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Families of differential equations and determinant forms of the generalized Legendre-Appell and related polynomials

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Abstract

In this study, we propose an extended form of the Legendre-based Appell polynomial families and examine their essential analytical properties. By employing the quasi-monomial approach, we establish the corresponding recurrence relations, multiplicative and derivative operators, together with the governing differential equations. Moreover, we formulate both the series and determinant representations for this newly constructed class of polynomials. Within this framework, we also introduce the generalized Legendre-Hermite Appell polynomials and derive their specific results. As special cases, the Legendre-Hermite-Bernoulli, Legendre-Hermite-Euler, and Legendre-Hermite-Genocchi polynomials are obtained, and their algebraic as well as operational features are analyzed. The findings presented herein enhance the theoretical development of special polynomial sequences and expand their potential applications in mathematical physics and differential equation analysis.

1 Introduction and preliminary results

Special polynomials occupy a central role in mathematical analysis, combinatorics, and mathematical physics due to their wide-ranging structural and operational properties. They often arise as solutions to differential, difference, or integral equations and serve as fundamental tools in approximation theory, orthogonal expansions, and numerical analysis. Families such as Hermite, Laguerre, Legendre, Bernoulli, and Euler polynomials exhibit deep interconnections through generating functions, recurrence relations, and operational frameworks, thereby providing a unified approach to diverse mathematical and physical problems (see [1, 6, 9, 10, 11, 8]).

Appell polynomials constitute one of the most significant subclasses of special polynomials, distinguished by their defining property that the derivative of each polynomial in the sequence is proportional to the preceding one. This differential property endows them with elegant algebraic and analytical characteristics, enabling their application in areas such as differential equations, combinatorial enumeration, and probability theory. Through their generating functions and operational formulations, Appell sequences offer a versatile foundation for constructing and generalizing other polynomial families, thereby enriching the theory of special functions and their practical applications. The family of Appell polynomial sequences [2] arises extensively across various domains of applied mathematics, theoretical physics, approximation theory, and related analytical disciplines. These polynomial sequences are characterized by the following exponential generating function:

$$\mathcal{A}(x,t) := \mathcal{A}(t)e^{xt} = \sum_{n=0}^{+\infty} \mathcal{A}_n(x)\frac{t^n}{n!}, \quad \mathcal{A}_n := \mathcal{A}_n(0), \tag{1}$$

where $\mathcal{A}(t)$ is an analytic function at $t=0$, expressed as:

$$\mathcal{A}(t) = \sum_{n=0}^{\infty} \mathcal{A}_n \frac{t^n}{n!}, \quad \mathcal{A}_0 \neq 0, \quad \mathcal{A}_i \ (i=0,1,2,\cdots) \text{ being real coefficients.} \tag{2}$$

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S.No.	Name of polynomials	$\mathcal{Q}(t)$	Generating function	Series definition
I.	Bernoulli	$\frac{t}{e^t-1}$	$\left(\frac{t}{e^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$	$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}$
	polynomials		$\left(\frac{t}{e^t-1}\right) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$	
	and numbers [22]		$B_n(:= B_n(0) = B_n(1))$	
II.	Euler	$\frac{2}{e^t+1}$	$\left(\frac{2}{e^t+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$	$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}$
	polynomials		$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$	
	and numbers [23]		$E_n := 2^n E_n\left(\frac{1}{2}\right)$	
III.	Genocchi	$\frac{2t}{e^t+1}$	$\left(\frac{2t}{e^t+1}\right)e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}$	$G_n(x) = \sum_{k=0}^{n}\binom{n}{k}G_k x^{n-k}$
	polynomials		$\frac{2t}{e^t+1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}$	
	and numbers [13, 14]		$G_n := G_n(0)$	

Table 1: Certain members belonging to the Appell family

n	0	1	2	3	4
B_n	1	$\pm\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
E_n	1	0	-1	0	5
G_n	0	1	-1	0	1

Table 2: First few values of B_n, E_n and G_n

The Appell polynomials $\mathcal{A}_n(x)$ are explicitly given by the series expansion [12]:

$$\mathcal{A}_n(x) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{A}_{n-k}\, x^k, \quad \mathcal{A}_n^{'}(x) = n\, \mathcal{A}_{n-1}(x). \tag{3}$$

By appropriately selecting $\mathcal{A}(t)$, various members of the Appell polynomial family can be derived. These are listed in Table 1 below:

For the purpose of simplifying subsequent calculations, we list the initial numerical values of the Bernoulli numbers B_n, Euler numbers E_n, and Genocchi numbers G_n in Table 2. These fundamental sequences frequently arise in the study of special functions, series expansions, and number-theoretic identities, and their early terms are crucial for verifying analytical results and constructing explicit examples. The corresponding values are tabulated as follows:

Note 1. From the preceding table, it can be observed that the polynomial sequence $G_n(x)$ possesses degree $n-1$, in contrast to the other Appell polynomial families, each of which has degree n. Consequently, $G_n(x)$ does not fall within the category of strongly Appell polynomial sequences (for a detailed discussion, see [23]).

Special polynomial systems involving two variables occupy a central position in mathematical physics, providing efficient analytical frameworks for solving diverse classes of partial differential equations frequently encountered in physical and engineering models. The 2-variable general polynomials (2VgP) denoted by $\mathcal{Y}_n(x,y)$ are specified by generating relation [19]:

$$\exp(xt)\Psi(y,t) = \sum_{n=0}^{\infty}\mathcal{Y}_n(x,y)\frac{t^n}{n!}, \quad (\mathcal{Y}_0(x,y) = 1), \tag{4}$$

where $\Psi(y,t)$ has (at least the formal) series expansion

$$\Psi(y,t)=\sum_{k=0}^{\infty}\Psi_k(y)\frac{t^k}{k!},\quad (\Psi_0(y)\neq 0,\quad k\geqslant 1). \tag{5}$$

The two-variable Legendre polynomials, denoted by $\mathcal{S}_n(x,y)$ and first introduced in [3], represent a natural and profound extension of the classical univariate Legendre polynomials. These polynomials encapsulate a rich algebraic and analytical structure, enabling a deeper understanding of multivariate orthogonal systems. Unlike their one-variable counterparts, the two-variable Legendre polynomials possess intricate recurrence relations, differential properties, and generating functions that reveal subtle interdependencies between the variables x and y. Their utility spans a wide range of mathematical and physical applications, including potential theory, where they aid in solving Laplace's equation in higher dimensions, and wave propagation problems, where they provide convenient bases for representing solutions in two-dimensional domains. The study of $\mathcal{S}_n(x,y)$ not only enriches the theory of special functions but also offers powerful tools for tackling complex, multivariate problems in applied mathematics and mathematical physics.

