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Asymptotic properties and quantitative results of the wavelet type Bernstein operators

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Abstract

This work is a continuation of the author's very recent study on the newly introduced wavelet type Bernstein operators [19]. The main goal of the present study is to obtain some asymptotic properties and quantitative results of the newly introduced wavelet type Bernstein operators by using the compactly supported Daubechies wavelets of the given function f. The basis used in this study are approximation theory and wavelet theory together with the rational sampling values of the function obtained by father wavelets. Later, we will examine some quantitative and Voronovskaya-type results in some function spaces.

Keywords: Bernstein operators, wavelets, compactly supported Daubechies wavelets, asymptotic approximation.

AMS Subject Classification: 42C40, 47A58, 94A20, 41A25, 41A35, 47G10, 47H30.

1 Introduction

For a bounded real valued function f defined on the interval $[0,1]$ ($f \in B[0,1]$), the Bernstein operators $B_n(f)$, $n \geq 1$ are defined by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \ , \quad n \geq 1, \tag{1}$$

where $p_{n,k}(x) = \begin{pmatrix} n \\ k \end{pmatrix} x^k (1-x)^{n-k}$ is the Bernstein basis $(0 \leq x \leq 1)$.

In approximation theory, these operators and some of their modifications are very well-known. Since the classical Bernstein operators (1) cannot be used for $L^p[0,1](1 \leq p < \infty)$ approximation, to obtain some positive results for these functions, their Kantorovich and Durrmeyer type versions were considered (see, e.g., [1, 6, 9, 27]).

Very recently, as an extension and generalization of the classical Bernstein operators, Karsli introduced in [19] the following wavelet type Bernstein operators $WB_n : B[0,1] \to C[0,1]$, $f \to WB_n f$, defined as

$$(WB_n f)(t) := n \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 f(x) w(nx-k) dx, \tag{2}$$

$$= \sum_{k=0}^{n} p_{n,k}(t) \int_0^{\lambda} f\left(\frac{x+k}{n}\right) w(x) dx,$$

with $t \in [0,1]$, specifying that $supp(w) \subseteq [0,\lambda]$, $0 < \lambda \leq 1$.

The author proved in [19] that the sequence $(WB_n f)$ converges pointwise and uniformly to f on $[0,1]$, and estimated the rate of these convergence results using the modulus of continuity, second order modulus of smoothness and Peetre's K-functionals. The author also obtained some results in L^p spaces for these operators.

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Based on the idea developed in [[4]-[5], [8]-[10] and [18]], the goal of this study is to obtain some asymptotic properties and quantitative results of the newly introduced wavelet type Bernstein operators WB_n by using the compactly supported Daubechies wavelets.

Unlike Fourier analysis (sinusoidal functions), wavelets can be tuned and adapted to the signals (target functions) and are inherently local. Since there is no single wavelet, they can be designed to suit individual applications. They are ideal for adaptive systems that adjust themselves to match the function.

Wavelet expansion, or reconstruction of signals via wavelets, allows for more accurate local identification and separation of signal features. A wavelet expansion coefficient represents a component that is itself local and easier to interpret. Wavelets can allow overlapping components of a signal to be separated in both time and frequency. Some detailed informations and advantages of the wavelets can be found in [7].

In addition, we will see that the results obtained for operators defined using some special cases of wavelets represent a natural extension to the classical Bernstein operators and their Kantorovich-type modifications ([15], [16]). It is also worth noting that the operators discussed here are closely related to hybrid type operators and quasi interpolation operators (see [4], [17], [23] and [26]).

Please also see the very recent studies of the author's on wavelet type Bezier operators, due to the advantage of the wavelet functions, which give some extensions of the previous results in the literature ([20] and [21]).

2 Preliminaries and auxiliary results

As usual, let $C[0,1]$ be the Banach space of continuous functions $u:[0,1]\to\mathbb{R}$ with the usual norm, and let $L^p[0,1](1\le p\le\infty)$ denote the space of Lebesgue measurable functions f satisfying some conditions related with the $p-$th power.

Let us consider two orthogonal functions: the scaling function (or father wavelet) $\phi(t)$ and the wavelet function (or mother wavelet) $\psi(t)$. By scaling and translation of these two orthogonal functions we obtain a complete basis set. These functions have the following important properties;

$$\int_{-\infty}^{\infty}\phi(t)dt=1,\int_{-\infty}^{\infty}\psi(t)dt=0,$$

$\phi,\psi\in L^2(\mathbb{R})$, and orthogonal. (see [11], [12])

In general, the wavelets refer to the set of family of orthonormal functions of the form

$$\psi_{a,b}(t)=\frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right),\quad a>0,b\in\mathbb{R}, \tag{3}$$

where ψ is the basic (mother) wavelet.

The simplest wavelet is known as the Haar wavelet given by:

$$\psi(x)=\begin{cases}1 & , & 0\le x<\frac{1}{2}\\ -1 & , & \frac{1}{2}\le x<1\\ 0 & , & e.w.\end{cases}$$

with the corresponding scaling function (father wavelet)

$$\phi(t)=\begin{cases}1 & , & 0\le x<1\\ 0 & , & e.w.\end{cases}.$$

Haar wavelets constitutes an orthonormal system for the space of square-integrable functions on the real line.

