



q –extensions for the U –Bernoulli and U –Euler polynomials

Semra Kuş^a · William Ramírez^{b,c} · Letelier Castilla^d · Mohammed Fadel^e

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Abstract

This paper defines the q – U –Bernoulli, q – U –Euler polynomials and numbers and obtains some of their feature relations. We also explore the q – U –Bernoulli polynomials $M_s(\gamma, q)$, Rogers-Szegő polynomials $H_s(\gamma, q)$ and the q –Bernoulli numbers $B_s(q)$ are related via some connection formulas. Further, we define various properties for the q –extensions U –Bernoulli and U –Euler polynomials like generating functions, q –partial derivatives and summation relations.

1 Introduction

Quantum calculus, often known as q –calculus is an important study issue in classical mathematical analysis. It aims to generalize differentiation and integration procedures. The use of q –calculus in several disciplines such as mathematics, mechanics, and physics is rapidly growing.

Scientists investigating q –special functions have invented and analyzed several functions, including their qualities. (see examples [1, 6, 9, 15, 20, 32])

Throughout this paper, we will use the following notation: \mathbb{N} for the set of all natural numbers, \mathbb{N}_0 for the set of all non-negative integers, \mathbb{Z} for the set of all integers, \mathbb{R} for the set of all real numbers, and \mathbb{C} for the set of all complex numbers. We briefly review several important definitions and notations from [2, 7, 8, 10] as an introduction to the q –calculus ($q \in \mathbb{C}$ with $0 < |q| < 1$).

In [10], the q –shifted factorial is defined by

$$(a; q)_0 = 1, \quad (a; q)_s = \prod_{j=0}^{s-1} (1 - q^j a), \quad s \in \mathbb{N},$$

$$(a; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j a), \quad |q| < 1, a \in \mathbb{C}.$$

The q –numbers and q –factorial are defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad q \neq 1, a \in \mathbb{C},$$

and

$$[s]_q! = \frac{(q; q)_s}{(1 - q)^s}, \quad q \neq 1, s \in \mathbb{N}, \quad \text{and} \quad [0]_q! = 1,$$

in [14].

The q –binomial coefficient is defined by

$$\binom{s}{u}_q = \frac{(q; q)_s}{(q; q)_{s-u} (q; q)_u} = \frac{[s]_q!}{[s-u]_q! [u]_q!}.$$

^aMucur Vocational High School, Kırşehir Ahi Evran University, 40100, Kırşehir, Turkey, email: skus@ahievran.edu.tr

^bDepartment of Natural and Exact Sciences, Universidad de la Costa, Calle 58, 55-66, 080002, Barranquilla, Colombia

^cSection of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186, Italy, email: wramirez4@cuc.edu.co

^dI.E.E. Normal Superior Nuestra Señora de Fatima, 610101, Sabanagrande, Colombia, email: lecas1977@hotmail.com

^eDepartment of Mathematics, University of Lahej, 73560, Lahej, Yemen, email: mohdfadel80@gmail.com

Some properties of the q -binomial coefficient are as follows:

$$\binom{s+1}{u}_q = q^u \binom{s}{u}_q + \binom{s}{u-1}_q$$

and

$$\binom{s+1}{u}_q = \binom{s}{u}_q + q^{s-u} \binom{s}{u-1}_q.$$

In [14], the Gauss's binomial formula is defined by

$$(\gamma + \beta)_q^s := \sum_{u=0}^s \binom{s}{u}_q q^{\frac{1}{2}u(u-1)} \gamma^{s-u} \beta^u, \quad s \in \mathbb{N}_0.$$

Specially,

$$(\gamma - 1)_q^s = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} q^{\frac{1}{2}(s-u)(s-u-1)} \gamma^u.$$

If $\beta\gamma = q\gamma\beta$, where q is a number commuting with both γ and β , then

$$(\gamma + \beta)_q^s = \sum_{u=0}^s \binom{s}{u}_q \gamma^u \beta^{s-u}$$

in [14].

In the standard approach to the q -calculus, two exponential functions are used

$$e_q(z) = \frac{1}{(1 - z(1 - q))_q^\infty} = \sum_{s=0}^\infty \frac{z^s}{[s]_q!}, \quad |z| < \frac{1}{|1 - q|} \tag{1}$$

$$E_q(z) = (1 + z(1 - q))_q^\infty = \sum_{s=0}^\infty \frac{q^{\frac{1}{2}s(s-1)} z^s}{[s]_q!}, \quad z \in \mathbb{C}.$$

Moreover, we have [12]

$$D_q e_q(z) = e_q(z), \tag{2}$$

$$D_q E_q(z) = E_q(qz), \tag{3}$$

where the q -derivative D_q is defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}, \quad 0 \neq z \in \mathbb{C}.$$

The above q -standard notation can be found in (see, [12]).

In [14], the Jackson integral is as follows

$$\int_a^b f(\gamma) d_q \gamma = \int_0^b f(\gamma) d_q \gamma - \int_0^a f(\gamma) d_q \gamma, \quad 0 < a < b$$

where

$$\int_0^a f(\gamma) d_q \gamma = a(1 - q) \sum_{i=0}^\infty q^i f(aq^i).$$

Specifically, we see that

$$\int_0^1 \gamma^s d_q \gamma = \frac{1}{[s+1]_q}. \tag{4}$$

The Rogers-Szegö polynomials $H_s(\gamma, q)$ and the Al-Salam Carlitz I polynomials $U_s^{(a)}(\gamma, q)$ [34, p. 1183], [16] are defined by the generating functions

$$e_q(z)e_q(\gamma z) = \sum_{s=0}^\infty H_s(\gamma, q) \frac{z^s}{[s]_q!} \tag{5}$$

and

$$\frac{e_q(\gamma z)}{e_q(z)e_q(az)} = \sum_{s=0}^\infty U_s^{(a)}(\gamma, q) \frac{z^s}{[s]_q!} \tag{6}$$

respectively.

Taking $a = -1$ in equation (6), we obtain the following relation

$$U_s^{(-1)}(\gamma, q) = h_s(\gamma, q),$$

where $h_s(\gamma, q)$ denotes the discrete q -Hermite I polynomials (see [17, Eq. 4.46]).

The inversion formulas for the polynomials $H_s(\gamma, q)$ [3, Eq. 1] and $U_s^{(a)}(\gamma, q)$ [34, p. 1180] are defined by

$$\gamma^u = \sum_{j=0}^u (-1)^{u-j} \binom{u}{j}_q q^{\binom{u-j}{2}} H_j(\gamma, q) \tag{7}$$

and

$$\gamma^u = \sum_{j=0}^u \binom{u}{j}_q \left(\sum_{s=0}^{u-j} \binom{u-j}{s}_q a^s \right) U_j^{(a)}(\gamma, q). \tag{8}$$

Taking $y = 0$ in [18, Eq. 2.8] gives the q -Fubini polynomials with two parameters by means of the following generating function

$$\frac{e_q(\gamma z)}{1 - w(e_q(z) - 1)} = \sum_{s=0}^{\infty} F_{s,q}(\gamma; w) \frac{z^s}{[s]_q!}. \tag{9}$$

When $w = 1$ and $\gamma = 0$ in the above equation, we obtain the s -th q -Fubini number $F_{s,q}$.