Recently Wani *et al.* [7] studied a hybrid family of Legendre-Sheffer polynomials, represented by ${}_S S_n(x,y)$, which is defined through the generating function

$$\mathcal{A}(t)\exp(yH(t))\mathcal{C}_0\left(-x(H(t))^2\right)=\sum_{n=0}^{\infty}{}_S S_n(x,y)\frac{t^n}{n!}. \tag{6}$$

When $H(t)=t$, in equation (6) yields the hybrid Legendre-Appell polynomials (LeAP), defined by

$$\mathcal{Q}(t)\ e^{yt}\ \mathcal{C}_0(2t\sqrt{-x})=\sum_{n=0}^{+\infty}{}_S\mathcal{A}_n(x,y)\frac{t^n}{n!}, \tag{7}$$

or, equivalently,

$$\mathcal{A}(t)\ e^{yt}\ \mathcal{C}_0(-xt^2)=\mathcal{Q}(t)\ e^{yt+D_x^{-1}t^2}=\sum_{n=0}^{\infty}{}_S\mathcal{A}_n(x,y)\frac{t^n}{n!}, \tag{8}$$

where $\mathcal{C}_0(x)$ denotes the 0^{th} order Bessel Tricomi function [5]. The n^{th}-order Tricomi functions $\mathcal{C}_n(x)$ are defined as

$$\mathcal{C}_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!(n+k)!}. \tag{9}$$

We also note that

$$\exp(-\alpha\widehat{D}_x^{-1})=\mathcal{C}_0(\alpha x),\quad \widehat{D}_x^{-n}\{1\}:=\frac{x^n}{n!}, \tag{10}$$

is the inverse differential operator.

The Legendre-Appell polynomials ${}_S\mathcal{A}_n(x,y)$ satisfies the series expansion:

$${}_S\mathcal{A}_n(x,y)=n!\sum_{k=0}^{[n/2]}\frac{\mathcal{A}_{n-2k}(y)x^k}{(n-2k)!(k!)^2}. \tag{11}$$

The 2-variable Legendre-Appell polynomials ${}_S\mathcal{A}_n(x,y)$ are also defined by the following operational rule:

$${}_S\mathcal{A}_n(x,y)=\exp\left(\widehat{D}_x^{-1}\frac{\partial^2}{\partial y^2}\right)\{\mathcal{A}_n(y)\}. \tag{12}$$

By appropriately choosing $\mathcal{A}(t)$, various members of the hybrid LeAP family can be derived. These are summarized in Table 3:

Note 2: Considering the observation made in Note 1, it follows that the hybrid LeGP ${}_S G_n(x,y)$ does not strongly belong to the class of hybrid LeAP ${}_S\mathcal{A}_n(x,y)$.

The notion of the monomiality principle originates from the pioneering work of Steffensen (1941) [15], who first formulated it through the framework of poweroids-an early operator-based approach to defining generalized polynomial systems. This foundational concept was later extended and systematized by Dattoli [16, 3, 4], whose contributions greatly advanced the operational treatment of special and orthogonal polynomials, thereby enriching the theoretical understanding and applications of such sequences in mathematical analysis and physics.

Within this principle, a pair of linear operators, denoted by $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$, are introduced to act as the multiplicative and derivative operators, respectively, for a polynomial sequence $q_n(x)_{n\in\mathbb{N}}$. These operators are constructed so as to satisfy the defining operational relations that emulate the behavior of monomials, namely:

$$q_{n+1}(x)=\widehat{\mathcal{M}}\{q_n(x)\}, \tag{13}$$

S. No.	Name of hybrid polynomials	$\mathcal{A}(t)$	Generating function	Series definition
I.	Hybrid Legendre-Bernoulli polynomials	$\frac{t}{e^t-1}$	$\left(\frac{t}{e^t-1}\right)e^{yt}\,\mathcal{C}_0(-xt^2)=\sum\limits_{n=0}^{\infty}{}_S B_n(x,y)\frac{t^n}{n!}$	${}_S B_n(x,y)=n!\sum\limits_{k=0}^{[n/2]}\frac{B_{n-2k}(y)x^k}{(n-2k)!(k!)^2}$
II.	Hybrid Legendre-Euler polynomials	$\frac{2}{e^t+1}$	$\left(\frac{2}{e^t+1}\right)e^{yt}\,\mathcal{C}_0(-xt^2)=\sum\limits_{n=0}^{\infty}{}_S E_n(x,y)\frac{t^n}{n!}$	${}_S E_n(x,y)=n!\sum\limits_{k=0}^{[n/2]}\frac{E_{n-2k}(y)x^k}{(n-2k)!(k!)^2}$
III.	Hybrid Legendre-Genocchi polynomials	$\frac{2t}{e^t+1}$	$\left(\frac{2t}{e^t+1}\right)e^{yt}\,\mathcal{C}_0(-xt^2)=\sum\limits_{n=0}^{\infty}{}_S G_n(x,y)\frac{t^n}{n!}$	${}_S G_n(x,y)=n!\sum\limits_{k=0}^{[n/2]}\frac{G_{n-2k}(y)x^k}{(n-2k)!(k!)^2}$

Table 3: Certain members belonging to the HLeAP family

and

$$n\,q_{n-1}(x)=\widehat{\mathcal{P}}\{q_n(x)\}. \tag{14}$$

A polynomial sequence that meets these conditions is termed a quasi-monomial set. Additionally, it must obey the fundamental commutation relation:

$$[\widehat{\mathcal{P}},\widehat{\mathcal{M}}]=\widehat{\mathcal{P}}\widehat{\mathcal{M}}-\widehat{\mathcal{M}}\widehat{\mathcal{P}}=\widehat{1}, \tag{15}$$

which is consistent with the structure of the Weyl algebra.

