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Kantorovich Variant of α-Bernstein Operators using Contagion Distribution

Kanita[a] · Naokant Deo[a]

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Abstract

This study puts forth an enhancement to the α-Bernstein operators, with the help of the Pólya-Eggenberger distribution, also known as the contagion distribution for $m \geq 2$. In order to deal with integrable functions we utilize the parameter γ to be of the order of $1/m$, to define and introduce the Kantorovich variation of the above said operators. First, we give some auxiliary properties and then we give an upper bound of our proposed operators on the space of Lebesgue integrable functions, $L^1[0,1]$, and the space of continuous functions, $C[0,1]$. Next, we study its asymptotic results with the help of Taylor's expansion. Modulus of continuity is also used to provide the asymptotic properties of Kantorovich operators, both for Lebesgue and continuous spaces of functions. Moreover, like the classical Kantorovich-Bernstein operators, we will see that the Kantorovich variant defined in this paper also does not preserve e_1, that is, $K_{m,\alpha}^{\langle\gamma\rangle}(t;u) \neq u$. Rather, we get an expression which tends to u as m tends to ∞. As an opening at finding a better approximating linear operators, we try and preserve these operators at e_1 and propose a genuine-type modification for same. We have also included graphical illustrations to help analyze and compare the approximation results and properties of both the Kantorovich variant of the α-Bernstein operators using contagion distribution and its genuine-type modification.

1 Introduction

The Bernstein operators play a significant role in approximation theory, numerical analysis, computer graphics, and various other fields where functions need to be approximated or interpolated efficiently and accurately. They find applications in computer graphics, particularly in the generation of Bézier curves and surfaces which are widely used in computer-aided design (CAD), computer animation, and modeling. To enhance the accuracy and versatility of approximation techniques, researchers have pursued numerous modifications of Bernstein operators (see [1, 3, 15, 17]). Among these modifications and generalizations, a notable stride has been made by Chen et al. [5] in 2017, with the introduction of α-Bernstein operators, for any function f having its domain as $[0,1]$. The α-Bernstein operators are defined by:

$$B_{n,\alpha}(f;u) = \sum_{k=0}^{n} p_{n,k}^{(\alpha)}(u) f\left(\frac{k}{n}\right), \tag{1}$$

where $p_{1,0}^{(\alpha)}(u) = 1-u$, $p_{1,1}^{(\alpha)}(u) = u$ and for $n \geq 2$,

$$p_{n,k}^{(\alpha)}(u) = \left[\binom{n-2}{k}(1-\alpha)u + \binom{n-2}{k-2}(1-\alpha)(1-u) + \binom{n}{k}\alpha u(1-u)\right] u^{k-1}(1-u)^{n-k-1}, \qquad u \in [0,1].$$

Simultaneously that year, Mohiuddine et al. [16] put forth the following Kantorovich form of the operators defined in (1):

$$K_{n,\alpha}(f;u) = (n+1)\sum_{k=0}^{n} p_{n,k}^{(\alpha)}(u) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt. \tag{2}$$

The Kantorovich variant is an important modification of linear positive operators that involves taking the integral of the operator rather than just the point-wise evaluation, which can lead to better approximation of integrable functions. Deo and Pratap [9]

[a]Department of Applied Mathematics, Delhi Technological University, India

conducted a thorough analysis of several auxiliary properties of the operators (2), which included examining the Voronovskaya-type asymptotic behaviour, the direct local approximation theorem and functions of bounded variation.

In the year 2018, Cai and Xu [4] extended the research by introducing a q-analog of the operators (1) and investigated some convexity and shape preserving properties, such as monoticity, with respect to $f(u)$. In a separate work, Pratap and Deo [18] proposed the q-analog of the Kantorovich form of the α-Bernstein operators, as defined in equation (2). They computed the moments and analyzed its convergence rate by using the concept of modulus of continuity.

One such pivotal advancement in this field was achieved by D.D. Stancu [19, 20], who utilized the Pólya-Eggenberger urn model and proposed a significant generalization of the Bernstein operators using contagion distribution [11]. The contagion distribution represents a straightforward procedure: An urn holds U red balls and V blue balls, and one ball is selected randomly from it. The colour of the ball is recorded and is substituted along with with X identical, same coloured balls. The aforementioned procedure is iterated m times. The occurrence of obtaining a red (or blue) ball in the j^{th} draw is represented by the random variable Y_j, which takes the value 1 (or 0) to indicate the event. The probability of observing a total of k red drawings, denoted as $k=\sum Y_j$, can be determined by the following expression:

$$\rho_{m,k} = {}^{m}C_k \frac{\prod_{i=0}^{k-1}(U+iX)\prod_{i=0}^{m-k-1}(V+iX)}{\prod_{i=0}^{m-1}(U+V+iX)}. \tag{3}$$

Several mathematicians studied the Bernstein and Baskakov operators with the help of (3) and gave their variants (see [6, 7, 8, 10, 12]). Recently, the α-Bernstein operators with respect to the contagion distribution [13] were introduced as,

$$\mathcal{P}_{m,\alpha}^{\langle\gamma\rangle}(h;u) = \sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) h\left(\frac{k}{m}\right), \tag{4}$$

where

$$\begin{aligned} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) = {}^{m-2}C_k\,(1-\alpha)\,&\frac{u^{[k,-\gamma]}(1-u)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}} \\ &+ {}^{m-2}C_{k-2}\,(1-\alpha)\,\frac{u^{[k-1,-\gamma]}(1-u)^{[m-k,-\gamma]}}{1^{[m-1,-\gamma]}} \\ &+ {}^{m}C_k\,\alpha\,\frac{u^{[k,-\gamma]}(1-u)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}} \end{aligned}$$

such that, $u^{[n,\mu]} = u(u-\mu)(u-2\mu)\cdots(u-(n-1)\mu)$, $u^{[0,\mu]}=1$, and ${}^{m}C_r$ is the binomial coefficient.

