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Bivariate Approximation through Szász-Chlodowsky Operators with General-Appell Polynomials

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With great respect and admiration, we dedicate this work to Prof. M. Mursaleen on the occasion of his 72nd birthday.

Abstract

The present article deals with the approximation properties of the linear positive operators, including general-Appell polynomials. We establish some results for the convergence of the operators and their order of approximation with the help of the modulus of continuity, Lipschitz class, and Voronovskaja-type theorem. Moreover, the statistical approximation properties of these operators are established by employing a universal Korovkin-type statistical approximation theorem. This article concludes with some numerical examples, their graphical representation, and some remarks.

Keywords: Szász operators, Tensor product, Bivariate operator, General-Appell polynomials, Statistical convergence.

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1 Introduction

Approximation theory has broad applications in mathematical physics. The subject first gained attention in 1885 when Weierstrass [18] addressed the problem of approximating continuous functions. He demonstrated that any function on a closed interval $[a, b]$ can be approximated by polynomials, inspiring many mathematicians to further explore polynomial approximation. Building on this foundation, Szász [16] later introduced the concept of linear positive operators for $\eta \in \mathbb{N}$ as follows:

$$\hat{\mathcal{S}}_\eta(f, t) = e^{-\eta t} \sum_{k=0}^{\infty} \frac{(\eta t)^k}{k!} f\left(\frac{k}{\eta}\right), \tag{1}$$

where $t \in [0, \infty)$ and $f \in C[0, \infty)$ once the sum (1) converges. In recent years, numerous operators with specific modifications that preserve the test functions have emerged in this area, leading to significant advancements in achieving more accurate approximations.

In 1969, Jakimovski and Leviatan [8] utilized Appell polynomials to develop a generalization of Szász operators, which they defined as follows:
The polynomials $\mathcal{B}_k(x)$ of degree k ($k \in \mathbb{N} \cup \{0\}$), form Appell set if there exists a generating function of the following form

$$\sum_{k=0}^{\infty} \mathcal{B}_k(x) t^k = \mathcal{B}(t) \exp(xt), \tag{2}$$

where $\mathcal{B}(t)$ can be written as [1]:

$$\mathcal{B}(t) = \sum_{k=0}^{\infty} \mathcal{B}_k t^k, \quad \mathcal{B}_0 \neq 0 \tag{3}$$

be an analytic function in the disc $|t| < \mathcal{R}$ ($\mathcal{R} > 1$), where $\mathcal{R}$ is the radius of convergence and $\mathcal{B}(1) \neq 0$.

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Under the restriction $\mathcal{B}_k(x) \geq 0$ for $x \in [0, \infty)$, the linear positive operators $\mathcal{P}_n(f; x)$ were established by Jakimovski and Leviatan as [8]:

$$\mathcal{P}_n(f; x) = \frac{e^{-nx}}{\mathcal{B}(1)} \sum_{k=0}^{\infty} \mathcal{B}_k(nx) f\left(\frac{k}{n}\right), \quad \text{for } n \in \mathbb{N} \tag{4}$$

are obtained the approximation properties of this sequence operator. In particular, if we take $\mathcal{B}(1) = 1$ in (4), then we have the classical Szász operators defined in equation (1).

Recent research has extensively explored generalizations of Szász operators using special polynomials, particularly those derived from generating functions; see, for instance, [6, 12]. These generalizations have introduced a variety of new operator sequences, significantly expanding the scope of approximation theory.

Special functions are a crucial part of mathematics and physics, frequently appearing in the solutions of differential equations, integrals, and other complex problems. Examples include Gamma and Beta functions, which generalize factorials and are essential in complex analysis and probability, and Bessel functions, which arise in wave propagation and heat conduction. Legendre, Hermite, and Laguerre polynomials are pivotal in quantum mechanics, particularly in solving problems with spherical or radial symmetry. Hypergeometric functions provide a unifying framework for many of these classical functions. These functions enable the modeling of a wide range of physical phenomena and are foundational in approximation theory and mathematical analysis.

In 2013, Khan and Raza [9] introduced a family of 2-variable general polynomials $p_n(\xi, \zeta)$, which they defined using the following generating function:

$$e^{\xi t} \psi(\zeta, t) = \sum_{k=0}^{\infty} p_k(\xi, \zeta) \frac{t^k}{k!}, \qquad (p_0(\xi, \zeta) = 1),$$

where $\psi(\zeta, t)$ can be written as

$$\psi(\zeta, t) = \sum_{k=0}^{\infty} \psi_k(\zeta) \frac{t^k}{k!}, \qquad \psi_0(\zeta) \neq 0. \tag{5}$$

Khan and Raza [9] also introduced the 2-variable general-Appell polynomial family ${}_p\mathcal{B}_n(\xi, \zeta)$, defined by the following generating function:

$$\mathcal{B}(t) e^{\xi t} \psi(\zeta, t) = \sum_{k=0}^{\infty} {}_p\mathcal{B}_k(\xi, \zeta) \frac{t^k}{k!} \tag{6}$$

where $\mathcal{B}(t)$ and $\psi(\zeta, t)$ are given by equation (3) and (5), respectively.

Recent Contemporary work (for example, in [12, 3, 10, 2, 14, 13, 19]) concentrated on Chlodowsky variants of generalized Szász-type operators, and some convergence characteristics of these operators are explained by means of a weighted Korovkin-type theorem.

After these studies, Raza *et. al.* [13] presented the Chlodowsky [4] type generalization of Szász operators:

$$\mathcal{G}_\nu^{\mathcal{B}}(\hat{f}, \xi) = \frac{e^{-\frac{\nu}{\beta_\nu}\xi}}{\mathcal{B}(1)\psi(l; 1)} \sum_{k=0}^{\infty} \frac{{}_p\mathcal{B}_k(\frac{\nu}{\beta_\nu}\xi; l)}{k!} \hat{f}\left(\frac{k}{\nu}\beta_\nu\right), \qquad \xi \in [0, \infty), \tag{7}$$

where $l \geq 0$, β_ν is a positive increasing sequence with the properties

$$\lim_{\nu \to \infty} \beta_\nu = \infty, \qquad \lim_{\nu \to \infty} \frac{\beta_\nu}{\nu} = 0 \tag{8}$$

and ${}_p\mathcal{B}_k$ are the general-Appell polynomials defined in equation (6). If we set $\frac{\nu}{\beta_\nu} = n$, then the operator defined in equation (7) reduces to the classical approximation operators associated with the general-Appell polynomials of the Chlodowsky variant.

1.1 Preliminaries

In this section, we review some fundamental concepts that will be employed in the development of the main results.

Definition 1.1. *Assume that I is an arbitrary compact interval of the real line, and also assume that $L : C(I) \to C(I), M : C(I) \to C(I)$ are defined operators in a discrete sense.*

$$L\hat{f}(\xi) = \sum_{i=0}^{n} \hat{f}(\xi_i) p_i(\xi), \qquad \hat{f} \in C(I)$$

and

$$M\hat{f}(\zeta) = \sum_{k=0}^{m} \hat{f}(\zeta_k) q_k(\zeta), \qquad \hat{f} \in C(I)$$

where $\xi_i, \zeta_k \in I$ *are mutually distinct, and* $p_i, q_k \in C(I)$.

