PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Special issue dedicated to Norm Levenberg on the occasion of his 60th birthday, Volume 11 · 2018 · Pages 51–61

On the metric space of pluriregular sets

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Dedicated to Norm Levenberg on the occasion of his 60th birthday.

Communicated by F. Piazzon

Abstract

The metric space of pluriregular sets was introduced over two decades ago but to this day most of its topological properties remain a mystery. The purpose of this short survey is to present the current state of knowledge concerning this space.

2010 AMS subject classification: 32U35; 32U15; 32H50.
Keywords: extremal plurisubharmonic functions, pluricomplex Green functions, iteration of polynomial mappings.

1 Introduction

A pseudometric measuring the distance between pluriregular sets was introduced in [18] in 1995. The initial motivation for introducing such a concept came a couple of years earlier from two quite different mathematical directions, both of which were linked to pluripotential theory.

First came the observation that the invariance formula for pluricomplex Green functions shown in [15] (see also [16] and Theorem 5.3.1 in [17]) makes it possible to treat proper polynomial mappings in $\mathbb{C}^N$ as contractions acting on the set of continuous pluricomplex Green functions. Consequently, the construction of classical filled-in Julia sets in the complex plane can be reduced to a simple application of Banach's Contraction Principle.

The other source of motivation was an inequality shown in [4] (see Theorem 2.1). In that paper, Aron, Beauzamy and Enflo, using techniques from classical analysis, were comparing real and complex supremum norms for multivariate polynomials, with focus on estimates independent of the dimension of the underlying space. In particular, they compared the supremum norm of polynomials with real coefficients on the closed unit polydisc $\mathbb{P} \subset \mathbb{C}^N$, with the supremum norm of the same polynomial on the hypercube $\mathbb{H} := [-1,1]^N = \mathbb{P} \cap \mathbb{R}^N$, concluding that for any polynomial p with real coefficients, $\|p\|_{\mathbb{P}} \leq C\|p\|_{\mathbb{H}}$, for explicitly given constant C depending only on the degree of the polynomial p. It was shown in [18] (and already acknowledged in [4] – see *added in proof* p. 197) that by comparing the pluricomplex Green functions for $\mathbb{P}$ and $\mathbb{H}$ one can get considerably better constants for all complex polynomials of degree at least 2. In fact, the improvement gets better exponentially as the degree of the polynomial increases.

The aim of this paper is to collect and interpret the results concerning the metric space of pluriregular sets, as these results have been scattered in literature over the last two decades. Most of them come from papers authored or co-authored by one or both of the authors of this survey. Those results have also found uses and extensions in the work of other researchers — see [5], [11], [40], [12], [41], [31] and [14].

We will close the paper with some general comments and a few open problems.

2 Preliminaries and Notation

For any non-empty sets X, Y, we will denote by Y^X the set of all functions from X to Y. If $\mathcal{F}$ is a non-empty collection of subsets of a set X, then

$$\bigcup \mathcal{F} := \bigcup_{F \in \mathcal{F}} F \subset X.$$

We will be using several metric spaces and so we have to establish convenient notational conventions. First of all, the default metric in $\mathbb{C}^N$ is the Euclidean metric. If (X,d) is a metric space, then $B_d(a,r)$ and $\overline{B}_d(a,r)$ will denote, respectively, the open and closed balls with centre at $a \in X$ and radius $r > 0$. For the Euclidean balls, the subscript d will be dropped. By $\kappa(X)$ we will denote the family of all non-empty compact subsets of X. The symbol $\mathcal{C}_b(X)$ will stand for the vector space of bounded

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complex-valued continuous functions on X, furnished with the metric $d_\infty(f,g) = \sup_X |f-g|$. If $E \in \kappa(X)$, then $\|f\|_E = \sup_E |f|$ for any $f \in \mathcal{C}_b(X)$. We define

$$\delta_E(z) := \operatorname{dist}(z,E) = \inf\{d(z,w) : w \in E\}$$

for any closed set $E \subset X$ and the *ε-dilation* of E

$$E^\varepsilon := \{z \in X : \delta_E(z) \leq \varepsilon\}, \qquad \varepsilon > 0.$$

The *Hausdorff distance* on $\kappa(X)$ is defined by

$$\chi_d(E,F) := \max\{\|\delta_E\|_F, \|\delta_F\|_E\} = \|\delta_E - \delta_F\|_X. \tag{2.1}$$

For the Hausdorff distance in $\kappa(\mathbb{C}^N)$, with d being the Euclidean metric, we will simply use the letter χ.

The dimension N will be fixed throughout the paper. The symbol $\mathfrak{p}_k$ will denote the family of all complex polynomials $p : \mathbb{C}^N \longrightarrow \mathbb{C}$ with $\deg p \leq k$. By $\mathfrak{p}$ we will denote the set of all polynomials. The symbol $\mathcal{P}_k$ will stand for the set of all complex polynomial mappings $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ of degree at most k, that is

$$\mathcal{P}_k = \underbrace{\mathfrak{p}_k \times \ldots \times \mathfrak{p}_k}_{N\text{-copies}}.$$

We say that $P \in \mathcal{P}_k$ is *regular*, if the zero set of the homogeneous part of P of degree k is a singleton.

For any set $E \in \kappa(\mathbb{C}^N)$, we will denote its *pluricomplex Green function* by V_E. For historical and mathematical background concerning pluricomplex Green functions we refer the reader to [17]. It is well-known that

$$V_E = \log \Phi_E, \tag{2.2}$$

where Φ_E is the *Siciak extremal function*

$$\Phi_E(z) := \sup\left\{|p(z)|^{1/\deg p} : p \in \mathfrak{p},\ \|p\|_E \leq 1,\ \deg p \geq 1\right\}, \qquad z \in \mathbb{C}^N.$$

It follows from (2.2) that for any compact set E, the polynomially convex hull $\widehat{E}$ of E coincides with the zero set of V_E. A compact set E is said to be *pluriregular* if V_E is continuous. We will use the symbol $\mathcal{R}_*$ to denote the family of all compact pluriregular subsets of $\mathbb{C}^N$. We also put

$$\mathcal{R} := \{E \in \mathcal{R}_* : E = \widehat{E}\}.$$

Siciak's extremal functions, and hence the pluricomplex Green functions, as well as pluriregular sets, have proved to be very useful concepts in pluricomplex analysis and most of all in approximation theory (see the excellent survey by Pleśniak [35] for an overview).

For any $E \in \kappa(\mathbb{C}^N)$ and any $\varepsilon > 0$, we define the *ε-augmentation* of E as the set

$$E(\varepsilon) := \{z \in \mathbb{C}^N : V_E(z) \leq \varepsilon\}.$$

Because of lower semicontinuity of V_E, all augmentations of E are closed. If $E \in \mathcal{R}_*$, then all $E(\varepsilon)$ are compact. Moreover $V_{E(\varepsilon)} = \max\{0, V_E - \varepsilon\}$ as shown by Mazurek [37, Proposition 5.11]. The relationship between the dilations and augmentations of E is in general far from clear.

Following Siciak [39], we say that a non-empty subset $\mathcal{E} \subset \mathcal{R}$ has the *equicontinuity property* if

$$\lim_{\varepsilon \to 0} \sup_{E \in \mathcal{E}} \|V_E\|_{E^\varepsilon} = 0. \tag{2.3}$$

Another natural concept linked to the pluricomplex Green functions is the *Robin constant*

$$\gamma(E) = \limsup_{\|z\| \to \infty} \left(V_E^*(z) - \log\|z\|\right), \qquad E \in \kappa(\mathbb{C}^N), \tag{2.4}$$

where V_E^* denotes the upper semicontinuous regularization of V_E.