However, since discrete operators are not suitable for approximating functions which are not continuous, we introduce the Kantorovich type generalization, which helps to approximate integrable functions. The Kantorovich form of $\mathcal{P}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ is defined as,

$$K_{m,\alpha}^{\langle\gamma\rangle}(h;u) = (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}\int_{k/m+1}^{(k+1)/m+1} h(t)\,dt \tag{5}$$

where $h \in L^1[0,1]$, the class of Lebesgue integrable functions with the norm

$$\|h\|_{L^1} = \int_0^1 |h(t)|\,dt.$$

When $\gamma=0$, the Kantorovich operators described above gets simplified and reduced to the α-Bernstein-Kantorovich operators presented earlier in (2). This observation is a special case of a more general mathematical concept known as a limiting case. In particular, if $m\to\infty$ and $\gamma = O(1/m)$, then $\gamma\to 0$. Consequently, for each fixed h and u, the difference between the operators $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ and $K_{m,\alpha}(h;u)$ tends to zero as $m\to\infty$.

The operators (5) indeed have a complicated form, however taking γ of order $1/m$ reduces the complexities upto some extend. Contagion distribution is particularly important and applicable in various real-world scenarios where the probability of an event depends on its past occurrences. For instance, researchers in the field of bio-mathematics can approximate current and future population growth functions by studying the past environmental activities. Researchers in the field of applied mathematics can use this distribution to approximate functions and curves as per their requirements and availability of data.

2 Approximation Results

Lemma 2.1. *Let α be any fixed real number in $[0,1]$ and $\gamma = O(1/m)$. Then for $0 \le u \le 1$,*

$$K_{m,\alpha}^{\langle\gamma\rangle}(h;u) = \frac{1}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\int_0^1 s^{\frac{u}{\gamma}-1}(1-s)^{\frac{1-u}{\gamma}-1}K_{m,\alpha}(h;s)\,ds,$$

where $K_{m,\alpha}(h;u)$ is the Kantorovich form of α-Bernstein operators (2).

Proof. The beta function $\beta(\mu,\nu)$ is described as:

$$\beta(\mu,\nu) = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)},$$

where $\Gamma(\alpha)$ represents the gamma function. Moreover, by the characteristics of gamma function we can write,

$$\Gamma(m+\alpha) = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+m-1)\Gamma(\alpha).$$

Thus,

$$\begin{aligned}
&\beta\left(\theta+\frac{u}{\gamma},\pi+\frac{1-u}{\gamma}\right)\\
&=\frac{\Gamma\left(\theta+\frac{u}{\gamma}\right)\Gamma\left(\pi+\frac{1-u}{\gamma}\right)}{\Gamma\left(\theta+\pi+\frac{1}{\gamma}\right)}\\
&=\frac{\frac{u}{\gamma}\left(\frac{u}{\gamma}+1\right)\ldots\left(\frac{u}{\gamma}+\theta-1\right)\Gamma\left(\frac{u}{\gamma}\right)\times\frac{1-u}{\gamma}\left(\frac{1-u}{\gamma}+1\right)\ldots\left(\frac{1-u}{\gamma}+\pi-1\right)\Gamma\left(\frac{1-u}{\gamma}\right)}{\frac{1}{\gamma}\left(\frac{1}{\gamma}+1\right)\ldots\left(\frac{1}{\gamma}+\theta+\pi-1\right)\Gamma\left(\frac{1}{\gamma}\right)}\\
&=\frac{u(u+\gamma)\ldots(u+(\theta-1)\gamma)(1-u)(1-u+\gamma)\ldots(1-u+(\pi-1)\gamma)}{(1+\gamma)\ldots(1+(\theta+\pi-1)\gamma)}\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)\\
&=\frac{u^{[\theta,-\gamma]}(1-u)^{[\pi,-\gamma]}}{1^{[\theta+\pi,-\gamma]}}\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right). \qquad (6)
\end{aligned}$$

By substituting the expression for the Kantorovich form of α-Bernstein operators (2) in the corresponding equation, and by utilizing the relation given by equation (6), we arrive at,

$$\begin{aligned}
&\frac{1}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\int_0^1 s^{\frac{u}{\gamma}-1}(1-s)^{\frac{1-u}{\gamma}-1}K_{m,\alpha}(h;s)\,ds\\
&=\frac{(m+1)}{\beta\left(\frac{u}{\gamma},\frac{1-u}{\gamma}\right)}\sum_{k=0}^{m}\Bigg[{}^{m-2}C_k(1-\alpha)\beta\left(k+\frac{u}{\gamma},m-k-1+\frac{1-u}{\gamma}\right)\\
&\quad+{}^{m-2}C_{k-2}(1-\alpha)\beta\left(k-1+\frac{u}{\gamma},m-k+\frac{1-u}{\gamma}\right)\\
&\quad+{}^{m}C_k\,\alpha\,\beta\left(k+\frac{u}{\gamma},m-k+\frac{1-u}{\gamma}\right)\Bigg]\int_{k/m+1}^{(k+1)/m+1}h(s)\,ds\\
&=(m+1)\sum_{k=0}^{m}\Bigg[{}^{m-2}C_k\frac{(1-\alpha)u^{[k,-\gamma]}(1-u)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}\\
&\quad+{}^{m-2}C_{k-2}\frac{(1-\alpha)u^{[k-1,-\gamma]}(1-u)^{[m-k,-\gamma]}}{1^{[m-1,-\gamma]}}\\
&\quad+{}^{m}C_k\frac{\alpha u^{[k,-\gamma]}(1-u)^{[m-k,-\gamma]}}{1^{[m,-\gamma]}}\Bigg]\int_{k/m+1}^{(k+1)/m+1}h(s)\,ds\\
&=(m+1)\sum_{k=0}^{m}p_{m,k}^{\langle\alpha,\gamma\rangle}\int_{k/m+1}^{(k+1)/m+1}h(s)\,ds\\
&=K_{m,\alpha}^{\langle\gamma\rangle}(h;u).
\end{aligned}$$

