}{e_q(z)-1}\right)^{r-1} e_q(-\gamma z)E_q(-\beta z) \\
&= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r-1)}(\gamma,\beta) \frac{z^s}{[s]_q!} \\
&= \sum_{s=0}^{\infty} \left(\sum_{u=0}^{s} \binom{s}{u}_q \mathfrak{M}_{s-u,q} \mathfrak{M}_{u,q}^{(r-1)}(\gamma,\beta)\right) \frac{z^s}{[s]_q!}.
\end{aligned}$$

□

Theorem 4.2. *The* $\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)$ *polynomials have the following relation*

$$\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta) = \sum_{u=0}^{s} \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r)} (\gamma+\beta)_q^{s-u}.$$

Proof. From (30), we have

$$\begin{aligned}
\sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(\gamma,\beta) \frac{z^s}{[s]_q!} &= \left(\frac{-ze_q(z)}{e_q(z)-1}\right)^r e_q(-\gamma z)E_q(-\beta z) \\
&= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (-1)^s \gamma^s \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (-1)^s q^{\frac{s(s-1)}{2}} \beta^s \frac{z^s}{[s]_q!} \\
&= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(\sum_{u=0}^{s} \binom{s}{u}_q q^{\frac{u(u-1)}{2}} \gamma^{s-u} \beta^u\right) \frac{(-1)^s z^s}{[s]_q!} \\
&= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (\gamma+\beta)_q^s \frac{(-1)^s z^s}{[s]_q!} \\
&= \sum_{s=0}^{\infty} \left(\sum_{u=0}^{s} \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r)} (\gamma+\beta)_q^{s-u}\right) \frac{z^s}{[s]_q!}.
\end{aligned}$$

□

From (30), respectively, we easily obtain

$$\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q \mathfrak{M}_{s-u,q}(x,0)\,\mathfrak{M}_{u,q}^{(r-1)}(0,\beta),$$

$$\mathfrak{M}_{s,q}^{(r-\beta)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u}\,\mathfrak{M}_{u,q}^{(r-\beta)}(\gamma+\beta)_q^{s-u},$$

and

$$\mathfrak{M}_{s,q}^{(r-\beta)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q \mathfrak{M}_{u,q}^{(r)}(x,0)\,\mathfrak{M}_{s-u,q}^{(-\beta)}(0,\beta).$$

Proposition 4.3. *We have the following features*

$$\mathfrak{M}_{s,q}^{(r)}(\gamma,0)=\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u}\,\mathfrak{M}_{u,q}^{(r)}\gamma^{s-u} \tag{32}$$

and

$$\mathfrak{M}_{s,q}^{(r)}(0,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u}\,q^{\frac{(s-u)(s-u-1)}{2}}\,\mathfrak{M}_{u,q}^{(r)}\beta^{s-u}. \tag{33}$$

Proof. Firstly let's prove the equation (32)

$$\begin{aligned}\sum_{s=0}^{\infty}\mathfrak{M}_{s,q}^{(r)}(x,0)\frac{z^s}{[s]_q!}&=\left(\frac{-ze_q(z)}{e_q(z)-1}\right)^r e_q(-\gamma z)\\&=\sum_{s=0}^{\infty}\mathfrak{M}_{s,q}^{(r)}\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}(-1)^s\gamma^s\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q(-1)^{s-u}\,\mathfrak{M}_{u,q}^{(r)}\gamma^{s-u}\right)\frac{z^s}{[s]_q!}.\end{aligned}$$

Now let's show that equation (33) is true.

$$\begin{aligned}\sum_{s=0}^{\infty}\mathfrak{M}_{s,q}^{(r)}(0,\beta)\frac{z^s}{[s]_q!}&=\left(\frac{-ze_q(z)}{e_q(z)-1}\right)^r E_q(-\beta z)\\&=\sum_{s=0}^{\infty}\mathfrak{M}_{s,q}^{(r)}\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}(-1)^s q^{\frac{s(s-1)}{2}}\beta^s\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q(-1)^{s-u}q^{\frac{(s-u)(s-u-1)}{2}}\,\mathfrak{M}_{u,q}^{(r)}\beta^{s-u}\right)\frac{z^s}{[s]_q!}.\end{aligned}$$

