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On the Theory and Applications of q-Mittag-Leffler-Laguerre Polynomials

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Abstract

In this work, we develop the theory of 2-variable q-Mittag-Leffler-Laguerre polynomials by employing a generating function that incorporates 0^{th}- order q-Bessel Tricomi functions. We proceed to derive their series definition and various properties. By applying the extended monomiality principle for q-polynomials, we establish the quasi-monomiality characteristics of these polynomials and explore additional features. Additionally, we determine the operational representations of the 2-variable q-Mittag-Leffler-Laguerre polynomials. We also introduce the m^{th} order 2-variable q-Mittag-Leffler-Laguerre polynomials. Finally, this research concludes with the derivation of the 1-variable q-Mittag-Leffler-Laguerre polynomials, an analysis of their zero distributions, and a graphical representation of their properties.

1 Introduction

In 1903, the Swedish mathematician Gösta Mittag-Leffler identified a special function, as detailed in [26, 27], defined as:

$$E_\alpha(u) := \sum_{r=0}^{\infty} \frac{u^r}{\Gamma(\alpha r + 1)}, \qquad u \in \mathbb{C}, \quad Re(\alpha) > 0 \tag{1}$$

where $\Gamma(\cdot)$ is a classical gamma function [28]. The Mittag-Leffler function is the name given to the peculiar function found in equation (1). Because of its use in many disciplines, including physics, chemistry, biology, engineering, and applied sciences, its importance has increased throughout the past 20 years. Integral and differential equations of fractional order often have the Mittag-Leffler function as their solution.

In 1905, Wiman [29] introduced, for the first time, a generalization of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \quad Re(\alpha) > 0. \tag{2}$$

Many people in fractional calculus and related fields have been interested in the Mittag-Leffler function lately. In this domain, some mathematicians even consider the classical Mittag-Leffler function a queen function. The theory of Mittag-Leffler functions has been the subject of much-published research. Anyone interested in learning more about the Mittag-Leffler function and its many uses should peruse the scholarly works.

Laguerre polynomials find applications in quantum mechanics, particularly in the radial section of the Schrödinger equation for a single-electron atom. They are a class of orthogonal polynomials with widespread applications in various fields, such as quantum group theory, harmonic oscillator theory, and coding theory.

Dattoli and Torre [9, 10] explored the theory of 2-variable Laguerre polynomials, demonstrating that ordinary Laguerre polynomials could be interpreted within the framework of quasi-monomials. The interest in 2-variable Laguerre polynomials stems from

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their significant mathematical relevance. These polynomials emerge as natural solutions to specific sets of partial differential equations, such as the heat diffusion equation, and find applications in radiation physics problems [24].
Dattoli and colleagues introduced the 2-variable Laguerre polynomials $L_n(x,y)$ as [9, 10]:

$$\sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!} = C_0(xt)\exp(yt), \qquad |yt|<1; \quad |x|<\infty. \tag{3}$$

An extension of the 2-variable Laguerre polynomial is the m^{th} order 2-variable Laguerre polynomial, which is represented as ${}_mL_n(x,y)$. The following equation provides its generating function as [12]:

$$C_0(-xt^m)\exp(yt) = \sum_{n=0}^{\infty} {}_mL_n(x,y)\frac{t^n}{n!}. \tag{4}$$

The q-calculus is an extension of ordinary calculus that incorporates a new parameter q. This theory has found modern applications in several domains, including ordinary fractional calculus, optimal control issues, solving q-difference and q-integral equations, and q-transform analysis.

In 2009, Mansoor [20] introduced a new q-analog of the Mittag-Leffler function, defined as follows:

$$E_q^{(\alpha,\beta)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(\alpha n+\beta)}, \qquad \alpha,\beta\in\mathbb{C}, \quad Re(\alpha)>0, \quad |z|<(1-q)^{-n}. \tag{5}$$

For additional analogs of the Mittag-Leffler functions on the quantum time scale, particularly in the context of linear Caputo q-fractional initial value problems and their closer approximation to the theory of time scales [1, 2].

Now, we briefly explore the definitions and notations of q-calculus given from [14].

The q-analogue of a complex number d is provided by:

$$[d]_q = \frac{1-q^d}{1-q}, \qquad 0<|q|<1; \quad d\in\mathbb{C}. \tag{6}$$

The q-factorial is defined as:

$$[n]_q! = \begin{cases} \prod_{r=1}^{n}[r]_q, & 0<|q|<1, \quad n\geq 1,\\ 1, & n=0. \end{cases} \tag{7}$$

Gauss's q-binomial formula is expressed as

$$(x\pm a)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n\\ k \end{bmatrix}_q q^{\binom{n-k}{2}} x^k (\pm a)^{n-k}. \tag{8}$$

The respective two q-exponential functions are defined as

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}, \qquad 0<|\, q \,|<1 \tag{9}$$

and

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{[n]_q!}, \qquad 0<|\, q \,|<1. \tag{10}$$

Their product is given by

$$e_q(x)E_q(y) = \sum_{n=0}^{\infty} \frac{(x\oplus y)_q^n}{[n]_q!}. \tag{11}$$

Thus, we have

$$e_q(x)E_q(-x) = 1. \tag{12}$$

The q-derivative of a function $f(x)$ is defined as:

$$\hat{D}_{q,x} f(x) = \frac{f(qx)-f(x)}{qx-x}, \qquad 0<|\, q \,|<1, \ x\neq 0, \tag{13}$$

and

$$\hat{D}_{q,x}^k e_q(\alpha x) = \alpha^k e_q(\alpha x), \qquad k\in\mathbb{N}, \alpha\in\mathbb{C}, \tag{14}$$

where $\hat{D}^k_{q,x}$ denotes the k^{th} order q-derivative with respect to x.
The following represents the q-derivative of the product of two functions, $f(x)$ and $g(x)$:

$$\hat{D}_{q,x}\big(f(x)g(x)\big) = f(x)\hat{D}_{q,x}g(x) + g(qx)\hat{D}_{q,x}f(x). \tag{15}$$

The q-definite integral of a function $f(x)$ is defined as:

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} q^n\, f(aq^n). \tag{16}$$

Hence, we can deduce that

$$\int_0^x \zeta^m d_q\zeta = \frac{x^{m+1}}{[m+1]_q}, \qquad m \in \mathbb{N}\cup\{0\}. \tag{17}$$

If we make use of the notation shown below [6]:

$$D^{-1}_{q,x}f(x) := \int_0^x f(\zeta)d_q\zeta, \tag{18}$$

then in view of equation (17), we have

$$D^{-1}_{q,x}\{x^m\} = \frac{x^{m+1}}{[m+1]_q}, \qquad m \in \mathbb{N}\cup\{0\}. \tag{19}$$

Consequently, [6]

$$\left(D^{-1}_{q,x}\right)^r\{1\} = \frac{x^r}{[r]_q!}, \qquad r \in \mathbb{N}\cup\{0\}. \tag{20}$$

Recently, Cao *et al.* [6] introduced the 2-variable q-Laguerre polynomials $\mathcal{L}_{n,q}(x,y)$ in the context of monomiality as follows:

$$C_{0,q}(xt)e_q(yt) = \sum_{n=0}^{\infty}\mathcal{L}_{n,q}(x,y)\frac{t^n}{[n]_q!}, \tag{21}$$

where the 0^{th} order q-Bessel Tricomi functions $C_{0,q}(x)$ defined as [6]:

$$C_{0,q}(xt) = e_q(-D^{-1}_{q,x}t)\{1\}. \tag{22}$$

Consequently, the series definition is given as

$$C_{0,q}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{([k]_q!)^2}, \tag{23}$$

which converges absolutely $\forall x$.

Additionally, the authors in [6] have also introduced the m-th order 2-variable q-Laguerre polynomials ${}_{[m]}\mathcal{L}_{n,q}(x,y)$ in the context of monomiality as follows:

$$C_{0,q}(xt^m)e_q(yt) = \sum_{n=0}^{\infty}{}_{[m]}\mathcal{L}_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{24}$$

The concept of monomiality is a significant tool for analyzing certain specific polynomials and their properties. The concept of monomiality dates back to the early 19th century, when J. F. Steffensen established the notion of the poweroid [22]. In 1999, Dattoli reformed and advanced the concept into what is known as quasi-monomials [7]. Recently, several scholars have presented and studied new hybrid special polynomial sequences and families employing the monomiality principle (see [3, 8, 16, 17, 18, 19, 21, 30, 31, 32]).
For a q-polynomials set $p_{n,q}(x)$ ($n \in \mathbb{N}, x \in \mathbb{C}$), the two q-operators $\hat{M}_q$ and $\hat{P}_q$, also termed q-multiplicative and q-derivative operators, respectively, are realized by [17]:

$$\hat{M}_q\ \{p_{n,q}(x)\} = p_{n+1,q}(x) \tag{25}$$

and

$$\hat{P}_q\{p_{n,q}(x)\} = [n]_q p_{n-1,q}(x). \tag{26}$$

The q-operators $\hat{M}_q$ and $\hat{P}_q$ satisfy the following commutation relation:

$$[\hat{P}_q, \hat{M}_q] = \hat{P}_q\hat{M}_q - \hat{M}_q\hat{P}_q. \tag{27}$$

If $\hat{M}_q$ and $\hat{P}_q$ have q-differential realizations, then the q-differential equations, satisfied by $p_{n,q}(x)$, are:

$$\hat{M}_q\hat{P}_q\ \{p_{n,q}(x)\} = [n]_q\ p_{n,q}(x) \tag{28}$$

and

$$\hat{P}_q\hat{M}_q\ \{p_{n,q}(x)\} = [n+1]_q\ p_{n,q}(x). \tag{29}$$

In particular

$$p_{n,q}(x) = \hat{M}_q^n\{p_{0,q}(x)\} = \hat{M}_q^n\{1\}, \tag{30}$$

where $p_{0,q}(x) = 1$ is the q-analogue of polynomial $p_{n,q}(x)$.