We now consider a special orthonormal bases, called wavelets. There is a scaling function (father wavelet) $\phi(t)$ with $\{\phi(t-n)\}$ are orthogonal and the mother wavelet $\psi(t)$ based on the father wavelet $\phi(t)$ gives rise to the orthonormal basis

$$\psi_{j,k}(t):=2^{j/2}\psi(2^jt-k). \tag{4}$$

of $L^2(\mathbb{R})$.

Moreover, a multiresolution analysis (MRA) is a sequence $(Vj)_{j\in Z}$ of closed subspaces of $L^2(\mathbb{R})$, whose elements are scaling functions (father wavelets).

It is well-known that, each $f\in L^2(\mathbb{R})$ has the following representation

$$f(x)=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}b_{j,k}\psi_{j,k}(x),$$

called wavelet expansion and $b_{j,k}$ are wavelet coefficients given by

$$b_{j,k} = \langle f(x), \psi_{j,k}(x)\rangle = 2^{j/2} \int\limits_{\mathbb{R}} f(x)\overline{\psi(2^j x - k)}dx.$$

(see [3], [13], [22], [24] and [28]).

Let us assume that father wavelets $w \in L_\infty(\mathbb{R})$ satisfies:

a$_1$) w is a compactly supported, namely there is a real constant $0 < \lambda \leq 1$ such that supp $w \subset [0, \lambda]$,
a$_2$)

$$\int\limits_{\mathbb{R}} w(x)dx = 1,$$

a$_3$) the first N moments of the father wavelet w satisfy

$$m_j^w(w) := \int\limits_{\mathbb{R}} x^j w(x)dx = 0, \quad j = 1, ..., N.$$

Obviously, the absolute moments of the father wavelet w

$$M_j^w(w) := \int\limits_{\mathbb{R}} |x|^j |w(x)| dx < +\infty$$

for every $j \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Wavelets that meet the above conditions are called compactly supported Daubechies wavelets. Daubechies wavelets have strong relations with the properties of continuity and differentiability.
They are supported with $[0, 2N - 1]$, in addition there exists a constant $r > 0$ such that for $N \geq 2$, $w \in C^{rN}(\mathbb{R})$ and have a given number of vanishing moments.

When $N = 1$, then the first Daubechies wavelet ψ will be the classical Haar basis. As N increases, the regularity of the wavelet increases (see [11], [12]).

This means that if we want to use Daubechies wavelets to reconstruct a function, it is more convenient to choose or construct wavelets based on the continuity or differentiability properties of the given function.

Owing to the above definitions, first of all we will recall the wavelet type Bernstein operators WB_n introduced by the author [19].

Definition 6 [19]. *Let $f \in B[0,1]$, and let $w \in L_\infty(R)$ be a father wavelet satisfying a_1)-a_3). Then the wavelet type Bernstein operators are defined by:*

$$(WB_n f)(t) := n\sum_{k=0}^{n} p_{n,k}(t) \int\limits_0^1 f(x) w(nx - k)dx,$$

with $t \in [0,1]$, specifying that supp$(w) \subseteq [0, \lambda]$, $0 < \lambda \leq 1$.

Remark 1 [19]. *If we choose the father wavelet w as the Haar scaling function, namely $w(x) = \chi_{[0,1]}(x)$, then clearly our wavelet type operators reduce to the Kantorovich form of the Bernstein operators. Indeed:*

$$\begin{aligned}(WB_n f)(t) &= n\sum_{k=0}^{n} p_{n,k}(t) \int\limits_0^1 f(x) w(nx-k)dx \\ &= \sum_{k=0}^{n} p_{n,k}(t) \int\limits_0^1 f\left(\frac{u+k}{n}\right) w(u)du \\ &= n\sum_{k=0}^{n} p_{n,k}(t) \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} f(z)\,dz = (K_n f)(t).\end{aligned}$$

This shows that the wavelet type Bernstein operators (2) are a natural extension of the Kantorovich type of the Bernstein operators.

3 Fundamental properties

The following approximate results for the wavelet type Bernstein operators need to be remembered for dealing with application and reconstruction of functions. In particular, the following convergence theorem applies when continuous signals (functions) are considered.

As presented and proved in [**19**] we have the followings.

Theorem 1 [19]. *Let $f \in B[0,1]$ and let $w \in L_\infty(R)$ be a father wavelet satisfies a_1)-a_3). Then the moments of wavelet type Bernstein operators, constructed by using the compactly supported Daubechies wavelets (2) and the Bernstein operators (1) are the same, namely*

$$(WB_n x^s)(t) = (B_n x^s)(t), \quad s = 0, 1, ..., K$$

holds true.

