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Ostrowski type inequalities for 3-convex functions

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Abstract

The main aim of this paper is to derive new Ostrowski type inequalities for 3-convex functions and for functions whose modulus of derivatives are convex, using the weighted Montgomery identity and the weighted Hermite-Hadamard inequalities. Additionally, certain Hermite-Hadamard inequalities for 3-convex functions are provided.

1 Introduction

The following inequality, which establishes an approximation of the integral

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t$$

with the value of a function at an arbitrary point, is known in the literature as the Ostrowski integral inequality (see [1]). That is

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \le \left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty ,$$

for all $x\in[a,b]$, where f is differentiable and $f'\in L_\infty[a,b]$. The above inequality can be proved by using the Montgomery identity ([2])

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b P(x,t)f'(t)dt, \tag{1}$$

where $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $P(x,t)$ is the Peano kernel defined by

$$P(x,t)=\begin{cases}\frac{t-a}{b-a} & \text{for } t\in[a,x],\\[2ex] \frac{t-b}{b-a} & \text{for } t\in(x,b].\end{cases} \tag{2}$$

With these observations in mind, we apply the weighted Montgomery identity to derive certain Ostrowski type inequalities for 3-convex functions.

The weighted Montgomery identity ([3]) states

$$f(x)=\int_a^b w(t)f(t)dt+\int_a^b P_w(x,t)f'(t)dt \tag{3}$$

where $w:[a,b]\to[0,\infty)$ is some nonnegative integrable weight function and $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t)=\begin{cases}W(t) & \text{for } t\in[a,x],\\[2ex] W(t)-1 & \text{for } t\in(x,b]\end{cases} \tag{4}$$

and $\int_a^b w(s)ds=1$, $W(t)=\int_a^t w(s)ds$ for $t\in[a,b]$, W(t)=0, for $t<a$ and $W(t)=1$, for $t>b$. For the uniform weight function $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ the weighted Montgomery idenity reduces to the Montgomery identity (1).

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The role of convexity, alongside its diverse generalizations and extensions, holds significant importance in modern analysis (see [4, 5, 6, 7], and the references cited therein).

In [6] and [5, p.145] the weighted Hermite-Hadamard inequalities for convex functions are given.

Theorem 1.1. *Let $p : [a, b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I, then we have*

$$f(\lambda) \leq \frac{1}{P(b)} \int_a^b p(x) f(x) dx \leq \frac{b-\lambda}{b-a} f(a) + \frac{\lambda - a}{b-a} f(b) \tag{5}$$

or

$$P(b) f(\lambda) \leq \int_a^b p(x) f(x) dx \leq P(b) \left[\frac{b-\lambda}{b-a} f(a) + \frac{\lambda - a}{b-a} f(b) \right], \tag{6}$$

where

$$P(t) = \int_a^t p(x) dx \text{ and } \lambda = \frac{1}{P(b)} \int_a^b x p(x) dx. \tag{7}$$

If f is a concave function, than the inequalities in (5) and (6) are reversed.

In 1934, Popoviciu [7], among others, introduced the term of higher order convexity by using the divided difference of a function.

Definition 1.1. Let f be a real-valued function defined on the segment $[a, b]$. The divided difference of order n of the function f at distinct points $x_0, ..., x_n \in [a, b]$, is defined recursively by

$$f[x_i] = f(x_i), \quad (i = 0, \ldots, n)$$

and

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0} = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=0, j \neq i}^{n} (x_i - x_j)}.$$

The value $f[x_0, \ldots, x_n]$ is independent of the order of the points $x_0, \ldots, x_n$. A function f is called n-convex if all divided differences $f[x_0, \ldots, x_n]$ are nonnegative.

One can notice that a convex function of order 0 is a nonnegative function, a convex function of order 1 is a nondecreasing function, and the class of 2-convex functions is identical to the class of convex functions. It is an established fact that if f is an n-convex function, then the function $f^{(k)}$ exists and is $(n-k)$-convex for $1 \leq k \leq n-2$ (see, for example, [5, p.16]).