For $n \in \mathbb{N}_0$, in [19, Eq. (10)], the U -Bernoulli polynomials $M_s(\gamma)$ are defined by the following generating function:

$$\left(\frac{z}{e^{-z} - 1} \right) e^{-\gamma z} = \sum_{s=0}^{\infty} M_s(\gamma) \frac{z^s}{s!}, \quad |z| < 2\pi. \tag{10}$$

Note that, if $\gamma = 0$ in (10), the U -Bernoulli numbers M_s are obtained.

Similarly, the authors of [19, Eq. (19)] defined the U -Euler polynomials $A_s(\gamma)$ by the following generating function:

$$\left(\frac{2}{e^{-\frac{z}{2}} + 1} \right) e^{-\frac{\gamma z}{2}} = \sum_{s=0}^{\infty} A_s(\gamma) \frac{z^s}{s!}, \quad |z| < 2\pi. \tag{11}$$

For $\gamma = 0$ in ((11) the U -Euler numbers M_s are obtained.

We motivated by the introducing of U -Bernoulli, U -Euler polynomials and numbers [19]. Also, motivated from the applications of q -special polynomials in different fields of mathematics and sciences. In section 2, we define $q-U$ -Bernoulli, $q-U$ -Euler polynomials and numbers and obtains some their feature relations. Then we explore the $q-U$ -Bernoulli polynomials $M_s(\gamma, q)$, Rogers-Szegö polynomials $H_s(\gamma, q)$ and the q -Bernoulli numbers $B_s(q)$ are related via some connection formulas in section 3. Moreover, in section 4, we derive various features for the q -extensions U -Bernoulli/Euler polynomials like generating functions, q -partial derivatives and summation relations.

2 $q-U$ -Bernoulli, $q-U$ -Euler polynomials and numbers

The q -Bernoulli polynomials and numbers, first defined by Carlitz [21], have attracted the attention of many mathematicians, and a variety of their properties have been investigated. Kupersmidt, introduced q -Bernoulli polynomials and established their reflection symmetry [22]. After that, Kim first introduced the concept of q -Euler numbers and new q -extensions of polynomials using Kupersmidt's method, and investigated the symmetry properties of these q -Euler polynomials using q -differentiation and q -integration, and introduced various applications and symmetries [23].

In recent studies, degenerate Bernoulli numbers and polynomials have been investigated from a generalized perspective of q -Bernoulli polynomials [25, 31, 33], and numerous papers have been published examining these polynomials by deriving moment values from probability theory and relating them to various identities and values of the zeta function [26, 27, 29].

Moreover, the relationship between these degenerate polynomials and harmonic numbers has been established [28, 30]. The significance of q -Bernoulli numbers in the q -Laplace or degenerate Laplace transforms for different values of the parameter λ used in these studies has been demonstrated [24].

In this paper, we describe the $q-U$ -Bernoulli, $q-U$ -Euler polynomials and numbers. We obtain various properties these polynomials and numbers.

The q -Bernoulli polynomials $B_s(\gamma, q)$ are defined by means of the following generating function

$$\frac{ze_q(\gamma z)}{e_q(z) - 1} = \sum_{s=0}^{\infty} B_s(\gamma, q) \frac{z^s}{[s]_q!}. \tag{12}$$

Clearly, in case $\gamma = 0$, $B_s(0, q) = B_s(q)$ are q -Bernoulli numbers [4, 5, 13, 11].

Moreover, in the studies conducted, we can observe various generalizations of these polynomials [35, 36, 37, 38].

Definition 2.1. Let n be non-negative integer. The $q-U$ -Bernoulli polynomials $M_s(\gamma, q)$ are defined by means of the following generating function

$$\sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} = \frac{-ze_q(z)e_q(-\gamma z)}{e_q(z) - 1}. \tag{13}$$

When written $\gamma = 0$ in (13), the $q-U$ -Bernoulli numbers $M_s(q)$ are defined by the generating function

$$\sum_{s=0}^{\infty} M_s(q) \frac{z^s}{[s]_q!} = \frac{-ze_q(z)}{e_q(z) - 1}. \tag{14}$$

Here, we see that

$$\lim_{q \rightarrow 1} M_s(\gamma, q) = M_s(\gamma)$$

and

$$\lim_{q \rightarrow 1} M_s(q) = M_s,$$

where the $M_s(\gamma)$ is s -th U -Bernoulli polynomial and M_s is s -th U -Bernoulli number.

Theorem 2.1. Let $B_s(q)$ be s -th q -Bernoulli numbers.

$$M_s(q) = -\sum_{u=0}^s \binom{s}{u}_q B_u(q).$$

Proof. If we use generating function of $M_s(q)$ numbers, we obtain

$$\begin{aligned} \sum_{s=0}^{\infty} M_s(q) \frac{z^s}{[s]_q!} &= \frac{-ze_q(z)}{e_q(z) - 1} \\ &= -\sum_{s=0}^{\infty} B_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \frac{z^s}{[s]_q!} \\ &= \left(-\sum_{u=0}^s \binom{s}{u}_q B_u(q) \right) \frac{z^s}{[s]_q!}. \end{aligned} \tag{15}$$

□

Example 2.1. The first few $q-U$ -Bernoulli numbers $M_s(q)$ are as follows:

$$\begin{aligned} M_0(q) &= -1, \\ M_1(q) &= -\frac{1}{2}, \\ M_2(q) &= -\frac{5[2]_q}{12} - 1, \\ M_3(q) &= -1 + \frac{[3]_q}{2} - \frac{[3]_q[2]_q}{12}, \\ M_4(q) &= -1 + \frac{[4]_q}{2} - \frac{[4]_q[3]_q}{12} + \frac{[4]_q!}{720}, \\ M_5(q) &= -1 + \frac{[5]_q}{2} - \frac{[5]_q[4]_q}{12} + \frac{[5]_q!}{720}. \end{aligned}$$

Theorem 2.2. Let $M_s(q)$ be s -th $q-U$ -Bernoulli numbers. Then

$$M_s(\gamma, q) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} M_u(q) \gamma^{s-u}, \tag{16}$$

or

$$M_s(\gamma, q) = (-1)^{s+1} \gamma^s + \sum_{u=1}^s \binom{s}{u}_q (-1)^{s-u} M_u(q) \gamma^{s-u}.$$

Proof. Expanding the right hand side of equation (13) via equations (15) and (1), we have

$$\begin{aligned} \sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} &= \frac{-ze_q(z)e_q(-\gamma z)}{e_q(z) - 1} \\ &= \sum_{s=0}^{\infty} M_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (-1)^s \gamma^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{k=0}^s \binom{s}{k}_q (-1)^{s-k} M_k(q) \gamma^{s-k} \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

