PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special issue dedicated to Mirosław Baran on the occasion of his 60th birthday, Volume 14 · 2021 · Pages 59–65

Joukowski and Green, Chebyshev and Julia

Marta Kosek[a]

Dedicated to Mirosław Baran on the occasion of his 60th birthday.

Abstract

We deal with the Joukowski transformation, the Green function (mainly that of the interval $[-1,1]$), the Chebyshev polynomials and the filled Julia sets for polynomial mappings. We show that the functional sequence

$$\left(\frac{1}{d_1}\cdot\ldots\cdot\frac{1}{d_k}\log^+\left|T_{d_k}\circ\ldots\circ T_{d_1}\right|\right)_{k=1}^{\infty},$$

where T_d is the d-th Chebyshev polynomial, is uniformly convergent to $\mathbb{C}\ni\zeta\longmapsto\log\left|\zeta+(\zeta^2-1)^{1/2}\right|\in\mathbb{C}$ for any sequence of integers $(d_k)_{k=1}^{\infty}\subset\{2,3,\ldots\}$.

2020 AMS subject classification: Primary 30C10; Secondary 30E10.
Keywords: Joukowski's transformation, Green's function, Chebyshev's polynomials, Julia's sets.

1 Introduction

Let us start with the main result

Main Result 1.1. For any sequence $(d_k)_{k=1}^{\infty}\subset\{2,3,\ldots.\}$ the functional sequence

$$\left(\frac{1}{d_1}\cdot\ldots\cdot\frac{1}{d_k}\log^+\left|T_{d_k}\circ\ldots\circ T_{d_1}\right|\right)_{k=1}^{\infty}$$

(where T_d is the d-th Chebyshev polynomial) is uniformly convergent to the complex Green function of the interval $[-1,1]$, i.e. to $\log|\mathbf{h}|$.

To define the function $\mathbf{h}$ consider first the Joukowski transformation

$$\mathbf{J}:\mathbb{C}\setminus\overline{D}(1)\ni\zeta\longmapsto\frac{1}{2}\left(\zeta+\zeta^{-1}\right)\in\mathbb{C}\setminus[-1,1] \tag{1.1}$$

(in this paper $\overline{D}(1)$ stands for the closed disk with origin at 0 and radius 1) and then its inverse is

$$\mathbf{h}:=\mathbf{J}^{-1}. \tag{1.2}$$

It is known (see e.g. [9, §5.4]) that $|\mathbf{h}|$ can be extended continuously to whole $\mathbb{C}$ via formula

$$|\mathbf{h}|\,\Big|_{[-1,1]}\equiv 1. \tag{1.3}$$

The fact that $\log|\mathbf{h}|$ is the Green function of $[-1,1]$ is due to Lundin ([13]).

Main Result 1.1 deals with notions connected with the names of Joukowski, Green and Tchebyshev. An inspiration for this paper was given by two articles [1] and [2] published by Baran in 1988 and 1989. In [1] the author used transformations of Joukowski's type to give another proof of Lundin's formula ([13]) for Green's function. In [2] he proposed a functional equation for the Joukowski transformation in which the Chebyshev polynomials are also present. That is why the first three names appear in the title of this note. The fourth one is added since we will use the theory of Julia sets of polynomial mappings.

Apart from the main result and its consequences we want to recall a couple of examples arising from the complex dynamics. The key fact will be that $[-1,1]$ is the (filled) Julia set of every Chebyshev polynomial (cf. [15, Problem 7-c]) and that – as mentioned in Main Result 1.1 – its complex Green function is $\log|\mathbf{h}|$ (see e.g. [9, Lemma 5.4.2 (d)]).

[a]Institute of Mathematics, Faculty of Mathematics and Computer Science, Jagiellonian University, Łojasiewicza 6, 30-348 Kraków, Poland, E-mail: Marta.Kosek@im.uj.edu.pl

2 The four names

2.1 Joukowski

Note that Joukowski's transformation $\mathbf{J}$ given by (1.1) is a conformal bijection.

Recall (see e.g. [9, §5.4]) that $\mathbf{h}$ given by (1.2) can be written as follows

$$\mathbf{h} : \mathbb{C} \setminus [-1,1] \ni \zeta \longmapsto \zeta + \left(\zeta^2 - 1\right)^{1/2} \in \mathbb{C} \setminus \overline{D}(1), \tag{2.1}$$

where the branch of the square root is chosen so that $t > 1 \Longrightarrow \mathbf{h}(t) > 1$.