Definition 1.2. *Suppose that* $(\xi,\zeta) \in I \times I$. *The parametric extensions of* L *and* M *to* $C(I \times I)$ *are given by*

$$_{\xi}L\hat{f}(\xi,\zeta) = \sum_{i=0}^{n} \hat{f}(\xi_i,\zeta) p_i(\xi)$$

and

$$_{\zeta}M\hat{f}(\xi,\zeta) = \sum_{k=0}^{m} \hat{f}(\xi,\zeta_k) q_k(\zeta).$$

The tensor product of L and M is given by

$$T\hat{f}(\xi,\zeta) := ({}_{\xi}L \otimes {}_{\zeta}M)\hat{f}(\xi,\zeta) = \sum_{i=0}^{n}\sum_{k=0}^{m} \hat{f}(\xi_i,\zeta_k) p_i(\xi) q_k(\zeta), \qquad \hat{f} \in C(I \times I).$$

In the space of continuous functions on the compact set $I_a \times I_b := [0,a] \times [0,b] \subset C(I \times I)$. In the case of a bivariate function $\hat{f} \in C(I_a \times I_b)$, the complete modulus of continuity is defined as follows:

$$\omega(\hat{f};\delta) = \sup\left\{|\hat{f}(u,v) - \hat{f}(\xi,\zeta)| : (u,v),(\xi,\zeta) \in I_a \times I_b \text{ and } \sqrt{(u-\xi)^2 + (v-\zeta)^2} \le \delta\right\}.$$

The partial moduli of continuity concerning y and z are given by

$$_{\xi}\omega(\hat{f};\delta) = \sup\left\{|\hat{f}(u_1,\zeta) - \hat{f}(u_2,\zeta)| : z \in I_b \text{ and } |u_1 - u_2| \le \delta\right\}$$

and

$$_{\zeta}\omega(\hat{f};\delta) = \sup\left\{|\hat{f}(\xi,v_1) - \hat{f}(\xi,v_2)| : y \in I_a \text{ and } |v_1 - v_2| \le \delta\right\},$$

respectively. It is evident that these moduli satisfy the properties typically associated with the usual modulus of continuity.

The sequence of operators defined in equation (7) is applicable only to continuous univariate functions. Motivated by this limitation and aiming to broaden the scope of approximation theory, we extend the existing framework by introducing a new sequence of positive linear operators suitable for approximating functions in the bivariate setting. This development enhances the applicability of the operators and aligns with the overall objectives of the present work.

The upcoming sections are organized as follows: In Section 2, we initially propose the operator involving general-Appell polynomials. Further, moments and lemmas are derived to support the primary findings, local approximation results are obtained, and the Volkov theorem is utilized to analyze the convergence of certain operators using the modulus of continuity, Lipschitz class, and Voronovskaja-type theorem. Section 3 is devoted to the statistical approximation properties of the proposed operators, which are derived by applying a universal Korovkin-type statistical approximation theorem. Section 4: This section presents the error analysis with some numerical examples and their graphical representation, and we conclude this article with some remarks in section 5.

2 Local Approximation

In this section, initially, we define the operator, and then we obtain moments and some lemmas to support our main results. After the above-mentioned studies, the aim of this article is to define the bivariate generalization of linear positive operators involving general-Appell polynomials.

Definition 2.1. *Let* $I \times I := \{(\xi,\zeta) : 0 \le \xi \le \infty, 0 \le \zeta \le \infty\}$, *and* $C(I \times I) := \{\hat{f} : I \times I \to \mathbb{R}$ *is continuous*$\}$. *For* $\hat{f} \in C(I \times I)$, *the tensor product of* $\mathcal{G}_{\nu}^{\mathcal{B}}, \mathcal{G}_{\mu}^{\mathcal{B}}$ *is given by*

$$\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f};\xi,\zeta) \;:=\; ({}_{\xi}\mathcal{G}_{\nu}^{\mathcal{B}} \otimes {}_{\zeta}\mathcal{G}_{\mu}^{\mathcal{B}})\hat{f}(\xi,\zeta) \;=\; \frac{e^{-\frac{\nu}{\beta_\nu}\xi - \frac{\mu}{\theta_\mu}\zeta}}{\mathcal{B}_1(1)\mathcal{B}_2(1)\psi_1(l;1)\psi_2(l;1)} \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} \frac{{}_{p}\mathcal{B}_r(\frac{\nu}{\beta_\nu}\xi;l)}{r!} \frac{{}_{p}\mathcal{B}_s(\frac{m}{\theta_m}\zeta;l)}{s!} \hat{f}\left(\frac{r}{\nu}\beta_\nu, \frac{s}{\mu}\theta_\mu\right), \quad (9)$$

where $\mathcal{B}_i$ *and* ψ_i *are given by* (3) *and* (5), *respectively. Moreover,* $\mathcal{B}_i(1) \ne 0$ $\psi_i(l;1) \ne 0$ *for* $i = 1,2,\ l \ge 0$ *and* θ_μ *is defined in same manners as* β_ν *is defined in equation* (8).

Using the test functions $e_{i,j} = u^i v^j$ we present the following elementary results:

Lemma 2.1. *The operators defined in* (9) *satisfy the following results:*

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{0,0};\xi,\zeta)=1,$$

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{1,0};\xi,\zeta)=\xi+\frac{\beta_\nu}{\nu}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right),$$

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{0,1};\xi,\zeta)=\zeta+\frac{\theta_\mu}{\mu}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right),$$

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{2,0};\xi,\zeta) = \xi^2+\frac{\beta_\nu}{\nu}\xi\left(1+2\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+2\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)+\frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+\frac{\psi_1''(l;1)}{\psi_1(l;1)}\right),$$

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{0,2};\xi,\zeta) = \zeta^2+\frac{\theta_\mu}{\mu}\zeta\left(1+2\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+2\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right)+\frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+\frac{\psi_2''(l;1)}{\psi_2(l;1)}\right),$$

$$\begin{aligned}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{3,0};\xi,\zeta)=\xi^3&+3\frac{\beta_\nu}{\nu}\xi^2\left(1+\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)+\frac{\beta_\nu^2}{\nu^2}\xi\Big(1+6\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+3\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+6\frac{\psi_1'(l;1)}{\psi_1(l;1)}+6\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}\\&+3\frac{\psi_1''(l;1)}{\psi_1(l;1)}\Big)+\frac{\beta_\nu^3}{\nu^3}\Big(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+3\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+3\frac{\mathcal{B}_1'''(1)}{\mathcal{B}_1(1)}+6\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+3\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+3\frac{\psi_1''(l;1)}{\psi_1(l;1)}\\&+3\frac{\mathcal{B}_1'(1)\psi_1''(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+\frac{\psi_1'''(l;1)}{\psi_1(l;1)}\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{0,3};\xi,\zeta)=\zeta^3&+3\frac{\theta_\mu}{\mu}\zeta^2\left(1+\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right)+\frac{\theta_\mu^2}{\mu^2}\zeta\Big(1+6\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+3\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+6\frac{\psi_2'(l;1)}{\psi_2(l;1)}+6\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}\\&+3\frac{\psi_2''(l;1)}{\psi_2(l;1)}\Big)+\frac{\theta_\mu^3}{\mu^3}\Big(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+3\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+3\frac{\mathcal{B}_2'''(1)}{\mathcal{B}_2(1)}+6\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+3\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+3\frac{\psi_2''(l;1)}{\psi_2(l;1)}\\&+3\frac{\mathcal{B}_2'(1)\psi_2''(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+\frac{\psi_2'''(l;1)}{\psi_2(l;1)}\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{4,0};\xi,\zeta)=\xi^4&+\frac{\beta_\nu}{\nu}\xi^3\left(6+4\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+4\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)+\frac{\beta_\nu^2}{\nu^2}\xi^2\Big(7+18\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+6\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+18\frac{\psi_1'(l;1)}{\psi_1(l;1)}+12\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}\\&+6\frac{\psi_1''(l;1)}{\psi_1(l;1)}\Big)+\frac{\beta_\nu^3}{\nu^3}\xi\Big(1+14\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+18\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+4\frac{\mathcal{B}_1'''(1)}{\mathcal{B}_1(1)}+14\frac{\psi_1'(l;1)}{\psi_1(l;1)}+36\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+12\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}\\&+18\frac{\psi_1''(l;1)}{\psi_1(l;1)}+12\frac{\mathcal{B}_1'(1)\psi_1''(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+4\frac{\psi_1'''(l;1)}{\psi_1(l;1)}\Big)+\frac{\beta_\nu^4}{\nu^4}\Big(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+7\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+6\frac{\mathcal{B}_1'''(1)}{\mathcal{B}_1(1)}+\frac{\mathcal{B}_1^{iv}(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\\&+14\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+18\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+4\frac{\mathcal{B}_1'''(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+7\frac{\psi_1''(l;1)}{\psi_1(l;1)}+18\frac{\mathcal{B}_1'(1)\psi_1''(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}\\&+6\frac{\mathcal{B}_1''(1)\psi_1''(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+6\frac{\psi_1'''(l;1)}{\psi_1(l;1)}+4\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'''(l;1)}{\psi_1(l;1)}+\frac{\psi_1^{iv}(l;1)}{\psi_1(l;1)}\Big),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{0,4};\xi,\zeta)=\zeta^4&+\frac{\theta_\mu}{\mu}\zeta^3\left(6+4\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+4\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right)+\frac{\theta_\mu^2}{\mu^2}\zeta^2\Big(7+18\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+6\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+18\frac{\psi_2'(l;1)}{\psi_2(l;1)}+12\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}\\&+6\frac{\psi_2''(l;1)}{\psi_2(l;1)}\Big)+\frac{\theta_\mu^3}{\mu^3}\zeta\Big(1+14\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+18\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+4\frac{\mathcal{B}_2'''(1)}{\mathcal{B}_2(1)}+14\frac{\psi_2'(l;1)}{\psi_2(l;1)}+36\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+12\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}\\&+18\frac{\psi_2''(l;1)}{\psi_2(l;1)}+12\frac{\mathcal{B}_2'(1)\psi_2''(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+4\frac{\psi_2'''(l;1)}{\psi_2(l;1)}\Big)+\frac{\theta_\mu^4}{\mu^4}\Big(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+7\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+6\frac{\mathcal{B}_2'''(1)}{\mathcal{B}_2(1)}+\frac{\mathcal{B}_2^{iv}(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}\\&+14\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+18\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+4\frac{\mathcal{B}_2'''(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+7\frac{\psi_2''(l;1)}{\psi_2(l;1)}+18\frac{\mathcal{B}_2'(1)\psi_2''(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}\\&+6\frac{\mathcal{B}_2''(1)\psi_2''(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+6\frac{\psi_2'''(l;1)}{\psi_2(l;1)}+4\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'''(l;1)}{\psi_2(l;1)}+\frac{\psi_2^{iv}(l;1)}{\psi_2(l;1)}\Big).\end{aligned}$$