□

Proposition 4.4. *From (30), we have*

$$\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u}\,q^{\frac{(s-u)(s-u-1)}{2}}\,\mathfrak{M}_{u,q}^{(r)}(x,0)\,\beta^{s-u} \tag{34}$$

and

$$\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u}\,\mathfrak{M}_{u,q}^{(r)}(0,\beta)\,\gamma^{s-u}. \tag{35}$$

From (34) and (35), we easily obtain the following property.

Proposition 4.5.

$$\mathfrak{M}_{s,q}^{(r)}(\gamma,1)=\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u}\,q^{\frac{(s-u)(s-u-1)}{2}}\,\mathfrak{M}_{u,q}^{(r)}(\gamma,0)$$

and

$$\mathfrak{M}_{s,q}^{(r)}(1,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q (-1)^{s-u}\,\mathfrak{M}_{u,q}^{(r)}(0,\beta).$$

Theorem 4.6. *The* $\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)$ *polynomials have the following property:*

$$\sum_{u=0}^{s}\binom{s}{u}_q \mathfrak{M}_{u,q}^{(r)}(\gamma,\beta)\,\mathfrak{M}_{s-u,q}^{(-r)} = (-1)^s(\gamma+\beta)_q^s .$$

Proof. From the Cauchy product rule, we have

$$\sum_{s=0}^{\infty}\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\mathfrak{M}_{s,q}^{(-r)}\frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q \mathfrak{M}_{u,q}^{(r)}(\gamma,\beta)\,\mathfrak{M}_{s-u,q}^{(-r)}\right)\frac{z^s}{[s]_q!} \tag{36}$$

On the other hand, we have

$$\begin{aligned}
\sum_{s=0}^{\infty}\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\mathfrak{M}_{s,q}^{(-r)}\frac{z^s}{[s]_q!} &= \left(\frac{-ze_q(z)}{e_q(z)-1}\right)^r e_q(-\gamma z)E_q(-\beta z)\left(\frac{-ze_q(z)}{e_q(z)-1}\right)^{-r}\\
&= \sum_{s=0}^{\infty}(-1)^s\gamma^s\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}(-1)^s q^{\frac{s(s-1)}{2}}\beta^s\frac{z^s}{[s]_q!}\\
&= \sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q q^{\frac{k(k-1)}{2}}\gamma^{s-u}\beta^u\right)(-1)^s\frac{z^s}{[s]_q!}\\
&= \sum_{s=0}^{\infty}(-1)^s(\gamma+\beta)_q^s\frac{z^s}{[s]_q!}.
\end{aligned}$$

From (36) and the last equality, we get the desired result. □

Theorem 4.7. *The* $\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)$ *polynomials satisfy the following property*

$$\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta) = \sum_{u=0}^{s}\binom{s}{u}_q \mathfrak{A}_{s-u,q}\mathfrak{A}_{u,q}^{(r-1)}(\gamma,\beta).$$

Proof. If we use (31), we obtain

$$\begin{aligned}
\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!} &= \left(\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\right)^r e_q(-\gamma z/2)E_q(-\beta z/2)\\
&= \frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\\
&\quad\times\left(\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\right)^r e_q(-\gamma z/2)E_q(-\beta z/2)\\
&= \sum_{s=0}^{\infty}\mathfrak{A}_{s,q}\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r-1)}(\gamma,\beta)\frac{z^s}{[s]_q!}\\
&= \sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q \mathfrak{A}_{s-u,q}\mathfrak{A}_{u,q}^{(r-1)}(\gamma,\beta)\right)\frac{z^s}{[s]_q!}.
\end{aligned}$$