Note also that both Joukowski's transformation $\mathbf{J}$ and its inverse $\mathbf{h}$ can be continuously prolonged to ∞ by $\mathbf{J}(\infty) = \mathbf{h}(\infty) = \infty$. Furthermore, the mapping $\mathbf{J}$ can be actually defined on the whole sphere, namely

$$\mathbf{J} : \widehat{\mathbb{C}} \ni \zeta \longmapsto \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right) \in \widehat{\mathbb{C}}$$

and in such a setting we have $\mathbf{J}(\partial D(1)) = [-1,1]$ and $\mathbf{J}(D(1)) = \widehat{\mathbb{C}} \setminus \overline{D}(1)$ (set $\overline{D}(1)$ is defined just below).

In this paper we put for $r > 0$

$$D(r) := \{\zeta \in \mathbb{C} : \quad |z| < r\} \quad \text{and} \quad \overline{D}(r) := \{\zeta \in \mathbb{C} : \quad |z| \le r\}$$

and for $r > 1$

$$E(r) := \left\{ z = (x,y) \in \mathbb{R}^2 = \mathbb{C} : \quad \frac{4x^2}{(r + r^{-1})^2} + \frac{4y^2}{(r - r^{-1})^2} \le 1 \right\}.$$

It is easy to check that for $r > 1$

$$\mathbf{J}(\partial D(r)) = \mathbf{J}\left(\partial D\left(r^{-1}\right)\right) = \partial E(r). \tag{2.2}$$

Note finally that $\mathbf{J}\big|_{\mathbb{R}\setminus\{0\}}$ and $|\mathbf{h}|\,\big|_{\mathbb{R}}$ are even and $|\mathbf{h}|\,\big|_{[1,+\infty)}$ is nonnegative and increasing, and in consequence the same is true for $\log|\mathbf{h}|\,\big|_{[1,+\infty)}$. We note also that

$$\forall x \in (-\infty,-1] \cup [1,+\infty) : \quad \log|\mathbf{h}(x)| \le \log|2x| \le |x|.$$

2.2 Green

Let $K \subset \mathbb{C}^N$ be a compact set. Its (pluricomplex) Green function is defined as follows

$$V_K(z) := \sup\left\{u(z) : \quad u \in \mathcal{L}(\mathbb{C}^N) \quad \text{and} \quad u|_K \le 0\right\} \quad \text{for } z \in \mathbb{C}^N,$$

where $\mathcal{L}(\mathbb{C}^N)$ is the Lelong class, i.e. the family of all plurisubharmonic functions on $\mathbb{C}^N$ with logarithmic grow at the infinity. For the background see [9, Chapter 5]. In the one dimensional case V_K is the complex Green function of the $\mathbb{C} \setminus \widehat{K}$, the unbounded component of $\mathbb{C} \setminus K$ (we use the standard notation $\widehat{K}$ for the polynomially convex hull of K) extended by 0 on $\widehat{K}$.

Recall that for any complex norm $\|\cdot\|$ in $\mathbb{C}^N$ and the closed ball $B := \left\{z \in \mathbb{C}^N : \ \|z - a\| \le R\right\}$ we have

$$V_B : \mathbb{C}^N \ni z \longmapsto \log^+ \frac{\|z - a\|}{R} \in \mathbb{R}.$$

In particular

$$V_{\overline{D}(1)} : \mathbb{C} \ni \zeta \longmapsto \log^+|z| \in \mathbb{R}. \tag{2.3}$$

Recall also (see e.g. [9, Lemma 5.4.2 (d)]) that

$$V_{[-1,1]} = \log|\mathbf{h}|. \tag{2.4}$$

Consider a polynomial mapping $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$. The Łojasiewicz exponent at infinity of P (see e.g. [17]) is defined as

$$\mathscr{L}_\infty(P) := \sup\left\{\delta \in \mathbb{R} : \quad \liminf_{\|z\|\to\infty} \frac{\|P(z)\|}{\|z\|^\delta} > 0\right\}.$$

If $N = 1$, the Łojasiewicz exponent is equal to $\deg(P)$, in higher dimensions it can be strictly smaller. If $\mathscr{L}_\infty(P) = \deg(P)$, we say that P is regular. The mapping P is proper if and only if $\mathscr{L}_\infty(P) > 0$. For a proper polynomial map the following transformation formula holds (see [9, Theorem 5.3.1])

$$\mathscr{L}_\infty(P) V_{P^{-1}(K)} \le V_K \circ P \le \deg(P) V_{P^{-1}(K)} \quad \text{for any compact } K \subset \mathbb{C}^N. \tag{2.5}$$

Note that if $\mathscr{L}_\infty(P) = \deg(P)$, in particular if $N = 1$, we have equalities in (2.5).