Remark 2 [19]. *By the properties a_2) and a_3), one gets*

$$\begin{aligned}(WB_n (x-t)^\beta)(t) &= \frac{1}{n^\beta} \sum_{k=0}^{n} p_{n,k}(t)(k-nt)^\beta \\ &= (B_n (x-t)^\beta)(t).\end{aligned}$$

Throughout this work, the first two central moments of the wavelet type Bernstein operators (2) satisfy

$$\begin{aligned}\mu_1(t) &:= \frac{1}{n}\sum_{k=0}^{n} p_{n,k}(t)(k-nt) = 0, \\ \mu_2(t) &:= \frac{1}{n^2}\sum_{k=0}^{n} p_{n,k}(t)(k-nt)^2 = \frac{t(1-t)}{n} \le \frac{1}{4n}\end{aligned} \tag{5}$$

for every $t \in [0,1]$.

It is also well-known that for each $s \in \mathbb{N}_0$ there is a constant A_s only depending upon s such that

$$0 \le \mu_{2s}(t) \le \frac{A_s}{n^s} < \infty$$

hold (page 15 eq (6) Lorentz [25], see also [2]).

Moreover, for every $t \in [0,1]$ and for some $\beta > 0$, the discrete absolute moments of order β satisfy

$$\widetilde{\mu}_\beta(t) := (B_n |x-t|^\beta)(t) \le 2\Gamma\left(\frac{\beta}{2}+1\right)\frac{1}{n^{\beta/2}} < \infty, \tag{6}$$

where $\Gamma(\bullet)$ stands for the Gamma function (see [2]**)**.

In [**19**] we have also proved the following:

Theorem 2 [19]. *Let $f \in B[0,1]$ and let $\psi \in L_\infty(R)$ be a father wavelet satisfying a_1)-a_3). Then*

$$\lim_{n\to\infty} (WB_n f)(t_0) = f(t_0)$$

holds true at each point t_0 of continuity of f.

As a consequence of the Theorem 2, we have also the following uniform convergence result.

Corollary 1. *The same arguments of Theorem 2 apply to the case when $f \in C[0,1]$. In this case the convergence is uniform with respect to $t \in [0,1]$, and hence one has*

$$\lim_{n\to\infty} \|(WB_n f) - f\|_{C[0,1]} = 0.$$

4 Asymptotic expansion and Voronovskaya-type theorems

This section provides the main approximation results of the paper. We are now ready to establish one of the first main results of this study, which gives a strong relation between Bernstein operators (1) and our new operators (2) constructed by wavelets.

We will give some asymptotic formulas, quantitative estimates and some Voronovskaya type theorems for the wavelet type Bernstein operators. We have the following result.

Theorem 5. *Let $f:[0,1]\to\mathbb{R}$ be a bounded function. Moreover, we assume that $f'(t)$ exists at a fixed point t. Then the following asymptotic formula holds:*

$$(WB_nf)(t)=f(t)+o\left(n^{-1/2}\right)\ ,\ (n\to\infty).$$

Proof. *Since $f'(t)$ exists at a point t, then by the local Taylor's formula we have*

$$f(x)=f(t)+f'(t)(x-t)+h(x-t)(x-t),$$

where *h is a bounded function such that $\lim\limits_{y\to 0}h(y)=0$.*

In view of (2) and by the local Taylor's formula, we can write

$$\begin{aligned}(WB_nf)(t)&=n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 f(x)w(nx-k)dx\\&=n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\left[f(t)+f'(t)(x-t)+h(x-t)(x-t)\right]w(nx-k)dx\end{aligned}$$

$$\begin{aligned}(WB_nf)(t)=n\sum_{k=0}^{n}p_{n,k}(t)&\int_0^1\left[f(t)+f'(t)(x-t)\right]w(nx-k)dx\\+n\sum_{k=0}^{n}p_{n,k}(t)&\int_0^1 h(x-t)(x-t)w(nx-k)dx\\=:I_1+R.&\end{aligned}$$

Let us analyze the terms I_1 and R, respectively. Let us consider the term I_1.

$$\begin{aligned}I_1&=n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\left[f(t)+f'(t)(x-t)\right]w(nx-k)dx\\&=f(t)+f'(t)n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1(x-t)w(nx-k)dx.\end{aligned}$$

Note that

$$x-t=\frac{nx-nt}{n}=\frac{nx-k-nt+k}{n}=\frac{nx-k}{n}-\frac{nt-k}{n},\qquad(7)$$

and hence

$$|x-t|\leq\left|\frac{nx-k}{n}\right|+\left|\frac{nt-k}{n}\right|\qquad(8)$$

holds true. So, one has from (7)

$$\begin{aligned}I_1&=f(t)+f'(t)n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1(x-t)w(nx-k)dx\\&=f(t)+f'(t)n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\left(\frac{nx-k}{n}-\frac{nt-k}{n}\right)w(nx-k)dx\\&=f(t)+f'(t)\sum_{k=0}^{n}p_{n,k}(t)\int_0^1(nx-k)w(nx-k)dx\\&-f'(t)\sum_{k=0}^{n}(nt-k)p_{n,k}(t)\int_0^1 w(nx-k)dx.\end{aligned}$$

Owing to the definition and the properties of the compactly supported Daubechies wavelets, one has

$$\begin{aligned} I_1 &= f(t) + f'(t)\frac{m_1^w(w)}{n} + f'(t)\mu_1(t) \\ &= f(t). \end{aligned}$$

Now we evaluate the remainder term R.