□

Lemma 2.2. *For the Kantorovich form (5), we get the following identities:*

(i) $K^{\langle\gamma\rangle}_{m,\alpha}(e_0;u)=1$

(ii) $K^{\langle\gamma\rangle}_{m,\alpha}(e_1;u)=\frac{m}{m+1}u+\frac{1}{2(m+1)}$

(iii) $K^{\langle\gamma\rangle}_{m,\alpha}(e_2;u)=\frac{m^2-m-2(1-\alpha)}{(m+1)^2}\left(\frac{u+\gamma}{1+\gamma}\right)u+\frac{2m+2(1-\alpha)}{(m+1)^2}u+\frac{1}{3(m+1)^2}$.

where $e_r=t^r$ *for* $r=0,1,2$.

From Lemma 2.2, we can say that, as $m\to\infty$, $K^{\langle\gamma\rangle}_{m,\alpha}(e_r;u)\to e_r$ for $r=0,1,2$ and $\gamma\to 0$. Using the Bohman-Korovkin Theorem [2, 14], we can hence state that for a Lebesgue integrable function h, the sequence of Kantorovich operators $K^{\langle\gamma\rangle}_{m,\alpha}(h;u)$ converges uniformly to h. That is,

$$\lim_{m\to\infty}K^{\langle\gamma\rangle}_{m,\alpha}(h;u)=h.$$

Lemma 2.3. *With respect to the Kantorovich operators* $K^{\langle\gamma\rangle}_{m,\alpha}(h;u)$*, we can determine the following expressions for the central moments,*

$$\begin{aligned}
\mu_1(u)&=K^{\langle\gamma\rangle}_{m,\alpha}((t-u);u)\\
&=\frac{1-2u}{2(m+1)}\\
\mu_2(u)&=K^{\langle\gamma\rangle}_{m,\alpha}((t-u)^2;u)\\
&=\frac{m^2\gamma-\gamma+m-2\alpha+1}{(m+1)^2(1+\gamma)}u(1-u)+\frac{1}{3(m+1)^2}\\
\mu_4(u)&=K^{\langle\gamma\rangle}_{m,\alpha}((t-u)^4;u)\\
&=\frac{3m^2-4(2+3\alpha)m-(131-132\alpha)}{(m+1)^4}\left(\frac{u+3\gamma}{1+3\gamma}\right)\left(\frac{u+2\gamma}{1+2\gamma}\right)\left(\frac{u+\gamma}{1+\gamma}\right)u\\
&\quad-\frac{3m^2-4(2+3\alpha)m-(131-132\alpha)}{(m+1)^4}\left(\frac{u+2\gamma}{1+2\gamma}\right)\left(\frac{u+\gamma}{1+\gamma}\right)u\\
&\quad+\frac{3m^2-(13+12\alpha)m-2(80-81\alpha)}{(m+1)^4}\left(\frac{u+\gamma}{1+\gamma}\right)u+\frac{5m+(33-32\alpha)}{(m+1)^4}u\\
&\quad+\frac{1}{5(m+1)^4}.
\end{aligned}$$

Theorem 2.4. *Let* $m\geq 2$*,* $\alpha\in[0,1]$*, and let* $\gamma=O\left({}^1/_m\right)$*. Then for every* $h\in L^1[0,1]$*, we have*

$$\left\|K^{\langle\gamma\rangle}_{m,\alpha}(h)\right\|_{L^1}\ \leq\ C_m\,\|h\|_{L^1},$$

where

$$C_m=1-(1-\alpha)\frac{2}{m(m-1)}=\alpha+(1-\alpha)\frac{(m+1)(m-2)}{m(m-1)}.$$

In particular, $C_m<1$ *for every* $m\geq 3$ *and* $\alpha<1$*, while* $C_m=1$ *when* $\alpha=1$*.*

Proof. Let us denote the contagion distribution function by

$$b^{\langle\gamma\rangle}_{m,k}(u)=\frac{u^{[k,-\gamma]}(1-u)^{[m-k-1,-\gamma]}}{1^{[m-1,-\gamma]}}.$$

From the properties of Pólya-Eggenberger distribution, we can say that for each fixed $m\geq 0$ the functions $\{b^{\langle\gamma\rangle}_{m,k}\}_{k=0}^{m}$ are non-negative on $[0,1]$ and form a partition of unity, that is,

$$\sum_{k=0}^{m}b^{\langle\gamma\rangle}_{m,k}(u)=1\quad\text{for all } u\in[0,1]. \tag{7}$$

Moreover, these functions are symmetric in their domain, that is,

$$b^{\langle\gamma\rangle}_{m,k}(u)=b^{\langle\gamma\rangle}_{m,m-k}(1-u)\quad\text{for } 0\leq k\leq m \text{ and } u\in[0,1]. \tag{8}$$

From Lemma 2.2, $p^{\langle\alpha,\gamma\rangle}_{m,k}(u)\geq 0$ and $\sum_{k=0}^{m}p^{\langle\alpha,\gamma\rangle}_{m,k}(u)=1$ for all $u\in[0,1]$. Thus for every u, we have

$$\left|K^{\langle\gamma\rangle}_{m,\alpha}(h;u)\right|\ \leq\ (m+1)\sum_{k=0}^{m}p^{\langle\alpha,\gamma\rangle}_{m,k}(u)\int_{k/(m+1)}^{(k+1)/(m+1)}|h(t)|\,dt.$$