A compact set $K \subset \mathbb{C}^N$ is called (pluri)regular if its Green function V_K is continuous. We use the following notation

$$\mathcal{R} = \mathcal{R}(\mathbb{C}^N) = \{K \subset \mathbb{C}^N : \quad K \text{ is nonempty, compact, pluriregular and polynomially convex}\}.$$

Klimek defined in [8] for $E, F \in \mathcal{R}$ their distance

$$\Gamma(E,F) := \sup_{z\in\mathbb{C}^N} |V_E(z) - V_F(z)| = \max\left(\sup_{z\in E} V_F(z), \sup_{z\in F} V_E(z)\right)$$

and showed that $(\mathcal{R},\Gamma)$ is a complete metric space. Note that a sequence $(E_n)_{n=1}^\infty$ is convergent to F in $(\mathcal{R},\Gamma)$ if and only if the functional sequence $(V_{E_n})_{n=1}^\infty$ is uniformly convergent to V_F.

2.3 Chebyshev

Chebyshev's polynomials can be defined recursively by the formula

$$T_0(z) = 1, \quad T_1(z) = z,$$
$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z) \qquad \text{for } z \in \mathbb{C}, n \in \{1,2,3,\dots\}.$$

It follows that we can write

$$T_n(z) = 2^{n-1}z^n + R_n(z) \qquad \text{for } z \in \mathbb{C}, n \in \{1,2,3,\dots\}, \tag{2.6}$$

where $\deg(R_n) < n$.

It is well known that

$$T_n \circ T_m = T_{nm} \qquad \text{for } n,m \in \{0,1,\dots\}. \tag{2.7}$$

Number 1 is a fixed point of T_n (for any n) just as it is that for Joukowski's transformation $\mathbf{J}$. Baran noted in [2] that

$$T_n(\mathbf{J}(z)) = \mathbf{J}(z^n), \qquad \text{for } z \in \mathbb{C} \setminus \{0\},\ n \in \{1,2,3,\dots\}. \tag{2.8}$$

Let us recall that (2.8) appears also in [5, §1.4], but Beardon does not mention Joukowski's name. Note also that the right hand side of the equation is a generalized Joukowski transformation (see [3]).

It is known that

$$T_n([-1,1]) = [-1,1] \tag{2.9}$$

and this together with (2.8) yield that

$$\mathbb{C} \setminus [-1,1] = \{z \in \mathbb{C} : \lim_{k\to\infty} |T_n^k(z)| = \infty\} \quad \text{for } n \geq 2. \tag{2.10}$$

Note also that

$$T_n^{-1}([-1,1]) = [-1,1] \tag{2.11}$$

and T_n and $-T_n$ are the only polynomials of degree n satisfying (2.9) and (2.11) in view of [5, Theorem 1.4.1].

We will need information on the image and inverse image of the unit disk under the Chebyshev polynomials. We have $T_1\left(\overline{D}(1)\right) = \overline{D}(1)$ and $-3 = T_2(i) \in T_2\left(\overline{D}(1)\right)$, hence $T_2\left(\overline{D}(1)\right)$ is not included in $\overline{D}(1)$.

Proposition 2.1.

$$\forall n \in \{1,2,\dots\} : \qquad T_n\left(\overline{D}(1)\right) \subset \overline{D}(3^{n-1}).$$

Proof: Assume that $|z| \leq 1$. We have $|T_1(z)| = |z| \leq 1 = 3^0$ and $|T_2(z)| \leq 2|z|^2 + 1 \leq 3 = 3^1$. By the recurrence formula if $|T_n(z)| \leq 3^{n-1}$ and $|T_{n-1}(z)| \leq 3^{n-2}$, then $|T_{n+1}(z)| \leq 3^n$ for $n \geq 2$. The assertion follows by complete induction on n. ∎

Now we turn to the inverse image. It is easy to check that $T_k^{-1}\left(\overline{D}(1)\right) \subset \overline{D}(1)$ for $k \in \{1,2,3,4\}$, but to our aim it is sufficient to show the following inclusion.

Proposition 2.2.

$$\forall n \in \{1,2,\dots\} : \quad T_n^{-1}\left(\overline{D}(1)\right) \subset \overline{D}(3).$$

Proof: The assertion is true for $n = 1$.