3 Definition and fundamental properties

Following [18], we can define the pseudometric

$$\Gamma(E,F) := \max\{\|V_E\|_F, \|V_F\|_E\} = \|V_E - V_F\|_{\mathbb{C}^N}, \qquad E, F \in \mathcal{R}_*. \tag{3.1}$$

When restricted to the sets from $\mathcal{R}$, the pseudometric Γ becomes a metric and we will refer to the pair $(\mathcal{R}, \Gamma)$ as the *space of pluriregular sets*. It turns out that $(\mathcal{R}, \Gamma)$ is a complete metric space (see [18, Theorem 1]). Consequently, if $\mathbb{B}$ denotes the closed unit ball in $\mathbb{C}^N$, then

$$\big(\mathcal{R}, \Gamma\big) \ni E \longmapsto V_E - V_{\mathbb{B}} \in \Big(\mathcal{C}_b(\mathbb{C}^N), d_\infty\Big)$$

is an isometric embedding with closed range.

One of the most attractive properties of the metric space of pluriregular sets is the fact that proper polynomial mappings lead to natural contractions of that space. In order to be more precise we need some extra terminology. The *Łojasiewicz exponent at infinity* of a polynomial mapping $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ is the real number defined as

$$\mathscr{L}_\infty(P) := \sup\left\{\delta \in \mathbb{R} : \liminf_{\|z\| \to \infty} \frac{\|P(z)\|}{\|z\|^\delta} > 0\right\}.$$

It turns out that the supremum is always achieved. Moreover, if $R > 0$ is sufficiently large, then there exists $M > 0$ such that

$$||P(z)|| \geq M||z||^{\mathscr{L}_\infty(P)}, \qquad ||z|| \geq R. \tag{3.2}$$

The mapping P is proper if and only if $\mathscr{L}_\infty(P) > 0$. If P is regular, then $\mathscr{L}_\infty(P) = \deg P$. The survey [29] provides comprehensive information concerning the Łojasiewicz exponent at infinity and its properties.

If $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ is a proper polynomial mapping, then

$$\mathscr{L}_\infty(P) V_{P^{-1}(E)} \leq V_E \circ P \leq \deg(P) V_{P^{-1}(E)} \tag{3.3}$$

for any $E \in \kappa\left(\mathbb{C}^N\right)$ (see [15], [16] or [17, Theorem 5.3.1]). In particular, if $E \in \mathcal{R}$, then also $P^{-1}(E) \in \mathcal{R}$. Moreover, it follows that the function defined by the formula

$$A_{\{P\}} : \mathcal{R} \ni E \longmapsto P^{-1}(E) \in \mathcal{R} \tag{3.4}$$

satisfies the Lipschitz condition with respect to Γ with the constant $1/\mathscr{L}_\infty(P)$ (see [18]). Furthermore, if P is regular, then $A_{\{P\}}$ is a similitude [3], that is

$$\Gamma\Big(A_{\{P\}}(E), A_{\{P\}}(F)\Big) = \frac{1}{\deg P}\Gamma(E,F), \qquad E, F \in \mathcal{R}.$$

An *escape radius* for a polynomial mapping $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$, provided that it exists, is a number $R > 0$ such that if $z \in \mathbb{C}^N \setminus \overline{B}(0,R)$, then $\lim_{n\to\infty} \|z_n\| = \infty$, where $z_0 = z$ and $z_n = P(z_{n-1})$ for $n \geq 1$. If $1 < \delta \leq \mathscr{L}_\infty(P)$, then there exists an escape radius $R > 1$ such that (see [19])

$$\inf\left\{ \frac{||P(z)||}{||z||^\delta} \,:\, ||z|| \geq R \right\} > R^{1-\delta}. \tag{3.5}$$

In this case

$$B(0,R) \supset P^{-1}\left(\overline{B}(0,R)\right) \tag{3.6}$$

and hence, because of (3.3), we get the practical estimate

$$\Gamma\Big(P^{-1}(\overline{B}(0,R)), \overline{B}(0,R)\Big) \leq \frac{||P||_{\partial B(0,R)}}{R\,\delta}. \tag{3.7}$$

For a survey of the concept of escape radii see [28].

Directly from the definitions of the metric Γ and the Siciak extremal function we get another estimate

$$\|p\|_E \leq \exp\Big[k\Gamma(E,F)\Big]\|p\|_F, \qquad p \in \mathfrak{p}_k,\ E, F \in \mathcal{R}, \tag{3.8}$$

the usefulness of which depends on our ability to calculate or estimate the exact value of $\Gamma(E,F)$. When $E, F \in \mathcal{R}$ are in some way simple, this can often be done easily (see e.g. [18]). For instance, for the augmentations of any $E \in \mathcal{R}$, we have the equality $\Gamma(E, E(\varepsilon)) = \varepsilon$, where $\varepsilon > 0$, and for the sets mentioned in the introduction $\Gamma(\mathbb{P}, \mathbb{H}) = \log(1 + \sqrt{2})$. The latter fact, together with (3.8), proved to be important in [14, Proposition 1.2]. However, in general, the exact calculation might be a difficult task. Nevertheless, for some classes of sets a viable approximation is possible.

Recall that if $E \in \mathcal{R}_*$ and μ is a positive Borel measure supported on E, then the pair (E, μ) is said to have the *Bernstein-Markov property* if the sequence

$$M_k := \sup\left\{ \frac{||p||_E}{||p||_{L^2(\mu)}} \,:\, p \in \mathcal{P}_k(\mathbb{C}^N) \right\}, \qquad k \geq 1,$$

has subexponential growth, that is

$$\limsup_{k\to\infty} \sqrt[k]{M_k} \leq 1.$$

A comprehensive overview of this property can be found in [9] and [32]. One of the attractive consequences of the Bernstein-Markov property is the possibility of convenient approximation of the pluricomplex Green function of E. Bloom and Shiffman [10] demonstrated that if $\{Q_j \in \mathfrak{p}_k \,:\, j = 1, \dots, \dim \mathfrak{p}_k\}$ is an orthonormal basis for $\mathfrak{p}_k \cap L^2(\mu)$ and

$$B_k^\mu(z) := \sum_{j=1}^{\dim \mathfrak{p}_k} |Q_j(z)|^2, \qquad z \in \mathbb{C}^N,$$

then

$$\lim_{k\to\infty} \frac{1}{2k} \log B_k^\mu(z) = V_E(z), \qquad z \in \mathbb{C}^N, \tag{3.9}$$

and the convergence is uniform on compact subsets of $\mathbb{C}^N$ (see [10, Lemma 3.4]). The function B_k^μ is called the *Bergman function* of E of order k. Bergman functions can be approximated numerically with the help of weakly admissible meshes [32] or Monte-Carlo simulation [2]. Whenever such an approximation is viable for both E and F, the approximation of $\Gamma(E,F)$ is a natural consequence.

The following theorem summarizes the most useful metric and topological characteristics of the metric space $(\mathcal{R}, \Gamma)$.