Thus, the proof is complete. □

Example 2.2. The first few $M_s(\gamma, q)$ polynomials are as follows:

$$\begin{aligned} M_0(\gamma, q) &= -1, \\ M_1(\gamma, q) &= \gamma - \frac{1}{2}, \\ M_2(\gamma, q) &= -\gamma^2 + \frac{[2]_q \gamma}{2} + \frac{5[2]_q}{12} - 1, \\ M_3(\gamma, q) &= \gamma^3 - \frac{[3]_q \gamma^2}{2} - [3]_q \left(\frac{5[2]_q}{12} - 1 \right) \gamma, \\ M_4(\gamma, q) &= -\gamma^4 + \frac{[4]_q \gamma^3}{2} + \frac{[4]_q [3]_q}{[2]_q} \left(\frac{5[2]_q}{12} - 1 \right) \gamma^2 - [4]_q \left(-1 + \frac{[3]_q}{2} - \frac{[3]_q [2]_q}{12} \right) \gamma \\ &\quad + \left(-1 + \frac{[4]_q}{2} - \frac{[4]_q [3]_q}{12} + \frac{[4]_q!}{720} \right). \end{aligned}$$

Theorem 2.3. If $s \in \mathbb{N}_0$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$M_s(\gamma + \beta, q) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} M_u(\gamma, q) \beta^{s-u}$$

for q -commuting variables γ and β .

Proof. We have

$$\begin{aligned} \sum_{u=0}^{\infty} M_s(\gamma + \beta, q) \frac{z^s}{[s]_q!} &= \frac{-ze_q(z)e_q(-(\gamma + \beta)z)}{e_q(z) - 1} \\ &= \frac{-ze_q(z)}{e_q(z) - 1} e_q(-\gamma z) e_q(-\beta z) \\ &= e_q(-\beta z) \sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} \end{aligned}$$

and

$$e_q(-\beta z) = \sum_{s=0}^{\infty} (-\beta)^s \frac{z^s}{[s]_q!}. \tag{17}$$

On the other hand,

$$\sum_{s=0}^{\infty} (-\beta)^s \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q M_u(\gamma, q) (-\beta)^{s-u} \right) \frac{z^s}{[s]_q!}. \tag{18}$$

Thus, we achieve the desired result from the equality of (17) and (18). □

Theorem 2.4. The q -derivative of $M_s(\gamma, q)$ polynomials is

$$D_{q,\gamma} M_s(\gamma, q) = -[s]_q M_{s-1}(\gamma, q).$$

Proof. We obtain

$$\begin{aligned} D_{q,\gamma}M_s(\gamma, q) &= D_{q,\gamma} \left(\sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} M_u(q) \gamma^{s-u} \right) \\ &= \sum_{u=0}^{s-1} \binom{s}{u}_q (-1)^{s-u} M_u(q) [s-u]_q \gamma^{s-1-u} \\ &= -[s]_q \sum_{u=0}^{s-1} \binom{s-1}{u}_q (-1)^{s-1-u} M_u(q) [s-u]_q \gamma^{s-1-u} \\ &= -[s]_q M_{s-1}(\gamma, q). \end{aligned}$$

□

Theorem 2.5. *The $M_s(\gamma, q)$ polynomials have the following integral property.*

$$\int_0^1 M_s(\gamma, q) d_q(\gamma) = -\frac{M_{s+1}(1, q) - M_{s+1}(q)}{[s+1]_q}.$$

Proof. If we apply (4) to (16), we obtain

$$\begin{aligned} \int_0^1 M_s(\gamma, q) d_q(\gamma) &= \int_0^1 \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} M_u(q) \gamma^{s-u} d_q(\gamma) \\ &= \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} M_u(q) \int_0^1 \gamma^{s-u} d_q(\gamma) \\ &= \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} M_u(q) \frac{1}{[s-u+1]_q} \\ &= \frac{-1}{[s+1]_q} \sum_{u=0}^s \binom{s+1}{u}_q (-1)^{s+1-u} M_u(q) \end{aligned}$$

From (16), we have

$$M_s(1, q) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} M_u(q)$$

and

$$M_s(0, q) = M_s(q).$$

Thus

$$\begin{aligned} \int_0^1 M_s(\gamma, q) d_q(\gamma) &= \frac{-1}{[s+1]_q} \left(\sum_{u=0}^{s+1} \binom{s+1}{u}_q (-1)^{s+1-u} M_u(q) - M_{s+1}(0, q) \right) \\ &= \frac{-(M_{s+1}(1, q) - M_{s+1}(q))}{[s+1]_q}. \end{aligned}$$

□

Theorem 2.6. *The $M_s(\gamma, q)$ polynomials have the following property.*

$$M_{s+1}(\gamma, q) + \gamma M_s(\gamma, q) = \sum_{u=0}^s q^{s-u-1} \binom{s}{u}_q (-1)^{s-u} M_{u+1}(q) \gamma^{s-u}.$$

Proof. From (16), we have

$$\begin{aligned}
 M_{s+1}(\gamma, q) &= \sum_{u=0}^{s+1} \binom{s+1}{u}_q (-1)^{s+1-u} M_u(q) \gamma^{s+1-u} \\
 &\quad \sum_{u=0}^{s+1} \left(\binom{s}{u}_q + q^{s-u} \binom{s}{u-1}_q \right) (-1)^{s+1-u} M_u(q) \gamma^{s+1-u} \\
 &= \sum_{u=0}^{s+1} \binom{s}{u}_q (-1)^{s+1-u} M_u(q) \gamma^{s+1-u} \\
 &\quad + \sum_{u=0}^{s+1} q^{s-u} \binom{s}{u-1}_q (-1)^{s+1-u} M_u(q) \gamma^{s+1-u} \\
 &= -\gamma \sum_{u=0}^{s+1} \binom{s}{u}_q (-1)^{s-u} M_u(q) \gamma^{s-u} \\
 &\quad + \sum_{u=1}^{s+1} q^{s-u} \binom{s}{u-1}_q (-1)^{s+1-u} M_u(q) \gamma^{s+1-u} \\
 &= -\gamma \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} M_u(q) \gamma^{s-u} \\
 &\quad + \sum_{u=0}^s q^{s-u-1} \binom{s}{u}_q (-1)^{s-u} M_{u+1}(q) \gamma^{s-u} \\
 &= -\gamma M_s(\gamma, q) + \sum_{u=0}^s q^{s-u-1} \binom{s}{u}_q (-1)^{s-u} M_{u+1}(q) \gamma^{s-u}.
 \end{aligned}$$

□

Now, we define $q-U$ -Euler polynomials $A_s(\gamma, q)$ and study some of their feature relations.