Moreover, the following is the generating function of $p_{n,q}(x)$:

$$e_q(\hat{M}_q t)\{1\} = \sum_{n=0}^{\infty} p_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{31}$$

The following is how the q-dilatation operator T_z operates on any function [15]:

$$T_z^k f(z) = f(q^k z), \qquad k \in \mathbb{R} \quad z \in \mathbb{C}, \tag{32}$$

satisfies the property:

$$T_z^{-1}T_z^1 f(z) = f(z). \tag{33}$$

The q-derivative of the exponential $e_q(yt^2)$ is defined by [23]:

$$\hat{D}_{q,t}\ e_q(yt^2) = yte_q\big(yt^2\big) + qyte_q\big(qyt^2\big). \tag{34}$$

The utilization of the symbolic method offers effective and efficient approaches to introduce and analyze both novel and existing special functions. The symbolic method originated in 1938 (see [5]). Recently, Babusci and his co-authors introduced a new approach to the symbolic method in 2013 to study special functions by deriving certain operators, which are called symbolic operators [4]. In their work, Dattoli et al. [13] introduced a symbolic operator denoted as $d_{(\alpha,\beta)}$, where $\alpha, \beta \in \mathbb{R}^+$. This operator acts on the vacuum function $\psi_z = \frac{\Gamma(z+1)}{\Gamma(\alpha z+\beta)}$ according to the following equation:

$$d^k_{(\alpha,\beta)}\psi_z = \frac{\Gamma(z+k+1)}{\Gamma(\alpha(z+k)+\beta)}, \qquad k \in \mathbb{R}, \quad (k+z) > -1. \tag{35}$$

In particular, one can notice

$$d^k_{(\alpha,\beta)}\psi_0 := d^k_{(\alpha,\beta)}\psi_z\Big|_{z=0} = \frac{\Gamma(k+1)}{\Gamma(\alpha k+\beta)}, \qquad k > -1. \tag{36}$$

Notably, when $k = 0$, equation (36) yields:

$$\psi_0 = \frac{1}{\Gamma(\beta)}. \tag{37}$$

Considering equations (2) and (35), the symbolic representation of the Mittag-Leffler function in terms of $d_{(\alpha,\beta)}$ can be expressed as [13]:

$$E_{\alpha,\beta}(z) = e^{zd_{(\alpha,\beta)}}\{\psi_0\}. \tag{38}$$

In recent years, the Mittag-Leffler function has been described as the “queen function" in the realm of fractional calculus due to its extensive applications and theoretical significance. This has spurred mathematicians to explore new extensions and generalizations of this function, particularly within the framework of quantum calculus and special functions. The present research aims to build on these developments by introducing the 2-variable q-Mittag-Leffler-Laguerre polynomials. These polynomials represent a novel hybrid family within the q-special functions framework, combining the characteristics of the Mittag-Leffler function and Laguerre polynomials through the quasi-monomiality principle. The motivation is to explore their properties, establish their operational identities, and demonstrate their potential applications in solving complex problems in mathematical and physical sciences. The rest of this article is organized as follows:

In section 2, we introduce the theory of 2-variable q-Mittag-Leffler-Laguerre polynomials by employing a generating function that involves 0^{th}-order q-Bessel Tricomi functions utilizing a symbolic approach. We proceed to derive their series definition and various properties. Further, we establish the quasi-monomiality characteristic and determine the operational representation of these polynomials. In section 3, we introduce m^{th}-order 2-variable q-Mittag-Leffler-Laguerre polynomials and their characteristics. Finally, we conclude this paper by discussing the several properties of the 1-variable q-Mittag-Leffler-Laguerre polynomials, an analysis of their zero distributions, and a graphical representation of their properties.

2 The 2-variable q-Mittag-Leffler-Laguerre polynomials

In this section, we frame the q-Mittag-Leffler function within the context of symbolic formalism. To achieve this, we introduce a q-symbolic operator $d_{(\alpha,\beta)_q}$, which acts on the vacuum state $\varphi_{z,q}=\frac{\Gamma_q(z+1)}{\Gamma_q(\alpha z+\beta)}$ in the following manner:

$$d^k_{(\alpha,\beta)_q}\varphi_{z,q}=\frac{\Gamma_q(z+k+1)}{\Gamma_q(\alpha(z+k)+\beta)},\quad k\in\mathbb{R},\qquad (k+z)>-1. \tag{39}$$

In particular, for $z=0$, we have

$$d^k_{(\alpha,\beta)_q}\varphi_{0,q}=d^k_{(\alpha,\beta)_q}\varphi_{z,q}|_{z=0}=\frac{\Gamma_q(k+1)}{\Gamma_q(\alpha k+\beta)},\qquad k>-1. \tag{40}$$

Thus in view of equations (39) and (40), we have

$$d^r_{(\alpha,\beta)_q}d^k_{(\alpha,\beta)_q}\varphi_{0,q}=\frac{\Gamma_q(r+k+1)}{\Gamma_q(\alpha(r+k)+\beta)},\quad k,r\in\mathbb{R},\qquad (k+r)>-1. \tag{41}$$

Notably, when $k=0$, equation (36) yields:

$$\varphi_{0,q}=\frac{1}{\Gamma_q(\beta)}. \tag{42}$$

Considering equations (5) and (39), the symbolic representation of the Mittag-Leffler function in terms of $d_{(\alpha,\beta)_q}$ can be expressed as:

$$E_q^{(\alpha,\beta)}(z)=e_q\left(zd_{(\alpha,\beta)_q}\right)\varphi_{0,q}. \tag{43}$$

Now, taking into account equations (21) and (43), we define 2-variable q-Mittag-Leffler-Laguerre polynomials (2VqMLLP) ${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$ in the following manner:

$$C_{0,q}(xt)e_q\left(d_{(\alpha,\beta)_q}yt\right)\varphi_{0,q}=\sum_{n=0}^{\infty}{}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)\frac{t^n}{[n]_q!}, \tag{44}$$

which leads to the following generating function for the 2VqMLLP ${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$:

$$C_{0,q}(xt)E_q^{(\alpha,\beta)}(yt)=\sum_{n=0}^{\infty}{}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)\frac{t^n}{[n]_q!}, \tag{45}$$

where $C_{0,q}(x)$ and $E_q^{(\alpha,\beta)}(z)$ are defined by equation (22) and (43), respectively. Using equations (22) and (43) to simplify the left-hand side of equation (45), we obtain the following series definition of the 2-variable q-Mittag-Leffler-Laguerre polynomials ${}_E\mathcal{L}^{(\beta,\alpha)}_{n,q}(x,y)$:

$${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)=[n]_q!\sum_{k=0}^{n}\frac{(-1)^kx^ky^{n-k}}{([k]_q!)^2\Gamma_q(\alpha(n-k)+\beta)}. \tag{46}$$

From equation (46), we get the following initial conditions:

$${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,0)=\frac{(-1)^nx^n}{\Gamma_q(\beta)[n]_q!}\qquad\text{and}\qquad {}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(0,y)=\frac{y^n\Gamma_q(n+1)}{\Gamma_q(\alpha n+\beta)}. \tag{47}$$

In view of equations (22) and (43), we have

$$e_q(-D^{-1}_{q,x}t)E_q^{(\alpha,\beta)}(yt)\{1\}=\sum_{n=0}^{\infty}{}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{48}$$

Theorem 2.1. The 2-variable q-Mittag-Leffler-Laguerre polynomials, denoted as ${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$, exhibit quasi-monomial behavior when subjected to the following q-multiplicative and q-derivative operators:

$$\hat{M}_{2Vq_EL}=yd_{(\alpha,\beta)_q}T_x-\hat{D}^{-1}_{q,x}, \tag{49}$$

or, alternatively

$$\hat{M}_{2Vq_EL}=yd_{(\alpha,\beta)_q}-\hat{D}^{-1}_{q,x}T_y \tag{50}$$

and

$$\hat{P}_{2Vq_EL}=-\hat{D}_{q,x}x\hat{D}_{q,x}=-\frac{\partial_q}{\partial_q D^{-1}_{q,x}}, \tag{51}$$

or, alternatively

$$\hat{P}_{2Vq_EL}=d^{-1}_{(\alpha,\beta)_q}\hat{D}_{q,y}, \tag{52}$$

respectively, where $D^{-1}_{q,x}$ is determined by equation (18), and T_x and T_y signify the q-dilatation operators given by equation (32).

Proof. Using equations (14), (15), and [6, Lemma 2.1], we obtain the q-derivative of both sides of equation (45) with respect to t, we have

$$\sum_{n=1}^{\infty} {}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\frac{t^{n-1}}{[n-1]_q!} = yd_{(\alpha,\beta)_q}C_{0,q}(qxt)e_q(yd_{(\alpha,\beta)_q}t) - \hat{D}_{q,x}^{-1}C_{0,q}(xt)e_q(yd_{(\alpha,\beta)_q}t)\{\varphi_{0,q}\}. \tag{53}$$

Equations (32) and (45) from the equation (53) are used to obtain

$$\sum_{n=1}^{\infty} {}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\frac{t^{n-1}}{[n-1]_q!} = \sum_{n=0}^{\infty}\left(yd_{(\alpha,\beta)_q}T_x - \hat{D}_{q,x}^{-1}\right){}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}\frac{t^n}{[n]_q!}. \tag{54}$$

Equation (54) can be solved by comparing the coefficients of equal powers of t on both sides.

$${}_E\mathcal{L}_{n+1,q}^{(\alpha,\beta)}(x,y) = \left(yd_{(\alpha,\beta)_q}T_x - \hat{D}_{q,x}^{-1}\right){}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{55}$$

which in view of equation (25), we get assertion (49). Similarly, we apply equation (45) after differentiating both sides of the equation with respect to t and applying equation (15) for $f_q(t) = C_{0,q}(xt)$ and $g_q(t) = e_q(yt)$. When we compare the equal powers of t coefficients on both sides of the resulting equation, we get

$${}_E\mathcal{L}_{n+1,q}^{(\alpha,\beta)}(x,y) = \left(yd_{(\alpha,\beta)_q} - \hat{D}_{q,x}^{-1}T_y\right){}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{56}$$

which in view of equation (25), we get assertion (50).

Using [6, equation (2.15)] and operating $\hat{D}_{q,x}x\hat{D}_{q,x}$ on both sides of equation (45), we have

$$\hat{D}_{q,x}x\hat{D}_{q,x}C_{0,q}(xt)E_q^{(\alpha,\beta)}(yt) = \frac{\partial_q}{\partial_q D_{q,x}^{-1}}C_{0,q}(xt)E_q^{(\alpha,\beta)}(yt) = \sum_{n=0}^{\infty}\hat{D}_{q,x}x\hat{D}_{q,x}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\frac{t^n}{[n]_q!}, \tag{57}$$

which on using [6, Lemma 2.1], we get

$$-tC_{0,q}(xt)E_q^{(\alpha,\beta)}(yt) = \sum_{n=0}^{\infty}\frac{\partial_q}{\partial_q D_{q,x}^{-1}}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\hat{D}_{q,x}x\hat{D}_{q,x}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\frac{t^n}{[n]_q!}. \tag{58}$$

Using equation (45) in the left hand side of equation (58) and comparing the coefficients of equal powers of t in the resultant equation, we have

$$-\hat{D}_{q,x}x\hat{D}_{q,x}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = -\frac{\partial_q}{\partial_q D_{q,x}^{-1}}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = [n]_q{}_E\mathcal{L}_{n-1,q}^{(\alpha,\beta)}(x,y), \tag{59}$$

which in view of equations (26) and [6, equation (2.15)], gives assertion (51).