□

Theorem 4.8. *The* $\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)$ *polynomials satisfy the following relation*

$$\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta) = \sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}\mathfrak{A}_{u,q}^{(r)}(\gamma+\beta)_q^{s-u}.$$

Proof. From (4.2), we obtain

$$\begin{aligned}\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}&=\left(\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\right)^r e_q(-\gamma z/2)E_q(-\beta z/2)\\&=\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^s\gamma^s\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^s q^{\frac{s(s-1)}{2}}\beta^s\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q q^{\frac{k(k-1)}{2}}\gamma^{s-u}\beta^u\right)\left(-\frac{1}{2}\right)^s\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^s(\gamma+\beta)_q^s\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}\mathfrak{A}_{u,q}^{(r)}(\gamma+\beta)_q^{s-u}\right)\frac{z^s}{[s]_q!}.\end{aligned}$$

□

From (4.2), we easily obtain the following property

$$\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q\mathfrak{A}_{s-u,q}(x,0)\mathfrak{A}_{u,q}^{(r-1)}(0,\beta),$$

$$\mathfrak{A}_{s,q}^{(r-\beta)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}\mathfrak{A}_{u,q}^{(r-\beta)}(\gamma+\beta)_q^{s-u}$$

and

$$\mathfrak{A}_{s,q}^{(r-\beta)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q\mathfrak{A}_{u,q}^{(r)}(x,0)\mathfrak{A}_{s-u,q}^{(-\beta)}(0,\beta).$$

Proposition 4.9.

$$\mathfrak{A}_{s,q}^{(r)}(x,0)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}\mathfrak{A}_{u,q}^{(r)}\gamma^{s-u}\tag{37}$$

and

$$\mathfrak{A}_{s,q}^{(r)}(0,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}q^{\frac{(s-u)(s-u-1)}{2}}\mathfrak{A}_{u,q}^{(r)}\beta^{s-u}.\tag{38}$$

Proof. We have

$$\begin{aligned}\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}(x,0)\frac{z^s}{[s]_q!}&=\left(\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\right)^r e_q(-\gamma z/2)\\&=\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^s\gamma^s\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}\mathfrak{A}_{u,q}^{(r)}\gamma^{s-u}\right)\frac{z^s}{[s]_q!}.\end{aligned}$$

and

$$\begin{aligned}\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}(0,\beta)\frac{z^s}{[s]_q!}&=\left(\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\right)^r E_q(-\beta z/2)\\&=\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^s q^{\frac{s(s-1)}{2}}\beta^s\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}q^{\frac{(s-u)(s-u-1)}{2}}\mathfrak{A}_{u,q}^{(r)}\beta^{s-u}\right)\frac{z^s}{[s]_q!}.\end{aligned}$$

Thus, the validity of equations (37) and (38) is clearly evident. □

Proposition 4.10. *From (4.2), we have*

$$\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}q^{\frac{(s-u)(s-u-1)}{2}}\mathfrak{A}_{u,q}^{(r)}(x,0)\beta^{s-u}\tag{39}$$

and

$$\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}\mathfrak{A}_{u,q}^{(r)}(0,\beta)\gamma^{s-u}.\tag{40}$$

From (39) and (40), we easily obtain the following property.

Proposition 4.11.

$$\mathfrak{A}_{s,q}^{(r)}(\gamma,1)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}q^{\frac{(s-u)(s-u-1)}{2}}\mathfrak{A}_{u,q}^{(r)}(x,0)$$

and

$$\mathfrak{A}_{s,q}^{(r)}(1,\beta)=\sum_{u=0}^{s}\binom{s}{u}_q\left(-\frac{1}{2}\right)^{s-u}\mathfrak{A}_{u,q}^{(r)}(0,\beta).$$

Theorem 4.12. *The $\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)$ polynomials satisfy the following relation:*

$$\sum_{u=0}^{s}\binom{s}{u}_q\mathfrak{A}_{u,q}^{(r)}(\gamma,\beta)\mathfrak{A}_{s-u,q}^{(-r)}=\left(-\frac{1}{2}\right)^{s}(\gamma+\beta)_q^s.$$