Proof. By using the generating function of general-Appell polynomials (6), the proof is relatively simple to follow, and therefore, the detailed steps can be omitted. □

In the following lemma, we obtain the central moments of the operators (7).

Lemma 2.2. *For the operators (7), we have the following equalities:*

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(u-\xi;\xi,\zeta)=\frac{\beta_\nu}{\nu}\left(\frac{\mathcal{B}'(1)}{\mathcal{B}(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right),$$

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(v-\zeta;\xi,\zeta)=\frac{\theta_\mu}{\mu}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right),$$

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2;\xi,\zeta)=\frac{\beta_\nu}{\nu}\xi+\frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+\frac{\psi_1''(l;1)}{\psi_1(l;1)}\right),$$

$$\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-\zeta)^2;\xi,\zeta)=\frac{\theta_\mu}{\mu}\zeta+\frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+\frac{\psi_2''(l;1)}{\psi_2(l;1)}\right),$$

$$\begin{aligned}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^4;\xi,\zeta)&=3\frac{\beta_\nu^2}{\nu^2}\xi^2+\frac{\beta_\nu^3}{\nu^3}\xi\Bigg(1+10\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+6\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+10\frac{\psi_1'(l;1)}{\psi_1(l;1)}+12\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+6\frac{\psi_1''(l;1)}{\psi_1(l;1)}\Bigg)\\&+\frac{\beta_\nu^4}{\nu^4}\Bigg(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+7\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+6\frac{\mathcal{B}_1'''(1)}{\mathcal{B}_1(1)}+\frac{\mathcal{B}_1^{iv}(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+14\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+18\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}+4\frac{\mathcal{B}_1'''(1)}{\mathcal{B}_1(1)}\frac{\psi_1'(l;1)}{\psi_1(l;1)}\\&+7\frac{\psi_1''(l;1)}{\psi_1(l;1)}+18\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1''(l;1)}{\psi_1(l;1)}+6\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}\frac{\psi_1''(l;1)}{\psi_1(l;1)}+6\frac{\psi_1'''(l;1)}{\psi_1(l;1)}+4\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}\frac{\psi_1'''(l;1)}{\psi_1(l;1)}+\frac{\psi_1^{iv}(l;1)}{\psi_1(l;1)}\Bigg),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-\zeta)^4;\xi,\zeta)&=3\frac{\theta_\mu^2}{\mu^2}\zeta^2+\frac{\theta_\mu^3}{\mu^3}\zeta\Bigg(1+10\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+6\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+10\frac{\psi_2'(l;1)}{\psi_2(l;1)}+12\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+6\frac{\psi_2''(l;1)}{\psi_2(l;1)}\Bigg)\\&+\frac{\theta_\mu^4}{\mu^4}\Bigg(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+7\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+6\frac{\mathcal{B}_2'''(1)}{\mathcal{B}_2(1)}+\frac{\mathcal{B}_2^{iv}(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+14\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+18\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}+4\frac{\mathcal{B}_2'''(1)}{\mathcal{B}_2(1)}\frac{\psi_2'(l;1)}{\psi_2(l;1)}\\&+7\frac{\psi_2''(l;1)}{\psi_2(l;1)}+18\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2''(l;1)}{\psi_2(l;1)}+6\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}\frac{\psi_2''(l;1)}{\psi_2(l;1)}+6\frac{\psi_2'''(l;1)}{\psi_2(l;1)}+4\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}\frac{\psi_2'''(l;1)}{\psi_2(l;1)}+\frac{\psi_2^{iv}(l;1)}{\psi_2(l;1)}\Bigg).\end{aligned}$$

Proof. By applying Lemma 2.1, the proof becomes easy to follow. Therefore, the detailed steps can be omitted. □

Throughout this paper, we consider $C(I_a\times I_b)$ is the space of all bounded and continuous functions on $I_a\times I_b=[0,a]\times[0,b]$ endowed with

$$\|\hat{f}\|_{C(I_a\times I_b)}=\sup_{(\xi,\zeta)\in I_a\times I_b}|\hat{f}(\xi,\zeta)|.$$

Theorem 2.3. *For $\hat{f}\in C(I\times I)\cap\mathbb{T}$, we obtain that*

$$\lim_{\nu,\mu\to\infty}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta)=\hat{f}(\xi,\zeta)$$

uniformly on each compact subset $I_a\times I_b$ of $I\times I$. Here, we consider the set of weighted functions as

$$\mathbb{T}=\left\{\hat{f}:|\hat{f}(\xi,\zeta)|\le\kappa_{\hat{f}}\zeta(\xi,\zeta),\zeta(\xi,\zeta)=1+\xi^2+\zeta^2\right\},$$

where $\kappa_{\hat{f}}$ is fixed and depends only on $\hat{f}$.

Proof. In view of Lemma 2.1, we have

$$\lim_{\nu,\mu\to\infty}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{i,j};\xi,\zeta)=e_{i,j},\qquad(i,j)\in\{(0,0),(1,0),(0,1)\},$$

$$\lim_{\nu,\mu\to\infty}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(e_{2,0}+e_{0,2};\xi,\zeta)=e_{2,0}+e_{0,2}$$

uniformly on $I_a\times I_b$. If we apply the Volkov theorem in [17], then we complete the proof. □

Now, in the following theorem, we establish the degree of approximation for the operators defined by (9).

Theorem 2.4. *For $\hat{f} \in C(I_a \times I_b)$ and for all $(\xi, \zeta) \in I_a \times I_b$, the following inequality holds:*

$$|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)| \leq 2\omega(\hat{f};\delta_{\nu,\mu}),$$

where $\delta_{\nu,\mu} := \left\{\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2;\xi,\zeta) + \mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-\zeta)^2;\xi,\zeta)\right\}^{\frac{1}{2}}$.