Similar to this, by applying equation (14) to both sides of equation (45), we have

$$d_{(\alpha,\beta)_q}tC_{0,q}(xt)E_q^{(\alpha,\beta)}(yt) = \sum_{n=0}^{\infty}\hat{D}_{q,y}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\frac{t^n}{[n]_q!}, \tag{60}$$

which on using equation (45) in the left-hand side of the above equation and then comparing the coefficients of equal powers of t from both sides of the resultant equation, we get

$$d_{(\alpha,\beta)_q}^{-1}\hat{D}_{q,y}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = [n]_q{}_E\mathcal{L}_{n-1,q}^{(\alpha,\beta)}(x,y). \tag{61}$$

The aforementioned equation yields (52) in the light of equation (26). □

Given a function $f(x)$, we get the following in light of equation (15):

$$\hat{D}_{q,x}x\hat{D}_{q,x}f(x) = (\hat{D}_{q,x} + qx\hat{D}^2{}_{q,x})f(x). \tag{62}$$

For 2vqMLLP ${}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ following q-partial differential equation is obtained in the light of equations [6, equation 2.15], (59), (61), and (62):

Corollary 2.2. *The 2vqMLLP ${}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ satisfy the following q-partial differential equation as:*

$$-\hat{D}_{q,x}x\hat{D}_{q,x}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = d_{(\alpha,\beta)_q}^{-1}\hat{D}_{q,y}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{63}$$

or, alternatively

$$-(\hat{D}_{q,x} + qx\hat{D}^2{}_{q,x}){}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = d_{(\alpha,\beta)_q}^{-1}\hat{D}_{q,y}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{64}$$

or, alternatively

$$-\frac{\partial_q}{\partial_q D_{q,x}^{-1}}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = d_{(\alpha,\beta)_q}^{-1}\hat{D}_{q,y}{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y). \tag{65}$$

Now, we derive the operational identities for the 2-variable q-Mittag-Leffler-Laguerre polynomials ${}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$:

Theorem 2.3. The following operational identities are satisfied by 2VqMLLP ${}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ as follows:

$$ {}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = e_q(-d_{(\alpha,\beta)_q}^{-1}\hat{D}_{q,x}^{-1}\hat{D}_{q,y})\{(d_{(\alpha,\beta)_q}y)^n\varphi_{0,q}\}, \tag{66} $$

or, equivalently

$$ {}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = C_{0,q}(xd_{(\alpha,\beta)_q}^{-1}\hat{D}_{q,y})\{d_{(\alpha,\beta)_q}^{n}y^n\}\{\varphi_{0,q}\}, \tag{67} $$

$$ {}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = E_q^{(\alpha,\beta)}\Big(-y\frac{\partial_q}{\partial_q D_{q,x}^{-1}}\Big)\Big\{\frac{-x^n}{[n]_q!}\Big\}, \tag{68} $$

$$ d_{(\alpha,\beta)_q}^{-n}E_q(\hat{D}_{q,x}^{-1}\hat{D}_{q,y})\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\{\varphi_{0,q}\} = \frac{\Gamma_q(n+1)y^n}{\Gamma_q(\alpha n+\beta)} \tag{69} $$

and

$$ \mathcal{E}_q^{(\alpha,\beta)}\Big(y\frac{\partial_q}{\partial_q D_{q,x}^{-1}}\Big)\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = \frac{-x^n}{[n]_q!}. \tag{70} $$

Proof. By taking the k^{th} q-derivative of y^n, and utilizing equation (46), followed by the application of equations (9) and (20) to the resulting expression, we obtain the desired assertion (66). In view of equation (22) and (66), we get the desired assertion (67). Using equations (23) and (45) and then comparing the coefficients of t on both sides of the resultant equation, we get the desired result (68). Applying $d_{(\alpha,\beta)_q}^{-n}E_q(\hat{D}_{q,x}^{-1}\hat{D}_{q,y})\{\varphi_{0,q}\}$ on both sides of equation (66) and then using equation (12), we get the assertion (69). Similarly, applying $\mathcal{E}_q^{(\alpha,\beta)}\Big(y\frac{\partial_q}{\partial_q D_{q,x}^{-1}}\Big)$ on both sides of equation (66) and then using equation (12), we get the desired assertion (70). The proof of the theorem is complete. □

Subsequently, in the following theorem, we obtain the q-integro-differential equations governing the 2VqMLLP ${}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$:

Theorem 2.4. *The following q-integro-differential equations for the 2-variable q-Mittag-Leffler-Laguerre polynomials ${}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ hold true:*

$$ \int_0^x \hat{D}_{q,u}\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(u,y)d_qu + q\int_0^x u\hat{D}_{q,u}^2\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(u,y)d_qu = \Big([n]_q + qyd_{(\alpha,\beta)_q}T_x\hat{D}_{q,x} + yd_{(\alpha,\beta)_q}xT_x\hat{D}_{q,x}^2\Big)\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) \tag{71} $$

and

$$ \int_0^x \hat{D}_{q,y}\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(u,y)d_qu = d_{(\alpha,\beta)_q}\Big(yT_x\hat{D}_{q,y} - [n]_q\Big)\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = 0. \tag{72} $$

Proof. Taking into account of equations (28), (49) and (51), we have

$$ (yd_{(\alpha,\beta)_q}T_x - \hat{D}_{q,x}^{-1})(-\hat{D}_{q,x}x\hat{D}_{q,x})\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = [n]_q\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y). \tag{73} $$

employing equation (62) in equation (73), we have

$$ (yd_{(\alpha,\beta)_q}T_x - \hat{D}_{q,x}^{-1})(-\hat{D}_{q,x} - qx\hat{D}^2{}_{q,x})\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y) = [n]_q\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{74} $$

taking into account of equation (18), the proceeding equation gives assertion (71).

Now, in light of equations (28), (49) and (52), we have

$$ (yT_x\hat{D}_{q,y} - d_{(\alpha,\beta)_q}^{-1}\hat{D}_{q,x}^{-1}\hat{D}_{q,y})\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)_q}(x,y) = [n]_q\,{}_E\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{75} $$

which, in view of equation (18), gives assertion (72). □

Now, we consider 10^{th} and 11^{th} degree 2VqMLLP:

$$ \begin{aligned} {}_E\mathcal{L}_{10,q}^{(\alpha,\beta)}(x,y) \quad = \quad & [10]_q!\Big[\frac{y^{10}}{\Gamma_q(10\alpha+\beta)} - \frac{xy^9}{\Gamma_q(9\alpha+\beta)} + \frac{x^2y^8}{([2]_q!)^2\Gamma_q(8\alpha+\beta)} - \frac{x^3y^7}{([3]_q!)^2\Gamma_q(7\alpha+\beta)} \\ & + \frac{x^4y^6}{([4]_q!)^2\Gamma_q(6\alpha+\beta)} - \frac{x^5y^5}{([5]_q!)^2\Gamma_q(5\alpha+\beta)} + \frac{x^6y^4}{([6]_q!)^2\Gamma_q(4\alpha+\beta)} - \frac{x^7y^3}{([7]_q!)^2\Gamma_q(3\alpha+\beta)} \\ & + \frac{x^8y^2}{([8]_q!)^2\Gamma_q(2\alpha+\beta)} - \frac{x^9y}{([9]_q!)^2\Gamma_q(\alpha+\beta)}\Big] + \frac{x^{10}}{[10]_q!\Gamma_q(\beta)}, \end{aligned} \tag{76} $$

and

$$
\begin{aligned}
{}_E\mathcal{L}^{(\alpha,\beta)}_{11,q}(x,y) \quad = \quad & [11]_q!\Big[\frac{y^{11}}{\Gamma_q(11\alpha+\beta)}-\frac{xy^{10}}{\Gamma_q(10\alpha+\beta)}+\frac{x^2y^9}{([2]_q!)^2\Gamma_q(9\alpha+\beta)}-\frac{x^3y^8}{([3]_q!)^2\Gamma_q(8\alpha+\beta)}\\
&+\frac{x^4y^7}{([4]_q!)^2\Gamma_q(7\alpha+\beta)}-\frac{x^5y^6}{([5]_q!)^2\Gamma_q(6\alpha+\beta)}+\frac{x^6y^5}{([6]_q!)^2\Gamma_q(5\alpha+\beta)}-\frac{x^7y^4}{([7]_q!)^2\Gamma_q(4\alpha+\beta)}\\
&+\frac{x^8y^3}{([8]_q!)^2\Gamma_q(3\alpha+\beta)}-\frac{x^9y^2}{([9]_q!)^2\Gamma_q(2\alpha+\beta)}+\frac{x^{10}y}{([10]_q!)^2\Gamma_q(\alpha+\beta)}\Big]-\frac{x^{10}}{[11]_q!\Gamma_q(\beta)}.
\end{aligned}
\tag{77}
$$

The subsequent visual depictions of the 2-variable q-Mittag-Leffler-Laguerre polynomials ${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$ can be computed using equations (76) and (77) in the program Wolfram Mathematica, with specific values assigned to α, β, and q in Figures 1 and 2, respectively:

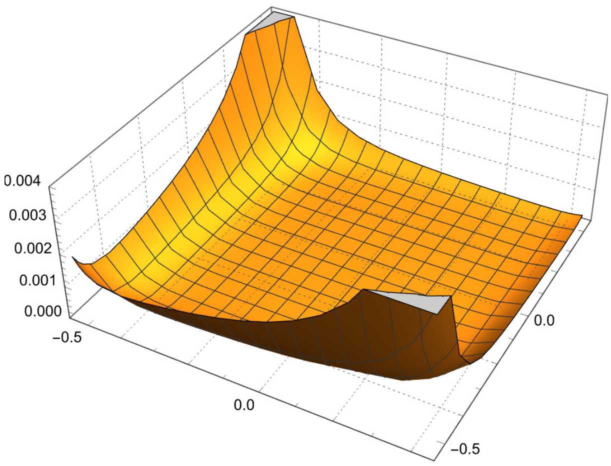

Figure 1: Surface plot of ${}_E\mathcal{L}^{(0.14,0.5)}_{10,0.004}(x,y)$

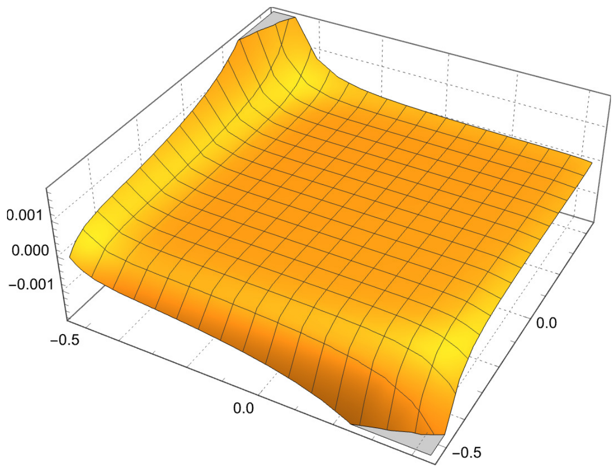

Figure 2: Surface plot of ${}_E\mathcal{L}^{(0.4,0.5)}_{11,0.004}(x,y)$

In the following section, we present the m^{th} order 2-variable q-Mittag-Leffler-Laguerre polynomials ${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$ and examine their properties.