Theorem 3.1. *We have the following properties:*

(*i*) *The metric space* $(\mathcal{R}, \Gamma)$ *is complete.*

(*ii*) *If* $E, F \in \kappa(\mathbb{C}^N)$ *and* $\alpha, \beta > 0$*, then*

$$\Gamma\left(E^\alpha, F^\beta\right) \le \log\left[\frac{\chi\left(E^\alpha, F^\beta\right)}{\min\{\alpha,\beta\}} + 1\right].$$

(*iii*) *If* $\mathcal{E} \in \kappa(\mathcal{R})$*, then* $\bigcup \mathcal{E} \in \kappa(\mathbb{C}^N)$.

(*iv*) *If* $\mathcal{E}, \mathcal{F} \in \kappa(\mathcal{R})$*, then*

$$\Gamma\Big(\bigcup \mathcal{E}, \bigcup \mathcal{F}\Big) \le \max\big\{\Gamma(E,F) \,:\, E \in \mathcal{E},\, F \in \mathcal{F}\big\}.$$

(*v*) *If* $E, F, G \in \mathcal{R}$ *and* $E \subset F \subset G$*, then* $\max\{\Gamma(E,F), \Gamma(F,G)\} \le \Gamma(E,G)$.

(*vi*) *If* $\mathcal{E}, \mathcal{F} \in \kappa(\mathcal{R})$*, then*

$$\Gamma\Big(\bigcup \mathcal{E}, \bigcup \mathcal{F}\Big) \le \chi_\Gamma(\mathcal{E}, \mathcal{F}).$$

(*vii*) *The space* $(\mathcal{R}, \Gamma)$ *is separable. Polynomially convex hulls of finite unions of closed balls with centres in* $(\mathbb{Q} + i\mathbb{Q})^N$ *and rational radii form a dense subset of* $\mathcal{R}$.

(*viii*) *Closed balls in* $(\mathcal{R}, \Gamma)$ *do not have to be compact. In other words, the space is not proper.*

(*ix*) *A set* $\mathcal{E}$ *is relatively compact in* $\mathcal{R}$ *if and only if* $\bigcup \mathcal{E}$ *is bounded in* $\mathbb{C}^N$ *and* $\mathcal{E}$ *has the equicontinuity property.*

(*x*) *Assume that* $E_n, E \in \mathcal{R}$*,* $n \in \mathbb{N}$*, and* $\lim_{n\to\infty} \chi(E_n, E) = 0$*. Then the sequence* $(E_n)_{n=1}^\infty$ *has the equicontinuity property if and only if* $\lim_{n\to\infty} \Gamma(E_n, E) = 0$.

(*xi*) *The Robin constant, treated as a real-valued function on* $\mathcal{R}$*, is non-expanding, that is,* $|\gamma(E) - \gamma(F)| \le \Gamma(E,F)$ *for all* $E, F \in \mathcal{R}$*. In particular, the logarithmic capacity* $\operatorname{Cap} : \mathcal{R} \ni E \mapsto \exp(-\gamma(E)) \in \mathbb{R}_+$ *is continuous.*

(*xii*) *If* (X, d) *is a metric space and* $f : X \longrightarrow \mathcal{R}$ *is continuous, then* f *is upper semicontinuous as a set-valued function, i.e. for any open subset* $U \subset \mathbb{C}^N$*, the set* $\{x \in X \,:\, f(x) \subset U\}$ *is open.*

Proof. For proofs we refer the reader to the literature. Property (i) was shown in [18, Theorem 1]). Proofs of (ii), (iv) and (xi) can also be found in [18]. Part (v) follows directly from the definitions. Parts (iii), (vi) and (vii) were shown in [1]. Property (viii) was shown in [1] using [39, Example 3.6]. Parts (ix) and (x) are due to Siciak [39]. Statement (xii) follows from the fact that given $E \in \mathcal{R}$, the family $\{\{V_E < \varepsilon\} \,:\, \varepsilon > 0\}$ forms a base of the filter of all neighbourhoods of the set E in $\mathbb{C}^N$ (see [18, Corollary 1]).

We would like to present here a more direct proof of (viii) (which also justifies Siciak's example [39, Example 3.6] in an elementary way). Let

$$E_{n,m} = \bigcup_{j=0}^{n-1}\left[\frac{j}{n}, \frac{j}{n} + \frac{1}{2mn}\right] \cup [1,2], \qquad n, m \in \mathbb{N},$$

and let $F_n = \{j/n \,:\, j = 0, \ldots, n-1\}$ for $n \in \mathbb{N}$. When n is fixed, then

$$\lim_{m\to\infty} V_{E_{n,m}} = V_{F_n \cup [1,2]}$$

pointwise, and there exists a polar set Z_n such that $V_{F_n\cup[1,2]}(z) = V_{[1,2]}(z)$ for all $z \in \mathbb{C} \setminus Z_n$. The set $Z = Z_1 \cup Z_2 \cup Z_3 \cup \ldots$ is polar. Let $a \in [0,1) \setminus Z$ be arbitrarily chosen. For each n, there exist m_n such that

$$0 < V_{[1,2]}(a) - V_{E_{n,m_n}}(a) < 1/n.$$

The choices can be made by induction so that $m_n < m_{n+1}$ for all n. Then

$$\limsup_{n\to\infty} \Gamma\big(E_{n,m_n}, [0,2]\big) \ge \lim_{n\to\infty} V_{E_{n,m_n}}(a) = V_{[1,2]}(a) > 0.$$

It is also clear that $\lim_{n\to\infty} \chi(E_{n,m_n}, [0,2]) = 0$. In view of (x) the sequence $(E_{n,m_n})_{n=1}^\infty$ is not relatively compact in the metric space $(\mathcal{R}, \Gamma)$. Because of (v), $\Gamma(E_{n,m_n}, [0,2]) \le \Gamma([1,2],[0,2])$. Consequently, the closed ball in $\mathcal{R}$ with center at $[0,2]$ and radius $\Gamma([1,2],[0,2])$ is not compact. If $N > 1$, we simply consider the N-th Cartesian powers of the sets E_{n,m_n}, $[1,2]$ and $[0,2]$. The product property of the pluricomplex Green functions (see [37, Proposition 5.9] or [17, Theorem 5.1.8]) implies that

$$\limsup_{n\to\infty} \Gamma\Big((E_{n,m_n})^N, [0,2]^N\Big) = \limsup_{n\to\infty} \Gamma(E_{n,m_n}, [0,2]) > 0.$$

Also, since $\chi\left((E_{n,m_n})^N, [0,2]^N\right) \to 0$ as $n \to \infty$, Part (x) yields the same conclusion as in the one-dimensional case. □

4 Julia sets and iteration problems

Different classes of Julia sets have been studied extensively in complex dynamics over the last few decades (see e.g. references given in [1]). In this section we only look at the types of Julia sets that can be constructed with the help of the metric Γ, as such sets have proven to be a rich source of examples of pluriregular sets, and in fact it was shown in [20] that the so-called composite filled-in Julia sets generated by quadratic polynomial mappings form a dense subset of $\mathcal{R}$.