Proof. With the help of the complete modulus of continuity, we get

$$|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)| \leq \mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|\hat{f}(u,v) - \hat{f}(\xi,\zeta)|;\xi,\zeta)$$

$$|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)| \leq \mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\omega(\hat{f};\sqrt{(u-\xi)^2+(v-\zeta)^2});\xi,\zeta),$$

which implies

$$|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)| \leq \omega(\hat{f};\delta_{\nu,\mu})\left\{1 + \frac{1}{\delta_{\nu,\mu}}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\sqrt{(u-\xi)^2+(v-\zeta)^2};\xi,\zeta)\right\}.$$

By making use of Cauchy-Schwartz inequality and Lemma 2.2, we have

$$|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)| \leq \omega(\hat{f};\delta_{\nu,\mu})\left\{1 + \frac{1}{\delta_{\nu,\mu}}\left\{\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2+(v-\zeta)^2;\xi,\zeta)\right\}^{\frac{1}{2}}\right\},$$

due to the linearity of operators, we have

$$|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)| \leq \omega(\hat{f};\delta_{\nu,\mu})\left\{1 + \frac{1}{\delta_{\nu,\mu}}\left\{\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2;\xi,\zeta) + \mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-\zeta)^2;\xi,\zeta)\right\}^{\frac{1}{2}}\right\}.$$

Choosing $\delta_{\nu,\mu} = \left\{\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2;\xi,\zeta) + \mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-\zeta)^2;\xi,\zeta)\right\}^{\frac{1}{2}}$, we get assertion (2.4). □

Theorem 2.5. *For $\hat{f} \in C(I_a \times I_b)$ and for all $(\xi, \zeta) \in I_a \times I_b$, the following inequality holds:*

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)\right| \leq 2({}_{\xi}\omega(\hat{f};\delta_{\nu}) + {}_{\zeta}\omega(\hat{f};\delta_{\mu})), \tag{10}$$

where

$$\delta_{\nu}^2 := \frac{\beta_\nu}{\nu}\xi + \frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)} + \frac{\psi_1'(l;1)}{\psi_1(l;1)} + 2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)} + \frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)} + \frac{\psi_1''(l;1)}{\psi_1(l;1)}\right)$$

and

$$\delta_{\mu}^2 := \frac{\theta_\mu}{\mu}\zeta + \frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)} + \frac{\psi_2'(l;1)}{\psi_2(l;1)} + 2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)} + \frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)} + \frac{\psi_2''(l;1)}{\psi_2(l;1)}\right).$$

Proof. Applying the definition of the moduli of partial continuity, along with Lemma 2.2 and the Cauchy-Schwarz inequality, we obtain:

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)\right| \leq \mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|\hat{f}(u,v) - \hat{f}(\xi,\zeta)|;\xi,\zeta)$$

or, equivalently

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)\right| \leq \mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|\hat{f}(u,v) - \hat{f}(\xi,v)|;\xi,\zeta) + \mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|\hat{f}(\xi,v) - \hat{f}(\xi,\zeta)|;\xi,\zeta)$$

because of the definition of partial moduli of continuity, we get

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)\right| \leq \mathcal{G}^{\mathcal{B}}_{\nu,\mu}({}_{\xi}\omega(\hat{f};|u-\xi|);\xi,\zeta) + \mathcal{G}^{\mathcal{B}}_{\nu,\mu}({}_{\zeta}\omega(\hat{f};|v-\zeta|);\xi,\zeta)$$

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)\right| \leq {}_{\xi}\omega(\hat{f};\delta_\nu)\left[1 + \frac{1}{\delta_\nu}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|u-\xi|;\xi,\zeta)\right] + {}_{\zeta}\omega(\hat{f},\delta_\mu)\left[1 + \frac{1}{\delta_\mu}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|v-\zeta|;\xi,\zeta)\right]$$

by using Cauchy-Schwartz inequality, we have

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)\right| \leq {}_{\xi}\omega(\hat{f};\delta_\nu)\left[1 + \frac{1}{\delta_\nu}(\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2;\xi,\zeta))^{\frac{1}{2}}\right] + {}_{\zeta}\omega(\hat{f},\delta_\mu)\left[1 + \frac{1}{\delta_\mu}(\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-\zeta)^2;\xi,\zeta))^{\frac{1}{2}}\right].$$

By choosing

$$\delta_{\nu}^2 := \frac{\beta_\nu}{\nu}\xi + \frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)} + \frac{\psi_1'(l;1)}{\psi_1(l;1)} + 2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)} + \frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)} + \frac{\psi_1''(l;1)}{\psi_1(l;1)}\right)$$

and

$$\delta_{\mu}^2 := \frac{\theta_\mu}{\mu}\zeta + \frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)} + \frac{\psi_2'(l;1)}{\psi_2(l;1)} + 2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)} + \frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)} + \frac{\psi_2''(l;1)}{\psi_2(l;1)}\right),$$

we obtain assertion (10). □

We will now determine the degree of approximation for the bivariate operators (9) using the concept of the Lipschitz class. For $0<\Theta_1\le 1$ and $0<\Theta_2\le 1$, and for $\hat f\in C(I\times I)$, we define the Lipschitz class $Lip_{\mathcal{M}}(\Theta_1,\Theta_2)$ for the bivariate case as follows:

$$|\hat f(u,v)-\hat f(\xi,\zeta)|\le \mathcal{M}|u-\xi|^{\Theta_1}|v-\zeta|^{\Theta_2}. \tag{11}$$

Theorem 2.6. *For $\hat f\in Lip_{\mathcal{M}}(\Theta_1,\Theta_2)$, we have the following inequality:*

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((\hat f;\xi,\zeta)-\hat f(\xi,\zeta)\right|\le \mathcal{M}\delta_\nu^{\Theta_1}\delta_\mu^{\Theta_2}, \tag{12}$$

where δ_ν and δ_μ are defined in the same manner as in the above theorem.

Proof. Since $\hat f\in Lip_{\mathcal{M}}(\Theta_1,\Theta_2)$, we may write

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat f;\xi,\zeta)-\hat f(\xi,\zeta)\right|\le \mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|\hat f(u,v)-\hat f(\xi,\zeta)|;\xi,\zeta)$$

in view of (11), we have

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat f;\xi,\zeta)-\hat f(\xi,\zeta)\right|\le \mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\mathcal{M}|u-\xi|^{\Theta_1}|v-\zeta|^{\Theta_2};\xi,\zeta)$$

or, equivalently

$$\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat f;\xi,\zeta)-\hat f(\xi,\zeta)\right|\le \mathcal{M}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|u-\xi|^{\Theta_1};\xi,\zeta)\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|v-\zeta|^{\Theta_2};\xi,\zeta).$$

By applying Hölder's inequality with exponents $\left(\frac{2}{\Theta_1},\frac{2}{2-\Theta_1}\right)$ and $\left(\frac{2}{\Theta_2},\frac{2}{2-\Theta_2}\right)$, we can obtain the following inequality:

$$\begin{aligned}\left|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat f;\xi,\zeta)-\hat f(\xi,\zeta)\right|&\le \mathcal{M}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2;\xi,\zeta)^{\Theta_1/2}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2;\xi,\zeta)^{(2-\Theta_1)/2}\\&\times\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-\zeta)^2;\xi,\zeta)^{\Theta_2/2}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-\zeta)^2;\xi,\zeta)^{(2-\Theta_2)/2}.\end{aligned}$$

Choosing δ_ν and δ_μ same as in the previous theorem, proves assertion (12). □

We will now establish the Voronovskaja-type theorem for the operators $\mathcal{G}^{\mathcal{B}}_{\nu,\mu}$. For this theorem, we define the space $\mathbb{E}(I\times I)$ as the set of all functions on $I\times I$ satisfying $|\hat f(\xi,\zeta)|\le\kappa_{\hat f}(1+\xi^2+\zeta^2)$. We denote $C_E(I\times I)$ as the space of all continuous functions within $\mathbb{E}(I\times I)$, and $C_B(I\times I)$ as the subspace of all $\hat f\in C_E(I\times I)$ for which $\lim_{(\xi,\zeta)\to\infty}\frac{\hat f(\xi,\zeta)}{1+\xi^2+\zeta^2}$ is finite.