Proof. We have

$$\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(-r)}\frac{z^s}{[s]_q!}=\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q\mathfrak{A}_{u,q}^{(r)}(\gamma,\beta)\mathfrak{A}_{s-u,q}^{(-r)}\right)\frac{z^s}{[s]_q!}$$

On the other hand, we obtain

$$\begin{aligned}\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\mathfrak{A}_{s,q}^{(-r)}\frac{z^s}{[s]_q!}&=\left(\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\right)^{r}e_q(-\gamma z/2)E_q(-\beta z/2)\\&\times\left(\frac{2e_q(z/2)\left(e_q(z/2)-1\right)}{e_q(z)-1}\right)^{-r}\\&=\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^{s}\gamma^s\frac{z^s}{[s]_q!}\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^{s}q^{\frac{s(s-1)}{2}}\beta^s\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\left(\sum_{u=0}^{s}\binom{s}{u}_q q^{\frac{k(k-1)}{2}}\gamma^{s-u}\beta^u\right)\left(-\frac{1}{2}\right)^{s}\frac{z^s}{[s]_q!}\\&=\sum_{s=0}^{\infty}\left(-\frac{1}{2}\right)^{s}(\gamma+\beta)_q^s\frac{z^s}{[s]_q!}.\end{aligned}$$

□

The theorem that is employed for demonstrating the $q-$partial derivatives for the $\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)$ and $\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)$:

Theorem 4.13. *For $\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)$ and $\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)$, the subsequent $q-$partial derivatives are valid:*

$$D_{q,\beta}\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)=-[s]_q\mathfrak{M}_{s-1,q}^{(r)}(\gamma,\beta),\qquad s\geq 1,\tag{41}$$

$$D_{q,\beta}\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)=[s]_q\mathfrak{M}_{s-1,q}^{(r)}(\gamma,q\beta),\qquad s\geq 1.\tag{42}$$

$$D_{q,\beta}\mathfrak{A}_{s-1,q}^{(r)}(\gamma,\beta)=\frac{-1}{2}[s]_q\mathfrak{A}_{s-1,q}^{(r)}(\gamma,\beta),\qquad s\geq 1,\tag{43}$$

$$D_{q,\beta}\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)=\frac{-1}{2}[s]_q\mathfrak{A}_{s-1,q}^{(r)}(\gamma,q\beta),\qquad s\geq 1.\tag{44}$$

Proof. We calculate the $q-$partial derivative for every value of the formula (30) in terms of γ and β using equations (2) and (3), respectively, this offers us

$$\sum_{s=0}^{\infty}D_{q,\gamma}\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}=-z\left(\frac{-ze_q(z)}{e_q(z)-1}\right)^{r}e_q(-\gamma z)E_q(-\beta z).$$

$$\sum_{s=0}^{\infty}D_{q,\beta}\mathfrak{M}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}=-z\left(\frac{-ze_q(z)}{e_q(z)-1}\right)^{r}e_q(-\gamma z)E_q(-q\beta z).$$

We calculate the $q-$partial derivative for every value of the formula (31) in terms of γ and β using equations (2) and (3), respectively, this offers us

$$\sum_{s=0}^{\infty}D_{q,\gamma}\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}=\frac{-z}{2}\left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1}\right)^{r}e_q(-\gamma z/2)E_q(-\beta z/2).$$

$$\sum_{s=0}^{\infty} D_{q,\beta}\mathfrak{A}_{s,q}^{(r)}(\gamma,\beta)\frac{z^s}{[s]_q!}=\frac{-z}{2}\left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1}\right)^r e_q(-\gamma z/2)E_q(-q\beta z/2).$$

Then, to produce assertions (43) as well as (44), respectively, we replicate the procedure described in the equation's proofs (41) as well as (42).

Theorem 4.13 has been fully proved. □

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