Let $\varepsilon > 0$ be fixed. Since $h(y)$ is a bounded function such that $\lim_{y\to 0} h(y) = 0$, there exists $\delta > 0$ such that $|h(y)| \le \varepsilon$ for every $|y| \le \delta$. Hence one obtains

$$\begin{aligned} R &= n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 h(x-t)(x-t)w(nx-k)dx \\ &= n\sum_{\left|\frac{k}{n}-t\right|\ge\delta} p_{n,k}(t)\int_0^1 h(x-t)(x-t)w(nx-k)dx \\ &+ n\sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\int_0^1 h(x-t)(x-t)w(nx-k)dx \\ &:= R_1 + R_{2.} \end{aligned}$$

In view of (8), we obtain

$$\begin{aligned} |R_2| &= \left| n\sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\int_0^1 h(x-t)(x-t)w(nx-k)dx \right| \\ &\le \varepsilon \sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\int_0^1 [|nx-k| + |nt-k|]\,|w(nx-k)|\,dx \\ &= \varepsilon \sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\int_0^1 |nx-k|\,|w(nx-k)|\,dx \\ &+ \varepsilon \sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\,|nt-k|\int_0^1 |w(nx-k)|\,dx \\ &\le \varepsilon\left(\frac{M_1^w(w)}{n} + \widetilde{\mu}_1(t)M_0^w(w)\right) = o\left(n^{-1/2}\right) \end{aligned}$$

for $n \to \infty$. Moreover, choosing a constant $B > 0$ such that $|h(y)| \le B$ we have

$$|R_1| \le B\left(\frac{M_1^w(w)}{n} + \widetilde{\mu}_1(t)M_0^w(w)\right) = o\left(n^{-1/2}\right)$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty} |R| = 0,$$

and hence the claim follows.

Theorem 6. *Let $f \in B[0,1]$ and let $t \in [0,1]$ be fixed. If for a certain $r \in N$, $f \in C^r$ locally at the point t, then the following asymptotic formula holds:*

$$(WB_n f)(t) = f(t) + \sum_{i=1}^{r} \frac{f^{(i)}(t)}{i!}\mu_i(t) + o\left(n^{-r/2}\right),\ (n \to \infty),$$

where μ_i is the $i-th$ order algebraic moment.

Proof. *Since $f^{(r)}(t)$ exists at a point t, then there exists a bounded function h such that $\lim_{y\to 0} h(y) = 0$. By the local Taylor's formula we have*

$$f(x) = \sum_{i=0}^{r} \frac{f^{(i)}(t)}{i!}(x-t)^i + h(x-t)(x-t)^r. \tag{9}$$

In view of (2) and (9), we can write

$$(WB_n f)(t) = n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 f(x)w(nx-k)dx$$
$$= n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 \left(\sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i + h(x-t)(x-t)^r\right)w(nx-k)dx$$
$$= n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 \left(\sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i\right)w(nx-k)dx$$
$$+ n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 h(x-t)(x-t)^r w(nx-k)dx$$
$$=: I_1 + R.$$

Let us analyze the terms I_1 and R, respectively. Let us consider the term I_1.

$$I_1 = n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 \left(\sum_{i=0}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i\right)w(nx-k)dx$$
$$= f(t) + n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 \left(\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}(x-t)^i\right)w(nx-k)dx$$
$$= f(t) + n\sum_{k=0}^{n} p_{n,k}(t)\left(\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\right)\int_0^1 (x-t)^i w(nx-k)dx.$$

As in the proof of Theorem 5, note that

$$(x-t)^i = \left(\frac{nx-nt}{n}\right)^i = \left(\frac{nx-k}{n} - \frac{nt-k}{n}\right)^i,$$

and applying Binomial expansion, one has

$$(x-t)^i = \sum_{v=0}^{i}\binom{i}{v}\left(\frac{nx-k}{n}\right)^v\left(-\frac{nt-k}{n}\right)^{i-v}.$$

So we can write

$$I_1$$
$$= f(t) + n\sum_{k=0}^{n} p_{n,k}(t)\left(\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\right)\int_0^1 \left(\sum_{v=0}^{i}\binom{i}{v}\left(\frac{nx-k}{n}\right)^v\left(-\frac{nt-k}{n}\right)^{i-v}\right)w(nx-k)dx$$
$$= f(t) + n\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!n^i}\sum_{k=0}^{n} p_{n,k}(t)\sum_{v=0}^{i}\binom{i}{v}(k-nt)^{i-v}\int_0^1 (nx-k)^v w(nx-k)dx$$
$$= f(t) + n\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!n^i}\sum_{v=0}^{i}\binom{i}{v}\sum_{k=0}^{n} p_{n,k}(t)(k-nt)^{i-v}\int_0^1 (nx-k)^v w(nx-k)dx.$$

In view of $\mathbf{a}_2$**)**, *Remark 2 and using the condition* $\mathbf{a}_3$**)** *when $N = n$, we have*

$$I_1 = f(t) + \sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_i(t).$$

Now we evaluate the term remainder term R.