Theorem 2.7. *If $\hat f\in C_B(I\times I)$ such that $\hat f',\hat f''\in C_B(I\times I)$ and (ξ,ζ) be in each compact subset of $I_a\times I_b$ of $I\times I$, then we get*

$$\lim_{\nu\to\infty}\nu(\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat f;\xi,\zeta)-\hat f(\xi,\zeta))=\hat f_\xi(\xi,\zeta))\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)+\hat f_\zeta(\xi,\zeta)\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right)+\frac12\left(\xi\hat f_{\xi\xi}(\xi,\zeta)+\zeta\hat f_{\zeta\zeta}(\xi,\zeta)\right). \tag{13}$$

Proof. Let $(\xi,\zeta)\in I_a\times I_b$. From the Taylor's expansion formula, we have

$$\begin{aligned}\hat f(u,v)&=\hat f(\xi,\zeta)+\hat f_\xi(\xi,\zeta)(u-\xi)+\hat f_\zeta(\xi,\zeta)(v-\zeta)+\frac12\Big\{\hat f_{\xi\xi}(\xi,\zeta)(u-\xi)^2+2\hat f_{\xi\zeta}(\xi,\zeta)(u-\xi)(v-\zeta)\\&+\hat f_{\zeta\zeta}(\xi,\zeta)(v-\zeta)^2\Big\}+\sigma(u,v;\xi,\zeta)\sqrt{(u-\xi)^4+(v-\zeta)^4},\end{aligned}$$

where $(u,v)\in I\times I$, $\sigma(u,v;\xi,\zeta)\in C(I\times I)$ and $\sigma(u,v;\xi,\zeta)\to 0$ as $(u,v)\to(\xi,\zeta)$.
Operating $\mathcal{G}^{\mathcal{B}}_{\nu,\nu}(\hat f;\xi,\zeta)$ on the proceeding equation, we obtain

$$\begin{aligned}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}(\hat f;\xi,\zeta)&=\hat f(\xi,\zeta)+\hat f_\xi(\xi,\zeta)\mathcal{G}^{\mathcal{B}}_{\nu,\nu}(u-\xi;\xi,\zeta)+\hat f_\zeta(\xi,\zeta)\mathcal{G}^{\mathcal{B}}_{\nu,\nu}(v-\zeta;\xi,\zeta)\\&+\frac12\Big[\hat f_{\xi\xi}(\xi,\zeta)\mathcal{G}^{\mathcal{B}}_{\nu,\nu}((u-\xi)^2;\xi,\zeta)+2\hat f_{\xi\zeta}(\xi,\zeta)\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)(v-\zeta);\xi,\zeta)\\&+\hat f_{\zeta\zeta}(\xi,\zeta)\mathcal{G}^{\mathcal{B}}_{\nu,\nu}((v-\zeta)^2;\xi,\zeta)\Big]+\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left(\sigma(u,v;\xi,\zeta)\sqrt{(u-\xi)^4+(v-\zeta)^4};\xi,\zeta\right)\end{aligned}$$

by multiplying with $\frac{\nu}{\beta_\nu}$ and taking limit as $\nu\to\infty$ and in view of Lemma 2.2, we get

$$\begin{aligned}\lim_{\nu\to\infty}\frac{\nu}{\beta_\nu}\left(\mathcal{G}^{\mathcal{B}}_{\nu,\nu}(\hat f;\xi,\zeta)-\hat f(\xi,\zeta)\right)&=\hat f_\xi(\xi,\zeta)\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)+\hat f_\zeta(\xi,\zeta)\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right)\\&+\frac12\left[y\hat f_{\xi\xi}(\xi,\zeta)+2\hat f_{\xi\zeta}(\xi,\zeta)\lim_{\nu\to\infty}\frac{\nu}{\beta_\nu}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}((u-\xi)(v-\zeta);\xi,\zeta)+z\hat f_{\zeta\zeta}(\xi,\zeta)\right]\\&+\lim_{\nu\to\infty}\frac{\nu}{\beta_\nu}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left(\sigma(u,v;\xi,\zeta)\sqrt{(u-\xi)^4+(v-\zeta)^4};\xi,\zeta\right).\end{aligned} \tag{14}$$

Additionally, in view of [13, Lemma 2.4], we have

$$\lim_{\nu\to\infty}\frac{\nu}{\beta_\nu}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}((u-\xi)(v-\zeta);\xi,\zeta)=\lim_{\nu\to\infty}\frac{\nu}{\beta_\nu}\left(\mathcal{G}^{\mathcal{B}}_{\nu}(u-\xi;x)\mathcal{G}^{\mathcal{B}}_{\nu}(v-\zeta;\xi)\right)=0. \tag{15}$$

From the Cauchy-Schwartz inequality, we get

$$\begin{aligned}&\frac{\nu}{\beta_\nu}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left(\sigma(u,v;\xi,\zeta)\sqrt{(u-\xi)^4+(v-\zeta)^4};\xi,\zeta\right)\\&\leq\left(\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left(\sigma^2(u,v;\xi,\zeta);\xi,\zeta\right)\right)^{\frac{1}{2}}\left(\frac{\nu^2}{\beta_\nu^2}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left((u-\xi)^4+(v-\zeta)^4;\xi,\zeta\right)\right)^{\frac{1}{2}}\\&=\left(\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left(\sigma^2(u,v;\xi,\zeta);\xi,\zeta\right)\right)^{\frac{1}{2}}\left(\frac{\nu^2}{\beta_\nu^2}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left((u-\xi)^4;\xi,\zeta\right)+\frac{\nu^2}{\beta_\nu^2}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left((v-\zeta)^4;\xi,\zeta\right)\right)^{\frac{1}{2}}.\end{aligned}$$

Since, $\sigma^2(u,v;\xi,\zeta)\to 0$ as $(u,v)\to(\xi,\zeta)$. Hence, by Theorem 2.3, we have that

$$\lim_{\nu\to\infty}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left(\sigma^2(u,v;\xi,\zeta)\right)=0$$

uniformaly with respect to $(\xi,\zeta)\in I_a\times I_b$. Also, with the aid of Lemma 2.2, we obtain

$$\lim_{\nu\to\infty}\frac{\nu^2}{\beta_\nu^2}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left((u-\xi)^4;\xi,\zeta\right)=3x^2$$

and

$$\lim_{\nu\to\infty}\frac{\nu^2}{\beta_\nu^2}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left((v-\zeta)^4;\xi,\zeta\right)=3\xi^2.$$

Hence,

$$\lim_{\nu\to\infty}\frac{\nu}{\beta_\nu}\mathcal{G}^{\mathcal{B}}_{\nu,\nu}\left(\sigma(u,v;\xi,\zeta)\sqrt{(u-\xi)^4+(v-\zeta)^4};\xi,\zeta\right)=0. \tag{16}$$

Finally, using equations (16) and (15), in Equation (14), we get assertion (13). □

In the following section, we present an alternative notion of convergence, namely A-Statistical Convergence.

3 A-Statistical Convergence

In this section, first, we recall the concept of A-statistical convergence for double sequences, then we establish a Korovkin-type theorem for A-statistical convergence of operators $\mathcal{G}^{\mathcal{B}}_{\nu,\nu}$.

Let $A=(\alpha_{i,j,m,n})$ be a four-dimensional summability matrix. For a double sequence $\xi=(\xi_{m,n})$, the A-transform of ξ denoted by $A\xi=((A\xi)_{i,j})$, is given by

$$(A\xi)_{i,j}=\sum_{(m,n)\in\mathbb{N}^2}\alpha_{i,j,m,n}\xi_{m,n},$$

provided that the double series converges in Pringsheim's sense for every $(i,j)\in\mathbb{N}^2$.
A two-dimensional matrix transformation is called regular if it maps every convergent sequence to a convergent sequence having the same limit. The definition and characterization of regularity for four-dimensional matrices is known as the Robison-Hamilton conditions, or, more briefly, RH-regularity, see([7, 11, 15, 20, 21]).