Fix an integer $n \geq 2$ and put $Z_n : \mathbb{C} \ni \zeta \longmapsto \zeta^n \in \mathbb{C}$.

Fix a point $z \in \mathbb{C} \setminus [-1,1]$. Then there exists a number $r = r_z > 1$ such that $z \in \partial E(r)$. It follows from (2.2) that $\mathbf{h}(z) \in \partial D(r)$ and therefore $\mathbf{J}(Z_n(\mathbf{h}(z))) \in \partial E(r^n)$. Since by (2.8)

$$T_n(z) = \mathbf{J}(Z_n(\mathbf{h}(z))),$$

we obtain the following implication

$$|T_n(z)| \leq 1 \quad \Longrightarrow \quad r^n \leq 1 + \sqrt{2},$$

because $\partial E(\varrho) \cap \overline{D}(1) \neq \emptyset$ if and only if $\varrho \in \left(1, 1+\sqrt{2}\right]$. Thus

$$|z| \leq r + r^{-1} \leq \sqrt[n]{1+\sqrt{2}} + \left(\sqrt[n]{1+\sqrt{2}}\right)^{-1} \leq 3.$$

∎

Let us recall that a compact set $E \subset \mathbb{C}$ is said to have Markov's property if there exist constants $M > 0, m > 0$ such that

$$\sup_{z\in E} |p'(z)| \leq M(\deg p)^m \sup_{z\in E} |p(z)|$$

for any polynomial p. Markov showed in [14] that for $E = [-1,1]$ one can take $M = 1$ and $m = 2$. Moreover these constants are best possible and here the Chebyshev polynomials play an important role, since for them the inequality becomes equality, as

$$T_n'(1) = n^2 \tag{2.12}$$

for any integer n.

Note also that if $E \subset \mathbb{C}$ has Markov's property, then the exponent m is not smaller than 1.

2.4 Julia

If $P:\mathbb{C}^N \longrightarrow \mathbb{C}^N$ is a polynomial mapping and k is an integer, then P^k denotes the k-th iteration given recursively by $P^1 = P, P^{k+1} = P \circ P^k$ for $k \in \{1,2,\ldots\}$.

Let P be a polynomial mapping. The (autonomous filled) Julia set of P is defined as

$$K[P] := \left\{ z \in \mathbb{C}^N : \quad \left(P^k(z)\right)_{k=1}^{\infty} \quad \text{is bounded} \right\}.$$

If $\mathscr{L}_\infty(P) > 1$, then $K[P] \in \mathcal{R}$ (see [8]). Moreover, the Green function $V_{K[P]}$ of the Julia set $K[P]$ is Hölder continuous (see [11]). By the standard proof of Banach's Contraction Principle

$$\forall E \in \mathcal{R}: \quad \lim_{k\to\infty} \Gamma\left(\left(P^k\right)^{-1}(E), K[P]\right) = 0 \tag{2.13}$$

and moreover if $R > 0$ is big enough, then

$$K[P] = \bigcap_{k=1}^{\infty} (P^k)^{-1}(\{z \in \mathbb{C}^N : \|z\| \leq R\}).$$

It follows from (2.5) that $K[P] = \mathbb{C}^N \setminus \left\{z \in \mathbb{C}^N : \lim_{k\to\infty} \left\|P^k(z)\right\| = \infty\right\}$. Thus by (2.10) we have $[-1,1] = K[T_n]$ for $n \geq 2$ (cf. [15, Problem 7-c]). In consequence e.g. $K[T_n \times T_d] = [-1,1]^2$ if $n, d \geq 2$.

3 Preliminaries

We recall some results which we will use later.

First we want to have a limit of a sequence of polynomial inverse images of one set.

Proposition 3.1. ([**10**, **Proposition 1**]) *Let $P_n : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ be a regular polynomial mapping of degree $d_n \geq 2$ for $n \in \{1,2,\ldots\}$. Let $K \in \mathcal{R}$ and define $E_n := (P_n \circ \ldots \circ P_1)^{-1}(K)$ for $n \in \{1,2,\ldots\}$. If*

$$\sum_{n=1}^{\infty} \frac{\Gamma(P_{n+1}^{-1}(K), K)}{d_1 d_2 \cdots d_n} < \infty, \tag{3.1}$$

then the sequence $(E_n)_{n=1}^{\infty}$ is convergent in $(\mathcal{R}, \Gamma)$ to a set E. Any other choice of $K \in \mathcal{R}$ for which (3.1) is satisfied, results in the same limit E.