3 mth-order 2-variable q-Mittag-Leffler-Laguerre polynomials

This section presents the m^{th} order 2-variable q-Mittag-Leffler-Laguerre polynomials ($m2Vq$MLLP) using an exponential generating function that incorporates 0^{th} order q-Tricomi functions. In addition, we determine specific attributes of these polynomials, such as their series definition and quasi-monomiality properties.

Considering equations (4) and (43), we establish the definition of the m^{th} order 2-variable q-Mittag-Leffler-Laguerre polynomials. ${}_{[m]E}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$ in the following manner:

$$
C_{0,q}(-xt^m)e_q(d_{(\alpha,\beta)_q}yt)=\sum_{n=0}^{\infty}{}_{[m]E}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)\frac{t^n}{[n]_q!},\qquad m\in\mathbb{N},\tag{78}
$$

which results in the following generating function:

$$
C_{0,q}(-xt^m)E^{(\alpha,\beta)}_q(yt)=\sum_{n=0}^{\infty}{}_{[m]E}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)\frac{t^n}{[n]_q!},\qquad m\in\mathbb{N}.\tag{79}
$$

We derive the series definition of $m2VqMLLP$ as follows. Equations (9) and (23) are used to extend the left-hand side of equation (79), yielding ${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$. The coefficients of equal powers of t from both sides of the resultant equation are then compared:

$$
{}_{[m]E}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)=[n]_q!\sum_{k=0}^{[n/m]}\frac{x^ky^{n-mk}}{([k]_q!)^2\Gamma_q(\alpha(n-mk)+\beta)}.\tag{80}
$$

Now, we obtain the following theorem for the operational identities of m2VqMLLP ${}_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$:

Theorem 3.1. *The following operational identities are satisfied by m2VqMLLP ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$:*

$$ {}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=e_q(\hat{D}_{q,x}^{-1}d_{(\alpha,\beta)_q}^{-m}\hat{D}_{q,y}^{m})(d_{(\alpha,\beta)_q}y)^n\varphi_{0,q}, \tag{81}$$

or, alternatively

$$ {}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=C_{0,q}(-xd_{(\alpha,\beta)_q}^{-m}\hat{D}_{q,y}^{m})(d_{(\alpha,\beta)_q}y)^n\varphi_{0,q} \tag{82}$$

and

$$E_q\left(-D_{q,x}^{-1}d_{(\alpha,\beta)_q}^{-m}\hat{D}_{q,y}^{m}\right){}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=\frac{\Gamma_q(n+1)y^n}{\Gamma_q(\alpha n+\beta)}. \tag{83}$$

Proof. By taking the mk-times q-derivative of y^n with respect to y, and utilizing equation (80), followed by the application of equations (9) and (20) to the resulting expression, we obtain the desired assertion (81). In view of equation (22) and (81), we get the desired assertion (82). Applying $E_q\left(-D_{q,x}^{-1}d_{(\alpha,\beta)_q}^{-m}\hat{D}_{q,y}^{m}\right)$ on both sides of equation (81) and then using equation (12), we get the assertion (83). □

In view of equations (22), (32) and (82), we have the following inequality:

$$T^r_{\hat{D}_{q,x}^{-1}}\,{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=T^r_x\,{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y). \tag{84}$$

Now, we have the following q-multiplicative and q-derivative operators of the m2VqMLLP ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$:

Theorem 3.2. *The m^{th} order 2-variable q-Mittag-Leffler-Laguerre polynomials ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ are quasi-monomials under the action of the following q-multiplicative and q-derivative operators:*

$$\hat{M}_{2Vq_{[m]E}L}=yd_{(\alpha,\beta)_q}T_x+D_{q,x}^{-1}T_{(x;m)}d_{(\alpha,\beta)_q}^{1-m}\hat{D}_{q,y}^{m-1}, \tag{85}$$

or, alternatively

$$\hat{M}_{2Vq_{[m]E}L}=yd_{(\alpha,\beta)_q}+D_{q,x}^{-1}T_{(x;m)}d_{(\alpha,\beta)_q}^{1-m}\hat{D}_{q,y}^{m-1}T_y \tag{86}$$

and

$$\hat{P}_{2Vq_{[m]E}L}=d_{(\alpha,\beta)_q}^{-1}\hat{D}_{q,y}, \tag{87}$$

where $T_{(x;m)}$ defined as [6]:

$$T_{(x;m)}:=\frac{1-q^mT_x^m}{1-qT_x}=1+qT_x+\cdots+q^{m-1}T_x^{m-1}. \tag{88}$$

Proof. Differentiating (79) with respect to t and in view of [6, (3.16)-(3.21)], we have

$$\sum_{n=1}^{\infty}{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\frac{t^{n-1}}{[n-1]_q!}=\sum_{n=0}^{\infty}\left(yd_{(\alpha,\beta)_q}T_x+D_{q,x}^{-1}T_{(x;m)}d_{(\alpha,\beta)_q}^{1-m}\hat{D}_{q,y}^{m-1}\right)\frac{t^n}{[n]_q!},\qquad m\in\mathbb{N}. \tag{89}$$

Comparing the powers of t of the equation (89) and taking into account equation (25), we get the assertion (85).
Similarly, by taking the q-derivative of equation (79) with respect to t for $\tilde{f}_q(t)=C_{0,q}(-xt^m)$ and $\tilde{g}_q(t)=E_q^{(\alpha,\beta)}(yt)$, and subsequently employing equations (32) and (79), followed by comparing the coefficients of t, we obtain an equivalent representation of the q-multiplicative operator corresponding to the m2VqMLLP ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$, as given in equation (86).
Using equation (79) and [6, (3.21)] and comparing the coefficient of t, we have

$$\hat{D}_{q,y}\,{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=[n]_q\,{}_{[m]E}\mathcal{L}_{n-1,q}^{(\alpha,\beta)}(x,y), \tag{90}$$

which, in view of equation (26) gives the assertion (87). Hence, the proof of the theorem is complete. □

Now, we demonstrate the partial differential equation for m2VqMLLP ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$:

Theorem 3.3. *The m^{th} order 2-variable q-Mittag-Leffler-Laguerre polynomials ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ satisfy the following q-partial differential equation:*

$$d_{(\alpha,\beta)_q}^{-m}\hat{D}_{q,y}^{m}\,{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=\hat{D}_{q,x}x\hat{D}_{q,x}\,{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{91}$$

or, alternatively

$$\left(d_{(\alpha,\beta)_q}^{-m}\hat{D}_{q,y}^{m}-x\hat{D}_{q,x}^{2}-q\hat{D}_{q,x}\right){}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=0, \tag{92}$$

or, alternatively

$$d_{(\alpha,\beta)_q}^{-m}\hat{D}_{q,y}^{m}\,{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=-\frac{\partial_q}{\partial_q D_{q,x}^{-1}}\,{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y). \tag{93}$$

Proof. With reference to [6, Lemma 2.1] and equation (14), and upon applying equation (79), equating the resulting expressions and comparing the coefficients of t yield the required assertion (91). In view of [6, Lemma 2.1], and equation (91) gives the assertion (92). Similarly, in view of [6, Remark 2.1] and then applying equation (79) on both sides of the resultant equation and comparing the coefficients of t, we get the assertion (93). Hence, the proof of the theorem is complete. □

Next, we prove the following theorem to establish the q-integro-differential equation of m2VqMLLP ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$:

Theorem 3.4. *The following q-integro-differential equation of m2VqMLLP ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ holds true:*

$$\int_0^x d_{(\alpha,\beta)_q}^{-m}\hat{D}_{q,y}^{m-1}T_{(u;m)}\hat{D}_{q,y}\ {}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(u,y)d_qu=\Big([n]_q-yT_x\hat{D}_{q,y}\Big)_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{94}$$

or, alternatively

$$d_{(\alpha,\beta)_q}^{-m}\sum_{k=0}^{m-1}q^k\int_0^x \hat{D}_{q,y}^m{}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(q^ku,y)d_qu=\Big([n]_q-yT_x\hat{D}_{q,y}\Big)\ {}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y), \tag{95}$$

where $T_{(x;m)}$ is defined by equation (88).

Proof. In view of equation (28), (85), and (87), then applying equation (18), we get the assertion (94). Applying $T_{(x;m)}$ on ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ given in equation (88) and then using resultant equation in equation (94), we get assertion (95). The proof of the theorem is complete. □

Remark 1. For $m=2$, equations (79) and (80) simplify to the generating function and series definition of the 2^{nd}-order two-variable q-Mittag-Leffler-Laguerre polynomials ${}_{[2]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$:

$$C_{0,q}(-xt^2)E_q^{(\alpha,\beta)}(yt)=\sum_{n=0}^{\infty}{}_{[2]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)\frac{t^n}{[n]_q!} \tag{96}$$

and

$${}_{[2]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)=[n]_q!\sum_{k=0}^{[n/2]}\frac{x^ky^{n-2k}}{([k]_q!)^2\Gamma_q(\alpha(n-2k)+\beta)}, \tag{97}$$

respectively.

Remark 2. For $m=1$ the m2VqMLLP reduce to 2VqMLLP and for $q\to1^-$ the m2VqMLLP reduce to 2-variable Mittag-Leffler-Laguerre polynomials. Moreover, for $\alpha=1,\beta=1$, the m2VqMLLP and 2VqMLLP reduce to the m^{th} order 2-variable q-Laguerre polynomials and 2-variable q-Laguerre polynomials, respectively [6].

In view of the above remark for $m=2$, Theorems 3.1-3.4 reduces for the second order two variable q-Mittag-Leffler-Laguerre polynomials ${}_{[2]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$.