For a quasi-monomial sequence $q_n(x)_{n\in\mathbb{N}}$, its properties are fully determined by the operators $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$. In particular, the following hold:

(i) The polynomials $q_n(x)$ satisfy the differential relation:

$$\widehat{\mathcal{M}}\widehat{\mathcal{P}}\{q_n(x)\}=n\,q_n(x), \tag{16}$$

provided appropriate differential forms of $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{P}}$ exist.

(ii) An explicit expression for $q_n(x)$ is:

$$q_n(x)=\widehat{\mathcal{M}}^n\,\{1\}, \tag{17}$$

with the initial condition $q_0(x)=1$.

(iii) Its exponential generating function is:

$$e^{t\widehat{\mathcal{M}}}\{1\}=\sum_{n=0}^{\infty}q_n(x)\frac{t^n}{n!}\qquad(|t|<\infty), \tag{18}$$

which follows directly from (17). See [16, 18, 17] for additional details.

These operational techniques have broad applicability in areas such as classical optics, quantum theory, and mathematical physics, providing effective tools for studying polynomial families.

Motivated by this framework, we introduce a new class of generalized Legendre-based Appell polynomials ($gLeAP$), denoted by ${}_{\mathcal{Y}}{}_{\mathcal{S}}\mathcal{A}_n(x,y,z)$. Section 2 develops this generalization, establishing core properties including recurrence relations, operator forms, and differential equations. Section 3 presents series expansions and determinant representations. Section 4 examines significant subfamilies and their features, followed by concluding remarks and future perspectives.

2 The new generalization of Legendre and Legendre-based Appell polynomials

This section presents an extended generalization of the $gLeAP$, denoted by ${}_{\mathcal{Y}}{}_{\mathcal{S}}\mathcal{A}_n(x,y,z)$. The analysis includes the establishment of their series form, quasi-monomial framework, operational identities, and the associated differential equations. The formulation begins with the $gLeP$ ${}_{\mathcal{Y}}\mathcal{S}_n(x,y,z)$, defined in view of relations (4) and (7) as

$$e^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2) = \sum_{n=0}^{\infty} {}_y\mathcal{S}_n(x,y,z)\frac{t^n}{n!}, \quad ({}_y\mathcal{S}_0(x,y,z)=1). \tag{19}$$

Simplifying the left-hand side of equation (19) using equations (5) and (9) yields the following series representation for $gLeP$ ${}_y\mathcal{S}_n(x,y,z)$:

$${}_y\mathcal{S}_n(x,y,z) = \sum_{m=0}^{n} \binom{n}{m} \Psi_m(y)\mathcal{S}_{n-m}(x,z). \tag{20}$$

We now derive the quasi-monomial identities for $gLeP$, denoted as ${}_y\mathcal{S}_n(x,y,z)$.

Theorem 2.1. *The newly defined $gLeP$ ${}_y\mathcal{S}_n(x,y,z)$ satisfies quasi-monomial properties under the following multiplicative and derivative operators:*

$$\widehat{M}_{gLeP} = x + \frac{\Psi'(y,\widehat{D}_x)}{\Psi(y,\widehat{D}_x)} + 2n\widehat{D}_z^{-1}, \tag{21}$$

and

$$\widehat{P}_{gLeP} = \widehat{D}_x, \tag{22}$$

respectively.

Proof. Differentiating equation (19) with respect to t, we obtain

$$\sum_{n=0}^{\infty} {}_y\mathcal{S}_{n+1}(x,y,z)\frac{t^n}{n!} = xe^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2) + \frac{\Psi'(y,t)}{\Psi(y,t)}e^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2) + \left(\sum_{n=0}^{\infty} \frac{z^n 2nt^{2n-1}}{([n]!)^2}\right)e^{xt}\Psi(y,t).$$

Thus,

$$\sum_{n=0}^{\infty} {}_y\mathcal{S}_{n+1}(x,y,z)\frac{t^n}{n!} = \left(x + \frac{\Psi'(y,t)}{\Psi(y,t)}\right)e^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2) + \left(\sum_{n=0}^{\infty} \frac{z^{n+1} 2(n+1)t^{2n+1}}{([n+1]!)^2}\right)e^{xt}\Psi(y,t).$$

Using equation (19), we have

$$\sum_{n=0}^{\infty} {}_y\mathcal{S}_{n+1}(x,y,z)\frac{t^n}{n!} = \left(x + \frac{\Psi'(y,t)}{\Psi(y,t)}\right)\sum_{n=0}^{\infty} {}_y\mathcal{S}_n(x,y,z)\frac{t^n}{n!} + \left(\sum_{n=0}^{\infty} \frac{z^{n+1} 2(n+1)t^{n+1}}{([n+1]!)^2}\right)e^{xt}\Psi(y,t). \tag{23}$$

Differentiating equation (23) with respect to z, we get

$$\sum_{n=0}^{\infty} D_z {}_y\mathcal{S}_{n+1}(x,y,z)\frac{t^n}{n!} = \left(x + \frac{\Psi'(y,t)}{\Psi(y,t)}\right)\sum_{n=0}^{\infty} D_z {}_y\mathcal{S}_n(x,y,z)\frac{t^n}{n!} + 2n\sum_{n=0}^{\infty} {}_y\mathcal{S}_n(x,y,z)\frac{t^n}{n!}. \tag{24}$$

Consequently

$$\widehat{D}_x\left\{e^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2)\right\} = te^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2), \tag{25}$$

and $\frac{\Psi'(y,t)}{\Psi(y,t)}$ posses power series expansion in t with $\Psi(y,t)$ being the invertible series of t.

Applying the inverse operator D_z^{-1} to equation (24), we get

$$\sum_{n=0}^{\infty} {}_y\mathcal{S}_{n+1}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(x + \frac{\Psi'(y,t)}{\Psi(y,t)} + 2nD_z^{-1}\right){}_y\mathcal{S}_n(x,y,z)\frac{t^n}{n!}. \tag{26}$$

Using equations (25) and (26), we get

$$\sum_{n=0}^{\infty} {}_y\mathcal{S}_{n+1}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(x + \frac{\Psi'(y,\widehat{D}_x)}{\Psi(y,\widehat{D}_x)} + 2nD_z^{-1}\right){}_y\mathcal{S}_n(x,y,z)\frac{t^n}{n!}. \tag{27}$$

By comparing with equation (13), we deduce the operator identity (21).