Let $P:\mathbb{C}^N\to\mathbb{C}^N$ be a polynomial mapping. Since $(\mathcal{R},\Gamma)$ is a complete metric space, if $\mathcal{L}_\infty(P)>1$, then the mapping $A_{\{P\}}$ defined by (3.4) has a unique fixed point $\mathbb{J}[P]$, by Banach's Contraction Principle. This happens to be the *(autonomous) filled-in Julia set* of P (see [18]) which can be defined as follows

$$\mathbb{J}[P]:=\left\{z\in\mathbb{C}^N:\ (P^n(z))_{n=1}^\infty\quad\text{is bounded}\right\},$$

where P^n denotes P composed with itself n times. The standard proof of Banach's Contraction Principle implies that

$$\lim_{n\to\infty}\Gamma\Big((P^n)^{-1}(E),\mathbb{J}[P]\Big)=0,\qquad E\in\mathcal{R}.$$

Moreover, if $a\in\mathbb{C}^N$ and $R>0$ is big enough, then in view of (3.6)

$$\mathbb{J}[P]=\bigcap_{n=1}^\infty(P^n)^{-1}\left(\overline{B}(a,R)\right). \tag{4.1}$$

Note that the last equality implies the convergence of the inverse images of the ball to the filled-in Julia set also in the Hausdorff metric.

The following result, [22, Lemma 4.5], enables us to generalize the notion of the autonomous filled-in Julia set.

Theorem 4.1 (Enhanced version of Banach's Contraction Principle, [22])**.** *Let (X,d) be a complete metric space and let $(H_n)_{n\ge1}$ be a sequence of contractions of X with contraction ratios not greater than $L<1$. If*

$$\sup_{n\ge1}d(H_n(x),x)<\infty \tag{4.2}$$

for each $x\in X$, then there exists a unique point c in X such that the sequence $(H_1\circ\ldots\circ H_n)_{n\ge1}$ converges pointwise to c.

Consider a sequence $(P_n)_{n=1}^\infty$ of polynomial mappings $P_n:\mathbb{C}^N\to\mathbb{C}^N$ such that $\inf_{n\in\mathbb{N}}\mathcal{L}_\infty(P_n)>1$ and such that there exists a common escape radius $R>0$ for all $P_n, n\in\mathbb{N}$, for which $\sup_{n\in\mathbb{N}}\|P_n\|_{\overline{B}(0,R)}<\infty$. Then the assumptions of the enhanced version of Banach's Contraction Principle for the sequence $(A_{\{P_n\}})_{n=1}^\infty$ are satisfied (for an argument justifying (4.2) see the proof of [22, Theorem 4.6]) and we get the unique point $\mathbb{J}[(P^n)_{n=1}^\infty]$ such that

$$\lim_{n\to\infty}\Gamma\Big((P_n\circ\ldots\circ P_1)^{-1}(E),\ \mathbb{J}[(P^n)_{n=1}^\infty]\Big)=0,\qquad E\in\mathcal{R}.$$

If $(P_n)_{n=1}^\infty$ is periodic, that is

$$\exists m\in\mathbb{N}\quad\forall k\in\mathbb{N},\ j\in\{0,\ldots,m-1\}:\quad P_{km+j}=P_j,$$

then $\mathbb{J}[(P^n(z))_{n=1}^\infty]$ is the (autonomous) filled-in Julia set of $P_m\circ\ldots\circ P_1$. However, if the sequence is not periodic, then we obtain the *non-autonomous filled-in Julia set* of $(P_n)_{n=1}^\infty$, i.e.

$$\mathbb{J}[(P_n)_{n=1}^\infty]=\left\{z\in\mathbb{C}^N:\ ((P_n\circ\ldots\circ P_1)(z))_{n=1}^\infty\quad\text{is bounded}\right\}.$$

One can also show that every non-autonomous filled-in Julia set can be approximated by the autonomous ones (cf. [3]), namely

$$\lim_{m\to\infty}\Gamma\Big(\mathbb{J}[(P_n)_{n=1}^\infty],\mathbb{J}[P_m\circ\ldots\circ P_1]\Big)=0. \tag{4.3}$$

It is also possible to consider a finite family $\mathcal{F}=\{P_1,\ldots,P_m\}$ of polynomial mappings $P_j:\mathbb{C}^N\to\mathbb{C}^N$ with Łojasiewicz exponents bigger than one and define the mappings

$$A_{\mathcal{F}}:\mathcal{R}_*\ni E\mapsto\bigcup_{j=1}^m P_j^{-1}(E)\in\mathcal{R}_*\qquad\text{and}\qquad\mathcal{R}\ni E\mapsto\widehat{A_{\mathcal{F}}(E)}\in\mathcal{R}.$$

These mappings are both contractions and the fixed point of the second one is called the *composite Julia set* of $\{P_1,\ldots,P_m\}$. Such sets were extensively studied in [18], [19], [25], [1], but we will go into more detail in a special setting a little further on.

From now on we will consider a simpler situation, namely the case of regular polynomial mappings. In this case, the Łojasiewicz exponent at infinity is equal to the degree of the mappings in question.

Before describing in detail a natural setting for composite Julia sets, it should be mentioned briefly that a more specialized generalization of the non-autonomous filled-in Julia sets was proposed in [20, Proposition 1, Corollaries 2 & 3].

For a fixed integer $k\ge2$, consider $\mathcal{P}_k^\star:=\{P\in\mathcal{P}_k: P\text{ is regular}\}$. It can be regarded as an open subset of the finite dimensional vector space $\mathcal{P}_k$ (see [20]). Moreover, [20, Lemma 1] gives a continuous dependence

$$\mathcal{P}_k^\star\ni P\mapsto r(P)\in(0,\infty)$$

of an escape radius $r(P)$ on P. Let $\mathcal{F}$ be a compact subfamily of $\mathcal{P}_k^\star$. Because of the continuous dependence, we can choose a common escape radius for all the mappings in this family, and in view of (3.7), we have

$$\sup_{P\in\mathcal{F}} \Gamma\Big(P^{-1}(\overline{B}(0,R)), \overline{B}(0,R)\Big) \leq \frac{1}{Rk} \sup_{P\in\mathcal{F}} ||P||_{\overline{B}(0,R)}. \tag{4.4}$$

Note that the supremum on the right hand side is finite because of the compactness of $\mathcal{F}$. If $(P_n)_{n=1}^\infty \subset \mathcal{F}$, it satisfies the conditions given above, hence we can consider its filled-in Julia set.

In this setting, we can define another type of Julia sets, namely two types of composite Julia sets. Recall that $A_{\{P\}}$ is a similitude, and in particular a continuous mapping. Moreover (see [3, Proposition 4.2]),

$$\mathcal{P}_k^\star \times \mathcal{R} \ni (P,K) \mapsto P^{-1}(K) \in \mathcal{R}$$

is continuous with respect to the product topology in the domain. It follows that for the compact family $\mathcal{F}$, the mapping

$$A_{\mathcal{F}} : \mathcal{R}_* \ni K \mapsto \bigcup_{P\in\mathcal{F}} P^{-1}(K) \in \mathcal{R}_*$$

is also a contraction with ratio $1/k$. We are now ready to define the *partly filled-in composite Julia set* of $\mathcal{F}$ as follows:

$$\mathbb{J}_{\mathrm{tr}}[\mathcal{F}] = \bigcap_{m\in\mathbb{N}} \left[\bigcup_{P_1,\ldots,P_m\in\mathcal{F}} (P_m \circ \ldots \circ P_1)^{-1}\big(\overline{B}(0,R)\big) \right] = \bigcap_{m\in\mathbb{N}} A_{\mathcal{F}}^m\big(\overline{B}(0,R)\big), \tag{4.5}$$

where R is once again a common escape radius for all the mappings from $\mathcal{F}$. The partly filled-in composite Julia set is closely related to the autonomous and non-autonomous filled-in Julia sets, namely (see [3, Theorem 5.2])

$$\mathbb{J}_{\mathrm{tr}}[\mathcal{F}] = \bigcup \left\{ J[(P_n)_{n=1}^\infty] : (P_n)_{n=1}^\infty \in \mathcal{F}^{\mathbb{N}} \right\}. \tag{4.6}$$

The subscript “tr” stands for the word “truncated” and is related to the truncated orbits of points staying in the ball $\overline{B}(0,R)$. The partly filled-in composite Julia set is compact and pluriregular (cf. [22, proof of Theorem 4.6]) but not necessarily polynomially convex, as was shown in [16, Example 2] and [26, Example 4.1]. Its polynomially convex hull $\mathbb{J}[\mathcal{F}]$ is the fixed point of the contraction

$$\mathcal{R} \ni K \mapsto \widehat{A_{\mathcal{F}}(K)} \in \mathcal{R}$$

and it is referred to as the *(filled-in) composite Julia set* generated by $\mathcal{F}$.