Now we turn to some compact subsets of algebraic sets. We start with a comparison of the Green functions of a set and its holomorphic image contained in an algebraic set.

Proposition 3.2. ([**4**, **Proposition 1.3**]) *Let k be an integer not greater than n and let $\mathbb{M}$ be an algebraic set in $\mathbb{C}^n$ of pure dimension k. Assume that $E \subset \mathbb{C}^k$ is compact and non-pluripolar, W is an open neighbourhood of $\widehat{E}$ and $f : W \longrightarrow \mathbb{M}$ is a holomorphic mapping. If $\operatorname{rank}_E f = k$, then there exist $C > 0$ and $\delta_0 > 0$ such that*

$$V_{f(E)}(f(z)) \leq C V_E(z) \quad \text{if } \operatorname{dist}(z, E) \leq \delta_0.$$

For some more regular subsets of algebraic sets one has a so called tangential Markov's property.

Theorem 3.3. ([**4**, **Theorem 2.1**]) *Let $k \in \{1,\ldots,n\}$ and let $\mathbb{M}$ be an algebraic set in $\mathbb{C}^n$ of pure dimension k. Assume that the Green function V_E of a compact set $E \subset \mathbb{C}^k$ is Hölder continuous with exponent $\frac{1}{m}$. Let W be an open connected neighbourhood of $\widehat{E}$ and $f : W \longrightarrow \mathbb{M}$ be a holomorphic mapping. If $\operatorname{rank}_E f = k$, then there exists $M > 0$ such that*

$$|D_{T(t,v)} p(f(t))| \leq M d^m \sup_{w \in f(t)} |p(w)| \quad \text{if } t \in E, v \in \mathbb{S}^{k-1} \text{ and } p \in \mathbb{C}_d[z_1,\ldots,z_n],$$

where $\mathbb{S}^{k-1}$ denotes the unit sphere in $\mathbb{C}^k$ and $T(t,v) := D_v f(t)$.

4 The main result and its consequence

Proof of Main Result 1.1:

We have $\Gamma\left(T_{d_{k+1}}^{-1}([-1,1]), [-1,1]\right) = 0$ in view of (2.11), hence by Proposition 3.1 the sequence $\left((T_{d_k} \circ \ldots \circ T_{d_1})^{-1}([-1,1])\right)_{k=1}^{\infty}$ is convergent to a set E. It follows from (2.7) and (2.11) that this sequence is constant and $E = [-1,1]$.

Fix an integer $d \geq 2$. In view of (2.5) and (2.3)

$$V_{T_d^{-1}(\overline{D}(1))} = \frac{1}{d} \log^+ |T_d|.$$

Thus by Proposition 2.1

$$\sup_{z \in \overline{D}(1)} V_{T_d^{-1}(\overline{D}(1))}(z) \leq \frac{d-1}{d} \log 3 \leq 2.$$

On the other hand (2.3) and Proposition 2.2 imply

$$\sup_{z\in T_d^{-1}(\overline{D}(1))} V_{\overline{D}(1)}(z) \le \sup_{z\in\overline{D}(3)} \log^+|z| = \log 3 \le 2.$$

Combining these estimations we obtain

$$\Gamma\big(T_d^{-1}(\overline{D}(1)), \overline{D}(1)\big) \le 2.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{\Gamma\big(T_{d_{n+1}}^{-1}(\overline{D}(1)), \overline{D}(1)\big)}{d_1 d_2 \cdots d_n} < \infty.$$

Therefore by Proposition 3.1 the sequence $\big((T_{d_k}\circ \dots \circ T_{d_1})^{-1}(\overline{D}(1))\big)_{k=1}^{\infty}$ is also convergent to $[-1,1]$ and this means that

$$\sup_{z\in\mathbb{C}^N}\left|\frac{1}{d_1}\cdot\ldots\cdot\frac{1}{d_k}\log^+\left|(T_{d_1}\circ\ldots\circ T_{d_k})(z)\right| - V_{[-1,1]}(z)\right| \longrightarrow 0 \quad \text{if } k\to\infty$$

once again thanks to (2.5) and (2.3). ∎

Corollary 4.1. *For any $d\ge 2$ the sequence*

$$\left(\frac{1}{d^k}\log^+\left|T_d^k\right|\right)_{k=1}^{\infty}$$

is uniformly convergent to the complex Green function of $[-1,1]$, i.e. to $\log|\mathbf{h}|$.