Similarly, by using equation (25) in (19), we obtain

$$D_x\left\{\sum_{n=0}^{\infty} {}_y\mathcal{S}_n(x,y,z)\frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_y\mathcal{S}_{n-1}(x,y,z)\frac{t^n}{(n-1)!}. \tag{28}$$

Matching the coefficients of equal powers of t on both sides of (27), we find

$$D_x\left\{{}_y\mathcal{S}_n(x,y,z)\right\} = n\,{}_y\mathcal{S}_{n-1}(x,y,z), \quad n \geqslant 1. \tag{29}$$

Hence, in view of (14) and (29), we obtain the assertion (22). □

Theorem 2.2. *The $gLeP$ ${}_{\mathcal{Y}}\mathcal{S}_n(x,y,z)$ satisfies the following differential equation:*

$$\left(x\widehat{D}_x + \frac{\Psi^{'}(y,\widehat{D}_x)}{\Psi(y,\widehat{D}_x)}\widehat{D}_x + 2n\widehat{D}_z^{-1}\widehat{D}_x - n\right){}_{\mathcal{Y}}\mathcal{S}_n(x,y,z) = 0. \tag{30}$$

Proof. By substituting equations (21) and (22) into (16), we get

$$\left(x\widehat{D}_x + \frac{\Psi^{'}(y,\widehat{D}_x)}{\Psi(y,\widehat{D}_x)}\widehat{D}_x + 2n\widehat{D}_z^{-1}\widehat{D}_x\right){}_{\mathcal{Y}}\mathcal{S}_n(x,y,z) = n\ {}_{\mathcal{Y}}\mathcal{S}_n(x,y,z). \tag{31}$$

Upon solving the above equation, we get the assertion (30) □

Remark 1. Since $\mathcal{Y}_0(x,y) = 1$, it follows from the monomiality principle (17) that

$${}_{\mathcal{Y}}\mathcal{S}_n(x,y,z) = \left(x + \frac{\Psi^{'}(y,\widehat{D}_x)}{\Psi(y,\widehat{D}_x)} + 2n\widehat{D}_z^{-1}\right)^n \{1\}, \quad (\mathcal{Y}_0(x,y) = 1).$$

Furthermore, using equations (17), (19), and (21), we can write

$$\exp\left(\widehat{M}_{gLeP}\right)\{1\} = e^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2) = \sum_{v=0}^{\infty} {}_{\mathcal{Y}}\mathcal{S}_n(x,y,z)\frac{t^n}{n!}. \tag{32}$$

We now extend the construction to introduce a generalized form of $gLeAP$. To obtain its generating function, we employ the exponential generating framework of Appell polynomials. Substituting x in the left-hand side of (1) by the multiplicative operator ${}_{\mathcal{Y}}\mathcal{S}_n(x,y,z)$ from (21), we obtain ${}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)$:

$$\mathcal{A}(t)\exp\left(\widehat{M}_{gLeP}\right)\{1\} = \sum_{n=0}^{\infty} {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{33}$$

Using equation (21), we obtain two equivalent forms:

$$\mathcal{A}(t)\exp\left(x + \frac{\Psi^{'}(y,\widehat{D}_x)}{\Psi(y,\widehat{D}_x)} + 2n\widehat{D}_z^{-1}\right)\{1\} = \sum_{n=0}^{\infty} {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{34}$$

Applying relation (33) to the left-hand side of (34), we derive the generating function for the $gLeAP$ ${}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)$ as follows:

$$\mathcal{A}(t)e^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2) = \sum_{n=0}^{\infty} {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{35}$$

where

$$\mathcal{A}(t) = \sum_{k=0}^{\infty}\alpha_k\frac{t^k}{k!}, \quad \alpha_0 \neq 0, \quad \Psi(y,t) = \sum_{k=0}^{\infty}\psi_k(y)\frac{t^k}{k!}, \quad \psi_0 \neq 0. \tag{36}$$

3 Series representation and determinant form

Hybrid classes of special polynomials occupy a fundamental position in mathematical analysis owing to their rich algebraic and structural properties. Their series representation offers explicit analytical expressions and recurrence relations that prove instrumental in addressing various differential and functional equations. Meanwhile, the determinant form provides a concise and elegant algebraic framework for exploring their combinatorial and operational features. This representation facilitates the examination of orthogonality, symmetry, and transformation identities, while also establishing a connection between classical and modern polynomial families. Such formulations extend their utility to applications in mathematical physics, computational modeling, and engineering analysis. Moreover, the determinant structure serves as a powerful tool for efficiently computing higher-order coefficients. Hybrid polynomial systems are further recognized for their importance in approximation theory and numerical computation. Collectively, these attributes underscore their substantial contribution to both theoretical development and applied mathematical research.

Theorem 3.1. *The three-variable $gLeAP$ ${}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)$ can be expressed through the following series representation:*

$${}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{A}_k\ {}_{\mathcal{Y}}\mathcal{S}_{n-k}(x,y,z), \tag{37}$$

where $\mathcal{A}_k$ is defined by equation (2).

Proof. From equation (33), we can express

$$\sum_{n=0}^{\infty} {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!} = \mathcal{A}(t)\sum_{n=0}^{\infty} {}_{\mathcal{Y}}\mathcal{S}_n(x,y,z)\frac{t^n}{n!}. \tag{38}$$

By substituting the expansion (1) of $\mathcal{A}(t)$ into the left-hand side of (38) and equating coefficients of identical powers of t on both sides, we obtain the result stated in (37). □

Theorem 3.2. *The gLeAP* ${}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)$ *has the following determinant representation*

$${}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_{n,q}(x,y,z) = \frac{(-1)^n}{(\beta_0)^{n+1}}\begin{vmatrix} 1 & {}_{\mathcal{Y}}\mathcal{S}_1(x,y,z) & {}_{\mathcal{Y}}\mathcal{S}_2(x,y,z) & \dots & {}_{\mathcal{Y}}\mathcal{S}_{n-1}(x,y,z) & {}_{\mathcal{Y}}\mathcal{S}_n^{(m)}(x,y,z) \\ \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \dots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \dots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & \beta_0 & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{39}$$

where $\sum_{n=0}^{\infty} {}_{\mathcal{Y}}\mathcal{S}_n(x,y,z)\frac{t^n}{n!} = e^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2)$, $\frac{1}{\mathcal{A}(t)} = \sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}$, $\quad \beta_0 \neq 0$.