It is known that filled-in composite Julia sets generated by finite families of quadratic polynomial mappings are dense in $(\mathcal{R},\Gamma)$, see [20, Theorem 3]. In particular, the following result can be stated.

Proposition 4.2. *Let $E \in \mathcal{R}$ and $\varepsilon > 0$. Then there exists a finite family $\mathcal{F}_\varepsilon \subset \mathcal{P}_2^\star$ such that*

$$E \subset A_{\mathcal{F}_\varepsilon}(E) \qquad \text{and} \qquad \Gamma(E, A_{\mathcal{F}_\varepsilon}^n(E)) < \varepsilon \ \text{ for } \ n \geq 1.$$

In particular, $\Gamma\Big(E, \mathbb{J}[\mathcal{F}_\varepsilon]\Big) \leq \varepsilon$.

Proof. See the proof of [20, Theorem 3]. □

Julia type sets are of interest for their own sake and the usual way to approximate them is by other sets which can, for instance, be easier to plot on the computer (e.g. with use of formulas like (4.1) or (4.5), cf. [19] and [1]). We will recall here a recent result where the composite Julia set of a compact family of polynomial mappings can be approximated by sets obtained by using only one well chosen sequence of elements of this family.

Theorem 4.3 ([3, Corollary 7.2], cf. [1, Theorem 2 (c)] for finite families)**.** *Let $\mathcal{F}$ be a non-empty compact subset of $\mathcal{P}_k^\star$ and $\mathcal{F}_0 = \{\pi_n : n \in \mathbb{N}\}$ – a dense countable subset of $\mathcal{F}$. Let $\tau : \mathbb{N} \to \mathbb{N}$ be generated according to probabilities $p_1, p_2 \ldots > 0$ such that $\sum_{n=1}^\infty p_n = 1$, i.e. the values $\tau(j)$ of τ are chosen at random, independently from each other, so that $\mathbb{P}[\tau(j)=i] = p_i$ for $i, j \in \mathbb{N}$.*

Then for any $E \in \mathcal{R}$,

$$\lim_{m\to\infty} \Gamma\left(\mathbb{J}[\mathcal{F}], \overline{\bigcup \mathcal{E}_m}\right) = 0$$

with probability one, where $\mathcal{E}_m = \left\{ \left(\pi_{\tau(1)} \circ \ldots \circ \pi_{\tau(n)}\right)^{-1}(E) \,:\, n \geq m \right\}$.

A deterministic version of this theorem is also true and uses the notion of disjunctive sequences (see [3, Theorem 7.1] for details).

Theorem 4.3 concerns the approximation of Julia type sets. Recently an interest has arisen in the opposite problem, namely in the approximation of other compact sets by means of Julia type sets (see [8], especially the first paragraph of the introduction and references given there). In particular, the authors of [30] wanted to approximate compact planar sets by filled-in Julia sets in the Hausdorff metric. One way of achieving this aim was given in [30, Theorems 3.2 & 4.12], where it was proved that for a planar set $E \in \mathcal{R}$ and any $\varepsilon > 0$, there exists a polynomial P such that $E \subset \mathbb{J}[P] \subset E(\varepsilon)$. The authors did not address the correspondence between the augmentation they used and the dilation they would need in order to be able to derive any conclusions about the approximation in the Hausdorff metric (for more detailed study see also [8]).

Let us get back to Proposition 4.2 and show one of its consequences.

Corollary 4.4. *Let $E \in \mathcal{R}$ and $\varepsilon > 0$. Then there exist a finite family $\mathcal{F}_\varepsilon \subset \mathcal{P}_2^\star$ such that*

$$E \subset \mathbb{J}_{\mathrm{tr}}[\mathcal{F}_\varepsilon] \subset \mathbb{J}[\mathcal{F}_\varepsilon] \subset E^\varepsilon.$$

In particular, $\chi(E, \mathbb{J}_{\mathrm{tr}}[\mathcal{F}_\varepsilon]) \le \varepsilon$ and $\chi(E, \mathbb{J}[\mathcal{F}_\varepsilon]) \le \varepsilon$.

Proof. Since the family $\{\{V_E < \eta\} : \eta > 0\}$ is a basis of neighbourhoods of E in $\mathbb{C}^N$ (see [18, Corollary 1]), there exists $\eta > 0$ such that

$$\{z \in \mathbb{C}^N : V_E(z) < 2\eta\} \subset E^\varepsilon.$$

Take a finite family $\mathcal{G}_\eta$ from Proposition 4.2 for this fixed η. Since $E \subset A_{\mathcal{G}_\eta}(E)$, the sequence $(A^j_{\mathcal{G}_\eta}(E))_{j=1}^\infty$ is increasing with respect to inclusion of sets. Moreover, inequality $\Gamma(E, A^j_{\mathcal{G}_\eta}(E)) < \eta$ yields that each of these sets is contained in the η-augmentation of E. In particular, $\bigcup_{j=1}^\infty A^j_{\mathcal{G}_\eta}(E)$ is bounded.

Take now a common escape radius R for all the mappings from $\mathcal{G}_\eta$ which is large enough for the inclusion $\bigcup_{j=1}^\infty A^j_{\mathcal{G}_\eta}(E) \subset \overline{B}(0,R)$ to be satisfied. If $j, l \in \mathbb{N}$, then $A^{j+l}_{\mathcal{G}_\eta}(E) \subset A^l_{\mathcal{G}_\eta}(\overline{B}(0,R))$. In particular, $E \subset A^l_{\mathcal{G}_\eta}(\overline{B}(0,R))$ for all $l \in \mathbb{N}$. Now, (4.5) implies $E \subset \mathbb{J}_{\mathrm{tr}}[\mathcal{G}_\eta]$, and therefore also $E \subset \mathbb{J}_{\mathrm{tr}}[\mathcal{G}_\eta] \subset E(\eta)$ in view of the last inequality in Proposition 4.2. Since $E(\eta)$ is polynomially convex, it follows from the choice of η that

$$E \subset \mathbb{J}_{\mathrm{tr}}[\mathcal{F}_\varepsilon] \subset \mathbb{J}[\mathcal{F}_\varepsilon] \subset E(\eta) \subset E^\varepsilon$$

with $\mathcal{F}_\varepsilon := \mathcal{G}_\eta$. □

Let us finish this section by making some comparisons. The result from [30] concerns only planar sets, while the approximation by composite Julia sets from [20] and its consequence, i.e. Corollary 4.4 above, is valid for sets in $\mathbb{C}^N$. In [30], planar sets are approximated by filled-in Julia sets of polynomials and the degree of the chosen polynomial is bigger when its filled-in Julia set is closer to the limit (with respect to the Hausdorff metric or to Γ). On the other hand, in the approximation by composite Julia sets, the degree of the polynomials can be chosen to be always 2, however the number of the polynomials in the family grows when we get closer to the target set.