Now, we consider 3^{rd}-order 10^{th} and 14^{th} degree m2VqMLLP:

$${}_{[3]E}\mathcal{L}_{10,q}^{(\alpha,\beta)}(x,y)=[10]_q!\Big[\frac{y^{10}}{\Gamma_q(10\alpha+\beta)}-\frac{xy^7}{\Gamma_q(7\alpha+\beta)}+\frac{x^2y^4}{([2]_q!)^2\Gamma_q(4\alpha+\beta)}-\frac{x^3y}{([3]_q!)^2\Gamma_q(\alpha+\beta)}\Big] \tag{98}$$

$${}_{[3]E}\mathcal{L}_{14,q}^{(\alpha,\beta)}(x,y)=[14]_q!\Big[\frac{y^{14}}{\Gamma_q(14\alpha+\beta)}-\frac{xy^{11}}{\Gamma_q(11\alpha+\beta)}+\frac{x^2y^8}{([2]_q!)^2\Gamma_q(8\alpha+\beta)}-\frac{x^3y^5}{([3]_q!)^2\Gamma_q(5\alpha+\beta)}+\frac{x^4y^2}{([4]_q!)^2\Gamma_q(2\alpha+\beta)}\Big]. \tag{99}$$

The following graphical representations of the m2VqMLLP ${}_{[m]E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ for certain values of α,β,m and q are obtained by using equations (98) and (99) in the software Wolfram Mathematica in Figures (3) and (4), respectively:

4 Concluding remarks and distributions of zeros

In this section, we first introduce another type of q-Mittag-Leffler function by utilizing a symbolic approach. in view of equations (10) and (39), we have another kind of q-Mittag Leffler function as:

$$\mathcal{E}_q^{(\alpha,\beta)}(z)=E_q\Big(zd_{(\alpha,\beta)_q}\Big)\varphi_{0,q}, \tag{100}$$

which becomes in view of equation (39)

$$\mathcal{E}_q^{(\alpha,\beta)}(z)=\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}z^n}{\Gamma_q(\alpha n+\beta)}. \tag{101}$$

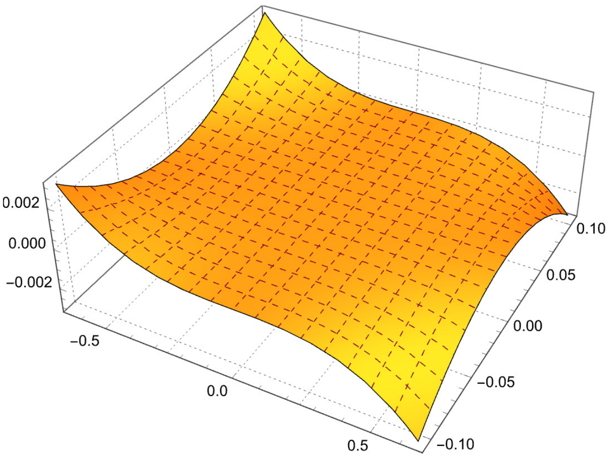

Figure 3: Surface plot of ${}_{[3]E}\mathcal{L}^{(0.14,0.5)}_{10,0.004}(x,y)$

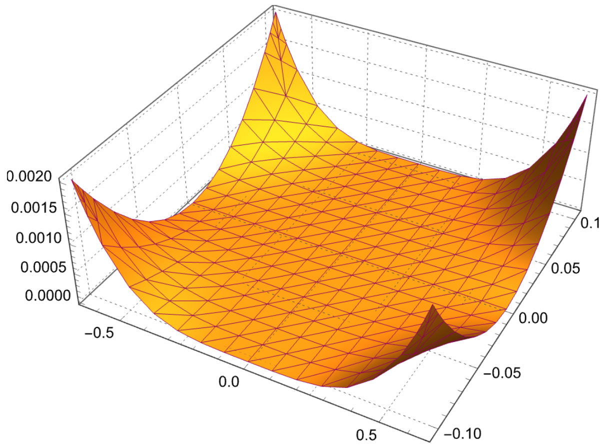

Figure 4: Surface plot of ${}_{[3]E}\mathcal{L}^{(0.4,0.5)}_{14,0.004}(x,y)$

Utilizing (100), we define a different type of two-variable q-Mittag-Leffler-Laguerre polynomials, denoted by ${}_{\mathcal{E}}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$:

$$C_{0,q}(xt)\mathcal{E}^{(\alpha,\beta)}_q(yt)=\sum_{n=0}^{\infty}{}_{\mathcal{E}}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)\frac{t^n}{[n]_q!}. \tag{102}$$

The series definition of ${}_{\mathcal{E}}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$ is derived by expanding the left-hand side of equation (102) using equations (10) and (23), and then equating the coefficients of corresponding powers of t from both sides of the resulting equation:

$${}_{\mathcal{E}}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)=[n]_q!\sum_{k=0}^{n}\frac{(-1)^k q^{\binom{n-k}{2}}x^k y^{n-k}}{([k]_q!)^2\Gamma_q(\alpha(n-k)+\beta)}. \tag{103}$$

Now, we consider 11^{th} and 12^{th} degree ${}_{\mathcal{E}}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$:

$$\begin{aligned}{}_{\mathcal{E}}\mathcal{L}^{(\alpha,\beta)}_{11,q}(x,y) \;=\;& [11]_q!\Big[\frac{q^{55}y^{11}}{\Gamma_q(11\alpha+\beta)}-\frac{q^{45}xy^{10}}{\Gamma_q(10\alpha+\beta)}+\frac{q^{36}x^2y^9}{([2]_q!)^2\Gamma_q(9\alpha+\beta)}-\frac{q^{28}x^3y^8}{([3]_q!)^2\Gamma_q(8\alpha+\beta)}\\ &+\frac{q^{21}x^4y^7}{([4]_q!)^2\Gamma_q(7\alpha+\beta)}-\frac{q^{15}x^5y^6}{([5]_q!)^2\Gamma_q(6\alpha+\beta)}+\frac{q^{10}x^6y^5}{([6]_q!)^2\Gamma_q(5\alpha+\beta)}-\frac{q^{6}x^7y^4}{([7]_q!)^2\Gamma_q(4\alpha+\beta)}\\ &+\frac{q^{3}x^8y^3}{([8]_q!)^2\Gamma_q(3\alpha+\beta)}-\frac{qx^9y^2}{([9]_q!)^2\Gamma_q(2\alpha+\beta)}+\frac{x^{10}y}{([10]_q!)^2\Gamma_q(\alpha+\beta)}\Big]-\frac{x^{11}}{[11]_q!\Gamma_q(\beta)},\end{aligned} \tag{104}$$

and

$$\begin{aligned}{}_{\mathcal{E}}\mathcal{L}^{(\alpha,\beta)}_{12,q}(x,y) \;=\;& [12]_q!\Big[\frac{q^{66}y^{12}}{\Gamma_q(11\alpha+\beta)}-\frac{q^{55}xy^{11}}{\Gamma_q(11\alpha+\beta)}+\frac{q^{45}x^2y^{10}}{([2]_q!)^2\Gamma_q(10\alpha+\beta)}-\frac{q^{36}x^3y^9}{([3]_q!)^2\Gamma_q(9\alpha+\beta)}\\ &+\frac{q^{28}x^4y^8}{([4]_q!)^2\Gamma_q(8\alpha+\beta)}-\frac{q^{21}x^5y^7}{([5]_q!)^2\Gamma_q(7\alpha+\beta)}+\frac{q^{15}x^6y^6}{([6]_q!)^2\Gamma_q(6\alpha+\beta)}-\frac{q^{10}x^7y^5}{([7]_q!)^2\Gamma_q(5\alpha+\beta)}\\ &+\frac{q^{6}x^8y^4}{([8]_q!)^2\Gamma_q(4\alpha+\beta)}-\frac{q^3x^9y^3}{([9]_q!)^2\Gamma_q(3\alpha+\beta)}+\frac{qx^{10}y^2}{([10]_q!)^2\Gamma_q(2\alpha+\beta)}-\frac{x^{11}y}{([11]_q!)^2\Gamma_q(\alpha+\beta)}\Big]\\ &+\frac{x^{12}}{[12]_q!\Gamma_q(\beta)},\end{aligned} \tag{105}$$

respectively.

The graphical representations of another type of the two-variable q-Mittag-Leffler-Laguerre polynomials ${}_{\mathcal{E}}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x,y)$ for specific values of α, β, and q are generated using the aforementioned expressions in equations (104) and (105) using software Wolfram Mathematica in Figures 5 and 6, respectively.

For $y=1$, equations (45) and (46) yield the following generating function and series definition of the q-Mittag-Leffler-Laguerre polynomials ${}_{E}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x)$:

$$C_{0,q}(xt)E^{(\alpha,\beta)}_q(t)=\sum_{n=0}^{\infty}{}_{E}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x)\frac{t^n}{[n]_q!} \tag{106}$$

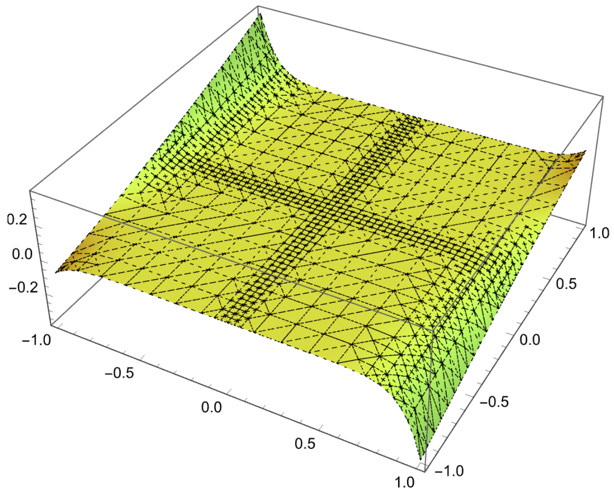

Figure 5: Surface plot of ${}_{\mathcal{E}}\mathcal{L}_{11,.104}^{(.14,.15)}(x,y)$

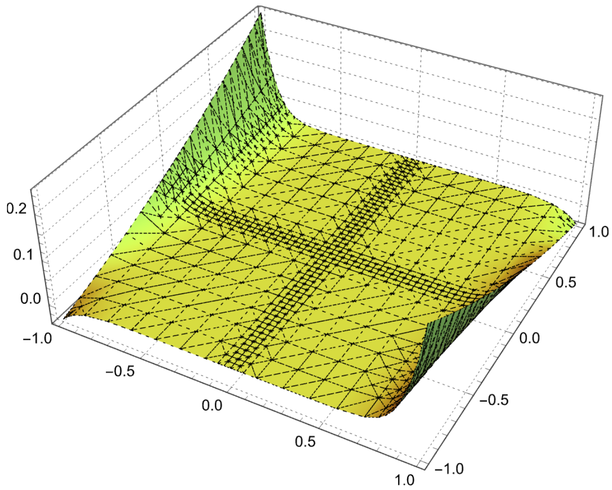

Figure 6: Surface plot of ${}_{\mathcal{E}}\mathcal{L}_{12,.14}^{(0.1,0.2)}(x,y)$

and

$$ {}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k x^k}{([k]_q!)^2 \Gamma_q(\alpha(n-k)+\beta)}. \tag{107} $$