Proof. Using the series representation of $\frac{1}{\mathcal{A}(t)}$ as follows:

$$[\mathcal{A}(t)]^{-1} = \sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!},$$

using the generation function (19), we get

$$e^{xt}\Psi(y,t)\mathcal{C}_0(-zt^2) = \left(\sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty} {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}\right).$$

Hence

$$\sum_{n=0}^{\infty} {}_{\mathcal{Y}}\mathcal{S}_n(x,y,z)\frac{t^n}{n!} = \left(\sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty} {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}\right).$$

Applying the Cauchy product, we have

$$\sum_{n=0}^{\infty} {}_{\mathcal{Y}}\mathcal{S}_n(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\beta_k\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_{n-k}(x,y,z)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ from the polynomial equation, we get

$${}_{\mathcal{Y}}\mathcal{S}_n(x,y,z) = \sum_{k=0}^{n}\binom{n}{k}\beta_k\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_{n-k}(x,y,z), \quad n \in \mathbb{N}_0.$$

So, we obtain the system of equations as follows:

$$\begin{aligned}
{}_{\mathcal{Y}}\mathcal{S}_0(x,y,z) &= \beta_0\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_0(x,y,z),\\
{}_{\mathcal{Y}}\mathcal{S}_1(x,y,z) &= \beta_0\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_1(x,y,z) + \beta_1\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_0(x,y,z),\\
{}_{\mathcal{Y}}\mathcal{S}_2(x,y,z) &= \beta_0\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_2(x,y,z) + \binom{2}{1}\beta_1\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_1(x,y,z) + \beta_2\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_0(x,y,z),\\
&\vdots\\
{}_{\mathcal{Y}}\mathcal{S}_{n-1}(x,y,z) &= \beta_0\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_{n-1}(x,y,z) + \binom{n-1}{1}\beta_1\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_{n-2}(x,y,z) + \cdots + \beta_{n-1}\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_0(x,y,z),\\
{}_{\mathcal{Y}}\mathcal{S}_n(x,y,z) &= \beta_0\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z) + \binom{n}{1}\beta_1\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_{n-1}(x,y,z) + \cdots + \beta_n\, {}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_0(x,y,z).
\end{aligned}$$

Applying Cramers' rule, we get

$$
{}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)=\frac{\begin{vmatrix} \beta_0 & 0 & \dots & 0 & {}_{\mathcal{Y}}\mathcal{S}_0(x,y,z)\\ \beta_1 & \beta_0 & \dots & 0 & {}_{\mathcal{Y}}\mathcal{S}_1(x,y,z)\\ \beta_2 & \binom{2}{1}\beta_1 & \dots & 0 & {}_{\mathcal{Y}}\mathcal{S}_2(x,y,z)\\ \beta_3 & \binom{3}{2}\beta_2 & \dots & 0 & {}_{\mathcal{Y}}\mathcal{S}_3(x,y,z)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \dots & \beta_0 & {}_{\mathcal{Y}}\mathcal{S}_{n-1}(x,y,z)\\ \beta_n & \binom{n}{1}\beta_{n-1} & \dots & \binom{n}{n-1}\beta_1 & {}_{\mathcal{Y}}\mathcal{S}_m(x,y,z) \end{vmatrix}}{\begin{vmatrix} \beta_0 & 0 & \dots & 0 & 0\\ \beta_1 & \beta_0 & \dots & 0 & 0\\ \beta_2 & \binom{2}{1}\beta_1 & \dots & 0 & 0\\ \beta_3 & \binom{3}{2}\beta_2 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \beta_{n-1} & \binom{n-1}{1}\beta_{n-2} & \dots & \beta_0 & 0\\ \beta_n & \binom{n}{1}\beta_{n-1} & \dots & \binom{n}{n-1}\beta_1 & \beta_0 \end{vmatrix}}.
$$

By taking the transpose in the last equation, we have

$$
{}_{\mathcal{Y}\mathcal{S}}\mathcal{A}_n(x,y,z)=\frac{1}{(\beta_0)^{n+1}}\begin{vmatrix} \beta_0 & \beta_1 & \dots & \beta_{n-1} & \beta_n\\ 0 & \beta_0 & \dots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1}\\ 0 & 0 & \dots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & \beta_0 & \binom{n}{n-1}\beta_1\\ {}_{\mathcal{Y}}\mathcal{S}_0(x,y,z) & {}_{\mathcal{Y}}\mathcal{S}_1(x,y,z) & \dots & {}_{\mathcal{Y}}\mathcal{S}_{n-1}(x,y,z) & {}_{\mathcal{Y}}\mathcal{S}_n(x,y,z) \end{vmatrix}.
$$

Thus, simple row operations are used to finish the proof. □

4 Applications

This study extends recent work by generalizing three-variable Legendre-based Appell polynomials. In particular, setting $\Psi(y,t)=e^{yt^2}$ in the generating function (33) yields the Legendre-Hermite-Appell polynomials ($LeHAP$) ${}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)$, defined by a specific generating function:

$$
\mathcal{A}(t)e^{xt+yt^2}\mathcal{C}_0(-zt^2)=\sum_{n=0}^{\infty}{}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{40}
$$

In other words, we note that

$$
{}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)=\exp\left(\widehat{D}_z^{-1}\frac{\partial^2}{\partial x^2}\right)\{{}_{H}\mathcal{A}_n(x,y)\}. \tag{41}
$$

Theorem 4.1. *The LeHAP are defined by the series:*

$$
{}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{A}_k\ {}_{\mathcal{S}}\mathcal{H}_{n-k}(x,y,z). \tag{42}
$$