Integrating with respect to u and using the norm of $L^1[0,1]$, we get

$$\begin{aligned}\|K_{m,\alpha}^{\langle\gamma\rangle}(h)\|_{L^1} &\leq (m+1)\sum_{k=0}^{m}\left(\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\,du\right)\left(\int_{k/(m+1)}^{(k+1)/(m+1)} |h(t)|\,dt\right)\\ &\leq (m+1)\Big[\max_{0\leq k\leq m}\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\,du\Big]\,\|h\|_{L^1}\\ &=: C_m\,\|h\|_{L^1}. \qquad (9)\end{aligned}$$

Now, let us define

$$I_{m,k} := \int_0^1 b_{m,k}^{\langle\gamma\rangle}(u)\,du \quad \text{for } 0\leq k\leq m.$$

From equation (7), we get

$$\sum_{k=0}^{m} I_{m,k} = 1. \qquad (10)$$

Using equation (8) and the change of variables $v = 1-u$, we obtain $I_{m,k} = I_{m,m-k}$ for all k. We will now show that all the $I_{m,k}$ are equal. Consider the moments of the Stancu operators introduced by D.D. Stancu [7] using the Pólya-Eggenberger distribution. The second moment gives,

$$\sum_{k=0}^{m}\frac{k}{m}\,b_{m,k}^{\langle\gamma\rangle}(u) = u \quad \text{for } u\in[0,1],\ m\geq 1. \qquad (11)$$

Integrating (11) over $u\in[0,1]$ yields

$$\sum_{k=0}^{m} k\,I_{m,k} = \frac{m}{2}. \qquad (12)$$

Similarly, using the third moment identity

$$\sum_{k=0}^{m}\frac{k(k-1)}{m(m-1)}\,b_{m,k}^{\langle\gamma\rangle}(u) = u^2, \qquad (13)$$

and integrating, we get

$$\sum_{k=0}^{m} k(k-1)\,I_{m,k} = \frac{m(m-1)}{3}. \qquad (14)$$

Now fix m and consider the second central moment of the weights $I_{m,k}$ about $m/2$, we get

$$\begin{aligned}\Sigma_m &:= \sum_{k=0}^{m}\left(k-\frac{m}{2}\right)^2 I_{m,k}\\ &= \sum_{k=0}^{m} k^2 I_{m,k} - m\sum_{k=0}^{m} kI_{m,k} + \frac{m^2}{4}\sum_{k=0}^{m} I_{m,k}.\end{aligned}$$

Using (10), (12) and (14) gives

$$\begin{aligned}\Sigma_m &= \frac{m(m-1)}{3} + \frac{m}{2} - \frac{m^2}{2} + \frac{m^2}{4}\\ &= \frac{m(m+2)}{12}. \qquad (15)\end{aligned}$$

Consider,

$$\sum_{k=0}^{m}\left(k-\frac{m}{2}\right)^2\frac{1}{m+1} = \frac{m(m+2)}{12}. \qquad (16)$$

Since the two central moments (15) and (16) are equal, the only possibility is that

$$I_{m,0} = I_{m,1} = \cdots = I_{m,m} = \frac{1}{m+1}. \qquad (17)$$

Since $I_{m,k} := \int_0^1 b_{m,k}^{\langle\gamma\rangle}(u)\,du = \frac{1}{m+1}$, we can write

$$\int_0^1 b_{m-1,k}^{\langle\gamma\rangle}(u)\,du = \frac{1}{m} \quad \text{and} \quad \int_0^1 b_{m,k}^{\langle\gamma\rangle}(u)\,du = \frac{1}{m+1}.$$

Thus, we get

$$\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\,du = (1-\alpha)\left(\frac{m-1-k}{m(m-1)}+\frac{k-1}{m(m-1)}\right)+\frac{\alpha}{m+1}$$
$$= (1-\alpha)\frac{m-2}{m(m-1)}+\frac{\alpha}{m+1},$$

which is independent of k. Therefore, in equation (9),

$$C_m = (m+1)\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\,du = (m+1)\left[(1-\alpha)\frac{m-2}{m(m-1)}+\frac{\alpha}{m+1}\right] = 1-(1-\alpha)\frac{2}{m(m-1)}.$$

Substituting this value of C_m in equation (9) yields the desired result. □

Theorem 2.4 gives us an upper bound of the proposed Kantorovich operators $K_{m,\alpha}^{\langle\gamma\rangle}(h)$ for $h \in L^1[0,1]$. Next we derive a Lemma and the Voronskaya type result of these operators, but for $h \in C[0,1]$.

Lemma 2.5. *Let $h \in C[0,1]$ and $0 \le \alpha \le 1$. Then $\left|K_{m,\alpha}^{\langle\gamma\rangle}(h;u)\right| \le \|h\|$, where $\|.\|$ is the supremum norm of a function defined as $\|h\| = \sup_{s\in[0,1]} |h(s)|$.*

Proof. From (5), we have,

$$\left|K_{m,\alpha}^{\langle\gamma\rangle}(h;u)\right| = \left|(m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\int_{k/m+1}^{(k+1)/m+1} h(s)ds\right|$$
$$\le \|h\|\,(m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\int_{k/m+1}^{(k+1)/m+1} ds$$
$$= \|h\|.$$

□

Theorem 2.6. *Let $h(u)$ be bounded on $[0,1]$ and $0 \le \alpha \le 1$. Then for any $u \in [0,1]$ and $\gamma = O(1/m)$ at which $h''(u)$ exists,*

$$\lim_{m\to\infty} m\left[K_{m,\alpha}^{\langle\gamma\rangle}(h(t);u)-h(u)\right] = \left(\frac{1-2u}{2}\right)h'(u)+\frac{u(1-u)}{2(1+\gamma)}h''(u).$$