The approach to investigations of Julia sets, which is built around properties of the space $(\mathcal{R}, \Gamma)$, allows to create classes of canonical examples of analytic set-valued functions. One simply looks at the Julia sets generated by finite or countable sequences of polynomial mappings as the values of such functions whereas the generating polynomials are treated as variables. Two kinds of analytic dependence were shown, and in particular, Theorem 3.1(xii) was of use in the proofs. Most of the results were presented in [20], [21], [22] and [23]. For a survey on this subject we refer the reader to [27].

5 Hölder continuity property and Łojasiewicz-Siciak inequality

Let us start with a simple observation that $\mathcal{R}$ is not a closed subset of $(\kappa(\mathbb{C}^N), \chi)$. E.g. $(\overline{B}(0,1/n))_{n=1}^\infty \subset \mathcal{R}$ and $\{0\} \notin \mathcal{R}$ (cf. [18, p.2767]). However, we have the following property.

Proposition 5.1. *If $(E_n)_{n=1}^\infty \subset \mathcal{R}$ is convergent with respect to the Hausdorff metric to the set $E := \bigcap_{n=1}^\infty E_n \in \mathcal{R}$, then*

$$\lim_{n\to\infty} \Gamma(E_n, E) = 0.$$

Proof.

$$\lim_{n\to\infty} \Gamma(E_n, E) = \lim_{n\to\infty} ||V_E||_{E_n} = 0$$

by continuity of V_E and the fact that the augmentations form a basis of neighbourhoods of any set $E \in \mathcal{R}$ ([18, Corollary 1]). □

Consider a subfamily $\mathcal{R}_{\text{Höl}}$ of $\mathcal{R}$ of all polynomially convex sets whose Green functions are Hölder continuous. All autonomous filled-in Julia sets (for the one-dimensional case see [13, Theorem VIII.3.2 and the subsequent remarks] and in general [24, Theorem 1.2]) and all composite Julia sets defined by finite families of polynomial mappings with Łojasiewicz exponents at infinity bigger than one (see [25, Theorem 4.1]) or by compact families of regular polynomial mappings of degree $k \ge 2$ (see, [23, Theorem 7.3]) belong to $\mathcal{R}_{\text{Höl}}$. Furthermore, if $\mathcal{F}$ is a compact subset of $\mathcal{P}_k^\star$ for some $k \ge 2$ and $E \in \mathcal{R}_{\text{Höl}}$, then also $\widehat{A_{\mathcal{F}}(E)} \in \mathcal{R}_{\text{Höl}}$ (see [22, Theorem 3.5]).

In view of [20, Corollaries 2 & 3] we know that $\mathcal{R}_{\text{Höl}}$ is not closed in $(\mathcal{R}, \Gamma)$. Let us recall here the first of those corollaries.

Proposition 5.2. *Consider $(\lambda_n)_{n=1}^\infty \subset (4, \infty)$ such that*

$$\sum_{n=1}^\infty \frac{\log \lambda_n}{2^n} < \infty \qquad \text{and} \qquad \limsup_{n\to\infty} (\lambda_1 \lambda_2 \ldots \lambda_n)^{1/n} = \infty.$$

Define

$$f_{\lambda_n} : \mathbb{C} \ni z \mapsto \lambda_n z(1-z) \in \mathbb{C} \qquad \text{and} \qquad E_n = \left(f_{\lambda_n} \circ \ldots \circ f_{\lambda_1}\right)^{-1}([0,1]).$$

Then $(E_n)_{n=1}^\infty$ converges in $(\mathcal{R}, \Gamma)$ to a set $E \in \mathcal{R} \setminus \mathcal{R}_{\text{Höl}}$, while $E_n \in \mathcal{R}_{\text{Höl}}$, $n \in \mathbb{N}$.

Proof. See [20, Corollary 2]. For the last statement observe that Hölder continuity of V_{E_n} in this proposition follows from the Hölder continuity of $V_{[0,1]}$ and (3.3). □

Now, let $f:\Omega\to(W,\|\cdot\|)$ be a mapping from a non-empty open set $\Omega\subset\mathbb{C}^N$ to a normed space and fix $\alpha\in(0,1]$. Recall that if we put

$$\mathrm{H\ddot{o}l}_\alpha(f):=\sup\left\{\frac{\|f(x)-f(y)\|}{\|x-y\|^\alpha}:\ x,y\in\Omega,x\neq y\right\},$$

then $H_\alpha(\Omega,W):=\left\{f\in W^\Omega:\mathrm{H\ddot{o}l}_\alpha(f)<\infty\right\}$ is a vector space and $\mathrm{H\ddot{o}l}_\alpha$ is a seminorm.

Define, for a fixed $\alpha\in(0,1]$,

$$\mathcal{R}_\alpha:=\left\{E\in\mathcal{R}:\ V_E\in H_\alpha\left(\mathbb{C}^N,\mathbb{R}\right)\right\}$$

and

$$\Gamma_\alpha(E,F):=\Gamma(E,F)+\mathrm{H\ddot{o}l}_\alpha(V_E-V_F),\qquad E,F\in\mathcal{R}_\alpha. \tag{5.1}$$

Proposition 5.3. *Fix $\alpha\in(0,1]$. Then $(\mathcal{R}_\alpha,\Gamma_\alpha)$ is a complete metric space.*

Proof. Let $(E_n)_{n=1}^\infty$ be a Cauchy sequence in $(\mathcal{R}_\alpha,\Gamma_\alpha)$. Then $(E_n)_{n=1}^\infty$ is a Cauchy sequence in $(\mathcal{R},\Gamma)$ as well and in view of Theorem 3.1(i), it is convergent to $E\in\mathcal{R}$. The definition of Γ yields uniform convergence of V_{E_n} to V_E.

Since $(E_n)_{n=1}^\infty$ is a Cauchy sequence, we also have

$$\forall\varepsilon>0\ \exists l\in\mathbb{N}\ \forall m,n\geq l\ \forall z\neq w:\qquad \frac{|(V_{E_n}-V_{E_m})(z)-(V_{E_n}-V_{E_m})(w)|}{||z-w||^\alpha}<\varepsilon.$$

Letting $m\to\infty$, we get

$$\frac{|(V_{E_n}-V_E)(z)-(V_{E_n}-V_E)(w)|}{||z-w||^\alpha}\leq\varepsilon,$$

which implies that

$$\frac{|V_E(z)-V_E(w)|}{||z-w||^\alpha}\leq\varepsilon+\mathrm{H\ddot{o}l}_\alpha(V_{E_n})$$

and the right hand side is finite. □

Note that Proposition 5.2 shows that $\mathcal{R}_\alpha$ does not have to be closed in $(\mathcal{R},\Gamma)$ (we have $[0,1]\in\mathcal{R}_{1/2}$ and it follows that $E_n\in\mathcal{R}_{1/2}$ too, cf. e.g. [22, Theorem 3.5]). Observe also that $\mathcal{R}_{\mathrm{H\ddot{o}l}}=\bigcup_{\alpha\in(0,1]}\mathcal{R}_\alpha$.