Let $\varepsilon > 0$ be fixed. Since $h(y)$ is a bounded function such that $\lim_{y\to 0} h(y) = 0$, there exists $\delta > 0$ such that $|h(y)| \le \varepsilon$ for every $|y| \le \delta$. We have

$$\begin{aligned}
|R| &\le n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 |h(x-t)|\,|x-t|^r\,|w(nx-k)|\,dx \\
&= n\left(\sum_{\left|\frac{k}{n}-t\right|\ge\delta} + \sum_{\left|\frac{k}{n}-t\right|<\delta}\right) p_{n,k}(t)\int_0^1 |h(x-t)|\,|x-t|^r\,|w(nx-k)|\,dx \\
&:= R_1 + R_2,
\end{aligned}$$

and hence

$$\begin{aligned}
R_2 &\le \varepsilon n \sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\int_0^1 |x-t|^r\,|w(nx-k)|\,dx \\
&= \varepsilon n \sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\int_0^1 \sum_{\nu=0}^{r}\binom{r}{\nu}\left|\frac{nx-k}{n}\right|^{\nu}\left|-\frac{nt-k}{n}\right|^{r-\nu}|w(nx-k)|\,dx \\
&= \varepsilon n \sum_{\nu=0}^{r}\binom{r}{\nu}\sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\left|-\frac{nt-k}{n}\right|^{r-\nu}\int_0^1\left|\frac{nx-k}{n}\right|^{\nu}|w(nx-k)|\,dx \\
&= \frac{\varepsilon}{n^r}\sum_{\nu=0}^{r}\binom{r}{\nu}\sum_{\left|\frac{k}{n}-t\right|<\delta} p_{n,k}(t)\,|k-nt|^{r-\nu}\,M_\nu^w(w) \\
&\le \varepsilon\sum_{\nu=0}^{r}\binom{r}{\nu}\frac{M_\nu^w(w)\widetilde{\mu}_{r-\nu}(t)}{n^\nu}.
\end{aligned}$$

Moreover, choosing a constant $B > 0$ such that $|h(y)| \le B$ we have

$$R_1 \le B\sum_{\nu=0}^{r}\binom{r}{\nu}\frac{M_\nu^w(w)\widetilde{\mu}_{r-\nu}(t)}{n^\nu}.$$

Thus

$$\lim_{n\to\infty}|R| = 0,$$

and hence the claim follows.

As a consequence of Theorems 5 and 6 we can establish the following first and second order Voronovskaya type theorems, respectively.

Theorem 7. *Let $f \in B[0,1]$ and let $t \in [0,1]$ be fixed. If $f \in C^1$ locally at the point t, then we have*

$$\lim_{n\to\infty} n^{1/2}\left[(WB_n f)(t) - f(t)\right] = 0.$$

Proof. *Applying the asymptotic formula of Theorem 6 with $r = 1$, and using (9) and assumption (5), we can write:*

$$(WB_n f)(t) = f(t) + f'(t)\mu_1(t) + o\left(n^{-1/2}\right),\ (n\to\infty).$$

Then the proof follows by passing to the limit for $n \to \infty$.

Theorem 8. *Let $f \in B[0,1]$ and let $t \in [0,1]$ be fixed. If $f \in C^2$ locally at the point t, then we have*

$$\begin{aligned}
\lim_{n\to\infty} n\left[(WB_n f)(t) - f(t)\right] &= \frac{1}{2}f''(t)\mu_2(t) \\
&= \frac{t(1-t)}{2}f''(t).
\end{aligned}$$

Proof. *As in the proof of Theorem 7, applying the asymptotic formula of Theorem 6 with $r = 2$, and using (9) and assumption (5), we can write:*

$$(WB_n f)(t) = f(t) + \sum_{i=1}^{2}\frac{f^{(i)}(t)}{i!}\mu_i(t) + o\left(n^{-2/2}\right),\ (n\to\infty).$$

Then the proof follows by passing to the limit for $n \to \infty$.

The above Theorems show that the order of pointwise approximation is at least of order $O(n^{-1/2})$, and $O(n^{-1})$, as $n \to +\infty$, respectively.

5 Quantitative Voronovskaya type estimates

In this section we will give some quantitative estimates for the Voronovskaya type results given in the previous section.

Let$f : I \to \mathbb{R}$ be function.

Let us denote by $L^{\infty}(I)$ the space of all essentially bounded functions endowed with the essential sup-norm $\|\bullet\|_{\infty}$.

By $C^0(I)$,we denote the space of all uniformly continuous and bounded functions $f : I \to \mathbb{R}$. For $m \geq 1$ by $C^m(I)$ the subspace of $C^0(I)$ whose elements $f : I \to \mathbb{R}$ are $m-$times continuously differentiable and $f^{(k)} \in C^0(I)$.

At first we recall some important relation and estimation on the Taylor remainder term obtained by Gonska et al. [14].

For $C[0,1]$, let us consider the following Peetre's $K-$functional:

$$K_1(f;\delta) := \inf_{g \in W^1} \{\|f-g\|_{\infty} + \delta \left\|g'\right\|_{\infty}\},$$

where $\delta > 0$ and $W^1 = \{g \in C[0,1] : g' \in C[0,1]\}$. Then there exists an absolute constant $C > 0$ such that

$$C^{-1}\omega(f;\sqrt{\delta}) \leq K_1(f;\delta) \leq C\omega(f;\sqrt{\delta}),$$

where

$$\omega(f;\sqrt{\delta}) := \sup_{0<h\leq\sqrt{\delta}} \sup_{x\in[0,1]} |f(x+h)-f(x)|$$

is the first order modulus of continuity of f.