The primary objective of this paper is to derive new Ostrowski type inequalities for 3-convex functions and for functions whose modulus of derivatives are convex, through the utilization of the weighted Montgomery identity and the weighted Hermite-Hadamard inequalities. Furthermore, Hermite-Hadamard inequalities for 3-convex functions and for functions whose modulus of derivatives are convex are obtained. This approach generalizes the results given by Cerone and Dragomir in [4] related to the Ostrowski integral inequality.

Extending the Hermite-Hadamard inequalities to 3-convex functions and functions whose modulus of derivatives are convex provides deeper insights into their behavior and gives more accurate estimates of the mean values of the functions based on their values at the ends of the interval and inner points. This extension is useful in optimization, approximation theory, and many other areas in mathematics and applied sciences.

Some newer results dealing with 3-convex functions and generalizations of the Hermite-Hadamard types of inequality for the weighted case can be found in [8] and [9].

Throughout the paper, it is assumed that all integrals under consideration exist and are finite.

2 Main result

In this section, applying the weighted Montgomery identity (3), the weighted Hermite-Hadamard inequalities (6), and the properties of 3-convex functions, we consider new Ostrowski type inequalities for 3-convex functions.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function and $f' \in L[a, b]$, and let $w : [a, b] \to [0, \infty)$ be some nonnegative integrable weight function, such that $\int_a^b w(s) ds = 1$, and $x \in [a, b]$. If f is a 3-convex function then:*

$$\begin{aligned} &\Omega_1(x) f'(\lambda_1(x)) - \Omega_2(x) \left[\frac{b - \lambda_2(x)}{b-x} f'(x) + \frac{\lambda_2(x) - x}{b-x} f'(b) \right] \\ &\leq f(x) - \int_a^b w(t) f(t) dt \\ &\leq \Omega_1(x) \left[\frac{x - \lambda_1(x)}{x-a} f'(a) + \frac{\lambda_1(x) - a}{x-a} f'(x) \right] - \Omega_2(x) f'(\lambda_2(x)), \end{aligned} \tag{8}$$

where

$$\Omega_1(x)=\int_a^x (x-s)w(s)ds, \quad \Omega_2(x)=\int_x^b (s-x)w(s)ds, \tag{9}$$

$$\lambda_1(x)=\frac{1}{2\int_a^x (x-s)w(s)ds}\int_a^x (x^2-s^2)w(s)ds \tag{10}$$

and

$$\lambda_2(x)=\frac{1}{2\int_x^b (s-x)w(s)ds}\int_x^b (s^2-x^2)w(s)ds. \tag{11}$$

If f is a 3-concave function, then the inequalities in (8) are reversed.

Proof. The weighted Montgomery identity states

$$f(x)-\int_a^b w(t)f(t)dt=\int_a^x W(t)f'(t)dt-\int_x^b (1-W(t))f'(t)dt. \tag{12}$$

Since f is a 3-convex function, the function f' exists and is convex on $[a,b]$, [5, p.16]. So, we apply (6) to get upper and lower estimates for integrals $\int_a^x W(t)f'(t)dt$ and $\int_x^b (1-W(t))f'(t)dt$:

$$\Omega_1(x)f'(\lambda_1(x))\le \int_a^x W(t)f'(t)dt\le \Omega_1(x)\left[\frac{x-\lambda_1(x)}{x-a}f'(a)+\frac{\lambda_1(x)-a}{x-a}f'(x)\right] \tag{13}$$

and

$$\Omega_2(x)f'(\lambda_2(x))\le \int_x^b (1-W(t))f'(t)dt\le \Omega_2(x)\left[\frac{b-\lambda_2(x)}{b-x}f'(x)+\frac{\lambda_2(x)-x}{b-x}f'(b)\right]. \tag{14}$$