Let us recall that $E\in\mathcal{R}_\alpha$ if and only if the so called Hölder continuity property, (HCP) for short, holds (which was shown by Błocki, see [38, Proposition 3.5]), i.e.

$$\exists A>0\ \forall z\in\mathbb{C}^N:\quad V_E(z)\leq A(\delta_E(z))^\alpha. \tag{5.2}$$

Moreover, it is enough to check this inequality for z close to E, e.g. such that $\delta_E(z)\leq 1$.

We have the following direct consequence of Proposition 5.1 and (HCP).

Corollary 5.4. *Let $(E_n)_{n=1}^\infty\subset\mathcal{R}$ be convergent with respect to the Hausdorff metric to the set $E:=\bigcap_{n=1}^\infty E_n\in\mathcal{R}_{\mathrm{H\ddot{o}l}}$ and let the rate of convergence be $\chi(E_n,E)=\varepsilon_n$. Then this sequence is convergent to the same limit with respect to Γ and there exist positive constants A,α such that the rate of convergence is estimated by*

$$\Gamma(E_n,E)\leq A\varepsilon_n^\alpha. \qquad\square$$

In particular, we can also consider the rate of convergence of the dilations.

Corollary 5.5. *Fix $E\in\mathcal{R}_{\mathrm{H\ddot{o}l}}$. Then there exist positive constants A,α such that*

$$\Gamma\left(E,\widehat{E^\varepsilon}\right)=\Gamma\left(E,E^\varepsilon\right)\leq A\varepsilon^\alpha. \qquad\square$$

Let us now turn our attention to another property. Belghiti and Gendre in their Ph.D. dissertations (cf. [6] and the references given there) considered the so called Łojasiewicz-Siciak inequality, which can be viewed as an opposite condition to (HCP), as it reverses the direction of (5.2). Namely, we say that a compact set E satisfies the *Łojasiewicz-Siciak inequality*, (ŁS) for short, if

$$\exists B,\beta>0:\quad V_E(z)\geq B(\delta_E(z))^\beta\qquad\text{if}\quad \delta_E(z)\leq 1.$$

In this case, the assumption that z should be close to E cannot be removed because of the logarithmic growth of V_E at infinity. The Łojasiewicz-Siciak inequality is also equivalent to the fact that for every bounded neighbourhood U of E,

$$\exists B(U),\beta(U)>0\quad\forall z\in U:\quad V_E(z)\geq B(U)(\delta_E(z))^{\beta(U)}. \tag{5.3}$$

Furthermore, it implies polynomial connectivity of E (see [34, Proposition 2.1]).

Put $\mathcal{R}_{\text{ŁS}}:=\{E\in\mathcal{R}:E\text{ satisfies (ŁS)}\}$. Let us list a few examples of classes of sets belonging to $\mathcal{R}_{\text{ŁS}}$. They include finite unions of pairwise disjoint balls or polydiscs, pluriregular subsets of $\mathbb{R}^N$ treated as subsets of $\mathbb{C}^N$ (see [33, Theorem 1.1]), some polynomially convex holomorphic polyhedra (see [34, Theorem 3.1]), planar filled-in Julia sets which have non-empty interior (see [36]) and some planar Cantor type sets (see [7]). $\mathcal{R}_{\text{ŁS}}$ is not closed in $(\mathcal{R},\Gamma)$, which can be shown by the following example in the complex plane:

Example 5.1. Consider
$$K_n := \overline{B}\left(-2, \sqrt{4+\frac{1}{n^2}}\right) \cup \overline{B}\left(2, \sqrt{4+\frac{1}{n^2}}\right) \subset \mathbb{C}, \quad n \in \mathbb{N}.$$
Then $K_n \in \mathcal{R}_{ŁS}$ for all $n \in \mathbb{N}$, and the sequence $(K_n)_{n=1}^{\infty}$ is convergent in $(\mathcal{R}, \Gamma)$ to a set $K \in \mathcal{R} \setminus \mathcal{R}_{ŁS}$.

Proof. It was shown that $K_n \in \mathcal{R}_{ŁS}$ in [33, Example 1]. Note that $(K_n)_{n=1}^{\infty}$ is decreasing with respect to inclusion of sets and
$$\bigcap_{n=1}^{\infty} K_n = \overline{B}(-2,2) \cup \overline{B}(2,2) =: K.$$
The set K is polynomially convex and regular but does not satisfy (ŁS), which was shown by Siciak (see [7, Example 1.1]). We have
$$\lim_{n\to\infty} \Gamma(K_n, K) = 0,$$
by Proposition 5.1. □

We have the following counterparts of Corollaries 5.4 and 5.5.

Corollary 5.6. *Let $(E_n)_{n=1}^{\infty} \subset \mathcal{R}$ be decreasing with respect to inclusion and let $E := \bigcap_{n=1}^{\infty} E_n \in \mathcal{R}_{ŁS}$. Let the rate of convergence with respect to Γ be given by $\Gamma(E_n, E) = \varepsilon_n$. Then this sequence is convergent to the same limit with respect to the Hausdorff metric and there exists positive constants B, β such that the rate of convergence is estimated by*
$$\chi(E_n, E) \le B\varepsilon_n^{\beta}. \qquad \square$$

In particular, we can also consider the rate of convergence of the augmentations.

Corollary 5.7. *Fix $E \in \mathcal{R}_{ŁS}$. Then there exist positive constants B, β such that*
$$\chi\Big(E, E(\varepsilon)\Big) \le B\varepsilon^{\beta}. \qquad \square$$

We also have

Corollary 5.8. *Let $P : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $k \ge 2$ such that its filled-in Julia set $\mathbb{J}[P]$ has a non-empty interior. Fix an $R > 0$ such that $\mathbb{J}[P] \subset \overline{B}(0,R)$. Then there exist positive constants B, β such that*
$$\chi\Big(\mathbb{J}[P], (P^n)^{-1}\left(\overline{B}(0,R)\right)\Big) \le B\, \frac{||P||_{\partial B(0,R)}^{\beta}}{(R(k-1))^{\beta} k^{n\beta}}.$$

Proof. It was shown in [36] that $\mathbb{J}[P]$ satisfies (ŁS). The chosen R is an escape radius for P. It can be calculated directly or deduced e.g. from [19, (1)] that
$$\Gamma\Big(\mathbb{J}[P], (P^n)^{-1}\left(\overline{B}(0,R)\right)\Big) \le \frac{\Gamma\Big(P^{-1}\left(\overline{B}(0,R)\right), \overline{B}(0,R)\Big)}{k^{n-1}(k-1)},$$
and from (3.7) that
$$\Gamma\Big(P^{-1}\left(\overline{B}(0,R)\right), \overline{B}(0,R)\Big) \le \frac{||P||_{\partial B(0,R)}}{Rk}.$$
□

Let us now consider sets which satisfy both (HCP) and (ŁS). Note that Example 5.1 shows that $\mathcal{R}_{Höl} \cap \mathcal{R}_{ŁS}$ is not closed in $(\mathcal{R}, \Gamma)$. Namely, $K_n \in \mathcal{R}_{Höl}$, since it is a connected infinite compact planar set (the Hölder continuity of the Green function follows from the classic Leja's Polynomial Lemma), for any $n \in \mathbb{N}$.