Proof. By Taylor's theorem, there exist ξ between u and t such that,

$$h(t) = h(u)+(t-u)h'(u)+\frac{(t-u)^2}{2!}h''(u)+\mathfrak{z}(t,u)(t-u)^2,$$

where $\mathfrak{z}(t,u) = \frac{h''(\xi)-h''(u)}{2} \to 0$ as $t \to u$. Hence,

$$\lim_{m\to\infty} m\left[K_{m,\alpha}^{\langle\gamma\rangle}(h(t)-h(u);u)\right] = h'(u)\lim_{m\to\infty} mK_{m,\alpha}^{\langle\gamma\rangle}((t-u);u)$$
$$+\frac{h''(u)}{2}\lim_{m\to\infty} mK_{m,\alpha}^{\langle\gamma\rangle}\left((t-u)^2;u\right)$$
$$+\lim_{m\to\infty} mK_{m,\alpha}^{\langle\gamma\rangle}\left(\mathfrak{z}(t,u)(t-u)^2;u\right).$$

Claim, $mK_{m,\alpha}^{\langle\gamma\rangle}\left(\mathfrak{z}(t,u)(t-u)^2;u\right) \to 0$ as $m \to \infty$.
For every $\varepsilon > 0$, we consider $\delta > 0$ such that $\mathfrak{z}(t,u) < \varepsilon$ for $|t-u| < \delta$.
And for $|t-u| \ge \delta$, we can say $\mathfrak{z}(t,u)$ is bounded above, say by M. Thus,

$$mK_{m,\alpha}^{\langle\gamma\rangle}\left(\mathfrak{z}(t,u)(t-u)^2;u\right) \le \varepsilon m\mu_2(u)+\frac{M}{\delta^2}m\mu_4(u).$$

where, $\mu_r = K_{m,\alpha}^{\langle\gamma\rangle}((t-u)^r;u)$. Hence our claim is proved.

From Lemma 2.3 we have,

$$\lim_{m\to\infty} m\left[K_{m,\alpha}^{\langle\gamma\rangle}(h(t)-h(u);u)\right]$$
$$= \lim_{m\to\infty} m\left[K_{m,\alpha}^{\langle\gamma\rangle}(h(t);u)-h(u)\right]$$
$$= \lim_{m\to\infty} m\left(\frac{1-2u}{2(m+1)}\right)h'(u)+\lim_{m\to\infty} m\left(\frac{m-2\alpha+1}{(m+1)^2(1+\gamma)}u(1-u)+\frac{1}{3(m+1)^2}\right)\frac{h''(u)}{2}$$
$$= \left(\frac{1-2u}{2}\right)h'(u)+\frac{u(1-u)}{2(1+\gamma)}h''(u).$$

□

Now we will consider some error estimates for the proposed Kantorovich operators using moduli of continuity.

Definition 2.1 (Integral Modulus of Continuity). Let $h \in L^1[0,1]$. The integral modulus of continuity of h (also called the L^1-modulus or Steklov modulus) is defined by

$$\omega_1(h;\delta) := \sup_{|r|\le\delta} \int_0^1 |h(t+r)-h(t)|\,dt, \qquad \delta>0.$$

Remark 1. The integral modulus of continuity, $\omega_1(h;\delta)$ of $h(u)$ is non-decreasing and satisfies the following properties:

- $\omega_1(h;\delta+\rho) \le \omega_1(h;\delta)+\omega_1(h;\rho)$, and
- $\omega_1(h;\delta) \to 0$ as $\delta \to 0^+$.

Theorem 2.7. *Let $h \in L^1[0,1]$, $0 \le \alpha \le 1$, and $\gamma = O({}^1/_m)$. Then for every $m \ge 2$,*

$$\|K_{m,\alpha}^{\langle\gamma\rangle}(h)-h\|_{L^1} \le 2\,\omega_1\Big(h;\ \frac{1}{2(m+1)}+\sqrt{\overline{\mu}_2}\Big),$$

where $\omega_1(h,\delta)$ is the integral modulus of continuity of h, and

$$\mu_2(u) = (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\int_{k/(m+1)}^{(k+1)/(m+1)} (s-u)^2\,ds, \qquad \overline{\mu}_2 = \int_0^1 \mu_2(u)\,du.$$

Proof. From the definition of the operators,

$$K_{m,\alpha}^{\langle\gamma\rangle}(h;u) = (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\int_{k/(m+1)}^{(k+1)/(m+1)} h(s)\,ds.$$

Then for all $u \in [0,1]$, we have

$$\|K_{m,\alpha}^{\langle\gamma\rangle}(h)-h\|_{L^1} \le (m+1)\sum_{k=0}^{m}\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\int_{k/(m+1)}^{(k+1)/(m+1)} |h(s)-h(u)|\,ds\,du.$$

Now, using the triangle inequality, we can write

$$|h(s)-h(u)| \le |h(s)-h(c_k)|+|h(c_k)-h(u)|,$$

where $c_k = \frac{2k+1}{2(m+1)}$. Thus,

$$\left\|K_{m,\alpha}^{\langle\gamma\rangle}(h)-h\right\|_{L^1} \le A+B, \tag{18}$$

where

$$A = (m+1)\sum_{k=0}^{m}\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\int_{k/(m+1)}^{(k+1)/(m+1)} |h(s)-h(c_k)|\ ds\,du,$$

and

$$B = (m+1)\sum_{k=0}^{m}\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\int_{k/(m+1)}^{(k+1)/(m+1)} |h(c_k)-h(u)|\ ds\,du.$$

For each k, since $|s-c_k| \le \frac{1}{2(m+1)}$ for all $s \in \big[\frac{k}{m+1},\frac{k+1}{m+1}\big]$, we obtain

$$\int_{k/(m+1)}^{(k+1)/(m+1)} |h(s)-h(c_k)|\ ds \le \frac{1}{m+1}\,\omega_1\Big(h;\frac{1}{2(m+1)}\Big).$$