Definition 2.2. Let n be non-negative integer. The $q-U$ -Euler polynomials $A_s(\gamma, q)$ are defined by means of the following generating function

$$\left(\frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} \right) e_q(-\gamma z/2) = \sum_{s=0}^{\infty} A_s(\gamma, q) \frac{z^s}{[s]_q!}. \tag{19}$$

For $\gamma = 0$ in (19) the $q-U$ -Euler numbers $A_s(q)$ are defined by the generating function

$$\frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} = \sum_{s=0}^{\infty} A_s(q) \frac{z^s}{[s]_q!}. \tag{20}$$

Moreover, we see that

$$\lim_{q \rightarrow 1} A_s(\gamma, q) = A_s(\gamma)$$

and

$$\lim_{q \rightarrow 1} A_s(q) = A_s,$$

where the $A_s(\gamma)$ is s -th U -Euler polynomial and A_s is s -th U -Euler number.

Theorem 2.7. Let $B_s(q)$ be s -th q -Bernoulli numbers and $A_s(q)$ be s -th $q-U$ -Euler numbers,

$$A_s(q) = \sum_{u=0}^s \left(\frac{1}{2} \right)_q^u \binom{s}{u}_q C_u(q) B_{s-u}(q), \tag{21}$$

where

$$C_u(q) = \sum_{i=0}^u \frac{1}{[i+1]_q} \binom{u}{i}_q.$$

Proof. Let's prove this using (20)

$$\begin{aligned}
 \sum_{s=0}^{\infty} A_s(q) \frac{z^s}{[s]_q!} &= \frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} \\
 &= \frac{2}{z} \sum_{s=0}^{\infty} B_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)^s \frac{z^s}{[s]_q!} \sum_{s=1}^{\infty} \left(\frac{1}{2}\right)^s \frac{z^s}{[s]_q!} \\
 &= \sum_{s=0}^{\infty} B_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)^s \frac{z^s}{[s]_q!} \sum_{s=1}^{\infty} \left(\frac{1}{2}\right)^{s-1} \frac{z^{s-1}}{[s]_q!} \\
 &= \sum_{s=0}^{\infty} B_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)^s \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)^s \frac{z^s}{[s+1]_q!} \\
 &= \sum_{s=0}^{\infty} B_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \frac{1}{[u+1]_q} \binom{s}{u}_q \right) \left(\frac{1}{2}\right)^s \frac{z^s}{[s]_q!} \\
 &= \sum_{s=0}^{\infty} B_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} C_s(q) \left(\frac{1}{2}\right)^s \frac{z^s}{[s]_q!} \\
 &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \left(\frac{1}{2}\right)^u \binom{s}{u}_q C_u(q) B_{s-u}(q) \right) \frac{z^s}{[s]_q!}.
 \end{aligned}$$

Thus the proof is completed. □

Example 2.3. First few the $A_s(q)$ numbers are as follows:

$$\begin{aligned}
 A_0(q) &= 1, \\
 A_1(q) &= \frac{1}{2[2]_q}, \\
 A_2(q) &= -\frac{[2]_q}{6} + \frac{1}{4[3]_q} + \frac{1}{4} \\
 A_3(q) &= \frac{[3]_q([2]_q - 5)}{24} + \frac{[5]_q + q^2 + 1}{8[4]_q} + \frac{1}{8}, \\
 A_4(q) &= -\frac{[4]_q!}{720} + \frac{[4]_q(2[3]_q - 5)}{48} - \frac{[4]_q([3]_q - 2)}{16[2]_q} + \frac{1}{16[5]_q} + \frac{1}{16}.
 \end{aligned}$$

Theorem 2.8. For every $s, k \in \mathbb{N}$,

$$A_s(\gamma, q) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(q) \gamma^{s-u}. \tag{22}$$

Proof. By virtue of (20), we get

$$\begin{aligned}
 \sum_{s=0}^{\infty} A_s(\gamma, q) \frac{z^s}{[s]_q!} &= \left(\frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} \right) e_q(-\gamma z/2) \\
 &= \sum_{s=0}^{\infty} A_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s \gamma^s \frac{z^s}{[s]_q!} \\
 &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(q) \gamma^{s-u} \right) \frac{z^s}{[s]_q!}.
 \end{aligned}$$

□

Some of the $A_s(\gamma, q)$ polynomials are as follows:

$$\begin{aligned}
 A_0(\gamma, q) &= 1, \\
 A_1(\gamma, q) &= -\frac{1}{2}\gamma + \frac{1}{2[2]_q}, \\
 A_2(\gamma, q) &= \frac{1}{4}\gamma^2 - \frac{1}{4}\gamma - \frac{[2]_q}{6} + \frac{1}{4[3]_q} + \frac{1}{4}, \\
 A_3(\gamma, q) &= -\frac{1}{8}\gamma^3 + \frac{[3]_q}{8[2]_q}\gamma^2 - \left(-\frac{[3]_q[2]_q}{12} + \frac{[3]_q + 1}{8}\right)\gamma \\
 &\quad + \frac{[3]_q([2]_q - 5)}{24} + \frac{[5]_q + q^2 + 1}{8[4]_q} + \frac{1}{8}, \\
 A_4(\gamma, q) &= \frac{1}{16}\gamma^4 - \frac{[4]_q}{16[2]_q}\gamma^3 + \left(-\frac{[4]_q[3]_q}{24} + \frac{[4]_q[3]_q + [4]_q}{16[2]_q}\right)\gamma^2 \\
 &\quad - \left(\frac{[4]_q[3]_q([2]_q - 5)}{48} + \frac{[5]_q + [4]_q + q^2 + 1}{16}\right)\gamma \\
 &\quad - \frac{[4]_q!}{720} + \frac{[4]_q(2[3]_q - 5)}{48} - \frac{[4]_q([3]_q - 2)}{16[2]_q} + \frac{1}{16[5]_q} + \frac{1}{16}.
 \end{aligned}$$

Theorem 2.9. The q -derivative of $A_s(\gamma, q)$ polynomials is

$$D_{q,\gamma}A_s(\gamma, q) = -\frac{1}{2}[s]_q A_{s-1}(\gamma, q).$$

Proof. We obtain

$$\begin{aligned}
 D_{q,\gamma}A_s(\gamma, q) &= D_{q,\gamma} \left(\sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(q) \gamma^{s-u} \right) \\
 &= \sum_{u=0}^{s-1} \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(q) [s-u]_q \gamma^{s-1-u} \\
 &= -\frac{1}{2} [s]_q \sum_{u=0}^{s-1} \binom{s-1}{u}_q \left(-\frac{1}{2}\right)^{s-1-u} A_u(q) [s-u]_q \gamma^{s-1-u} \\
 &= -\frac{1}{2} [s]_q A_{s-1}(\gamma, q).
 \end{aligned}$$

□

Theorem 2.10. For q -commuting variables γ and β , $s \in \mathbb{N}_0$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$A_s(\gamma + \beta, q) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(\gamma, q) \beta^{s-u}.$$