Proof: It is a consequence of Main Result 1.1 for a constant sequence $(d_k)_{k=1}^{\infty}$ of integers. ∎

The fact from this corollary is actually well known, since $[-1,1] = K[T_d]$. Let namely $d\ge 2$ and p be a polynomial of degree d. Consider the sequence

$$\left(\frac{1}{d^k}\log^+\left|p^k\right|\right)_{k=1}^{\infty}. \tag{4.1}$$

The locally uniform convergence of this sequence in $\mathbb{C}\setminus K[p]$ can be deduced from an analogue of Böttcher theorem (see e.g. [15, §9]). The locally uniform convergence in the whole complex plane can be also proved by means of currents, see e.g. [6, Proposition 4.2]. The uniform convergence of the sequence given in (4.1) follows from a rather complicated approach developed by Ueda [19] and Hubbard and Papadol [7]. Note however that the fact that the limit of this sequence is the Green function $V_{K[p]}$ was shown in [18, Proposition 3].

On the other hand, one can obtain the uniform convergence of the sequence given in (4.1) to the Green function $V_{K[p]}$ directly from (2.13), i.e. from the argument developed by Klimek in [8]. Note that articles [19], [7] and [8], which show the uniform convergence, were published almost simultanously.

5 A regular set without Markov's property

Because of the title of this section we would like first refer the reader to [16]. We will give here an example similar to that given in [10, Corollary 2].

Example 5.1. Fix a bounded sequence $(d_n)_{n=1}^{\infty}$ of integers not smaller than 2. Let $P_n := (n+1)T_{d_n} - n$ for any integer n. Then the sequence

$$\big((P_n\circ\ldots\circ P_1)^{-1}([-1,1])\big)_{n=1}^{\infty}$$

is convergent in $(\mathcal{R},\Gamma)$ to a set E, which does not have Markov's property.

Proof: It follows from (2.11) that $P_n^{-1}([-1,1])\subset[-1,1]$. On the other hand (2.9) yields $\max\{|P_n(x)| : x\in[-1,1]\} = 2n+1$. Therefore combining (2.5), (2.4) and the properties of function $|\mathbf{h}|$ (see the end of Subsection 2.1) we obtain

$$\Gamma\big(P_n^{-1}([-1,1]),[-1,1]\big) = \sup_{x\in[-1,1]} V_{P_n^{-1}([-1,1])}(x) = $$

$$= \sup_{x\in[-1,1]}\frac{1}{d_n}\log\left|\mathbf{h}(P_n(x))\right| \le \frac{2n+1}{d_n}.$$

Therefore

$$\sum_{n=1}^{\infty}\frac{\Gamma\big(P_{n+1}^{-1}([-1,1]),[-1,1]\big)}{d_1 d_2\cdots d_n} < \infty$$

and we may use Proposition 3.1 to obtain

$$E = \lim_{n\to\infty}(P_n\circ\ldots\circ P_1)^{-1}([-1,1]).$$

Moreover,

$$E = \bigcap_{n=1}^{\infty}(P_n\circ\ldots\circ P_1)^{-1}([-1,1]).$$

Note that 1 is a fixed point of every P_n and therefore $1 \in E$. For any integer n consider the polynomial $q_n := P_n \circ \ldots \circ P_1$. In view of (2.12) we have $q_n'(1) = (n+1)! d_1 \cdots d_n$. On the other hand $q_n^{-1}([-1,1])$ is decreasing (with respect to inclusion) to E, hence $q_n(E) \subset [-1,1]$. If E had Markov's property, there would exist positive constants M, m such that

$$(n+1)! d_1 \cdots d_n \leq M(d_1 \cdots d_n)^m. \tag{5.1}$$

The sequence $(d_n)_{n=1}^{\infty}$ is bounded, say $d = \max\{d_n : n \geq 1\}$. Note that (5.1) implies

$$(n+1)! \leq M d^{(m-1)n},$$

which is impossible. ∎

6 A Julia set on an algebraic set

Here we turn to [12]. This article was devoted to some examples of iteration on algebraic sets. Portions of Julia sets contained in algebraic sets turned up to be good examples for the setting dscribed in [4] (see Proposition 3.2 and Theorem 3.3 above). We will describe here the case of such iteration on algebraic sets. For proofs and details we refer the reader to [12].