Therefore, for $y=1$, all the outcomes related to the two-variable q-Mittag-Leffler-Laguerre polynomials ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x,y)$ simplify to the corresponding results for the one-variable q-Mittag-Leffler-Laguerre polynomials ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$. In the following equation, we focus on the 12^{th} degree ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$:

$$ \begin{aligned} {}_{E}\mathcal{L}_{12,q}^{(\alpha,\beta)}(x) \;=\; & [12]_q!\Big[\frac{1}{\Gamma_q(12\alpha+\beta)} - \frac{x}{\Gamma_q(11\alpha+\beta)} + \frac{x^2}{([2]_q!)^2\Gamma_q(10\alpha+\beta)} - \frac{x^3}{([3]_q!)^2\Gamma_q(9\alpha+\beta)} \\ & + \frac{x^4}{([4]_q!)^2\Gamma_q(8\alpha+\beta)} - \frac{x^5}{([5]_q!)^2\Gamma_q(7\alpha+\beta)} + \frac{x^6}{([6]_q!)^2\Gamma_q(6\alpha+\beta)} - \frac{x^7}{([7]_q!)^2\Gamma_q(5\alpha+\beta)} \\ & + \frac{x^8}{([8]_q!)^2\Gamma_q(4\alpha+\beta)} - \frac{x^9}{([9]_q!)^2\Gamma_q(3\alpha+\beta)} + \frac{x^{10}}{([10]_q!)^2\Gamma_q(2\alpha+\beta)} - \frac{x^{11}}{([11]_q!)^2\Gamma_q(\alpha+\beta)}\Big] \\ & + \frac{x^{12}}{[12]_q!\Gamma_q(\beta)}. \end{aligned} \tag{108} $$

Similarly, for $y=1$, equations (102) and (103) yield the following generating function and series definition for ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$:

$$ C_{0,q}(xt)\mathcal{E}_q^{(\alpha,\beta)}(t) = \sum_{n=0}^{\infty} {}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x) \frac{t^n}{[n]_q!} \tag{109} $$

and

$$ {}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x) = [n]_q! \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{n-k}{2}} x^k}{([k]_q!)^2 \Gamma_q(\alpha(n-k)+\beta)}, \tag{110} $$

respectively.

In the following equation, we consider 11^{th} degree ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$:

$$ \begin{aligned} {}_{\mathcal{E}}\mathcal{L}_{11,q}^{(\alpha,\beta)}(x) \;=\; & [11]_q!\Big[\frac{q^{55}}{\Gamma_q(11\alpha+\beta)} - \frac{q^{45}x}{\Gamma_q(10\alpha+\beta)} + \frac{q^{36}x^2}{([2]_q!)^2\Gamma_q(9\alpha+\beta)} - \frac{q^{28}x^3}{([3]_q!)^2\Gamma_q(8\alpha+\beta)} \\ & + \frac{q^{21}x^4}{([4]_q!)^2\Gamma_q(7\alpha+\beta)} - \frac{q^{15}x^5}{([5]_q!)^2\Gamma_q(6\alpha+\beta)} + \frac{q^{10}x^6}{([6]_q!)^2\Gamma_q(5\alpha+\beta)} - \frac{q^6x^7}{([7]_q!)^2\Gamma_q(4\alpha+\beta)} \\ & + \frac{q^3x^8}{([8]_q!)^2\Gamma_q(3\alpha+\beta)} - \frac{qx^9}{([9]_q!)^2\Gamma_q(2\alpha+\beta)} + \frac{x^{10}}{([10]_q!)^2\Gamma_q(\alpha+\beta)}\Big] - \frac{x^{11}}{[11]_q!\Gamma_q(\beta)}. \end{aligned} \tag{111} $$

The following graphical representations of ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$ and ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$ for certain values of α,β and q are obtained by using Equations (108) and (111) in the software Wolfram Mathematica in Figures 7 and 8, respectively:

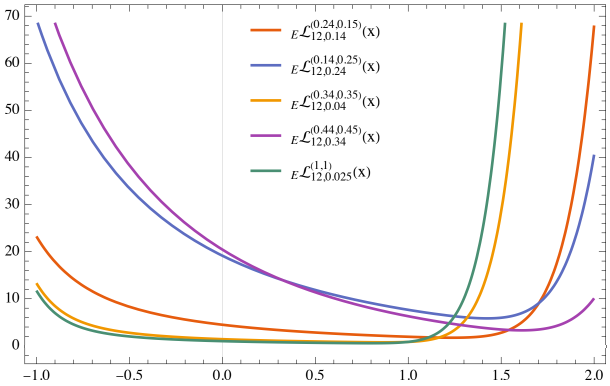

Figure 7: Graphical Representation of $_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x)$

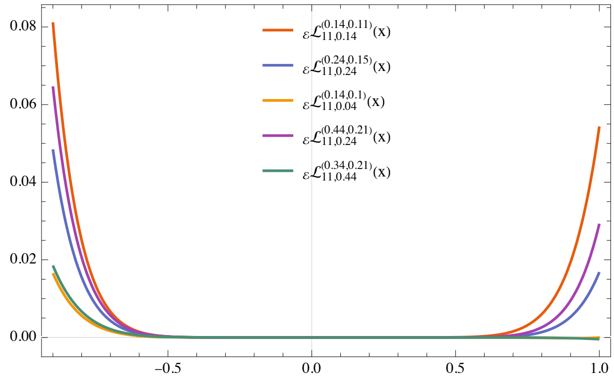

Figure 8: Graphical Representation of $_\mathcal{E}\mathcal{L}^{(\alpha,\beta)}_{n,q}(x)$

Now, we examine the zeros of $_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x)$. In Figure 9, we present the zeros of $_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x)$ for $n=30$, $\alpha=0.24$, $\beta=0.15$, $q=0.5$ (blue), for $n=40$, $\alpha=0.34$, $\beta=0.25$, $q=0.65$ (yellow), and for $n=50$, $\alpha=0.14$, $\beta=0.25$, $q=0.65$ (green).

Table 1: Approximate solutions of 1-variable q-Mittag-Leffler-Laguerre polynomials $_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x)=0$

n	**Zeros of 1-variable q-Mittag-Leffler-Laguerre polynomials $_E\mathcal{L}^{(\alpha,\beta)}_{n,q}(x)=0$**
5	$1.5563, 2.34197-9.96429i, 2.34197+9.96429i, 5.9176-2.43837i, 5.9176+2.43837i$
10	$1.50037, -11.041-26.6286i, -11.041+26.6286i, 4.59782-20.0026i, 4.59782+20.0026i,$ $7.9976, 11.3883-11.519i, 11.3883+11.519i, 13.1682-3.61071i, 13.1682+3.61071i$
15	$-29.9905-36.0576i, -29.9905+36.0576i, -8.33263-35.3824i, -8.33263+35.3824i, 1.46282$ $5.50499-29.7562i, 5.50499+29.7562i, 7.81177, 14.1384-21.9877i, 14.1384+21.9877i$ $18.5957, 18.7323-13.5782i, 18.7323+13.5782i, 19.9976-5.45678i, 19.9976+5.45678i$
20	$-47.159-39.1026i, -47.159+39.1026i, -24.4925-43.9222i, -24.4925+43.9222i, 7.66739,$ $-7.59638-42.8389i, -7.59638+42.8389i, 1.4352, 5.29604-38.3028i, 5.29604+38.3028i,$ $14.8032-31.6892i, 14.8032+31.6892i, 19.3176, 21.3051-23.9447i, 21.3051+23.9447i,$ $25.133-15.7825i, 25.133+15.7825i, 26.6772-7.75102i, 26.6772+7.75102i, 26.8166$
21	$-50.1500-39.1900i, -50.1500+39.1900i, -27.6500-44.9000i, -27.6500+44.9000i,$ $-10.5200-44.6900i, -10.5200+44.6900i, 1.4300, 2.8710-40.9000i, 2.8710+40.9000i,$ $7.6430, 13.0700-34.8600i, 13.0700+34.8600i, 19.2500, 20.3800-27.4900i, 20.3800+27.4900i,$ $25.0900-19.4800i, 25.0900+19.4800i, 27.4600-11.4000i, 27.4600+11.4000i, 28.0300-3.6970i, 28.0300+3.6970i$
22	$-52.9800-39.1500i, -52.9800+39.1500i, -30.7400-45.6900i, -30.7400+45.6900i,$ $-13.4700-46.3100i, -13.4700+46.3100i, 0.3258-43.2700i, 0.3258+43.2700i,$ $1.4260, 7.6190, 11.1200-37.8300i, 11.1200+37.8300i, 19.1700-30.8800i, 19.1700+30.8800i, 19.2000,$ $24.6900-23.1100i, 24.6900+23.1100i, 27.8900-15.0600i, 27.8900+15.0600i, 29.1000-7.2270i, 29.1000+7.2270i, 29.1600$
23	$-55.6500-39.0000i, -55.6500+39.0000i, -33.7300-46.2800i, -33.7300+46.2800i,$ $-16.4200-47.7000i, -16.4200+47.7000i, -2.3060-45.4000i, -2.3060+45.4000i, 1.4220,$ $7.5970, 8.9970-40.5800i, 8.9970+40.5800i, 17.7000-34.1100i, 17.7000+34.1100i, 19.1400,$ $23.9600-26.6400i, 23.9600+26.6400i, 27.9700-18.7100i, 27.9700+18.7100i,$ $29.9200-10.7800i, 29.9200+10.7800i, 30.2500-3.3920i, 30.2500+3.3920i$
24	$-58.1600-38.7500i, -58.1600+38.7500i, -36.6300-46.7200i, -36.6300+46.7200i,$ $-19.3300-48.8900i, -19.3300+48.8900i, -4.9950-47.3000i, -4.9950+47.3000i,$ $1.4180, 6.7330-43.1200i, 6.7330+43.1200i, 7.5750, 16.0100-37.1500i, 16.0100+37.1500i, 19.0900,$ $22.9500-30.0400i, 22.9500+30.0400i, 27.7000-22.3000i, 27.7000+22.3000i,$ $30.4200-14.3700i, 30.4200+14.3700i, 31.1800, 31.3100-6.7330i, 31.3100+6.7330i$
25	$-60.5204-38.4304i, -60.5204+38.4304i, -39.4209-47.0008i, -39.4209+47.0008i,$ $-22.2105-49.8788i, -22.2105+49.8788i, -7.7159-48.9909i, -7.7159+48.9909i,$ $4.35875-45.4396i, 4.35875+45.4396i, 14.1313-40.0022i, 14.1313+40.0022i$ $21.6898-33.2886i, 21.6898+33.2886i, 27.1344-25.8021i, 27.1344+25.8021i$ $30.5883-17.9702i, 30.5883+17.9702i, 32.1875-10.1876i, 32.1875+10.1876i$ $32.2072-3.08003i, 32.2072+3.08003i, 1.41369, 7.55437, 19.0376$
26	$-62.7300-38.0400i, -62.7300+38.0400i, -42.1000-47.1500i, -42.1000+47.1500i, -25.0300-50.6900i, -25.0300+50.6900i,$ $-10.4500-50.4700i, -10.4500+50.4700i, 1.4100, 1.9040-47.5500i, 1.9040+47.5500i, 7.5350, 12.1000-42.6500i, 12.1000+42.6500i,$ $18.9900, 20.2100-36.3700i, 20.2100+36.3700i, 26.2900-29.1900i, 26.2900+29.1900i,$ $30.4300-21.5200i, 30.4300+21.5200i, 32.7600-13.7300i, 32.7600+13.7300i, 32.9200, 33.3400-6.2300i, 33.3400+6.2300i$
27	$-64.8000-37.5900i, -64.8000+37.5900i, -44.6600-47.1900i, -44.6600+47.1900i,$ $-27.7800-51.3400i, -27.7800+51.3400i, -13.1700-51.7600i, -13.1700+51.7600i,$ $-0.6069-49.4400i, -0.6069+49.4400i, 1.4060, 7.5160, 9.9540-45.1000i, 9.9540+45.1000i,$ $18.5400-39.2700i, 18.5400+39.2700i, 18.9400, 25.2000-32.4400i, 25.2000+32.4400i,$ $29.9800-24.9900i, 29.9800+24.9900i, 32.9900-17.2700i, 32.9900+17.2700i, 33.9300-2.7060i,$ $33.9300+2.7060i, 34.3000-9.6140i, 34.3000+9.6140i$
28	$-66.7300-37.1000i, -66.7300+37.1000i, -47.1100-47.1200i, -47.1100+47.1200i,$ $-30.4600-51.8300i, -30.4600+51.8300i, -15.8700-52.8600i, -15.8700+52.8600i,$ $-3.1520-51.1400i, -3.1520+51.1400i, 1.4030, 7.4970, 7.7070-47.3400i, 7.7070+47.3400i,$ $16.7200-41.9900i, 16.7200+41.9900i, 18.9000, 23.8900-35.5300i, 23.8900+35.5300i,$ $29.2700-28.3400i, 29.2700+28.3400i, 32.9200-20.7700i, 32.9200+20.7700i, 34.3400, 34.9200-13.1100i,$ $34.9200+13.1100i, 35.2400-5.6880i, 35.2400+5.6880i$
29	$-68.5400-36.5600i, -68.5400+36.5600i, -49.4500-46.9600i, -49.4500+46.9600i,$ $-33.0500-52.1900i, -33.0500+52.1900i, -18.5300-53.7900i, -18.5300+53.7900i, -5.7130-52.6500i,$ $-5.7130+52.6500i, 1.3990, 5.3880-49.4000i, 5.3880+49.4000i, 7.4800, 14.7600-44.5300i, 14.7600+44.5300i,$ $18.8500, 22.3900-38.4600i, 22.3900+38.4600i, 28.3100-31.5800i, 28.3100+31.5800i,$ $32.5600-24.1900i, 32.5600+24.1900i, 35.2100-16.6000i, 35.2100+16.6000i,$ $35.4300-2.1770i, 35.4300+2.1770i, 36.2900-9.0540i, 36.2900+9.0540i$
30	$-70.2165-36.0005i, -70.2165+36.0005i, -51.6686-46.7161i, -51.6686+46.7161i$ $-35.5629-52.4271i, -35.5629+52.4271i, -21.1378-54.5706i, -21.1378+54.5706i$ $-8.27289-53.9833i, -8.27289+53.9833i, 3.01739-51.2594i, 3.01739+51.2594i$ $12.6969-46.8724i, 12.6969+46.8724i, 20.7417-41.2244i, 20.7417+41.2244i$ $27.1515-34.6684i, 27.1515+34.6684i, 31.9523-27.519i, 31.9523+27.519i$ $35.195-20.0566i, 35.195+20.0566i, 36.9391-12.5206i, 36.9391+12.5206i$ $37.0796-5.09704i, 37.0796+5.09704i, 1.39626, 7.4626, 18.8118, 35.3404$