Proof. In view of (40), we have

$$
\sum_{n=0}^{\infty}{}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}=\mathcal{A}(t)\sum_{n=0}^{\infty}{}_{\mathcal{S}}\mathcal{H}_n(x,y,z)\frac{t^n}{n!}. \tag{43}
$$

By incorporating the series expansion of $\mathcal{A}(t)$ in preceeding expression and equating the coefficients of same exponents of t on both sides, we arrive at (42). □

We now proceed to establish the determinant representation of ${}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)$, employing a method analogous to that used in [20, 21], with reference to equation (40). "

Theorem 4.2. *The determinant representation of LeHAP ${}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)$ of degree n is*

$$
{}_{\mathcal{S}\mathcal{H}}\mathcal{A}_0(x,y,z)=\frac{1}{\beta_0},
$$

$$
{}_{S\mathcal{H}}\mathcal{A}_n(x,y,z) = \frac{(-1)^n}{(\beta_0)^{n+1}} \begin{vmatrix} 1 & {}_S\mathcal{H}_1(x,y,z) & {}_S\mathcal{H}_2(x,y,z) & \ldots & {}_S\mathcal{H}_{n-1}(x,y,z) & {}_S\mathcal{H}_n(x,y,z) \\ \beta_0 & \beta_1 & \beta_2 & \ldots & \beta_{n-1} & \beta_n \\ 0 & \beta_0 & \binom{2}{1}\beta_1 & \ldots & \binom{n-1}{1}\beta_{n-2} & \binom{n}{1}\beta_{n-1} \\ 0 & 0 & \beta_0 & \ldots & \binom{n-1}{1}\beta_{n-3} & \binom{n}{2}\beta_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots & \\ 0 & 0 & 0 & \ldots & \beta_{0,q} & \binom{n}{n-1}\beta_1 \end{vmatrix}, \tag{44}
$$

$$
\beta_n = -\frac{1}{\mathcal{A}_0}\left(\sum_{k=1}^{n}\binom{n}{k}\mathcal{A}_k\beta_{n-k}\right), \quad n = 0,1,2,\ldots,
$$

where $\beta_0 \neq 0$, $\beta_0 = \frac{1}{\mathcal{A}_0}$ *and* ${}_S\mathcal{H}_n(x,y,z)$, $\quad n = 0,1,2,\cdots$, *are the LeHP.*

Proof. By substituting the series representation of the newly defined generalization of the *LeHP* into the generating function corresponding to the *LeHAP*, we derive the following relation:

$$
\mathcal{A}(t)\sum_{n=0}^{\infty} {}_S\mathcal{H}_n(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_{S\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{45}
$$

On multiplication of

$$
\frac{1}{\mathcal{A}(t)} = \sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}, \tag{46}
$$

of the preceeding expression, we find

$$
\sum_{n=0}^{\infty} {}_S\mathcal{H}_n(x,y,z)\frac{t^n}{n!} = \sum_{k=0}^{\infty}\beta_k\frac{t^k}{k!}\sum_{n=0}^{\infty} {}_{S\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{47}
$$

By utilizing the Cauchy product in (47), it follows that

$$
{}_S\mathcal{H}_n(x,y,z) = \sum_{k=0}^{n}\binom{n}{k}\beta_k \; {}_{S\mathcal{H}}\mathcal{A}_{n-k}(x,y,z). \tag{48}
$$

"This relation yields a system of n linear equations involving the unknowns $\mathcal{A}_n(x,y,z)$, where $n = 0,1,2,\cdots$.

To determine the solution through Cramer's rule, it is observed that the denominator corresponds to the determinant of a lower triangular matrix, whose determinant evaluates to $(\beta_0)^{n+1}$. By transposing the numerator and substituting the i^{th} row with the $(i+1)^{\text{th}}$ position for $i = 1,2,\cdots,n-1$, the required expression is obtained. □

Next, we establish the multiplicative and derivative operators associated with ${}_{S\mathcal{H}}\mathcal{A}_n(x,y,z)$. The result is stated in the following theorem.

Theorem 4.3. *The generalized LeHAP family satisfies the following multiplicative and derivative operator relations:*

$$
\hat{M} = x + \frac{\mathcal{A}'(\widehat{D}_x)}{\mathcal{A}(\widehat{D}_x)} + +2n\left(y + D_z^{-1}\right), \tag{49}
$$

and

$$
\hat{P} = D_x, \tag{50}
$$

respectively.

Proof. Utilizing the derivative with respect to t on both sides of equation (40), we find

$$
\begin{aligned}
\sum_{n=0}^{\infty} {}_{S\mathcal{H}}\mathcal{A}_{n+1}(x,y,z)\frac{t^n}{n!} &= \frac{\mathcal{A}'(t)}{\mathcal{A}(t)}\mathcal{A}(t)e^{xt+yt^2}\mathcal{C}_0(-zt^2) + x\mathcal{A}(t)e^{xt+yt^2}\mathcal{C}_0(-zt^2) + 2yt\mathcal{A}(t)e^{xt+yt^2}\mathcal{C}_0(-zt^2) \\
&\quad + \left(\sum_{n=0}^{\infty}\frac{z^n 2nt^{2n-1}}{([n]!)^2}\right)\mathcal{A}(t)e^{xt+yt^2}
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} {}_{S\mathcal{H}}\mathcal{A}_{n+1}(x,y,z)\frac{t^n}{n!} &= \left(x + \frac{\mathcal{A}'(t)}{\mathcal{A}(t)} + 2ny\right)\mathcal{A}(t)e^{xt+yt2}\mathcal{C}_0(-zt^2) \\
&\quad + \left(\sum_{n=0}^{\infty}\frac{z^{n+1}2(n+1)t^{2n+1}}{([n+1]!)^2}\right)\mathcal{A}(t)e^{xt+yt^2}.
\end{aligned} \tag{52}
$$

By using equation (40), we get

$$\sum_{n=0}^{\infty} {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_{n+1}(x,y,z)\frac{t^n}{n!} = \left(x + \frac{\mathcal{A}'(t)}{\mathcal{A}(t)} + 2ny\right)\sum_{n=0}^{\infty} {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}$$
$$+ \left(\sum_{n=0}^{\infty} \frac{z^{n+1}2(n+1)t^{2n+1}}{([n+1]!)^2}\right)\mathcal{A}(t)e^{xt+yt^2}. \tag{53}$$