Using $\sum_{k=0}^{m}\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\,du = 1$, we get

$$A \le \omega_1\Big(h;\frac{1}{2(m+1)}\Big).$$

Moreover, using Remark 1, for any fixed u,

$$|h(c_k)-h(u)| \le \omega_1(h;|c_k-u|).$$

Hence,

$$B \le (m+1)\sum_{k=0}^{m}\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\int_{k/(m+1)}^{(k+1)/(m+1)} \omega_1(h;|c_k-u|)\,ds\,du$$
$$\le \sum_{k=0}^{m}\int_0^1 p_{m,k}^{\langle\alpha,\gamma\rangle}(u)\,\omega_1(h;|c_k-u|)\ du.$$

Since, $\int_0^1 \sqrt{\mu_2(u)}\,du \le \sqrt{\int_0^1 \mu_2(u)\,du} = \sqrt{\overline{\mu}_2}$, and applying the Cauchy-Schwarz inequality, we arrive at

$$B \le \int_0^1 \omega_1\left(h; \sqrt{\mu_2(u)}\right) du \le \omega_1\left(h; \sqrt{\overline{\mu}_2}\right).$$

Finally, we obtain

$$\begin{aligned}\left\|K_{m,\alpha}^{\langle\gamma\rangle}(h) - h\right\|_{L^1} &\le \omega_1\left(h; \frac{1}{2(m+1)}\right) + \omega_1\left(h; \sqrt{\overline{\mu}_2}\right)\\ &\le 2\,\omega_1\left(h; \frac{1}{2(m+1)} + \sqrt{\overline{\mu}_2}\right).\end{aligned}$$

This completes the proof. □

Definition 2.2 (Usual Modulus of Continuity)**.** Let $h \in C[0,1]$. The modulus of continuity of h is defined as

$$\omega(h;\delta) := \sup_{\substack{u,s\in[0,1]\\ |u-s|\le\delta}} |h(u) - h(s)|, \qquad \delta > 0.$$

It measures the uniform continuity of h on $[0,1]$ and is non-decreasing in δ.

Remark 2. For modulus of continuity, $\omega(\delta)$ of $h(u)$ on the interval $[a,b]$ the following property holds: If $\rho > 0$, then $\omega(\rho\delta) \le (1+\rho)\,\omega(\delta)$.

Theorem 2.8. *Consider a continuous function h on $[0,1]$. For $\alpha \in [0,1]$ and $\gamma = O\left({}^1/_m\right)$ we have:*

$$\left|K_{m,\alpha}^{\langle\gamma\rangle}(h(t);u) - h(u)\right| \le 2\omega(h, \delta(u)),$$

where $\omega(h;\delta)$ is the modulus of continuity of h with $\delta(u) = (\mu_2(u))^{1/2}$.

Proof. Using Remark 2 and the definition of operators (5), we have,

$$\begin{aligned}
&\left|K_{m,\alpha}^{\langle\gamma\rangle}(h(s);u) - h(u)\right|\\
&\le (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} |h(s) - h(u)|\,ds\\
&\le (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} \omega(h;|s-u|)\,ds\\
&\le (m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} \left(1 + \frac{1}{\delta}|s-u|\right)\omega(h;\delta)\,ds\\
&\le \left[1 + \frac{1}{\delta}(m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} |s-u|\,ds\right]\omega(h;\delta)\\
&\le \left[1 + \frac{1}{\delta}(m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \left(\int_{k/m+1}^{(k+1)/m+1} 1^2 ds\right)^{\frac{1}{2}} \left(\int_{k/m+1}^{(k+1)/m+1} |s-u|^2 ds\right)^{\frac{1}{2}}\right]\omega(h;\delta)\\
&\le \left[1 + \frac{1}{\delta}\left((m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} ds\right)^{\frac{1}{2}}\right.\\
&\qquad \left.\times\left((m+1)\sum_{k=0}^{m} p_{m,k}^{\langle\alpha,\gamma\rangle}(u) \int_{k/m+1}^{(k+1)/m+1} (s-u)^2 ds\right)^{\frac{1}{2}}\right]\omega(h;\delta)\\
&= \left[1 + \frac{1}{\delta}\left(K_{m,\alpha}^{\langle\gamma\rangle}(1;u)\right)^{\frac{1}{2}}\left(K_{m,\alpha}^{\langle\gamma\rangle}\left((s-u)^2;u\right)\right)^{\frac{1}{2}}\right]\omega(h;\delta)\\
&= \left[1 + \frac{1}{\delta}(\mu_2(u))^{\frac{1}{2}}\right]\omega(h;\delta)\\
&= 2\omega(h;\delta); \qquad\qquad \text{where } \delta = (\mu_2(u))^{\frac{1}{2}}.
\end{aligned}$$

□

3 Genuine-Type Modification

Sticking to the assumption that $K_{m,\alpha}^{\langle\gamma\rangle}(e_1;u)=u$, we can define a new modification as:

$$\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)=(m+1)\sum_{k=0}^{m}p_{m,k}^{\langle\alpha,\gamma\rangle}(r_m(u))\int_{k/m+1}^{(k+1)/m+1}h(s)ds, \tag{19}$$

where,

$$r_m(u)=\left(\frac{m+1}{m}\right)u-\frac{1}{2m}.$$

We will now find the moments of this new modification; the calculation and simplification of which is left for the reader. For ease, let us denote the central moments of (19) with $\tilde{\mu}_1$ and $\tilde{\mu}_2$.