Let $f \in C^m(I)$ and consider the following version of the Taylor formula;

$$f(t) = \sum_{i=0}^{m} \frac{f^{(i)}(t_0)}{i!}(t-t_0)^i + R_m(f;t,t_0),$$

where $t, t_0 \in I$ and $R_m(f;t,t_0) = h(t_0-t)(t_0-t)^m$ is the remainder term satisfying

$$|R_m(f;t,t_0)| \leq \frac{|t-t_0|^m}{m!}\omega\left(f^{(m)};|t-t_0|\right)$$

and

$$|R_m(f;t,t_0)| \leq 2\frac{|t-t_0|^m}{m!}K_1\left(f^{(m)};\frac{|t-t_0|}{2(m+1)}\right) \tag{10}$$

(see [14]).

Here we study quantitative estimates of the convergence results given in Theorems 5 and 6.

Theorem 9. *Under the assumptions of Theorem 5, namely let f be a bounded function $f : [0,1] \to R$. Moreover, we also assume that $f \in C^1$ locally at a fixed point t. Then there holds*

$$|n((WB_nf)(t)-f(t))| \leq 2D_1K_1\left(f';\frac{D_2}{4nD_1}\right),$$

where

$$D_1 = M_1^w(w) + M_0^w(w)\widetilde{\mu}_1(t)$$
$$D_2 = M_2^w(w) + M_0^w(w)\widetilde{\mu}_2(t) + 2M_1^w(w)\widetilde{\mu}_1(t),$$

and $M_i^w(\bullet)$ $(i = 0,1,2)$ are the absolute moments.

Proof. *Using the Taylor formula of the first order as in Theorem 5*

$$f(x) = f(t) + f'(t)(x-t) + h(x-t)(x-t),$$

where h is a bounded function such that $\lim_{y\to 0} h(y) = 0$, we get

$$(WB_nf)(t) = n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 \left[f(t)+f'(t)(x-t)+h(x-t)(x-t)\right]w(nx-k)dx,$$

and hence

$$(WB_nf)(t)-f(t) = f'(t)n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 (x-t)w(nx-k)dx$$

$$+n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 h(x-t)(x-t)w(nx-k)dx$$

$$=f'(t)\frac{m_1^w(w)}{n}+n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 h(x-t)(x-t)w(nx-k)dx.$$

Recalling the properties of the moments of the wavelets, namely,

$$m_j^w(w):=\int_{\mathbb{R}} x^j w(x)dx=0, \quad j=1,...,N,$$

and the definition of the first order remainder term

$$R_1(f;t,x)=h(x-t)(x-t),$$

we can immediately obtain

$$(WB_nf)(t)-f(t)=n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 R_1(f;t,x)w(nx-k)dx.$$

The last equality yields

$$|(WB_nf)(t)-f(t)|$$
$$\leq n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 |R_1(f;t,x)|\,|w(nx-k)|\,dx.$$

According to (10), one has

$$|(WB_nf)(t)-f(t)|$$
$$\leq n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 2\,|t-x|\,K_1\left(f';\frac{|t-x|}{4}\right)|w(nx-k)|\,dx.$$

Let $g\in C^2$ be fixed. Then there holds

$$|(WB_nf)(t)-f(t)|$$
$$\leq 2n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 |t-x|\left\{\left\|f'-g'\right\|_\infty+\frac{|t-x|}{4}\left\|g''\right\|_\infty\right\}|w(nx-k)|\,dx$$
$$=2\left\|f'-g'\right\|_\infty n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 |t-x|\,|w(nx-k)|\,dx$$
$$+\frac{2\left\|g''\right\|_\infty}{4}n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 |t-x|^2\,|w(nx-k)|\,dx.$$

By (8) we also have

$$|x-t|^2\leq\left|\frac{nx-k}{n}\right|^2+\left|\frac{nt-k}{n}\right|^2+2\left|\frac{nx-k}{n}\right|\left|\frac{nt-k}{n}\right|,$$

yields

$$|(WB_nf)(t)-f(t)|$$
$$\leq 2\left\|f'-g'\right\|_\infty n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\left|\frac{nx-k}{n}\right||w(nx-k)|\,dx$$
$$+2\left\|f'-g'\right\|_\infty n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\left|\frac{nt-k}{n}\right||w(nx-k)|\,dx$$

$$+\frac{2\|g''\|_\infty}{4} n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 \left|\frac{nx-k}{n}\right|^2 |w(nx-k)|\,dx$$
$$+\frac{2\|g''\|_\infty}{4} n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 \left|\frac{nt-k}{n}\right|^2 |w(nx-k)|\,dx$$
$$+\frac{2\|g''\|_\infty}{4} n\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 2\left|\frac{nx-k}{n}\right|\left|\frac{nt-k}{n}\right| |w(nx-k)|\,dx$$
$$=\frac{2\|f'-g'\|_\infty}{n}\left\{\widetilde{\mu}_0(t)M_1^w(w)+M_0^w(w)\widetilde{\mu}_1(t)\right\}$$
$$+\frac{2\|g''\|_\infty}{4n^2}\left\{\widetilde{\mu}_0(t)M_2^w(w)+M_0^w(w)\widetilde{\mu}_2(t)+2M_1^w(w)\widetilde{\mu}_1(t)\right\}$$
$$=:\frac{2\|f'-g'\|_\infty}{n}D_1+\frac{2\|g''\|_\infty}{4n^2}D_2$$