We finish this section with an observation which is a direct consequence of the equivalent conditions for (HCP) and (ŁS), given above as (5.2) and (5.3):

Proposition 5.9. *If $E, F \in \mathcal{R}_{Höl} \cap \mathcal{R}_{ŁS}$, then there exists positive constants A, B, α, β such that*
$$\Gamma(E,F) \le A\Big(\chi(E,F)\Big)^{\alpha} \le B\Big(\Gamma(E,F)\Big)^{\beta}. \qquad \square$$

It would be natural to suspect that this proposition implies equivalence of the metrics Γ and χ restricted to $\mathcal{R}_{Höl} \cap \mathcal{R}_{ŁS}$. This however is not the case. To see this, consider the following example in $\mathbb{C}$ (cf. [18])

Example 5.2. Let $E_n := \{e^{it} :\ t \in [0, 2\pi - n^{-1}]\}$, $F := \{z \in \mathbb{C} : |z| = 1\}$ and $E := \widehat{F} = \overline{B}(0,1)$. Then $E_n, E \in \mathcal{R}_{Höl} \cap \mathcal{R}_{ŁS}$, $F \notin \mathcal{R}$.

Furthermore, $(E_n)_{n=1}^{\infty}$ is convergent in $(\mathcal{R}, \Gamma)$ to E, but in $(\kappa(\mathbb{C}), \chi)$ it is convergent to F. Therefore this sequence is not convergent with respect to the Hausdorff distance in $\mathcal{R}_{Höl} \cap \mathcal{R}_{ŁS}$.

Proof. The Łojasiewicz-Siciak inequality implies polynomial convexity, hence F does not satisfy (ŁS).

Fix $n \in \mathbb{N}$. The set E_n has Hölder continuous Green's function, since it is an infinite compact connected subset of $\mathbb{C}$ (we use once again the Leja Polynomial Lemma). Moreover, the Łojasiewicz-Siciak inequality holds for this set (it can be deduced e.g. from [33, Theorem 1.2]). □

Let us mention finally that both the Hölder Continuity Property and the Łojasiewicz-Siciak inequality were useful in [8, Section 5], where the estimation of the rate of approximation of compact planar sets by filled-in Julia sets was approached.

6 Final remarks and open problems

When studying different classes of sets in $\mathcal{R}_*$ one could consider modifications of the pseudometric Γ that would reflect specific properties of the class of sets being investigated.

For instance, in [31] and [12] Burns, Levenberg and Ma'u studied the class Π^1 of lineally convex sets in $\mathcal{R}$ whose images through $\mathbb{C}$-linear functionals are polynomially convex in the plane. If $E \in \Pi^1$, then the natural extremal function is

$$V_E^{(1)}(z) = \sup\{V_{\ell(E)}(\ell(z)) : \ell \text{ is } \mathbb{C}\text{-affine and non-constant}\}, \quad z \in \mathbb{C}^N.$$

They showed that the corresponding metric $\Gamma^1(E,F) = \|V_E^{(1)} - V_F^{(1)}\|_{\mathbb{C}^N}$, $E, F \in \Pi^1$, is dominated by Γ, and hence generates weaker topology than that induced from $(\mathcal{R}, \Gamma)$.

One way to generate a stronger topology than that given by Γ can be outlined as follows. Assume that $\mathcal{X} \subset \mathcal{R}_*$ is non-empty and is endowed with a pseudometric d. If either Γ or d furnish a metric on $\mathcal{X}$, then also

$$\Gamma_d(E,F) := \Gamma(E,F) + d(E,F), \qquad E, F \in \mathcal{X}, \tag{6.1}$$

is a metric on $\mathcal{X}$. Since the topology generated by Γ_d is finer that than induced from $(\mathcal{R}_*, \Gamma)$, we will refer to Γ_d as the *d-refinement* of Γ.

The metric defined in (5.1) is an example of a refinement of Γ. Another natural choice would be the χ-refinement of Γ on $\mathcal{R}_*$. Yet another possibility, this time on $\mathcal{R}$, would be to use $d(E,F) = \chi(\partial E, \partial F)$ for $E, F \in \mathcal{R}$. This can be potentially useful in studying conventional Julia sets in the complex plane. To use the terminology from this paper, such sets are the boundaries of filled-in Julia sets, for a polynomial in the autonomous case or for a sequence of polynomials in the more general case. (For background see e.g. [13]). At present, this refinement of Γ is not well understood. Incidentally, both refinements related to the Hausdorff distance coincide if we restrict our attention only to convex sets (see [42]).

Rather than modifying Γ one could also consider modifications of its definition. A natural idea would be to replace the L^∞ norm used in the definition of Γ (3.1) by another norm in $\mathcal{C}_b(\mathbb{C}^N)$. For example, one could use an L^p-norm with $p \in [1, \infty)$ or, in the case of $p \in (0,1)$, the metric $(f,g) \mapsto \int_{\mathbb{C}^N} |f-g|^p$. However, if $E \in \mathcal{R}$ is fixed and $N = 1$,

$$\bigcup_{0<p<\infty} \left\{ F \in \mathcal{R} : \int_{\mathbb{C}} |V_E(z) - V_F(z)|^p \, dz < \infty \right\} \subset \{F \in \mathcal{R} : \gamma(F) = \gamma(E)\},$$

which hints at the possibility that such classes of sets in $\mathcal{R}$ might be rather small.

It is also natural to speculate if there is a link between $\Gamma(E,F)$ and the equilibrium measures on E and F. Since these measures always have the total mass $(2\pi)^N$, they can be normalized to become probability measures and there are many techniques for comparing two probability distributions. Unfortunately it is usual to assume that the measures being compared are defined on the same set. Moreover, the equilibrium measures are supported on the boundaries of the considered sets E, F. This makes the situation even more difficult and at present it is unknown if any usable relationship connects the equilibrium measures and the metric Γ.

Despite the progress that has been made, a lot of questions concerning the topology of the space of pluriregular sets are still waiting for an answer. At the most basic level, specific sets which are not closed or open with respect to Γ, may have these properties with respect to a refinement of Γ. The topological link between Γ and χ is also more complicated than it may seem. For example, as we have seen in the previous section, the family of all polynomially convex sets with Hölder continuous Green function is not closed in $(\mathcal{R}, \Gamma)$ nor is it closed with respect to the Hausdorff metric. One could ask if the set is closed with repect to Γ_χ or some other natural refinement of Γ. Then, there is the issue of compactness in $(\mathcal{R}, \Gamma)$. The compact subsets of $\mathcal{R}$ seem to play quite an important role in some constructions and their unions can be quite useful too (e.g. in investigations of formula (4.6)). Some questions about such unions were answered in Theorem 3.1, in particular in (iii), (vi) and (ix). Note that in particular (iii) deals with compactness in $(\mathcal{R}, \Gamma)$ and with compactness in $\mathbb{C}^N$. In the latter case a set is compact if and only if it is bounded and closed, but according to Theorem 3.1 (viii), this is not the case in $(\mathcal{R}, \Gamma)$. Observe also that Part (iii) of the theorem implies that if $\mathcal{E} \subset \mathcal{R}$ is compact then

$$\bigcup \overline{\mathcal{E}} = \bigcup \mathcal{E} = \overline{\bigcup \mathcal{E}},$$

where the closures are taken respectively in $\mathcal{R}$ and $\mathbb{C}^N$. If $\mathcal{E}$ is only relatively compact, then $\overline{\bigcup \mathcal{E}} \subset \bigcup \overline{\mathcal{E}}$, but the opposite inclusion does not hold in general. Moreover, if $\mathcal{E}$ is only bounded, then even the last inclusion does not need to hold either (see [3]). Furthermore, the assumption about compactness of $\mathcal{E}$ in Theorem 3.1 (iii) cannot be replaced with the assumption that $\mathcal{E}$ is closed and bounded (see [3]).

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