Let $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ be a polynomial mapping. If a set $\mathbb{M} \subset \mathbb{C}^N$ satisfies $P(\mathbb{M}) \subset \mathbb{M}$, we can consider a Julia type set of the form

$$K[P|_{\mathbb{M}}] := \left\{ z \in \mathbb{M} \ : \ ((P|_{\mathbb{M}})^j(z))_{j=1}^{\infty} \ \text{ is bounded} \right\}.$$

It is obvious that $K[P|_{\mathbb{M}}] = K[P] \cap \mathbb{M}$. In our case $\mathbb{M}$ is an algebraic set, which is a natural choice since P is a polynomial mapping.

We consider now namely $k \in \{1, \ldots, N\}$ and let $\mathbb{M}$ be an algebraic set in $\mathbb{C}^N$ of pure dimension k. Let $P : \mathbb{C}^N \to \mathbb{C}^N$ be a polynomial mapping such that $P(\mathbb{M}) \subset \mathbb{M}$. Suppose that there exists a biholomorphic polynomial mapping $f : \mathbb{C}^k \ni \zeta \longmapsto f(\zeta) \in \mathbb{M}$ and write $Q := f^{-1} \circ P \circ f$. Then Q is a polynomial mapping and we have $K[P|_{\mathbb{M}}] = f(K[Q])$ (see [12, proof of Proposition 2.2]). If $\mathscr{L}_\infty(Q) > 1$, then $K[Q] \in \mathcal{R}$ and its Green function is Hölder continuous (see Subsection 2.4). Hence in this case we can apply both Proposition 3.2 and Theorem 3.3. The situation is more interesting if $\mathscr{L}_\infty(P) \leq 1$, since in this case $K[P]$ may be not compact even if its portion contained in $\mathbb{M}$ is. Recall that we defined the Green function only for compact sets. Even though a definition for noncompact sets is also possible, we do not have in general results about regularity of such sets like those we have for compact sets.

In [12] some examples were given. Here we propose another one and its generalization.

Example 6.1. Let

$$\mathbb{M} := \{(z,w) \in \mathbb{C}^2 : w = z^2\}, \quad f : \mathbb{C} \ni \zeta \longmapsto f(\zeta) := (\zeta, \zeta^2) \in \mathbb{M},$$
$$P : \mathbb{C}^2 \ni (z,w) \longmapsto P(z,w) := (2w-1, 4wz^2 - 4w + 1) \in \mathbb{C}^2.$$

Then $K[P]$ is not compact, but $K[P|_{\mathbb{M}}]$ is and there exist $C > 0$ and $\delta > 0$ such that

$$V_{K[P|_{\mathbb{M}}]}(f(\zeta)) \leq C \log|\mathbf{h}(\zeta)| \quad \text{if} \quad \operatorname{dist}(\zeta, [-1,1]) < \delta.$$

Furthermore, there exist positive constants M and m such that

$$|D_{T(\zeta,v)} p(f(\zeta))| \leq M d^m \sup_{w \in K[P|_{\mathbb{M}}]} |p(w)|,$$

where $\zeta \in [-1,1], v \in \partial D(1)$ and $p \in \mathbb{C}_d[z_1, z_2]$.

Proof: Since $P(z, 0) = (-1, 1)$ for any $z \in \mathbb{C}$, the Julia set $K[P]$ is unbounded.

One can check straightforward that $f^{-1} : \mathbb{M} \ni (z,w) \longmapsto z \in \mathbb{C}$ and $f^{-1} \circ P \circ f = T_2$.

We are ready to apply Proposition 3.2 and Theorem 3.3. ∎

In Example 6.1 we have $K[Q] = [-1,1]$ and this means that $f(K[Q])$ is actually very simple. Therefore the tangential Markov's inequality is not surprising here at all. The more interesting fact is that $K[P|_{\mathbb{M}}]$ is of this simple form, since $K[P]$ is unbounded.

In this example T_2 played a role. The following example is actually the generalization of the previous one, here we have T_d for any $d \geq 2$.

Example 6.2. Let $d \in \{2, 3, 4, \ldots\}$, R_d be as in (2.6) and let

$$\mathbb{M}_d := \{(z,w) \in \mathbb{C}^2 : w = z^d\}, \quad f_d : \mathbb{C} \ni \zeta \longmapsto f(\zeta) := (\zeta, \zeta^d) \in \mathbb{M},$$
$$P_d : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \qquad \text{be given by the formula}$$
$$P_d(z,w) := \left(2^{d-1}w + R_d(z), \sum_{j=1}^{d} 2^{(d-1)j} z^{d(j-1)} w \cdot (R_d(z))^{d-j} + (R_d(z))^d \right).$$