Proof. We have

$$\begin{aligned}
 \sum_{u=0}^{\infty} A_s(\gamma + \beta, q) \frac{z^s}{[s]_q!} &= \frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} e_q(-z(\gamma + \beta)/2) \\
 &= \frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} e_q(-\gamma z/2) e_q(-\beta z/2) \\
 &= e_q(-\beta z/2) \sum_{s=0}^{\infty} A_s(\gamma, q) \frac{z^s}{[s]_q!}
 \end{aligned}$$

and from (1) we write

$$e_q(-\beta z/2) = \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s \beta^s \frac{z^s}{[s]_q!}.$$

On the other hand,

$$\sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s \beta^s \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} A_s(\gamma, q) \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(\gamma, q) \beta^{s-u} \frac{z^s}{[s]_q!}. \tag{23}$$

Thus, the proof is obtained from the equality of (22) and (23). □

Theorem 2.11. *The $A_s(\gamma, q)$ polynomials have the following integral property.*

$$\int_0^1 A_s(\gamma, q) d_q(\gamma) = -\frac{2(A_{s+1}(1, q) - A_{s+1}(q))}{[s + 1]_q}.$$

Proof. If we apply (4) to (21), we obtain

$$\begin{aligned} \int_0^1 A_s(\gamma, q) d_q(\gamma) &= \int_0^1 \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(q) \gamma^{s-u} d_q(\gamma) \\ &= \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(q) \int_0^1 \gamma^{s-u} d_q(\gamma) \\ &= \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(q) \frac{1}{[s - u + 1]_q} \\ &= \frac{-1}{[s + 1]_q} \sum_{u=0}^s \binom{s+1}{u}_q \left(-\frac{1}{2}\right)^{s+1-u} A_u(q) \\ &= \frac{-1}{[s + 1]_q} \left(\sum_{u=0}^{s+1} \binom{s+1}{u}_q \left(-\frac{1}{2}\right)^{s+1-u} A_u(q) - A_{s+1}(q) \right). \end{aligned}$$

From (22), we have

$$A_s(1, q) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} A_u(q)$$

and

$$A_s(0, q) = A_s(q).$$

Thus

$$\begin{aligned} \int_0^1 A_s(\gamma, q) d_q(\gamma) &= \frac{-1}{[s + 1]_q} \left(\sum_{u=0}^{s+1} \binom{s+1}{u}_q \left(-\frac{1}{2}\right)^{s+1-u} A_u(q) - A_{s+1}(q) \right) \\ &= -\frac{2(A_{s+1}(1, q) - A_{s+1}(q))}{[s + 1]_q}. \end{aligned}$$

The proof is complete. □

3 Some connection formulas for the polynomials $M_s(\gamma, q)$

Next, we explore the $q-U$ -Bernoulli polynomials $M_n(\gamma, q)$, the Rogers-Szegö polynomials $H_n(\gamma, q)$, and the q -Bernoulli numbers $B_n(q)$, examining how they are related through various connection formulas.

Theorem 3.1. *The $q-U$ -Bernoulli polynomials $M_s(\gamma, q)$ are related with the Rogers-Szegö polynomials $H_s(\gamma, q)$ and the q -Bernoulli numbers $B_s(q)$ by the means of the following identities*

$$M_s(\gamma, q) = -\sum_{u=0}^s \binom{s}{u}_q B_{s-u}(q) H_u(-\gamma, q). \tag{24}$$

$$H_s(-\gamma, q) = -\sum_{u=0}^s \frac{1}{[s + 1]_q} \binom{s+1}{u} M_u(\gamma, q). \tag{25}$$

Proof. (24) Using Definition (13), (5) and the q -Bernoulli numbers (12) with $\gamma = 0$, we have

$$\begin{aligned} \sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} &= \frac{-ze_q(z)e_q(-\gamma z)}{e_q(z) - 1} \\ &= -\sum_{s=0}^{\infty} B_s(q) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} H_s(-\gamma, q) \frac{z^s}{[s]_q!} \\ &= -\sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q B_{s-u}(q) H_u(-\gamma, q) \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

□

Proof. (25) We have

$$\begin{aligned}
 & -\frac{e_q(z)-1}{z} \sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} = e_q(z)e_q(-\gamma z) \\
 & -\sum_{s=0}^{\infty} \frac{z^s}{[s+1]_q!} \sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} H_s(-\gamma, q) \frac{z^s}{[s]_q!} \\
 & -\sum_{s=0}^{\infty} \sum_{u=0}^s \frac{1}{[s+1]_q} \binom{s+1}{u}_q M_u(\gamma, q) \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} H_s(-\gamma, q) \frac{z^s}{[s]_q!}.
 \end{aligned}$$

Finally, equating the coefficients of the powers of z on both sides, we get (25). □

Theorem 3.2. The $q-U$ -Bernoulli polynomials $M_s(\gamma, q)$ are related with the q -Fubini polynomials $F_{s,q}(\gamma; \omega)$ by the means of the following identity

$$M_s(\gamma, q) = \sum_{u=0}^s \binom{s}{u}_q M_{s-u}(q) \left[(1 + \omega)F_{u,q}(-\gamma; \omega) - \omega \sum_{k=0}^u \binom{u}{k}_q F_{k,q}(-\gamma; \omega) \right]. \tag{26}$$

Proof. We have

$$\begin{aligned}
 \sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} &= \frac{-ze_q(z) \cdot e_q(-\gamma z)}{e_q(z)-1} \cdot \frac{1 - \omega(e_q(z)-1)}{1 - \omega(e_q(z)-1)} \\
 &= \frac{-ze_q(z)}{e_q(z)-1} \cdot \left[\frac{e_q(-\gamma z)}{1 - \omega(e_q(z)-1)} - \frac{e_q(-\gamma z) \cdot e_q(z) \cdot \omega}{1 - \omega(e_q(z)-1)} + \frac{\omega e_q(-\gamma z)}{1 - \omega(e_q(z)-1)} \right]
 \end{aligned}$$

using the equations (15) and (9), we obtain

$$\begin{aligned}
 \sum_{s=0}^{\infty} M_s(\gamma, q) \frac{z^s}{[s]_q!} &= \sum_{s=0}^{\infty} M_s(q) \frac{z^s}{[s]!} \\
 &\times \left[(1 + \omega) \sum_{s=0}^{\infty} F_{s,q}(-\gamma; \omega) \frac{z^s}{[s]_q!} - \omega \sum_{s=0}^{\infty} F_{s,q}(-\gamma; \omega) \frac{z^s}{[s]_q!} \cdot \sum_{s=0}^{\infty} \frac{z^s}{[s]_q!} \right] \\
 &= \sum_{s=0}^{\infty} M_s(q) \frac{z^s}{[s]!} \sum_{s=0}^{\infty} \left[(1 + \omega)F_{s,q}(-\gamma; \omega) - \omega \sum_{k=0}^s \binom{s}{k}_q F_{k,q}(-\gamma; \omega) \right] \frac{z^s}{[s]!} \\
 &= \sum_{s=0}^{\infty} \sum_{u=0}^s \binom{s}{u}_q M_{s-u}(q) \\
 &\times \left[(1 + \omega)F_{u,q}(-\gamma; \omega) - \omega \sum_{k=0}^u \binom{u}{k}_q F_{k,q}(-\gamma; \omega) \right] \frac{z^s}{[s]!}.
 \end{aligned}$$