On differentiating both sides of the last equation with respect to z, we obtain:

$$\sum_{n=0}^{\infty} \widehat{D}_z {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_{n+1}(x,y,z)\frac{t^n}{n!} = \left(x + \frac{\mathcal{A}'(t)}{\mathcal{A}(t)} + 2ny\right)\sum_{n=0}^{\infty} \widehat{D}_z {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}$$
$$+ 2n\sum_{n=0}^{\infty} {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{54}$$

Consequently

$$\widehat{D}_x\left\{\mathcal{A}(t)e^{xt+yt^2}\mathcal{C}_0(-zt^2)\right\} = t\mathcal{A}(t)e^{xt+yt^2}\mathcal{C}_0(-zt^2). \tag{55}$$

Applying $\widehat{D}_z^{-1}$ to both sides of the above equation, we get

$$\sum_{n=0}^{\infty} {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_{n+1}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(x + \frac{\mathcal{A}'(t)}{\mathcal{A}(t)} + 2n\left(y + \widehat{D}_z^{-1}\right)\right) {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{56}$$

By using equations (55) and (56), we get

$$\sum_{n=0}^{\infty} {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_{n+1}(x,y,z)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(x + \frac{\mathcal{A}'(\widehat{D}_x)}{\mathcal{A}(\widehat{D}_x)} + 2n\left(y + \widehat{D}_z^{-1}\right)\right) {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}. \tag{57}$$

In view of (13) and (57), we get the assertion (49).

Similarly, by using equations (55) in (40), we obtain

$$\widehat{D}_x\left\{\sum_{n=0}^{\infty} {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\frac{t^n}{n!}\right\} = \sum_{n=1}^{\infty} {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_{n-1}(x,y,z)\frac{t^n}{(n-1)!}. \tag{58}$$

Matching the coefficients of equal powers of t on both sides of (58), we find

$$\widehat{D}_x\left\{{}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)\right\} = n\ {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_{n-1}(x,y,z), \quad n \geqslant 1. \tag{59}$$

Hence, in view of (14) and (59), we obtain the assertion (50). □

Remark 2. By using (17), the generalized *LeHAP* have the following explicit representations:

$${}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z) = \widehat{M}^n\{1\}, \tag{60}$$

$${}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z) = \left(x + \frac{\mathcal{A}'(\widehat{D}_x)}{\mathcal{A}(\widehat{D}_x)} + +2n\left(y + D_z^{-1}\right)\right)^n\{1\}. \tag{61}$$

Theorem 4.4. *The following differential equation for ${}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)$ holds true:*

$$\left(x\widehat{D}_x + \frac{\mathcal{A}'(\widehat{D}_x)}{\mathcal{A}(\widehat{D}_x)} + 2n\left(y + \widehat{D}_z^{-1}\right)\widehat{D}_x - n\right) {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z) = 0. \tag{62}$$

Proof. Using (49) and (50) in (16), we get

$$\left(x\widehat{D}_x + \frac{\mathcal{A}'(\widehat{D}_x)}{\mathcal{A}(\widehat{D}_x)}\widehat{D}_x + 2n\left(y + \widehat{D}_z^{-1}\right)\widehat{D}_x\right) {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z) = n\ {}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z). \tag{63}$$

Upon the simplification, we get the assertion (62). □

5 Examples

The Appell polynomial family, characterized by a parameter function $\mathcal{A}(t)$, serves as a foundational framework for generating solutions to a variety of differential equations. Distinct selections of $\mathcal{A}(t)$ yield different subclasses of polynomials, thereby offering remarkable adaptability in mathematical modeling and analytical studies. This inherent flexibility underscores their importance in diverse scientific domains such as physics, engineering, and computational analysis.

Table 1 provides a systematic overview of their generating functions, series formulations, and corresponding numerical evaluations. The generating functions offer compact power series representations that facilitate symbolic manipulation and theoretical derivations, while the series definitions furnish explicit expressions essential for analytical computations and numerical implementation. The inclusion of numerical values enhances practical comprehension and supports real-world applications.

Due to their rich structural properties and wide applicability, Appell polynomials have found significant use in probability theory, quantum mechanics, and signal processing. Their ability to adapt to complex analytical frameworks renders them an indispensable tool for solving intricate mathematical problems and advancing modern scientific research.

"As a result, different members of ${}_{\mathcal{S}\mathcal{H}}\mathcal{A}_n(x,y,z)$ appear as Legendre-Hermite-based Bernoulli polynomials ${}_{\mathcal{S}\mathcal{H}}\mathcal{B}_n(x,y,z)$, Legendre-Hermite-Euler polynomials ${}_{\mathcal{S}\mathcal{H}}\mathcal{E}_n(x,y,z)$, and Legendre-Hermite-Genocchi polynomials ${}_{\mathcal{S}\mathcal{H}}\mathcal{G}_n(x,y,z)$. The following expressions can be used to cast these polynomials":

$$\frac{t}{e^t-1}e^{xt+yt^2}\mathcal{C}_0(-zt^2)=\sum_{n=0}^{\infty}{}_{\mathcal{S}\mathcal{H}}\mathcal{B}_n(x,y,z)\frac{t^n}{n!}, \tag{64}$$

$$\frac{2}{e^t+1}e^{xt+yt^2}\mathcal{C}_0(-zt^2)=\sum_{n=0}^{\infty}{}_{\mathcal{S}\mathcal{H}}\mathcal{E}_n(x,y,z)\frac{t^n}{n!}, \tag{65}$$

and

$$\frac{2t}{e^t+1}e^{xt+yt^2}\mathcal{C}_0(-zt^2)=\sum_{n=0}^{\infty}{}_{\mathcal{S}\mathcal{H}}\mathcal{G}_n(x,y,z)\frac{t^n}{n!}. \tag{66}$$