Now, we derive $\lambda_1(x)$, $\lambda_2(x)$, $\Omega_1(x)$ and $\Omega_2(x)$ from (7):

$$\Omega_1(x)=\int_a^x W(t)dt=\int_a^x\left(\int_a^t w(s)ds\right)dt=\int_a^x (x-s)w(s)ds,$$

$$\lambda_1(x)=\frac{1}{\int_a^x W(t)dt}\int_a^x tW(t)dt=\frac{1}{\int_a^x\left(\int_a^t w(s)ds\right)dt}\int_a^x t\left(\int_a^t w(s)ds\right)dt$$
$$=\frac{1}{2\int_a^x (x-s)w(s)ds}\int_a^x (x^2-s^2)w(s)ds,$$

$$\Omega_2(x)=\int_x^b (1-W(t))dt=\int_x^b\left(\int_t^b w(s)ds\right)dt=\int_x^b (s-x)w(s)ds$$

and

$$\lambda_2(x)=\frac{1}{\int_x^b (1-W(t))dt}\int_x^b t(1-W(t))dt=\frac{1}{\int_x^b\left(\int_t^b w(s)ds\right)dt}\int_x^b t\left(\int_t^b w(s)ds\right)dt$$
$$=\frac{1}{2\int_x^b (s-x)w(s)ds}\int_x^b (s^2-x^2)w(s)ds.$$

Finally, from (3), using (13) and (14), we get inequalities (8). Further, if f is a 3-concave function, the inequalities in (13) and (14) are reversed, so we obtain opposite inequalities in (8).

□

Corollary 2.2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function, $f'\in L[a,b]$ and $x\in[a,b]$. If f is 3-convex then:*

$$\frac{(x-a)^2}{2(b-a)}f'\left(\frac{a+2x}{3}\right)-\frac{(b-x)^2}{2(b-a)}\left[\frac{2}{3}f'(x)+\frac{1}{3}f'(b)\right]$$
$$\le f(x)-\frac{1}{b-a}\int_a^b f(t)dt$$
$$\le \frac{(x-a)^2}{2(b-a)}\left[\frac{1}{3}f'(a)+\frac{2}{3}f'(x)\right]-\frac{(b-x)^2}{2(b-a)}f'\left(\frac{2x+b}{3}\right). \tag{15}$$

If f is a 3-concave function, then the inequalities in (15) are reversed.

Proof. This is a special case of Theorem 2.1 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Now, we derive Ostrowski type inequalities for functions whose modulus of derivatives are convex.

Theorem 2.3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function and $f' \in L[a,b]$, and let $w : [a,b] \to [0,\infty)$ be some nonnegative integrable weight function, such that $\int_a^b w(s)ds = 1$, and $x \in [a,b]$. If $|f'|$ is convex then:*

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \\ \leq \frac{x|f'(a)| - a|f'(x)|}{x-a}\int_a^x (x-s)w(s)ds + \frac{|f'(x)| - |f'(a)|}{2(x-a)}\int_a^x (x^2-s^2)w(s)ds \\ + \frac{b|f'(x)| - x|f'(b)|}{b-x}\int_x^b (s-x)w(s)ds + \frac{|f'(b)| - |f'(x)|}{2(b-x)}\int_x^b (s^2-x^2)w(s)ds. \tag{16}$$

If $|f'|$ is concave then:

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| \\ \leq |f'(\lambda_1(x))|\int_a^x (x-s)w(s)ds + |f'(\lambda_2(x))|\int_x^b (s-x)w(s)ds, \tag{17}$$

where

$$\lambda_1(x) = \frac{1}{2\int_a^x (x-s)w(s)ds}\int_a^x (x^2-s^2)w(s)ds$$

and

$$\lambda_2(x) = \frac{1}{2\int_x^b (s-x)w(s)ds}\int_x^b (s^2-x^2)w(s)ds.$$

Proof. The weighted Montgomery identity states

$$f(x) - \int_a^b w(t)f(t)dt = \int_a^x W(t)f'(t)dt - \int_x^b (1-W(t))f'(t)dt.$$

Now, using properties of the modulus, we get

$$\left|f(x) - \int_a^b w(t)f(t)dt\right| = \left|\int_a^x W(t)f'(t)dt - \int_x^b (1-W(t))f'(t)dt\right| \\ \leq \left|\int_a^x W(t)f'(t)dt\right| + \left|\int_x^b (1-W(t))f'(t)dt\right| \\ \leq \int_a^x W(t)\left|f'(t)\right|dt + \int_x^b (1-W(t))\left|f'(t)\right|dt. \tag{18}$$