Equating the coefficients of $\frac{z^s}{[s]!}$, we derive asserted result. □

Theorem 3.3. Each of the following identities holds true

$$\begin{aligned}
 M_s(\gamma, q) &= \sum_{i=0}^s \left(\sum_{u=i}^s \binom{s}{u}_q \binom{u}{i}_q (-1)^{2u-i} q^{\binom{u-i}{2}} M_{s-u}(q) \right) H_i(\gamma, q). \\
 M_s(\gamma; q) &= \sum_{j=0}^s \left(\sum_{k=j}^s \binom{s}{k}_q \binom{k}{j}_q (-1)^k M_{s-k}(q) \sum_{u=0}^{k-1} \binom{k-j}{u}_q a^u \right) U_j^{(a)}(\gamma; q)
 \end{aligned}$$

Proof. The proof of this theorem uses (16), (7), (8) and the summation formula

$$\sum_{u=0}^s A_u \sum_{i=0}^u B_u = \sum_{i=0}^s \left(\sum_{u=i}^s A_u \right) B_i.$$

□

Now, we combine of the Rogers-Szegö polynomials $H_s(\gamma, q)$ and the Al-Salam Carlitz polynomials $U_s^{(a)}(\gamma, q)$ with q -Bernoulli polynomials $B_s(\gamma, q)$ to consider the q -Rogers-Szegö-Bernoulli polynomials ${}_H B_s(\gamma, q)$ and q -Al-Salam Carlitz-Bernoulli polynomials ${}_U B_s^{(a)}(\gamma, q)$ and establishes their series representation.

By employing expansion (1) in equation (12) and then replacing powers of $\gamma^0, \gamma^1, \gamma^2, \dots, \gamma^s$ of γ with the correlating polynomials $H_0(\gamma, q), H_1(\gamma, q), \dots, H_s(\gamma, q)$ of Rogers-Szegö polynomials $H_s(\gamma, q)$, and summing up the terms in the resulting equation and denoting the resulting on the right aspect as ${}_H B_s(\gamma, q)$, the following generating equation of q -Rogers-Szegö-Bernoulli polynomials is given:

$$\sum_{s=0}^{\infty} {}_H B_s(\gamma, q) \frac{z^s}{[s]_q!} = \frac{z e_q(\gamma z)}{e_q(z) - 1} e_q(z), \quad \gamma \in \mathbb{R}. \tag{27}$$

Theorem 3.4. For the ${}_H B_s(\gamma, q)$, the following series representation holds true:

$${}_H B_s(\gamma, q) = \sum_{u=0}^s \binom{s}{u}_q B_{s-u}(\gamma, q). \tag{28}$$

Proof. Using generating function (12) and expansion (1) in equation (27), we have

$$\sum_{s=0}^{\infty} {}_H B_s(\gamma, q) \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} \sum_{u=0}^{\infty} B_s(\gamma, q) \frac{z^{s+u}}{[s]_q! [u]_q!},$$

which on simplification becomes

$$\sum_{s=0}^{\infty} {}_H B_s(\gamma, q) \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} \sum_{u=0}^s B_s(\gamma, q) \frac{z^s}{[s-u]_q! [u]_q!}.$$

Equating the coefficients of like powers of z on both sides of the previous formula, we obtain assertion (28). □

Similarly, employing expansion (1) in equation (12) and then replacing powers of $\gamma^0, \gamma^1, \gamma^2, \dots, \gamma^s$ of γ with the correlating polynomials $U_0^{(a)}(\gamma, q), U_1^{(a)}(\gamma, q), \dots, U_s^{(a)}(\gamma, q)$ of q -Al-Salam Carlitz polynomials $U_s^{(a)}(\gamma, q)$, and summing up the terms in the resulting equation and denoting the resultant on the r.h.s. as ${}_U B_s^{(a)}(\gamma, q)$, the following generating equation of q -Al-Salam Carlitz-Bernoulli polynomials is obtained:

$$\frac{z}{e_q(z) - 1} \frac{e_q(\gamma z)}{e_q(z) e_q(az)} = \sum_{s=0}^{\infty} {}_U B_s^{(a)}(\gamma, q) \frac{z^s}{[s]_q!}, \quad \gamma \in \mathbb{R}. \tag{29}$$

Theorem 3.5. For the polynomials ${}_U B_s^{(a)}(\gamma, q)$, the following series representation is valid:

$${}_U B_s^{(a)}(\gamma, q) = \sum_{u=0}^s \binom{s}{u}_q B_u(q) U_{s-u}^{(a)}(\gamma, q).$$

Proof. Using generating function (6) and expansion for $-$ Bernoulli numbers in equation (29), we have

$$\sum_{s=0}^{\infty} {}_U B_s^{(a)}(\gamma, q) \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} \sum_{u=0}^{\infty} B_u(q) U_s^{(a)}(\gamma, q) \frac{z^{s+u}}{[s]_q! [u]_q!},$$

which on simplification becomes

$$\sum_{s=0}^{\infty} {}_U B_s^{(a)}(\gamma, q) \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} \sum_{u=0}^s B_u(q) U_{s-u}^{(a)}(\gamma, q) \frac{z^s}{[s-u]_q! [u]_q!}.$$

Equating the coefficients of same powers of z on each side of the previous formula, we obtain assertion (28). □

4 The Generalized $q-U$ -Bernoulli/Euler Polynomials and their properties

In this section, we derive some properties for the q -extensions U -Bernoulli and U -Euler polynomials like generating functions, q -partial derivatives and summation relations.

Definition 4.1. Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$. The generalized $q-U$ -Bernoulli numbers $\mathfrak{M}_{s,q}^{(\alpha)}$ and the generalized $q-U$ -Bernoulli polynomials $\mathfrak{M}_{s,q}^{(\alpha)}(x, y)$ in x, y of order α are defined by means of the generating functions:

$$\left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r = \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!}$$

and

$$\sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} = \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r e_q(-\gamma z) E_q(-\beta z). \tag{30}$$

It is obvious that $\mathfrak{M}_{s,q}^{(r)}(0, 0) = \mathfrak{M}_{s,q}^{(r)}$.

Definition 4.2. Let $q \in \mathbb{C}$, $\alpha \in \mathbb{N}$, $0 < |q| < 1$. The generalized $q-U$ -Bernoulli numbers $\mathfrak{A}_{s,q}^{(\alpha)}$ and the generalized $q-U$ -Bernoulli polynomials $\mathfrak{A}_{s,q}^{(\alpha)}(x, y)$ in x, y of order α are defined by means of the generating functions:

$$\left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \right)^r = \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)} \frac{z^s}{[s]_q!},$$

$$\sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} = \left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \right)^r e_q(-\gamma z/2) E_q(-\beta z/2), \tag{31}$$

It is obvious that $\mathfrak{A}_{s,q}^{(r)}(0, 0) = \mathfrak{A}_{s,q}^{(r)}$.