For instance the Legendre-Hermite-based Bernoulli polynomials ${}_{\mathcal{S}\mathcal{H}}\mathcal{B}_n(x,y,z)$, Legendre-Hermite-Euler polynomials ${}_{\mathcal{S}\mathcal{H}}\mathcal{E}_n(x,y,z)$, and Legendre-Hermite-Genocchi polynomials ${}_{\mathcal{S}\mathcal{H}}\mathcal{G}_n(x,y,z)$ are defined by the following operational identities:

$${}_{\mathcal{S}\mathcal{H}}\mathcal{B}_n(x,y,z)=\exp\left(\widehat{D}_z^{-1}\frac{\partial^2}{\partial x^2}\right)\{{}_H\mathcal{B}_n(x,y)\}, \tag{67}$$

$${}_{\mathcal{S}\mathcal{H}}\mathbb{E}_n(x,y,z)=\exp\left(\widehat{D}_z^{-1}\frac{\partial^2}{\partial x^2}\right)\{{}_H\mathcal{E}_n(x,y)\}. \tag{68}$$

and

$${}_{\mathcal{S}\mathcal{H}}\mathcal{G}_n(x,y,z)=\exp\left(\widehat{D}_z^{-1}\frac{\partial^2}{\partial x^2}\right)\{{}_H\mathcal{G}_n(x,y)\}. \tag{69}$$

The polynomials can be explored using the monomiality principle, explicit expressions, differential equations, and determinant forms-revealing their structure, interrelations, and connections to linear algebra and broader mathematical frameworks.

Furthermore, in view of expressions (44), the polynomials ${}_{\mathcal{S}\mathcal{H}}\mathcal{B}_n(x,y,z)$, ${}_{\mathcal{S}\mathcal{H}}\mathcal{E}_n(x,y,z)$ and ${}_{\mathcal{S}\mathcal{H}}\mathcal{G}_n(x,y,z)$ satisfy the following determinant representations:

$${}_{\mathcal{S}\mathcal{H}}\mathcal{B}_n(x,y,z)=(-1)^n\begin{vmatrix}1 & {}_{\mathcal{S}}\mathcal{H}_1(x,y,z) & {}_{\mathcal{S}}\mathcal{H}_2(x,y,z) & \cdots & {}_{\mathcal{S}}\mathcal{H}_{n-1}(x,y,z) & {}_{\mathcal{S}}\mathcal{H}_n(x,y,z)\\ \delta_0 & \delta_1 & \delta_2 & \cdots & \delta_{n-1} & \delta_n\\ 0 & \delta_0 & \binom{2}{1}\delta_1 & \cdots & \binom{n-1}{1}\delta_{n-2} & \binom{n}{1}\delta_{n-1}\\ 0 & 0 & \delta_0 & \cdots & \binom{n-1}{2}\delta_{n-3} & \binom{n}{2}\delta_{n-2}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & \delta_0 & \binom{n}{n-1}\delta_1\end{vmatrix}, \tag{70}$$

$$
{}_{\mathcal{S}\mathcal{H}}\mathcal{E}_n(x,y,z) = (-1)^n \begin{vmatrix} 1 & {}_{\mathcal{S}}\mathcal{H}_1(x,y,z) & {}_{\mathcal{S}}\mathcal{H}_2(x,y,z) & \cdots & {}_{\mathcal{S}}\mathcal{H}_{n-1}(x,y,z) & {}_{\mathcal{S}}\mathcal{H}_n(x,y,z) \\ \delta_0 & \delta_1 & \delta_2 & \cdots & \delta_{n-1} & \delta_n \\ 0 & \delta_0 & \binom{2}{1}\delta_1 & \cdots & \binom{n-1}{1}\delta_{n-2} & \binom{n}{1}\delta_{n-1} \\ 0 & 0 & \delta_0 & \cdots & \binom{n-1}{2}\delta_{n-3} & \binom{n}{2}\delta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \delta_0 & \binom{n}{n-1}\delta_1 \end{vmatrix}, \tag{71}
$$

and

$$
{}_{\mathcal{S}\mathcal{H}}\mathcal{G}_n(x,y,z) = (-1)^n \begin{vmatrix} 1 & {}_{\mathcal{S}}\mathcal{H}_1(x,y,z) & {}_{\mathcal{S}}\mathcal{H}_2(x,y,z) & \cdots & {}_{\mathcal{S}}\mathcal{H}_{n-1}(x,y,z) & {}_{\mathcal{S}}\mathcal{H}_n(x,y,z) \\ \delta_0 & \delta_1 & \delta_2 & \cdots & \delta_{n-1} & \delta_n \\ 0 & \delta_0 & \binom{2}{1}\delta_1 & \cdots & \binom{n-1}{1}\delta_{n-2} & \binom{n}{1}\delta_{n-1} \\ 0 & 0 & \delta_0 & \cdots & \binom{n-1}{2}\delta_{n-3} & \binom{n}{2}\delta_{n-2} \\ . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . \\ 0 & 0 & 0 & \cdots & \delta_0 & \binom{n}{n-1}\delta_1 \end{vmatrix}, \tag{72}
$$

6 Concluding remarks

In this study, we have introduced and rigorously analyzed a new extension of the Legendre and Legendre-associated Appell polynomial families. By employing a comprehensive approach, we have established their fundamental characteristics, including recurrence relations, operational structures involving multiplicative and derivative operators, and the governing differential equations derived through the framework of quasi-monomiality. The construction of both the series expansion and determinant formulation underscores the algebraic depth and intrinsic structure of this newly proposed class of polynomials. Moreover, the introduction of the generalized Legendre-Hermite Appell polynomials, together with their notable special cases corresponding to the Bernoulli-, Euler-, and Genocch-type sequences, considerably expands the theoretical foundation of special functions and their interconnections. Collectively, these findings contribute to a deeper understanding of polynomial systems and their analytical significance within the domains of mathematical physics and differential equations.

Prospective investigations may focus on the orthogonality properties and integral transform representations of these polynomials to elucidate their structural and analytical behavior further. Establishing connections with fractional calculus and advanced special function theory could yield innovative outcomes with potential applications in approximation theory, signal processing, and mathematical modeling. In addition, extending the present framework to the context of q-calculus and number-theoretic settings may reveal new algebraic and combinatorial attributes. The computational and numerical exploration of these polynomials also represents a promising avenue for future research, particularly in enhancing their practical relevance to scientific computation, modeling techniques, and engineering-based simulations.

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