Further, we apply (6) for convex function $|f'|$ to get upper estimates for integrals $\int_a^x W(t)|f'(t)|dt$ and $\int_x^b (1-W(t))|f'(t)|dt$:

$$\int_a^x W(t)\left|f'(t)\right|dt \leq \Omega_1(x)\left[\frac{x-\lambda_1(x)}{x-a}\left|f'(a)\right| + \frac{\lambda_1(x)-a}{x-a}\left|f'(x)\right|\right] \tag{19}$$

and

$$\int_x^b (1-W(t))\left|f'(t)\right|dt \leq \Omega_2(x)\left[\frac{b-\lambda_2(x)}{b-x}\left|f'(x)\right| + \frac{\lambda_2(x)-x}{b-x}\left|f'(b)\right|\right], \tag{20}$$

where

$$\Omega_1(x) = \int_a^x (x-s)w(s)ds, \quad \Omega_2(x) = \int_x^b (s-x)w(s)ds,$$

$$\lambda_1(x) = \frac{1}{2\int_a^x (x-s)w(s)ds} \int_a^x (x^2 - s^2)w(s)ds$$

and

$$\lambda_2(x) = \frac{1}{2\int_x^b (s-x)w(s)ds} \int_x^b (s^2 - x^2)w(s)ds.$$

So, from (18) using (19) and (20) we get inequality (16).
If $|f'|$ is concave then from Theorem 1.1 we get

$$\int_a^x W(t)\left|f'(t)\right|dt \le \Omega_1(x)\left|f'(\lambda_1(x))\right| \tag{21}$$

and

$$\int_x^b (1-W(t))\left|f'(t)\right|dt \le \Omega_2(x)\left|f'(\lambda_2(x))\right|. \tag{22}$$

Finally, from (18) using (21) and (22) we get inequality (17).

□

As a special case of Theorem 2.3, for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$, we obtain the same inequality as in paper [4].

Corollary 2.4. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function, $f' \in L[a,b]$ and $x \in [a,b]$. If $|f'|$ is convex then:*

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\le \frac{(x-a)^2|f'(a)| + (b-x)^2|f'(b)|}{6(b-a)} + \frac{(x-a)^2 + (b-x)^2}{3(b-a)}|f'(x)|.\end{aligned} \tag{23}$$

If $|f'|$ is concave then:

$$\begin{aligned}&\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \\ &\le \frac{(x-a)^2}{2(b-a)}\left|f'\left(\frac{2x+a}{3}\right)\right| + \frac{(b-x)^2}{2(b-a)}\left|f'\left(\frac{2x+b}{3}\right)\right|.\end{aligned} \tag{24}$$

Proof. This is a special case of Theorem 2.3 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$.

□

3 Hermite-Hadamard inequalities for 3-convex functions

Applying results from previous section we obtain Hermite-Hadamard inequalities for 3-convex functions and for functions whose modulus of derivatives are convex.

Theorem 3.1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function and $f' \in L[a,b]$, and let $w : [a,b] \to [0,\infty)$ be some nonnegative integrable weight function, such that $\int_a^b w(s)ds = 1$ and $\lambda = \int_a^b tw(t)dt$. If f is a 3-convex function then:*

$$\begin{aligned}&\Omega(\lambda)\left[f'(\lambda_1(\lambda)) - \frac{b-\lambda_2(\lambda)}{b-\lambda}f'(\lambda) - \frac{\lambda_2(\lambda)-\lambda}{b-\lambda}f'(b)\right] \\ &\le f(\lambda) - \int_a^b w(t)f(t)dt \\ &\le \Omega(\lambda)\left[\frac{\lambda-\lambda_1(\lambda)}{\lambda-a}f'(a) + \frac{\lambda_1(\lambda)-a}{\lambda-a}f'(\lambda) - f'(\lambda_2(\lambda))\right],\end{aligned} \tag{25}$$

where

$$\Omega(\lambda) = \int_a^\lambda (\lambda - s)w(s)ds, \tag{26}$$

$$\lambda_1(\lambda) = \frac{1}{2\int_a^\lambda (\lambda-s)w(s)ds}\int_a^\lambda (\lambda^2 - s^2)w(s)ds \tag{27}$$

and

$$\lambda_2(\lambda) = \frac{1}{2\int_\lambda^b (s-\lambda)w(s)ds} \int_\lambda^b (s^2-\lambda^2)w(s)ds. \tag{28}$$

If f is a 3 concave function, then the inequalities in (25) are reversed.