Theorem 4.1. The $\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta)$ polynomials satisfy the following property

$$\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q \mathfrak{M}_{s-u,q} \mathfrak{M}_{u,q}^{(r-1)}(\gamma, \beta).$$

Proof. If we use (30), we obtain

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} &= \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r e_q(-\gamma z) E_q(-\beta z) \\ &= \frac{-ze_q(z)}{e_q(z)-1} \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^{r-1} e_q(-\gamma z) E_q(-\beta z) \\ &= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r-1)}(\gamma, \beta) \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q \mathfrak{M}_{s-u,q} \mathfrak{M}_{u,q}^{(r-1)}(\gamma, \beta) \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

□

Theorem 4.2. The $\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta)$ polynomials have the following relation

$$\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r)}(\gamma + \beta)^{s-u}.$$

Proof. From (30), we have

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} &= \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r e_q(-\gamma z) E_q(-\beta z) \\ &= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (-1)^s \gamma^s \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (-1)^s q^{\frac{s(s-1)}{2}} \beta^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q q^{\frac{u(u-1)}{2}} \gamma^{s-u} \beta^u \right) \frac{(-1)^s z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (\gamma + \beta)_q^s \frac{(-1)^s z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r)}(\gamma + \beta)^{s-u} \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

□

From (30), respectively, we easily obtain

$$\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q \mathfrak{M}_{s-u,q}(x, 0) \mathfrak{M}_{u,q}^{(r-1)}(0, \beta),$$

$$\mathfrak{M}_{s,q}^{(r-\beta)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r-\beta)}(\gamma + \beta) \beta^{s-u},$$

and

$$\mathfrak{M}_{s,q}^{(r-\beta)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q \mathfrak{M}_{u,q}^{(r)}(x, 0) \mathfrak{M}_{s-u,q}^{(-\beta)}(0, \beta).$$

Proposition 4.3. *We have the following features*

$$\mathfrak{M}_{s,q}^{(r)}(\gamma, 0) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r)} \gamma^{s-u} \tag{32}$$

and

$$\mathfrak{M}_{s,q}^{(r)}(0, \beta) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} q^{\frac{(s-u)(s-u-1)}{2}} \mathfrak{M}_{u,q}^{(r)} \beta^{s-u}. \tag{33}$$

Proof. Firstly let's prove the equation (32)

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(x, 0) \frac{z^s}{[s]_q!} &= \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r e_q(-\gamma z) \\ &= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (-1)^s \gamma^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r)} \gamma^{s-u} \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

Now let's show that equation (33) is true.

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(0, \beta) \frac{z^s}{[s]_q!} &= \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r E_q(-\beta z) \\ &= \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (-1)^s q^{\frac{s(s-1)}{2}} \beta^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} q^{\frac{(s-u)(s-u-1)}{2}} \mathfrak{M}_{u,q}^{(r)} \beta^{s-u} \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

□

Proposition 4.4. *From (30), we have*

$$\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} q^{\frac{(s-u)(s-u-1)}{2}} \mathfrak{M}_{u,q}^{(r)}(x, 0) \beta^{s-u} \tag{34}$$

and

$$\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r)}(0, \beta) \gamma^{s-u}. \tag{35}$$

From (34) and (35), we easily obtain the following property.

Proposition 4.5.

$$\mathfrak{M}_{s,q}^{(r)}(\gamma, 1) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} q^{\frac{(s-u)(s-u-1)}{2}} \mathfrak{M}_{u,q}^{(r)}(\gamma, 0)$$

and

$$\mathfrak{M}_{s,q}^{(r)}(1, \beta) = \sum_{u=0}^s \binom{s}{u}_q (-1)^{s-u} \mathfrak{M}_{u,q}^{(r)}(0, \beta).$$

Theorem 4.6. The $\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta)$ polynomials have the following property:

$$\sum_{u=0}^s \binom{s}{u}_q \mathfrak{M}_{u,q}^{(r)}(\gamma, \beta) \mathfrak{M}_{s-u,q}^{(-r)} = (-1)^s (\gamma + \beta)_q^s.$$

Proof. From the Cauchy product rule, we have

$$\sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(-r)} \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q \mathfrak{M}_{u,q}^{(r)}(\gamma, \beta) \mathfrak{M}_{s-u,q}^{(-r)} \right) \frac{z^s}{[s]_q!} \tag{36}$$

On the other hand, we have

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \mathfrak{M}_{s,q}^{(-r)} \frac{z^s}{[s]_q!} &= \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r e_q(-\gamma z) E_q(-\beta z) \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^{-r} \\ &= \sum_{s=0}^{\infty} (-1)^s \gamma^s \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} (-1)^s q^{\frac{s(s-1)}{2}} \beta^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q q^{\frac{k(k-1)}{2}} \gamma^{s-u} \beta^u \right) (-1)^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} (-1)^s (\gamma + \beta)_q^s \frac{z^s}{[s]_q!}. \end{aligned}$$

From (36) and the last equality, we get the desired result. □

Theorem 4.7. The $\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta)$ polynomials satisfy the following property

$$\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q \mathfrak{A}_{s-u,q} \mathfrak{A}_{u,q}^{(r-1)}(\gamma, \beta).$$

Proof. If we use (31), we obtain

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} &= \left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \right)^r e_q(-\gamma z/2) E_q(-\beta z/2) \\ &= \frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \\ &\quad \times \left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \right)^r e_q(-\gamma z/2) E_q(-\beta z/2) \\ &= \sum_{s=0}^{\infty} \mathfrak{A}_{s,q} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r-1)}(\gamma, \beta) \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q \mathfrak{A}_{s-u,q} \mathfrak{A}_{u,q}^{(r-1)}(\gamma, \beta) \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

□

Theorem 4.8. The $\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta)$ polynomials satisfy the following relation

$$\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2} \right)^{s-u} \mathfrak{A}_{u,q}^{(r)}(\gamma + \beta)_q^{s-u}.$$

Proof. From (4.2), we obtain

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} &= \left(\frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} \right)^r e_q(-\gamma z/2) E_q(-\beta z/2) \\ &= \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s \gamma^s \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s q^{\frac{s(s-1)}{2}} \beta^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q q^{\frac{k(k-1)}{2}} \gamma^{s-u} \beta^u \right) \left(-\frac{1}{2}\right)^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s (\gamma + \beta)_q^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} \mathfrak{A}_{u,q}^{(r)} (\gamma + \beta)_q^{s-u} \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