Finally we have

$$|(WB_nf)(t)-f(t)|\le\frac{2D_1}{n}\{\|f-g\|_\infty+\frac{D_2}{4nD_1}\left\|g'\right\|_\infty\}$$
$$\le\frac{2D_1}{n}K_1\left(f';\frac{D_2}{4nD_1}\right).$$

Similarly we have the following quantitative estimates for the $r-th$ order asymtotic expansion obtained in Theorem 6.

Theorem 10. *Under the assumptions of Theorem 6 we also assume that $f\in C^r$ locally at a fixed point t,* then *we have*

$$\left|n^r\left((WB_nf)(t)-f(t)-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_i(t)\right)\right|\le\frac{2L_r}{r!}K_1\left(f^{(r)};\frac{J_r}{2n(r+1)L_r}\right),$$

where

$$L_r=\sum_{k=0}^{r}\binom{r}{k}M_k^w(w)\widetilde{\mu}_{r-k}(t)$$
$$J_r=\sum_{k=0}^{r+1}\binom{r+1}{k}M_k^w(w)\widetilde{\mu}_{r+1-k}(t).$$

Proof. *Using the Taylor formula of the $r-th$ order as in Theorem 6, we get*

$$\left|(WB_nf)(t)-f(t)-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_i(t)\right|$$
$$\le n\sum_{k=-\infty}^{\infty}p_{n,k}(t)\int_0^1|h(x-t)|\,|x-t|^r\,|w(nx-k)|\,dx.$$

Since the remainder term $R_r(f;t,x)=h(x-t)(x-t)^r$ satisfies (10), we can write

$$\left|(WB_nf)(t)-f(t)-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_i(t)\right|$$
$$\le n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1 2\frac{|t-x|^r}{r!}K_1\left(f^{(r)};\frac{|t-x|}{2(r+1)}\right)|w(nx-k)|\,dx.$$

Let $g \in C^r$ be fixed. Then there holds

$$\left|(WB_nf)(t)-f(t)-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_i(t)\right|$$
$$\leq 2n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\frac{|t-x|^r}{r!}\left\{\left\|f^{(r)}-g^{(r)}\right\|_\infty+\frac{|t-x|}{2(r+1)}\left\|g^{(r+1)}\right\|_\infty\right\}|w(nx-k)|\,dx$$
$$=2\left\|f^{(r)}-g^{(r)}\right\|_\infty n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\frac{|t-x|^r}{r!}|w(nx-k)|\,dx$$
$$+\frac{2\left\|g^{(r+1)}\right\|_\infty}{2(r+1)}n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\frac{|t-x|^{r+1}}{r!}|w(nx-k)|\,dx.$$

By Binomial expansion we have

$$|x-t|^i\leq\sum_{k=0}^{i}\binom{i}{k}\left|\frac{nx-k}{n}\right|^k\left|\frac{nt-k}{n}\right|^{i-k}.$$

This yields

$$\left|(WB_nf)(t)-f(t)-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_i(t)\right|$$
$$\leq\frac{2\left\|f^{(r)}-g^{(r)}\right\|_\infty}{r!}n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\sum_{k=0}^{r}\binom{r}{k}\left|\frac{nx-k}{n}\right|^k\left|\frac{nt-k}{n}\right|^{r-k}|w(nx-k)|\,dx$$
$$+\frac{2\left\|g^{(r+1)}\right\|_\infty}{2(r+1)!}n\sum_{k=0}^{n}p_{n,k}(t)\int_0^1\sum_{k=0}^{r+1}\binom{r+1}{k}\left|\frac{nx-k}{n}\right|^k\left|\frac{nt-k}{n}\right|^{r+1-k}|w(nx-k)|\,dx.$$
$$=:\frac{2\left\|f^{(r)}-g^{(r)}\right\|_\infty}{r!n^r}L_r+\frac{\left\|g^{(r+1)}\right\|_\infty}{(r+1)!n^{r+1}}J_r,$$

where

$$L_r=\sum_{k=0}^{r}\binom{r}{k}M_k^w(w)\widetilde{\mu}_{r-k}(t)$$
$$J_r=\sum_{k=0}^{r+1}\binom{r+1}{k}M_k^w(w)\widetilde{\mu}_{r+1-k}(t)$$

Finally we have the following quantitative estimate

$$\left|(WB_nf)(t)-f(t)-\sum_{i=1}^{r}\frac{f^{(i)}(t)}{i!}\mu_i(t)\right|$$
$$\leq\frac{2L_r}{r!n^r}\{\left\|f^{(r)}-g^{(r)}\right\|_\infty+\frac{J_r}{2n(r+1)L_r}\left\|g'\right\|_\infty\}$$
$$\leq\frac{2L_r}{r!n^r}K_1\left(f^{(r)};\frac{J_r}{2n(r+1)L_r}\right).$$

This completes the proof.