Proof. Applying (8) with $\lambda = \frac{1}{\int_a^b w(t)dt}\int_a^b tw(t)dt = \int_a^b tw(t)dt$ instead of x, for a 3-convex function f, we obtain the inequalities (25). Further, by elementary calculations from (9), (10) and (11) we get $\Omega_1(\lambda) = \Omega_2(\lambda) = \int_a^\lambda (\lambda - s)w(s)ds$ and

$$\lambda_2(\lambda) = \frac{\int_a^b (s^2-\lambda^2)w(s)ds}{2\int_a^\lambda (\lambda-s)w(s)ds} + \lambda_1(\lambda),$$

where

$$\lambda_1(\lambda) = \frac{1}{2\int_a^\lambda (\lambda-s)w(s)ds} \int_a^\lambda (\lambda^2-s^2)w(s)ds,$$

$$\lambda_2(\lambda) = \frac{1}{2\int_\lambda^b (s-\lambda)w(s)ds} \int_\lambda^b (s^2-\lambda^2)w(s)ds.$$

Similarly, if f is a 3 concave function, then using Theorem 2.1, we derive the reversed inequalities in (25).

□

Corollary 3.2. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $f'\in L[a,b]$. If f is a 3-convex function then:*

$$\begin{aligned}
&\frac{b-a}{8}\left[f'\left(\frac{2a+b}{3}\right) - \frac{2}{3}f'\left(\frac{a+b}{2}\right) - \frac{1}{3}f'(b)\right] \\
&\le f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \\
&\le \frac{b-a}{8}\left[\frac{1}{3}f'(a) + \frac{2}{3}f'\left(\frac{a+b}{2}\right) - f'\left(\frac{a+2b}{3}\right)\right].
\end{aligned} \tag{29}$$

If f is a 3-concave function, then the inequalities in (29) are reversed.

Proof. This is a special case of Theorem 3.1 for $w(t) = \frac{1}{b-a}$, $t\in[a,b]$.

□

Theorem 3.3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $f'\in L[a,b]$, let $w:[a,b]\to[0,\infty)$ be some nonnegative integrable weight function, such that $\int_a^b w(s)ds = 1$ and $\lambda = \int_a^b tw(t)dt$. If $|f'|$ is convex then:*

$$\begin{aligned}
&\left| f(\lambda) - \int_a^b w(t)f(t)dt \right| \\
&\le \left(\frac{\lambda|f'(a)| - a|f'(\lambda)|}{\lambda - a} + \frac{b|f'(\lambda)| - \lambda|f'(b)|}{b-\lambda}\right)\int_a^\lambda (\lambda-s)w(s)ds \\
&+ \frac{|f'(\lambda)| - |f'(a)|}{2(\lambda-a)}\int_a^\lambda (\lambda^2-s^2)w(s)ds + \frac{|f'(b)| - |f'(\lambda)|}{2(b-\lambda)}\int_\lambda^b (s^2-\lambda^2)w(s)ds.
\end{aligned} \tag{30}$$

If $|f'|$ is concave then:

$$\begin{aligned}
&\left| f(\lambda) - \int_a^b w(t)f(t)dt \right| \\
&\le \left(|f'(\lambda_1(\lambda))| + |f'(\lambda_2(\lambda))|\right)\int_a^\lambda (\lambda-s)w(s)ds,
\end{aligned} \tag{31}$$

where $\lambda_1(\lambda)$ and $\lambda_2(\lambda)$ are given by (27) and (28).

Proof. Applying (16) and (17) with $\lambda = \frac{1}{\int_a^b w(t)dt}\int_a^b tw(t)dt = \int_a^b tw(t)dt$ instead of x, we obtain inequalities (30) and (31), respectively.