□

From (4.2), we easily obtain the following property

$$\begin{aligned} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) &= \sum_{u=0}^s \binom{s}{u}_q \mathfrak{A}_{s-u,q}(x, 0) \mathfrak{A}_{u,q}^{(r-1)}(0, \beta), \\ \mathfrak{A}_{s,q}^{(r-\beta)}(\gamma, \beta) &= \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} \mathfrak{A}_{u,q}^{(r-\beta)}(\gamma + \beta)_q^{s-u} \end{aligned}$$

and

$$\mathfrak{A}_{s,q}^{(r-\beta)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q \mathfrak{A}_{u,q}^{(r)}(x, 0) \mathfrak{A}_{s-u,q}^{(-\beta)}(0, \beta).$$

Proposition 4.9.

$$\mathfrak{A}_{s,q}^{(r)}(x, 0) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} \mathfrak{A}_{u,q}^{(r)} \gamma^{s-u} \tag{37}$$

and

$$\mathfrak{A}_{s,q}^{(r)}(0, \beta) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} q^{\frac{(s-u)(s-u-1)}{2}} \mathfrak{A}_{u,q}^{(r)} \beta^{s-u}. \tag{38}$$

Proof. We have

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)}(x, 0) \frac{z^s}{[s]_q!} &= \left(\frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} \right)^r e_q(-\gamma z/2) \\ &= \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s \gamma^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} \mathfrak{A}_{u,q}^{(r)} \gamma^{s-u} \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

and

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)}(0, \beta) \frac{z^s}{[s]_q!} &= \left(\frac{2e_q(z/2)(e_q(z/2) - 1)}{e_q(z) - 1} \right)^r E_q(-\beta z/2) \\ &= \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)} \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s q^{\frac{s(s-1)}{2}} \beta^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} q^{\frac{(s-u)(s-u-1)}{2}} \mathfrak{A}_{u,q}^{(r)} \beta^{s-u} \right) \frac{z^s}{[s]_q!}. \end{aligned}$$

Thus, the validity of equations (37) and (38) is clearly evident. □

Proposition 4.10. From (4.2), we have

$$\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} q^{\frac{(s-u)(s-u-1)}{2}} \mathfrak{A}_{u,q}^{(r)}(x, 0) \beta^{s-u} \tag{39}$$

and

$$\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} \mathfrak{A}_{u,q}^{(r)}(0, \beta) \gamma^{s-u}. \tag{40}$$

From (39) and (40), we easily obtain the following property.

Proposition 4.11.

$$\mathfrak{A}_{s,q}^{(r)}(\gamma, 1) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} q^{\frac{(s-u)(s-u-1)}{2}} \mathfrak{A}_{u,q}^{(r)}(x, 0)$$

and

$$\mathfrak{A}_{s,q}^{(r)}(1, \beta) = \sum_{u=0}^s \binom{s}{u}_q \left(-\frac{1}{2}\right)^{s-u} \mathfrak{A}_{u,q}^{(r)}(0, \beta).$$

Theorem 4.12. The $\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta)$ polynomials satisfy the following relation:

$$\sum_{u=0}^s \binom{s}{u}_q \mathfrak{A}_{u,q}^{(r)}(\gamma, \beta) \mathfrak{A}_{s-u,q}^{(-r)} = \left(-\frac{1}{2}\right)^s (\gamma + \beta)_q^s.$$

Proof. We have

$$\sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(-r)} \frac{z^s}{[s]_q!} = \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q \mathfrak{A}_{u,q}^{(r)}(\gamma, \beta) \mathfrak{A}_{s-u,q}^{(-r)} \right) \frac{z^s}{[s]_q!}$$

On the other hand, we obtain

$$\begin{aligned} \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \mathfrak{A}_{s,q}^{(-r)} \frac{z^s}{[s]_q!} &= \left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \right)^r e_q(-\gamma z/2) E_q(-\beta z/2) \\ &\quad \times \left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \right)^{-r} \\ &= \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s \gamma^s \frac{z^s}{[s]_q!} \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s q^{\frac{s(s-1)}{2}} \beta^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(\sum_{u=0}^s \binom{s}{u}_q q^{\frac{k(k-1)}{2}} \gamma^{s-u} \beta^u \right) \left(-\frac{1}{2}\right)^s \frac{z^s}{[s]_q!} \\ &= \sum_{s=0}^{\infty} \left(-\frac{1}{2}\right)^s (\gamma + \beta)_q^s \frac{z^s}{[s]_q!}. \end{aligned}$$

□

The theorem that is employed for demonstrating the q -partial derivatives for the $\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta)$ and $\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta)$:

Theorem 4.13. For $\mathfrak{M}_{s,q}^{(r)}(\gamma, \beta)$ and $\mathfrak{A}_{s,q}^{(r)}(\gamma, \beta)$, the subsequent q -partial derivatives are valid:

$$D_{q,\beta} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) = -[s]_q \mathfrak{M}_{s-1,q}^{(r)}(\gamma, \beta), \quad s \geq 1, \tag{41}$$

$$D_{q,\beta} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) = [s]_q \mathfrak{M}_{s-1,q}^{(r)}(\gamma, q\beta), \quad s \geq 1. \tag{42}$$

$$D_{q,\beta} \mathfrak{A}_{s-1,q}^{(r)}(\gamma, \beta) = \frac{-1}{2} [s]_q \mathfrak{A}_{s-1,q}^{(r)}(\gamma, \beta), \quad s \geq 1, \tag{43}$$

$$D_{q,\beta} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) = \frac{-1}{2} [s]_q \mathfrak{A}_{s-1,q}^{(r)}(\gamma, q\beta), \quad s \geq 1. \tag{44}$$

Proof. We calculate the q -partial derivative for every value of the formula (30) in terms of γ and β using equations (2) and (3), respectively, this offers us

$$\begin{aligned} \sum_{s=0}^{\infty} D_{q,\gamma} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} &= -z \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r e_q(-\gamma z) E_q(-\beta z). \\ \sum_{s=0}^{\infty} D_{q,\beta} \mathfrak{M}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} &= -z \left(\frac{-ze_q(z)}{e_q(z)-1} \right)^r e_q(-\gamma z) E_q(-q\beta z). \end{aligned}$$

We calculate the q -partial derivative for every value of the formula (31) in terms of γ and β using equations (2) and (3), respectively, this offers us

$$\sum_{s=0}^{\infty} D_{q,\gamma} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} = \frac{-z}{2} \left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \right)^r e_q(-\gamma z/2) E_q(-\beta z/2).$$

$$\sum_{s=0}^{\infty} D_{q,\beta} \mathfrak{A}_{s,q}^{(r)}(\gamma, \beta) \frac{z^s}{[s]_q!} = \frac{-z}{2} \left(\frac{2e_q(z/2)(e_q(z/2)-1)}{e_q(z)-1} \right)^r e_q(-\gamma z/2) E_q(-q\beta z/2).$$

Then, to produce assertions (43) as well as (44), respectively, we replicate the procedure described in the equation's proofs (41) as well as (42).

Theorem 4.13 has been fully proved. □

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