Lemma 3.1. *Let $\alpha\in[0,1]$ and $\gamma=O(1/m)\in[0,1]$. Then the moments of $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ are as follows:*

(i) $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(1;u)=1$

(ii) $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(t;u)=u$

(iii) $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}\left(t^2;u\right)=\frac{1}{(m+1)(1+\gamma)}\left[(m-2)(r_m(u))^2+(m\gamma+2)r_m(u)\right]$
$-\frac{2\alpha}{(m+1)^2(1+\gamma)}r_m(u)(1-r_m(u))+\frac{1}{3(m+1)^2}$

(iv) $\tilde{\mu}_1=0$

(v) $\tilde{\mu}_2=\frac{1}{1+\gamma}\left[\frac{(m-2)(m+1)}{m^2}u^2-\frac{m-2}{m^2}u+\frac{m-2}{4m^2(m+1)}+\frac{m\gamma+2}{m}u-\frac{m\gamma+2}{2m(m+1)}\right]$
$-\frac{2\alpha}{1+\gamma}\left[\frac{u}{m(m+1)}-\frac{1}{2m(m+1)^2}-\frac{u^2}{m^2}+\frac{u}{m^2(m+1)}-\frac{1}{4m^2(m+1)^2}\right]+\frac{1}{3(m+1)^2}-u^2$

The moments and central moments of the previously defined Kantorovich operators (5) can be utilized to demonstrate the proof of the aforementioned lemma. Building upon this, we can establish the Voronovskya-type result for the newly modified operators $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ as well.

Theorem 3.2. *Let $h(u)$ be bounded on $[0,1]$ and $0\le\alpha\le1$. Then for any $u\in[0,1]$ and $\gamma=O(1/m)$ at which $h''(u)$ exists,*

$$\lim_{m\to\infty}m\left[\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h(t);u)-h(u)\right]=\frac{1}{2(1+\gamma)}cu(1-u)h''(u),$$

where $c=\lim_{m\to\infty}(m\gamma+1)$.

Proof. Following the procedure as done in Theorem 2.6, we arrive at the following:

$$\lim_{m\to\infty}m\left[\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h(t)-h(u));u\right]=h'(u)\lim_{m\to\infty}m\tilde{\mu}_1+\frac{h''(u)}{2}\lim_{m\to\infty}m\tilde{\mu}_2+\lim_{m\to\infty}m\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}\left(\wr(t,u)(t-u)^2;u\right),$$

where $m\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}\left(\wr(t,u)(t-u)^2;u\right)\to0$ as $m\to\infty$. □

$$\begin{aligned}\lim_{m\to\infty}m\tilde{\mu}_2&=\frac{1}{1+\gamma}\left[\left(-1-\frac{2}{m}-m\gamma\right)u^2+\left(m\gamma+\frac{2}{m}+1\right)u+\frac{m-2}{4m(m+1)}-\frac{m\gamma+2}{2(m+1)}\right]\\&\quad-\frac{2\alpha}{1+\gamma}\left[\frac{u}{m+1}-\frac{1}{2(m+1)^2}-\frac{u^2}{m}+\frac{u}{m(m+1)}-\frac{1}{4m(m+1)^2}\right]+\frac{m}{3(m+1)^2}\\&=\frac{1}{1+\gamma}cu(1-u),\end{aligned}$$

where $c=\lim_{m\to\infty}(m\gamma+1)$.

Thus,

$$\lim_{m\to\infty}m\left[\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h(t)-h(u);u)\right]=\frac{1}{2(1+\gamma)}cu(1-u)h''(u).$$

4 Graphical Illustrations

In this section, we demonstrate the approximation properties of the Kantorovich operators (5) and their genuine-type modification (19), based on the parameters α and γ, when approximating a continuous function $h(u)$. To better understand these concepts, numerical illustrations are provided that demonstrate the performance of the two operators. Furthermore, this section presents graphically, the error of approximation for each operators, allowing for a more comprehensive understanding of their relative approximation.

Now, from the moments of the proposed Kantorovich operators, given in Lemma 2.2, we claimed that for the operators to uniformly converge to the desired function, γ should tend to 0 as m tends to ∞. We verify this graphically in the subsequent subsection.

4.1 Effect of parameter γ

In this subsection, we investigate the influence of the Pólya parameter γ on the behaviour of the proposed Kantorovich operator $K_{m,\alpha}^{\langle\gamma\rangle}(h;x)$ for a fixed value of α and m. The parameter γ plays a crucial role in shaping the generalized factorial terms appearing in the definition of the operator, and thereby controls the skewness and spread of the associated basis functions.

To illustrate this dependence, we consider the piece-wise continuous function

$$h(u) = \begin{cases} u, & 0 \le u < 0.6 \\ 1-u, & 0.6 \le u < 0.8 \\ 0.2, & 0.8 \le u \le 1, \end{cases}$$

and compute $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ for a fixed value of $m = 200$ and $\alpha = 0.8$, while varying $\gamma \in \{0.001, 0.01, 0.1\}$.

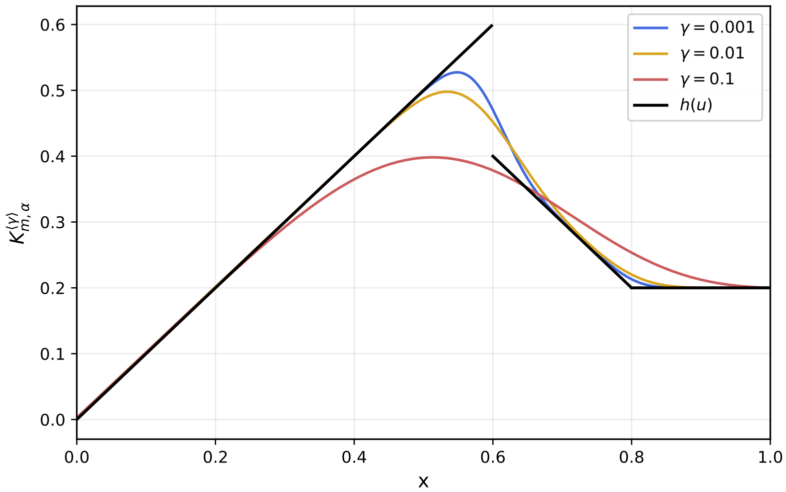

Figure 1: Effect of γ on the approximation process, with $\alpha = 0.8$ and $m = 200$.