□

Corollary 3.4. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $f'\in L[a,b]$. If $|f'|$ is convex then:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{24}\left[\left|f'(a)\right|+4\left|f'\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right]. \tag{32}$$

If $|f'|$ is concave then:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8}\left[\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right]. \tag{33}$$

Proof. This is a special case of Theorem 3.3 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Remark 1. The inequality (32) was first proved by Cerone and Dragomir in [4].

Theorem 3.5. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $f'\in L[a,b]$, and let $w:[a,b]\to[0,\infty)$ be some non-negative integrable weight function such that $\int_a^b w(s)ds=1$ and $\lambda=\int_a^b tw(t)dt$. If f is 3-convex then:*

$$\begin{aligned}&\frac{(b-\lambda)(\lambda-a)}{b-a}\left[f'(\lambda_1(b))-\frac{b-\lambda_2(a)}{b-a}f'(a)-\frac{\lambda_2(a)-a}{b-a}f'(b)\right]\\ &\leq \frac{b-\lambda}{b-a}f(a)+\frac{\lambda-a}{b-a}f(b)-\int_a^b w(t)f(t)dt\\ &\leq \frac{(b-\lambda)(\lambda-a)}{b-a}\left[\frac{b-\lambda_1(b)}{b-a}f'(a)+\frac{\lambda_1(b)-a}{b-a}f'(b)-f'(\lambda_2(a))\right],\end{aligned} \tag{34}$$

where

$$\lambda_1(b)=\frac{1}{2(b-\lambda)}\int_a^b (b^2-s^2)w(s)ds \tag{35}$$

and

$$\lambda_2(a)=\frac{1}{2(\lambda-a)}\int_a^b (s^2-a^2)w(s)ds. \tag{36}$$

If f is a 3-concave function, then the inequalities in (34) are reversed.

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 3.6. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $f'\in L[a,b]$. If f is 3-convex then:*

$$\begin{aligned}&\frac{b-a}{4}\left[f'\left(\frac{a+2b}{3}\right)-\frac{2}{3}f'(a)-\frac{1}{3}f'(b)\right]\\ &\leq \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\\ &\leq \frac{b-a}{4}\left[\frac{1}{3}f'(a)+\frac{2}{3}f'(b)-f'\left(\frac{2a+b}{3}\right)\right].\end{aligned} \tag{37}$$

If f is a 3-concave function, than the inequalities in (37) are reversed.

Proof. This is a special case of Theorem 3.5 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

Theorem 3.7. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $f'\in L[a,b]$, and let $w:[a,b]\to[0,\infty)$ be some non-negative integrable weight function such that $\int_a^b w(s)ds=1$ and $\lambda=\int_a^b tw(t)dt$. If $|f'|$ is convex then:*

$$\begin{aligned}&\left|\frac{b-\lambda}{b-a}f(a)+\frac{\lambda-a}{b-a}f(b)-\int_a^b w(t)f(t)dt\right|\\ &\leq \frac{(b-\lambda)(\lambda-a)}{b-a}\left(\frac{2b-\lambda_1(b)-\lambda_2(a)}{b-a}|f'(a)|+\frac{\lambda_1(b)+\lambda_2(a)-2a}{b-a}|f'(b)|\right).\end{aligned} \tag{38}$$

If $|f'|$ is concave then:

$$\left|\frac{b-\lambda}{b-a}f(a)+\frac{\lambda-a}{b-a}f(b)-\int_a^b w(t)f(t)dt\right|$$
$$\leq \frac{(b-\lambda)(\lambda-a)}{b-a}\left(|f'(\lambda_1(b))|+|f'(\lambda_2(a))|\right), \tag{39}$$

where $\lambda_1(b)$ and $\lambda_2(a)$ are given by (35) and (36).

Proof. The proof is similar to the proof of Theorem 2.3. □

Corollary 3.8. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function and $f'\in L[a,b]$. If $|f'|$ is convex then:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{b-a}{4}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \tag{40}$$

If $|f'|$ is concave then:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{b-a}{4}\left[\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right]. \tag{41}$$

Proof. This is a special case of Theorem 3.7 for $w(t)=\frac{1}{b-a}$, $t\in[a,b]$. □

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