Then $\mathscr{L}_\infty(P_d) \leq 1$, but $K[P_d|_{\mathbb{M}_d}]$ is compact and there exist $C > 0$ and $\delta > 0$ such that

$$V_{K[P_d|_{\mathbb{M}_d}]}(f_d(\zeta)) \leq C \log|\mathbf{h}(\zeta)| \quad \text{if} \quad \operatorname{dist}(\zeta, [-1,1]) < \delta.$$

Furthermore, there exist positive constants M and m such that

$$|D_{T(\zeta,v)} p(f_d(\zeta))| \leq M d^m \sup_{w \in K[P_d|_{\mathbb{M}_d}]} |p(w)|,$$

where $\zeta \in [-1,1], v \in \partial D(1)$ and $p \in \mathbb{C}_d[z_1, z_2]$.

Proof: Since $P_d(0,w) = \left(2^{d-1}w + R_d(0),\ 2^{d-1}w \cdot (R_d(0))^{d-1} + (R_d(0))^d\right)$ is a polynomial mapping of degree 1 with respect to w, it follows from the definition of the Łojasiewicz exponent that $\mathscr{L}_\infty(P_d) \le 1$.

As in the previous case $f_d^{-1} : \mathbb{M}_d \ni (z,w) \longmapsto z \in \mathbb{C}$. Moreover, $f_d^{-1} \circ P_d \circ f_d = T_d$.

Apply Proposition 3.2 and Theorem 3.3. ∎

Acknowledgements

The author wishes to thank Małgorzata Stawiska for the discussion on the Böttcher coordinate and the uniform convergence of iterates of polynomial maps in the basin of attraction of infinity as well as for bringing articles [19], [7], [6] to her attention.

References

[1] M. Baran, Siciak's extremal function of convex sets in $\mathbb{C}^N$, *Ann. Polon. Math.* 48:275–280, 1988.

[2] M. Baran, A functional equation for the Joukowski transformation, *Proc. Amer. Math. Soc.* 106:423–426, 1989.

[3] M. Baran, H. Haruki, On two new functional equations for generalized Joukowski transformation, *Ann. Polon. Math.* 56:79–84, 1991.

[4] M. Baran, W. Pleśniak, Polynomial inequalities on algebraic sets, *Studia Math.* 141:209–219, 2000.

[5] A. F. Beardon, Iteration of Rational Functions, Springer-Verlag, New York, 1991.

[6] R. Dujardin, Some problems of arithmetic origin in rational dynamics, in Arakelov Geometry and Diophantine Applications, eds. E. Peyre, G. Rémond, 341–374, Springer, Cham, 2021.

[7] J. H. Hubbard, P. Papadol, Superattractive fixed points in $\mathbb{C}^n$, *Indiana U. Math. J.* 43:321–365, 1994.

[8] M. Klimek, Metrics associated with extremal plurisubharmonic functions, *Proc. Amer. Math. Soc.* 123:2763–2770, 1995.

[9] M. Klimek, Pluripotential Theory, London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[10] M. Klimek, Iteration of analytic multifunctions, *Nagoya Math. J.* 162:19–40, 2001.

[11] M. Kosek, Hölder continuity property of filled-in Julia sets in $\mathbb{C}^n$, *Proc. Amer. Math. Soc.* 125:2029–2032, 1997.

[12] M. Kosek, Iteration of polynomial mappings on algebraic sets, *Complex Variables Theory Appl.* 43:187–197, 2000.

[13] M. Lundin, The extremal PSH for the complement of convex, symmetric subsets of $\mathbb{R}^N$, *Michigan Math. J.* 32:197–201, 1985.

[14] A. A. Markov, On a problem posed by D. I. Mendeleev, *Izv. Akad. Nauk. St-Petersbourg* 62:1–24, 1889 (in Russian).

[15] J. Milnor, Dynamics in One Complex Variable, Third Edition, Annals of Mathematic Studies, Princeton University Press, Princeton, 2006.

[16] W. Pleśniak, A Cantor regular set which does not have Markov's property, *Ann. Polon. Math.* 51:39–46, 1990.

[17] A. Płoski, On the growth of proper polynomial mappings, *Ann. Polon. Math.* 45:297–309, 1985.

[18] M. Stawiska, Plurisubharmonic Lyapunov functions, *Michigan Math. J.* 52:131–140, 2004.

[19] T. Ueda, Fatou sets in complex dynamics on projective spaces, *J. Math. Soc. Japan* 46:545–555, 1994.