Recall that a four-dimensional matrix $A=(\alpha_{i,j,m,n})$ is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The Robison-Hamilton conditions assert that a four-dimensional matrix $A=(\alpha_{i,j,m,n})$ is RH-regular if and only if

- $P-\lim_{i,j}\alpha_{i,j,m,n}=0$ for each $(m,n)\in\mathbb{N}^2$,
- $P-\lim_{i,j}\sum_{(m,n)\in\mathbb{N}^2}\alpha_{i,j,m,n}=1$,
- $P-\lim_{i,j}\sum_{m\in\mathbb{N}}|\alpha_{i,j,m,n}|=0$ for each $n\in\mathbb{N}$,
- $P-\lim_{i,j}\sum_{n\in\mathbb{N}}|\alpha_{i,j,m,n}|=0$ for each $m\in\mathbb{N}$,
- $\sum_{(m,n)\in\mathbb{N}^2}|\alpha_{i,j,m,n}|$ is P-convergent for each $i,j\in\mathbb{N}$,
- there exist finite positive integers A and B such that $\sum_{m,n>B}|a_{j,k,m,n}|<A$ for every $(i,j)\in\mathbb{N}^2$.

Now, let $A=(\alpha_{i,j,m,n})$ be a non-negative RH-regular summability matrix, and let $\mathcal{K}\subset\mathbb{N}^2$. Then A-density of $\mathcal{K}$ is given by

$$\sigma_A^{(2)}\{\mathcal{K}\}:=P-\lim_{i,j}\sum_{(m,n)\in\mathcal{K}}\alpha_{i,j,m,n}$$

provided that the limit on the right-hand side exists in the sense of Pringsheim.
A real double sequence $\xi=(\xi_{m,n})$ is called A-statistically convergent to a number ℓ if, for each $\varepsilon>0$,

$$\sigma_A^{(2)}\{(m,n)\in\mathbb{N}^2:|\xi_{m,n}-\ell|\geq\varepsilon\}=0.$$

In this case, we denote $st_A^{(2)}-\lim_{m,n}\xi_{m,n}=\ell$. It is evident that every P-convergent double sequence is also A-statistically convergent to the same limit; however, the converse does not necessarily hold.

Now, we establish the following Korovkin-type theorem for A-statistical convergence:

Theorem 3.1. *Let $A=(\alpha_{i,j,m,n})$ be a non-negative RH-regular summability matrix. For $\hat{f}\in C(I_a\times I_b)$ and for all $(\xi,\zeta)\in I_a\times I_b$ the following result holds:*

$$st_A^{(2)}-\lim_{\nu,\mu}\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f})-\hat{f}\|_{C(I_a\times I_b)}=0.$$

Proof. In light of the criteria provided in [5, Theorem 2.1], we assert that

$$st_A^{(2)}-\lim_{\nu,\mu}\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_l)-\hat{f}_l\|_{C(I_a\times I_b)}=0, \tag{17}$$

where $\hat{f}_0=1$, $\hat{f}_1=\xi$, $\hat{f}_2=\zeta$, and $\hat{f}_3=\xi^2+\zeta^2$.
The following equality holds by virtue of Lemma 2.1.

$$st_A^{(2)}-\lim_{\nu,\mu}\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_0)-\hat{f}_0\|_{C(I_a\times I_b)}=0.$$

Consequently, the result guarantees the validity of equation (17) when $l=0$.
Now, by Lemma 2.1, we obtain

$$st_A^{(2)}-\lim_{\nu,\mu}\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_1)-\hat{f}_1\|_{C(I_a\times I_b)}=\frac{\beta_\nu}{\nu}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right).$$

For $\varepsilon>0$, we define the following sets

$$U_1:=\left\{(\nu,\mu):\left\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_1)-\hat{f}_1\right\|_{C(I_a\times I_b)}\geq\varepsilon'\right\},\quad U_2:=\left\{(\nu,\mu):\frac{\beta_\nu}{\nu}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)\geq\varepsilon-\varepsilon'\right\}.$$

Because $U_1\subset U_2$, we deduce that

$$\sigma_A^{(2)}\left(\left\{(\nu,\mu):\left\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_1)-\hat{f}_1\right\|_{C(I_a\times I_b)}\geq\varepsilon'\right\}\right)\leq\sigma_A^{(2)}\left(\left\{(\nu,\mu):\frac{\beta_\nu}{\nu}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)\geq\varepsilon-\varepsilon'\right\}\right).$$

Since

$$st_A^{(2)}-\lim_{\nu,\mu}\frac{\beta_\nu}{\nu}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)=0,$$

which implies

$$\sigma_A^{(2)}\left(\left\{(\nu,\mu):\left\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_1)-\hat{f}_1\right\|_{C(I_a\times I_b)}\geq\varepsilon'\right\}\right)=0.$$

Hence

$$st_A^{(2)}-\lim_{\nu,\mu}\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_1)-\hat{f}_1\|_{C(I_a\times I_b)}=0.$$

Similarly

$$st_A^{(2)}-\lim_{\nu,\mu}\|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_2)-\hat{f}_2\|_{C(I_a\times I_b)}=0.$$

Thus, we conclude that equation (17) holds for $l=1,2$. Finally, noting that

$$\begin{aligned}\left|\mathcal{G}_{\nu,\mu}^{\mathcal{B}}(\hat{f}_3)-\hat{f}_3\right|&\leq\left|\frac{\beta_\nu}{\nu}\left(1+2\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+2\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)\right|\xi\\&+\left|\frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+\frac{\psi_1''(l;1)}{\psi_1(l;1)}\right)\right|+\left|\frac{\theta_\mu}{\mu}\left(1+2\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+2\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right)\right|\zeta\\&+\left|\frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+\frac{\psi_2''(l;1)}{\psi_2(l;1)}\right)\right|.\end{aligned}$$

Now,

$$\left\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f}_3)-\hat{f}_3\right\|_{C(I_a\times I_b)} \leq \left|\frac{\beta_\nu}{\nu}\left(1+2\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+2\frac{\psi_1'(l;1)}{\psi_1(l;1)}\right)\right|a$$
$$+\left|\frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+\frac{\psi_1''(l;1)}{\psi_1(l;1)}\right)\right|+\left|\frac{\theta_\mu}{\mu}\left(1+2\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+2\frac{\psi_2'(l;1)}{\psi_2(l;1)}\right)\right|b$$
$$+\left|\frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+\frac{\psi_2''(l;1)}{\psi_2(l;1)}\right)\right|.$$

Applying the A-statistical limit to both sides of the preceding inequality, it follows that

$$st_A^{(2)}-\lim_{\nu,\mu}\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f}_3)-\hat{f}_3\|_{C(I_a\times I_b)}=0.$$

Thus, equation (17) remains valid in the case $l=3$. Since, $\mathcal{G}^{\mathcal{B}}_{\nu,\mu}$ is a double sequence of positive linear operators. Thus, the desired result follows directly from [5, Theorem 2.1]. □

Using the framework of the four-dimensional summability matrix A, we obtain the following result describing the rate of convergence.

Theorem 3.2. *Let $A=(\alpha_{i,j,m,n})$ be a non-negative RH-regular summability matrix. For $\hat{f}\in C(I_a\times I_b)$, $\forall(\xi,\zeta)\in I_a\times I_b$, and $(\xi_{\mu,\nu})$ be a positive double sequence so that $\omega(\hat{f};\delta^{a,b}_{\nu,\mu})=st_A^{(2)}-o(\xi_{\nu,\mu})$, then the following result holds:*

$$\left\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f})-\hat{f}\right\|_{C(I_a\times I_b)}\leq st_A^{(2)}-o(\xi_{\nu,\mu}), \tag{18}$$

where

$$\delta^{a,b}_{\nu,\mu}:=\Bigg\{\frac{\beta_\nu}{\nu}a+\frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+\frac{\psi_1''(l;1)}{\psi_1(l;1)}\right)$$
$$+\frac{\theta_\mu}{\mu}b+\frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+\frac{\psi_2''(l;1)}{\psi_2(l;1)}\right)\Bigg\}^{\frac{1}{2}}.$$