As corollaries of the Theorem 10 for $r = 1$ and $r = 2$, we obtain quantitative estimates for the first and second order Voronovskaya Theorems 7 and 8 proved in the previous section, respectively.

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References

[1] Acu A.M., Buscu I.C.,Raşa I., Generalized Kantorovich modifications of positive linear operators, Mathematical Foundations of Computing, 6(1) (2023), 54-62.

[2] Adell J.A., Bustamante J., Quesada J.M. Estimates for the moments of Berntein polynomials, J. Math. Anal. Appl., 432 (2015), pp. 114-128.

[3] Agratini, O., Construction of Baskakov-type operators by wavelets. Rev. Anal. Numér. Théor. Approx. 26 (1997), no. 1-2, 3–11.

[4] Bardaro C. and Mantellini I., Asymptotic expansion of generalized Durrmeyer sampling type series, Jaen J. Approx. 6(2), (2014) 143-165.

[5] Bardaro C., Faina L. and Mantellini I., Quantitative Voronovskaja formulae for generalized Durrmeyer sampling type series, Math. Nachr. 289, No. 14-15, (2016) 1702-1720.

[6] Berdysheva E., Heilmann M., Hennings K., Pointwise convergence of the Bernstein–Durrmeyer operators with respect to a collection of measures, J. Approx. Theory, 251 (2020).

[7] Burrus, C. S., Gopinath, R. A., & Guo, H.,. Introduction to wavelets and wavelet transforms. A primer. Prentice Hall. (1998)

[8] Butzer, P. L. and Nessel, R. J., Fourier Analysis and Approximation, V.1, Academic Press, New York, London, 1971.

[9] Costarelli D., Piconi M., Vinti G., On the convergence properties of sampling Durrmeyer-type operators in Orlicz spaces, Math. Nachr., 296(2) (2023), 588-609.

[10] Costarelli D, Vinti G., Rate of approximation for multivariate sampling Kantorovich operators on some functions spaces, J. Integral Equations Applications 2014; 26 (4): 455-481.

[11] Daubechies I., Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math. 41 (1988), 909-996.

[12] Daubechies I., Ten Lectures on Wavelets, CBMS-NSF Series in Appl. Math. 61, SIAM Publ. Philadelphia, 1992.

[13] Gonska H. H. and Zhou D. X., Using wavelets for Szász-type operators., Rev. Anal. Numér. Théor. Approx. 24 (1995), no. 1-2, 131–145

[14] Gonska H., Pitul P., Rasa I., On Peano's form of the Taylor remainder, Voronovskaja's theorem and the commutator of positive linear operators, in: Proc. Int. Conf. on Numerical Analysis and Approximation Theory, Cluj Napoca, Romania July 5–8 2006, pp. 55–80.

[15] Gupta, V., The bezier variant of Kantorovitch operators, Compt & Math Appl., Vol. 47, Issues 2–3, 2004, 227-232.

[16] Gupta, V., An estimate on the convergence of Baskakov–Bezier operators., J. Math. Anal. Appl. 312, 280–288 (2005).

[17] Gupta V. and Acu A. M., On difference of operators with different basis functions, Filomat 33 (10) (2019), 3023-3034.

[18] Karsli H., On Urysohn type Generalized Sampling Operators, Dolomites Research Notes on Approximation, Volume 14, (2021), Pages 58–67

[19] Karsli H. On Wavelet Type Bernstein Operators. Carpathian Math. Publ. 2023, 15 (1), 212-221.

[20] Karsli, H., Extension of the generalized Bezier operators by wavelet, General Math., Vol. 30, No. 2 (2022), 3–15.

[21] Karsli, H., On wavelet type generalized Bézier operators, Mathematical Foundations of Computing, 2023, 6(3): 439-452.

[22] Kelly S., Kon M, and Raphael L., Pointwise convergence of wavelet expansions, Bull. Amer. Math. Soc., Vol. 30 (1994), pp. 87-94.

[23] Kolomoitsev, Y., & Skopina, M. (2017). Approximation by multi-variate Kantorovich-Kotelnikov operators. J. Math. Anal. Appl., 456(1), 195-213.

[24] Łenski W., Szal B., Approximation of Integrable Functions by Wavelet Expansions. Results Math 72, 1203–1211 (2017).

[25] Lorentz G.G., Bernstein Polynomials, University of Toronto Press,Toronto, 1953.

[26] Lupas A., The approximation by means of some linear positive operators. In: Approximation Theory (M.W. Muller others, eds), pp. 201–227. Akademie-Verlag, Berlin (1995).

[27] Savaş E., Mursaleen M., Bézier Type Kantorovich q-Baskakov Operators via Wavelets and Some Approximation Properties. Bull. Iran. Math. Soc. 49, 68 (2023).

[28] Walter G.G, Pointwise convergence of wavelet expansions. J. Approx. Theory 80(1), 108–118 (1995)