Similarly, we examine the zeros of ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$. In Figure 10, we present the zeros of ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$ for $n=30$, $\alpha=0.09$, $\beta=0.09$, $q=0.89$ (blue), for $n=40$, $\alpha=0.09$, $\beta=0.09$, $q=0.9$ (yellow), and for $n=50$, $\alpha=0.09$, $\beta=0.09$, $q=0.92$ (green).

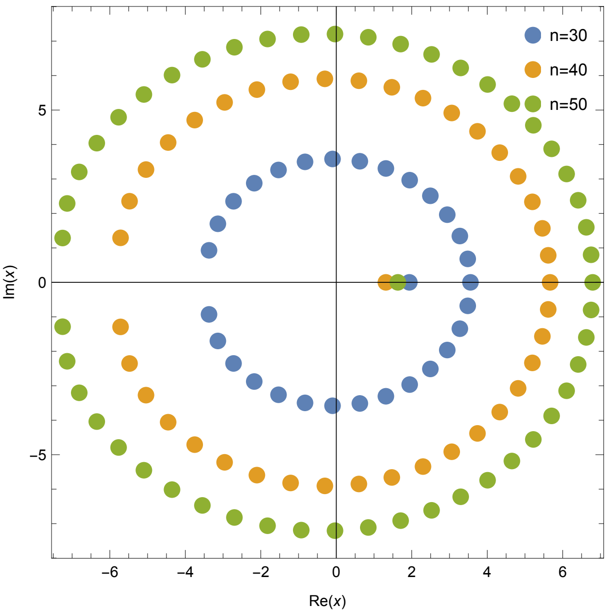

Figure 9: Zeros of ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$

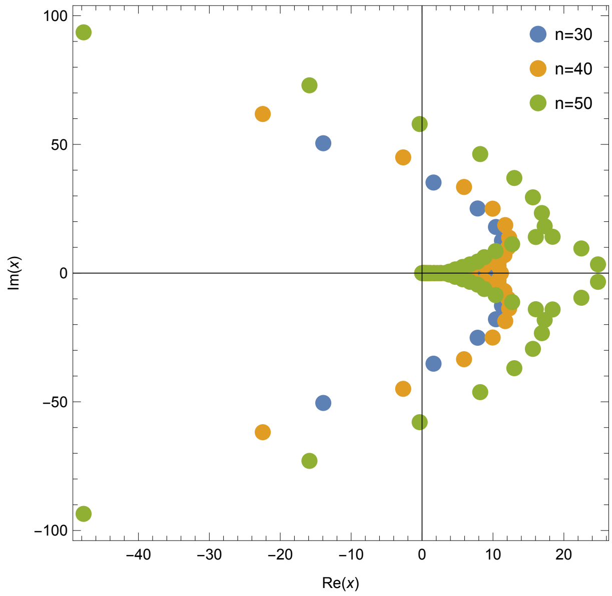

Figure 10: Zeros of ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$

The plot of real zeros of the ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$. for $1 \le n \le 30$, $\alpha=0.19$, $\beta=0.18$, and $q=0.9$ structure are presented in Figure 11. Stacks of zeros of the ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$ for $1 \le n \le 30$, $\alpha=0.09$, $\beta=0.8$, and $q=0.9$ from a 3D structure are presented in Figure 13.

Similarly, the plot of real zeros of the ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$. for $1 \le n \le 30$, $\alpha=0.09$, $\beta=0.8$, and $q=0.9$ structure are presented in Figure 11. Stacks of zeros of the ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$ for $1 \le n \le 30$, $\alpha=0.09$, $\beta=0.8$, and $q=0.9$ from a 3D structure are presented in Figure 13.

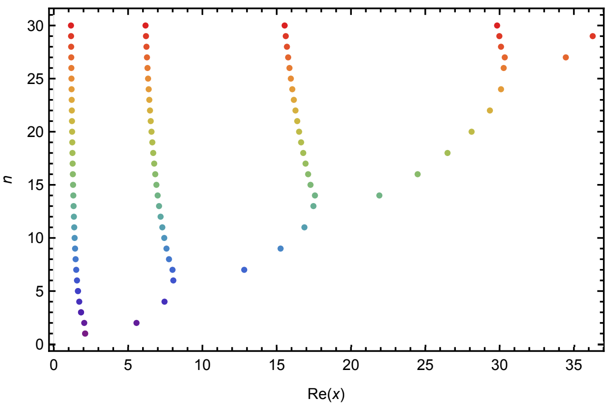

Figure 11: Real zeros of ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$

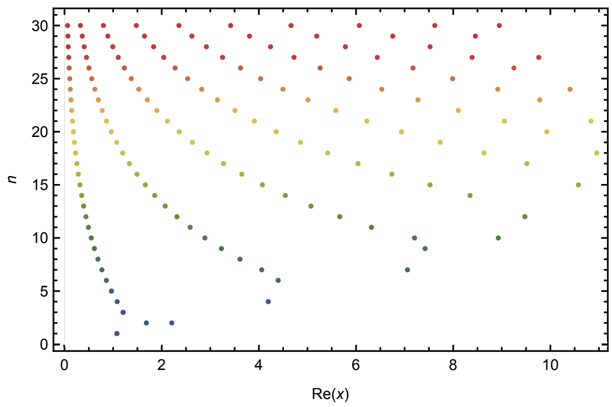

Figure 12: Real zeros of ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$

Our numerical results for the solutions satisfying 1-variable q-Mittag-Leffler-Laguerre polynomials ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)=0$, for $n=5, 10, 15, 20, 25, 30$, $\alpha=0.09$, $\beta=0.8$, and $q=0.9$ are listed in Table 1.

Similarly, our numerical results for the solutions satisfying another kind of 1-variable q-Mittag-Leffler-Laguerre polynomials ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)=0$, for $n=5, 10, 15, 20, 25, 30$, $\alpha=0.09$, $\beta=0.8$, and $q=0.9$ are listed in Table 2.