Proof. With the help of the complete modulus of continuity, we get

$$|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta)-\hat{f}(\xi,\zeta)|\leq\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(|\hat{f}(u,v)-\hat{f}(\xi,\zeta)|;\xi,\zeta)$$

which implies

$$|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f};\xi,\zeta)-\hat{f}(\xi,\zeta)| \leq \omega(\hat{f};\delta_{\nu,\mu})\left\{1+\frac{1}{\delta^2_{\nu,\mu}}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2+(v-\zeta)^2;\xi,\zeta)\right\}$$
$$\leq \omega(\hat{f};\delta_{\nu,\mu})+\frac{\omega(\hat{f};\delta_{\nu,\mu})}{\delta^2_{\nu,\mu}}\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-\xi)^2+(v-\zeta)^2;\xi,\zeta).$$

Now, taking the supremum over $(\xi,\zeta)\in I_a\times I_b$, we have

$$\left\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f})-\hat{f}\right\|_{C(I_a\times I_b)} \leq \omega(\hat{f};\delta_{\nu,\mu})+\frac{\omega(\hat{f};\delta_{\nu,\mu})}{\delta^2_{\nu,\mu}}\left\{\left\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((u-.)^2)\right\|+\left\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}((v-.)^2)\right\|\right\},$$
$$\leq \omega(\hat{f};\delta_{\nu,\mu})+\frac{\omega(\hat{f};\delta_{\nu,\mu})}{\delta^2_{\nu,\mu}}\Bigg\{\frac{\beta_\nu}{\nu}a+\frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+\frac{\psi_1''(l;1)}{\psi_1(l;1)}\right)$$
$$+\frac{\theta_\mu}{\mu}b+\frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+\frac{\psi_2''(l;1)}{\psi_2(l;1)}\right)\Bigg\}.$$

Now, choosing $\delta_{\nu,\mu}$ as

$$\delta^{a,b}_{\nu,\mu}:=\Bigg\{\frac{\beta_\nu}{\nu}a+\frac{\beta_\nu^2}{\nu^2}\left(\frac{\mathcal{B}_1'(1)}{\mathcal{B}_1(1)}+\frac{\psi_1'(l;1)}{\psi_1(l;1)}+2\frac{\mathcal{B}_1'(1)\psi_1'(l;1)}{\mathcal{B}_1(1)\psi_1(l;1)}+\frac{\mathcal{B}_1''(1)}{\mathcal{B}_1(1)}+\frac{\psi_1''(l;1)}{\psi_1(l;1)}\right)$$
$$+\frac{\theta_\mu}{\mu}b+\frac{\theta_\mu^2}{\mu^2}\left(\frac{\mathcal{B}_2'(1)}{\mathcal{B}_2(1)}+\frac{\psi_2'(l;1)}{\psi_2(l;1)}+2\frac{\mathcal{B}_2'(1)\psi_2'(l;1)}{\mathcal{B}_2(1)\psi_2(l;1)}+\frac{\mathcal{B}_2''(1)}{\mathcal{B}_2(1)}+\frac{\psi_2''(l;1)}{\psi_2(l;1)}\right)\Bigg\}^{\frac{1}{2}},$$

we obtain the following inequality for any positive integers ν, μ

$$\left\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f})-\hat{f}\right\|_{C(I_a\times I_b)}\leq 2\omega(\hat{f};\delta^{a,b}_{\nu,\mu}).$$

Hence, we have

$$\frac{1}{\xi_{\nu,\mu}} \sum_{\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f})-\hat{f}\|_{C(I_a\times I_b)}\geq\epsilon} \alpha_{i,j,m,n} \leq \frac{1}{\xi_{\nu,\mu}} \sum_{\omega(\hat{f};\delta^{a,b}_{\nu,\mu})\geq\frac{\epsilon}{2}} \alpha_{i,j,m,n}.$$

For any $\sigma > 0$, the assumption made above ensures that

$$\left\|\mathcal{G}^{\mathcal{B}}_{\nu,\mu}(\hat{f}) - \hat{f}\right\|_{C(I_a\times I_b)} \leq st_A^{(2)} - o(\xi_{\nu,\mu}).$$

□

In the next section, we present numerical examples to illustrate and verify the convergence behavior of the operators.

4 Numerical Examples

Example 4.1. In this example, we take the tensor product of the operators defined in [13, Example 4.2]. So, after taking the tensor product, we have the following operator:

$$\mathcal{G}^{_eH}_{\nu,\mu}(\hat{f};\xi,\zeta) := ({}_\xi\mathcal{G}^{_eH}_{\nu} \otimes {}_\zeta\mathcal{G}^{_eH}_{\mu})\hat{f}(\xi,\zeta) = \left(1-\left(\frac{1}{2}\right)^r\right)^2 \exp\left(-\frac{\nu}{2\beta_\nu}\xi - \frac{\mu}{2\theta_\mu}\zeta - \frac{2l}{2^{d+1}}\right)$$

$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{{}_eH_p^{(d+1,r)}(\frac{\nu}{\beta_\nu}\xi;l)}{2^p p!} \frac{{}_eH_q^{(d+1,r)}(\frac{\mu}{\theta_\mu}\zeta;l)}{2^q q!} \hat{f}\left(\frac{p}{\nu}\beta_\nu, \frac{q}{\mu}\theta_\mu\right). \quad (19)$$

For $\nu = \mu = 20, 30, 40$; $\beta_\nu = \nu^{\frac{1}{2}}$, $\theta_\mu = \mu^{\frac{1}{2}}$, $l = 2.9$, $d = 3$ and $r = 2$ Figure-1 illustrates the convergence of the operator (19) to the function

$$\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)e^{\xi-\zeta},$$

and Figure-3 illustrates the convergence of the operator (19) to the function

$$\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)\cos(\xi).$$

Further, we estimate the absolute error $\mathfrak{E}_{\nu,\mu} = \left|\mathcal{G}^{_eH}_{\nu,\mu}(\hat{f};\xi,\zeta) - \hat{f}(\xi,\zeta)\right|$ for different values of ν,μ shown in Figure-2 and Figure-4.

Tables-1 and 2, illustrate the error of approximation of $\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)e^{\xi-\zeta}$ and $\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)\cos(\xi)$ at different points of intervals.

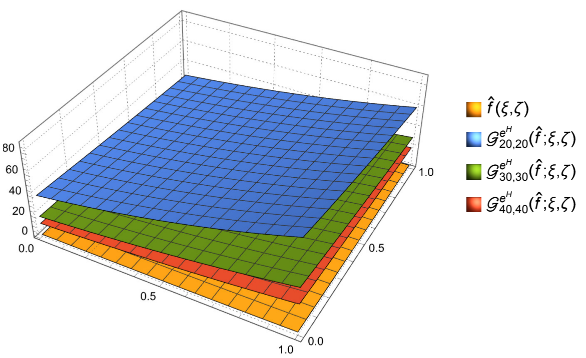

Figure 1: The convergence of the operators $\mathcal{G}^{_eH}_{\nu,\mu}(\hat{f};\xi,\zeta)$ to $\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)e^{\xi-\zeta}$

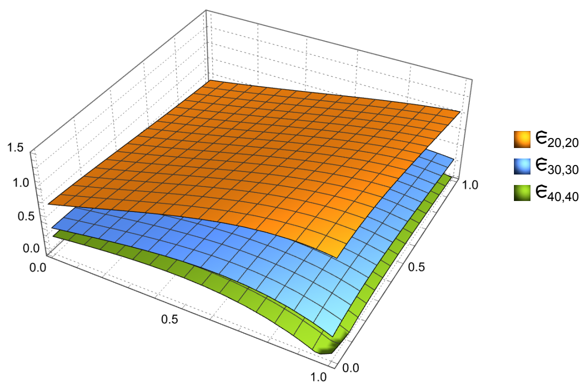

Figure 2: Absolute error of operators $\mathcal{G}^{_eH}_{\nu,\mu}(\hat{f};\xi,\zeta)$ to $\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)e^{\xi-\zeta}$

Both the visual insights from Figures 1-4 and the numerical results in Tables 1 and 2 confirm the expected behavior that increasing ν and μ leads to a marked reduction in approximation error.