In this article, we set out to unveil the q-symbolic approach to the Mittag-Leffler function. By leveraging the q-symbolic definition, we present the two-variable q-Mittag-Leffler-Laguerre polynomials and delve into their properties through a symbolic lens. The

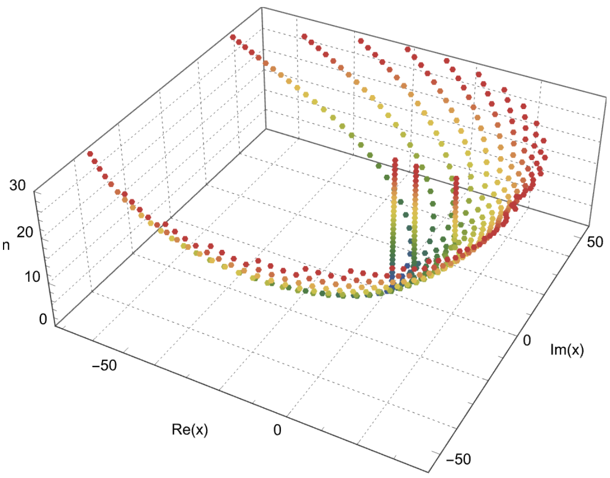

Figure 13: Zeros of ${}_{E}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$

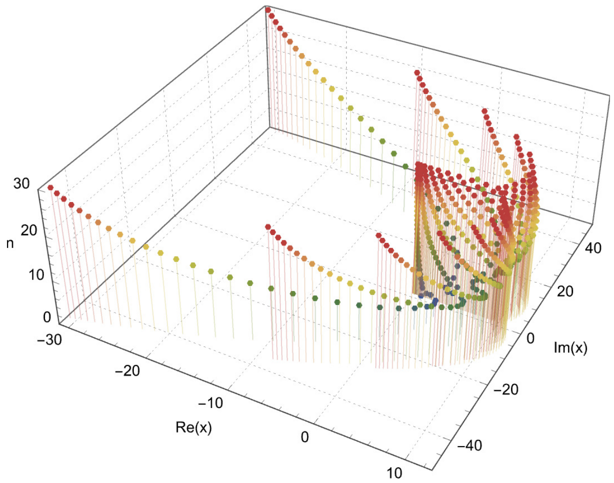

Figure 14: Zeros of ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)$

Table 2: Approximate solutions of another kind of 1-variable q-Mittag-Leffler-Laguerre polynomials ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)=0$

n	**Zeros of another kind of 1-variable q-Mittag-Leffler-Laguerre polynomials** ${}_{\mathcal{E}}\mathcal{L}_{n,q}^{(\alpha,\beta)}(x)=0$
5	$0.965684, 3.53999-8.7904i, 3.53999+8.7904i, 5.01489-1.30641i, 5.01489+1.30641i$
10	$-3.46115-24.4367i, -3.46115+24.4367i, 7.09787-13.5726i, 7.09787+13.5726i,$ $0.549511, 2.88872, 7.20417, 8.92742, 9.44072-5.8136i, 9.44072+5.8136i$
15	$-13.2332-35.4015i, -13.2332+35.4015i, 3.16655-24.2692i, 3.16655+24.2692i$ $0.316317, 1.66314, 4.07179, 7.52223, 9.3094-15.5416i, 9.3094+15.5416i$ $10.5768, 11.3191-3.6335i, 11.3191+3.6335i, 11.3482-8.79407i, 11.3482+8.79407i$
20	$-21.25-41.9701i, -21.25+41.9701i, -1.61722-31.0461i, -1.61722+31.0461i$ $0.183241, 0.963526, 2.3593, 4.35701, 6.93654, 9.92745$ $10.3762-15.3355i, 10.3762+15.3355i, 11.3097-1.91834i, 11.3097+1.91834i$ $11.8078-9.82054i, 11.8078+9.82054i, 11.9444-5.42725i, 11.9444+5.42725i$
21	$-22.5500-42.9000i, -22.5500+42.9000i, -2.4740-32.0100i, -2.4740+32.0100i,$ $0.1644, 0.8643, 2.1160, 3.9090, 6.0740-23.2600i, 6.0740+23.2600i, 6.2230, 9.0530,$ $10.0200-16.3200i, 10.0200+16.3200i, 10.8400, 11.4900-2.8570i, 11.4900+2.8570i,$ $11.6400-10.8000i, 11.6400+10.8000i, 11.9600-6.3870i, 11.9600+6.3870i$
22	$-23.7500-43.7400i, -23.7500+43.7400i, -3.2830-32.8800i, -3.2830+32.8800i,$ $0.1475, 0.7754, 1.8990, 3.5070, 5.5110-24.1400i, 5.5110+24.1400i,$ $8.1050, 9.6440-17.2000i, 9.6440+17.2000i, 10.7400-0.9241i, 10.7400+0.9241i,$ $5.5830, 11.4300-11.6900i, 11.4300+11.6900i, 11.5900-3.7160i, 11.5900+3.7160i,$ $11.9100-7.2680i, 11.9100+7.2680i$
23	$-24.8600-44.4800i, -24.8600+44.4800i, -4.0400-33.6500i, -4.0400+33.6500i,$ $0.1323, 0.6958, 1.7040, 3.1470, 4.9680-24.9200i, 4.9680+24.9200i,$ $5.0100, 7.2750, 9.2640-18.0000i, 9.2640+18.0000i, 9.7800, 10.9400-1.6280i,$ $10.9400+1.6280i, 11.1900-12.4800i, 11.1900+12.4800i, 11.6200-4.5120i,$ $11.6200+4.5120i, 11.8000-8.0690i, 11.8000+8.0690i$
24	$-25.8700-45.1500i, -25.8700+45.1500i, -4.7460-34.3400i, -4.7460+34.3400i,$ $0.1187, 0.6244, 1.5290, 2.8240, 4.4510-25.6200i, 4.4510+25.6200i,$ $4.4960, 6.5290, 8.8880-18.7000i, 8.8880+18.7000i, 8.9170, 10.4100, 10.9400-13.2000i,$ $10.9400+13.2000i, 11.0200-2.3630i, 11.0200+2.3630i, 11.5800-5.2400i,$ $11.5800+5.2400i, 11.6500-8.7930i, 11.6500+8.7930i$
25	$-26.8015-45.7467i, -26.8015+45.7467i, -5.40108-34.9532i, -5.40108+34.9532i$ $0.106574, 0.560416, 1.37234, 2.53464, 3.96207-26.2368i, 3.96207+26.2368i$ $5.86038, 8.52015-19.3276i, 8.52015+19.3276i, 10.2257-0.733129i, 10.2257+0.733129i,$ $4.03579, 10.6697-13.8361i, 10.6697+13.8361i, 11.0422-3.02503i, 11.0422+3.02503i,$ $7.98772, 11.4816-9.44187i, 11.4816+9.44187i, 11.5048-5.89931i, 11.5048+5.89931i$
26	$0.0621531, 0.326838, 0.800395, 1.47838, 2.35415, 3.41883, 4.66113, 6.0673, 7.62053,$ $-30.3656-47.9078i, -30.3656+47.9078i, -7.97091-37.1553i, -7.97091+37.1553i$ $-27.6500-46.2800i, -27.6500+46.2800i, -6.0060-35.5000i, -6.0060+35.5000i,$ $0.0957, 0.5030, 1.2320, 2.2750, 3.5040-26.7900i, 3.5040+26.7900i,$ $3.6230, 5.2610, 7.1720, 8.1670-19.8800i, 8.1670+19.8800i, 9.2490, 10.3400-1.2870i,$ $10.3400+1.2870i, 10.4000-14.4000i, 10.4000+14.4000i, 11.0100-3.6290i, 11.0100+3.6290i,$ $11.2900-10.0200i, 11.2900+10.0200i, 11.3900-6.4910i, 11.3900+6.4910i$
27	$-28.4300-46.7600i, -28.4300+46.7600i, -6.5630-35.9900i, -6.5630+35.9900i,$ $0.0859, 0.4516, 1.1060, 2.0420, 3.0760-27.2800i, 3.0760+27.2800i,$ $3.2520, 4.7230, 6.4380, 7.8310-20.3800i, 7.8310+20.3800i, 8.3880,$ $9.7610, 10.1400-14.9000i, 10.1400+14.9000i, 10.3300-1.8520i, 10.3300+1.8520i,$ $10.9500-4.1730i, 10.9500+4.1730i, 11.0900-10.5300i, 11.0900+10.5300i,$ $11.2600-7.0190i, 11.2600+7.0190i$
28	$-29.1400-47.1800i, -29.1400+47.1800i, -7.0750-36.4200i, -7.0750+36.4200i,$ $0.0771, 0.4054, 0.9928, 1.8340, 2.6790-27.7100i, 2.6790+27.7100i,$ $2.9200, 4.2400, 5.7810, 7.5140-20.8100i, 7.5140+20.8100i, 7.5240, 9.4960-0.4786i, 9.4960+0.4786i,$ $9.8840-15.3400i, 9.8840+15.3400i, 10.3000-2.3530i, 10.3000+2.3530i,$ $10.8500-4.6580i, 10.8500+4.6580i, 10.8900-10.9900i, 10.8900+10.9900i, 11.1100-7.4870i, 11.1100+7.4870i$
29	$-29.7800-47.5700i, -29.7800+47.5700i, -7.5430-36.8100i, -7.5430+36.8100i,$ $0.0692, 0.3640, 0.8914, 1.6460, 2.3130-28.1000i, 2.3130+28.1000i, 2.6220, 3.8070,$ $5.1910, 6.7560, 7.2180-21.2000i, 7.2180+21.2000i, 8.4570, 9.5730-0.9286i, 9.5730+0.9286i,$ $9.6410-15.7400i, 9.6410+15.7400i, 10.2400-2.8020i, 10.2400+2.8020i, 10.6900-11.3800i, 10.6900+11.3800i,$ $10.7300-5.0880i, 10.7300+5.0880i, 10.9500-7.8980i, 10.9500+7.8980i$
30	$1.97581-28.4441i, 1.97581+28.4441i, 8.94894, 6.94265-21.5443i, 6.94265+21.5443i$ $9.41184-16.0796i, 9.41184+16.0796i, 9.5118-1.34485i, 9.5118+1.34485i$ $10.1468-3.19998i, 10.1468+3.19998i, 10.5006-11.7337i, 10.5006+11.7337i$ $10.608-5.46474i, 10.608+5.46474i, 10.7901-8.25949i, 10.7901+8.25949i$

key achievements of our study encompass the derivation of generating functions, series definitions, quasi-monomiality properties, q-partial differential equations, operational identities, and q-integro-differential equations for the newly introduced two-variable q-Mittag-Leffler-Laguerre polynomials. Moreover, we extend these results to the mth-order two-variable q-Mittag-Leffler-Laguerre polynomials. The research culminates with the introduction of an additional type of two-variable q-Mittag-Leffler-Laguerre polynomials. Additionally, we derive the one-variable counterparts of these polynomials. Finally, we provide graphical representations and explore the symmetric structure of their approximate zeros for various values of α, β, and q using computer-aided tools.

The findings presented in this paper may encourage readers and scholars to study these q-special polynomials in more detail. These findings could have applications in engineering, mathematics, and mathematical physics.

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