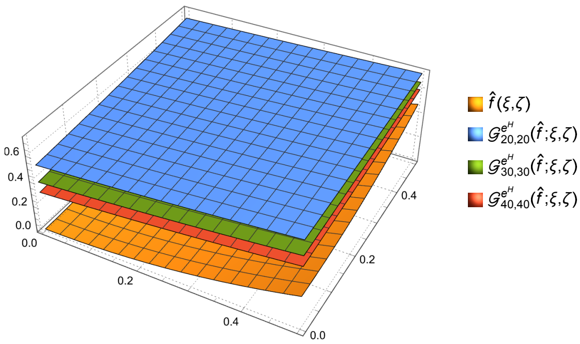

Figure 3: The convergence of the operators $\mathcal{G}^{e^H}_{\nu,\mu}(\hat{f};\xi,\zeta)$ to $\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)\cos(\xi)$

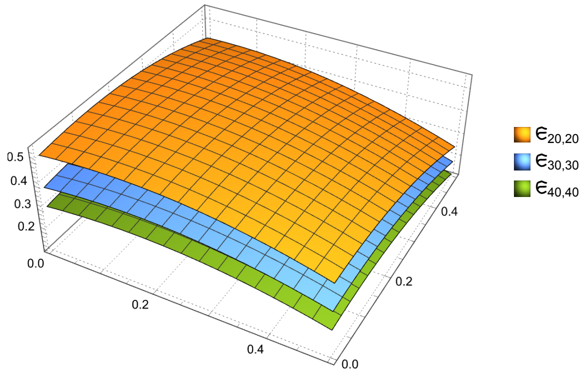

Figure 4: Absolute error of operators $\mathcal{G}^{e^H}_{\nu,\mu}(\hat{f};\xi,\zeta)$ to $\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)\cos(\xi)$

Table 1: Error of approximation process for $\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)e^{\xi-\zeta}$

Error at $(\xi,\zeta) =$	$\mathfrak{E}_{20,20}$	$\mathfrak{E}_{30,30}$	$\mathfrak{E}_{40,40}$
(0.1,0.1)	0.81629	0.42823	0.28521
(0.2,0.2)	0.88353	0.46381	0.30737
(0.3,0.3)	0.94010	0.48694	0.31634
(0.4,0.4)	0.98612	0.49774	0.31224
(0.5,0.5)	1.02180	0.49635	0.29517
(0.6,0.6)	1.04720	0.48291	0.26524
(0.7,0.7)	1.06270	0.45755	0.22256
(0.8,0.8)	1.06840	0.42040	0.16722
(0.9,0.9)	1.06450	0.37159	0.09930
(1.0,1.0)	1.05110	0.31122	0.01885

Table 2: Error of approximation process for $\hat{f}(\xi,\zeta) = (\xi^2+\zeta^2)\cos(\xi)$

Error at $(\xi,\zeta) =$	$\mathfrak{E}_{20,20}$	$\mathfrak{E}_{30,30}$	$\mathfrak{E}_{40,40}$
(0.05,0.05)	0.54271	0.41088	0.32853
(0.10,0.10)	0.55920	0.43042	0.34818
(0.15,0.15)	0.57046	0.44567	0.36411
(0.20,0.20)	0.57623	0.45641	0.37611
(0.25,0.25)	0.57626	0.46239	0.38399
(0.30,0.30)	0.57027	0.46336	0.38752
(0.35,0.35)	0.55801	0.45903	0.38646
(0.40,0.40)	0.53917	0.44913	0.38052
(0.45,0.45)	0.51348	0.43333	0.36942
(0.50,0.50)	0.48065	0.41132	0.35285

5 Concluding Remarks

In this article, we introduced the approximation properties of the linear positive operators associated with general-Appell polynomials. Recent research focuses on Appell polynomials, which have practical applications in pure and applied mathematics. From these studies and the results discussed above, we obtained more general impressions, and we predict that this article will contribute to many mathematical processes. After this study, the authors may study the integral modification of the operators defined in equation (9), including general-Appell polynomials and other generalizations of the operator.

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References

[1] P. Appell, Sur une classe de polynômes. (French) Ann. Sci. École Norm. Sup. (2) 9 (1880), 119–144.

[2] R. Aslan, M. Mursaleen, Approximation by bivariate Chlodowsky type Szász–Durrmeyer operators and associated GBS operators on weighted spaces. J. Inequal. Appl. 2022, 26 (2022). https://doi.org/10.1186/s13660-022-02763-7

[3] Büyükyazıcı, H. Tanberkan, S.K. Serenbay, C. Atakut, Approximation by Chlodowsky type Jakimovski-Leviatan operators. J. Comput. Appl. Math. 259 (2014), part A, 153–163.

[4] I. Chlodovsky, Sur le développement des fonctions définies dans un intervalle infini en séries de polynomes de M. S. Bernstein. (French) Compositio Math. 4 (1937), 380–393.

[5] F. Dirik, K. Demirci, Korovkin-type approximation theorem for functions of two variables in statistical sense. Turkish J. Math. 34 (2010), no. 1, 73–83.

[6] G. İçöz, S. Varma, S. Sucu, Approximation by operators including generalized Appell polynomials. Filomat 30 (2016), no. 2, 429–440.

[7] H.J. Hamilton, Transformations of multiple sequences, Duke Math. J. 2 (1936), 29–60.

[8] A. Jakimovski, D. Leviatan, Generalized Szász operators for the approximation in the infinite interval. Mathematica (Cluj) (1969), 11(34), 97–103.

[9] S. Khan, N. Raza, General-Appell polynomials within the context of monomiality principle. Int. J. Anal. 2013, Art. ID 328032, 11 pp.

[10] M. Mursaleen, A. A. H. Al-Abied, Blending type approximation by Stancu-Kantorovich operators associated with the inverse Pólya-Eggenberger distribution. Tbilisi Math. J. 11 (2018), no. 4, 79–91.

[11] M. Mursaleen, S. A. Mohiuddine, Korovkin-type approximation theorem for functions of two variables via statistical summability $(C,1,1)$. Analytic number theory, approximation theory, and special functions, 549–560, Springer, New York, 2014.

[12] M. Mursaleen, K. J. Ansari, On Chlodowsky variant of Szász operators by Brenke type polynomials. Appl. Math. Comput. 271 (2015), 991–1003.

[13] N. Raza, M. Kumar, M. Mursaleen, Approximation with Szász-Chlodowsky operators employing general-Appell polynomials. J. Inequal. Appl. 2024, 26 (2024). https://doi.org/10.1186/s13660-024-03105-5

[14] N. Raza, M. Kumar, M. Mursaleen, On Approximation Operators Involving Tricomi Function. Bull. Malays. Math. Sci. Soc. 47, 154 (2024). https://doi.org/10.1007/s40840-024-01750-z

[15] G.M. Robison, Divergent double sequences and series, Amer. Math. Soc. Transl. 28 (1926), 50–73.

[16] O. Szász, Generalization of S. Bernstein's polynomials to the infinite interval. J. Research Nat. Bur. Standards 45 (1950), 239–245.

[17] VI. Volkov, On the convergence of sequence of linear positive operators in the space of continuous functions of two variables. (Russian) Doklady Akademii Nauk SSSR (N.S) 1957; 115 (1): pp. 17–19.

[18] KG. Weierstrass Uber die analytische Darstellbarkeit sogenannter willkurlicher Functionen einer reellen Veranderlichen. Sitzungsberichte der Koniglich Preussischen Akademie der Wissenschaften zu Berlin 1885; pp. 633–639 & pp. 789–805 (in German).

[19] R. Yadav, VN. Mishra, R. Meher, M. Mursaleen, Statistical convergence of Szász-Mirakjan-Kantorovich-type operators and their bivariate extension. Filomat 36 (2022), no. 17, 5895–5912.

[20] F. Dirik, K. Demirci, Szász-Mirakjan type operators of two variables providing a better estimation on $[0,1]\times[0,1]$. Mat. Vesnik 63 (2011), no. 1, 59–66.

[21] KJ. Ansari, S. Karakılıç, F. Özger, Bivariate Bernstein-Kantorovich operators with a summability method and related GBS operators. Filomat 36 (2022), no. 19, 6751–6765.