From Figure 1 we can conclude that for smaller values of γ, the approximation follows the desired function, $h(u)$ more closely and for $\gamma = 0$, $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ will coincide with the classical α-Kantorovich-Bernstein form.

This confirms that γ acts as a shape control parameter. In other words, increasing its value leads to poorer approximations, whereas smaller γ values yield more accurate estimation of the desired function.

4.2 Illustrations

Illustration 1. The error functions are defined as $E_{m,\alpha}^{\langle\gamma\rangle}(u) = \left|K_{m,\alpha}^{\langle\gamma\rangle}(h;u) - h(u)\right|$ and $\tilde{E}_{m,\alpha}^{\langle\gamma\rangle}(u) = \left|\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u) - h(u)\right|$. Let us consider the case when $\alpha = 0.8$ and $\gamma = 0.1$. For $h(u) = \sin 2\pi u + \cos 2\pi u$, Figure 2 illustrates the graphs of the operators $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ and the error function $E_{m,\alpha}^{\langle\gamma\rangle}(u)$ for different values of m, respectively. Likewise, we graph the operators $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ and the error function $\tilde{E}_{m,\alpha}^{\langle\gamma\rangle}(u)$ in Figure 3.

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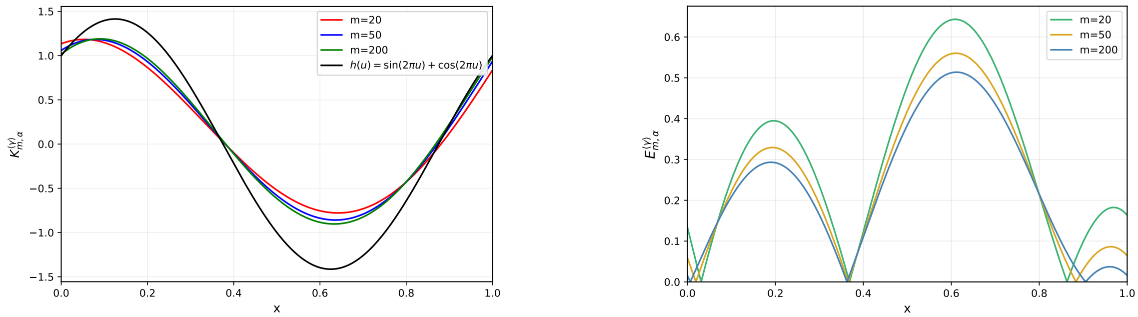
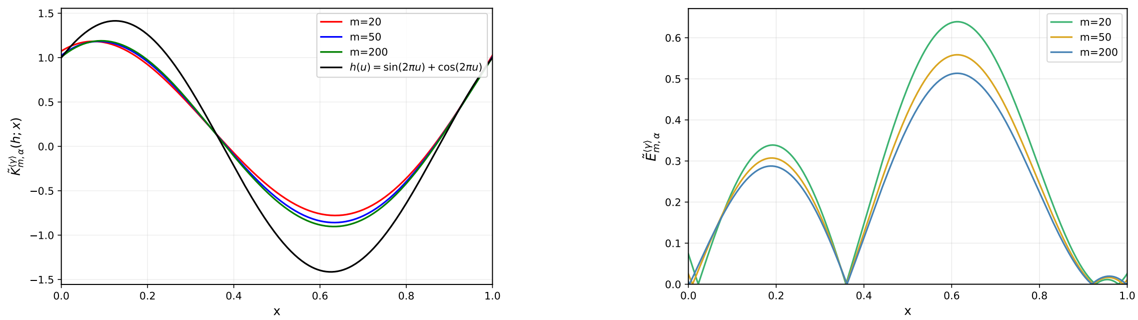


(a) Approximation Process

(b) Error Function

Figure 2: Kantorovich Operators, $K^{\langle\gamma\rangle}_{m,\alpha}(h;x)$

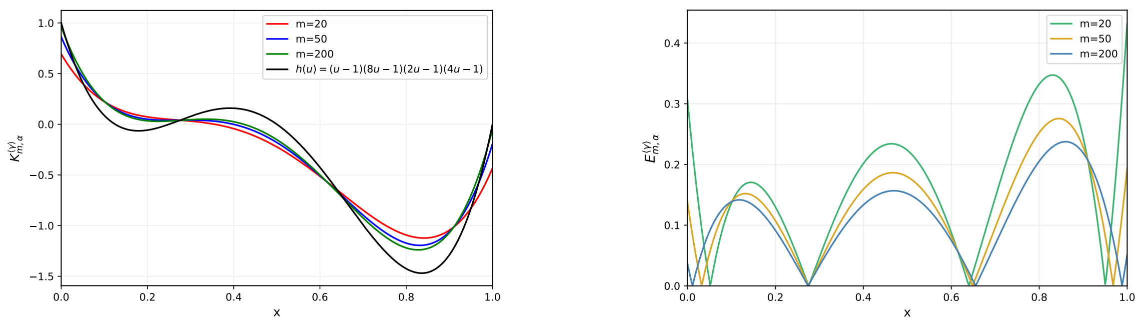


(a) Approximation Process

(b) Error Function

Figure 3: Genuine-type Modification, $\tilde{K}^{\langle\gamma\rangle}_{m,\alpha}(h;x)$

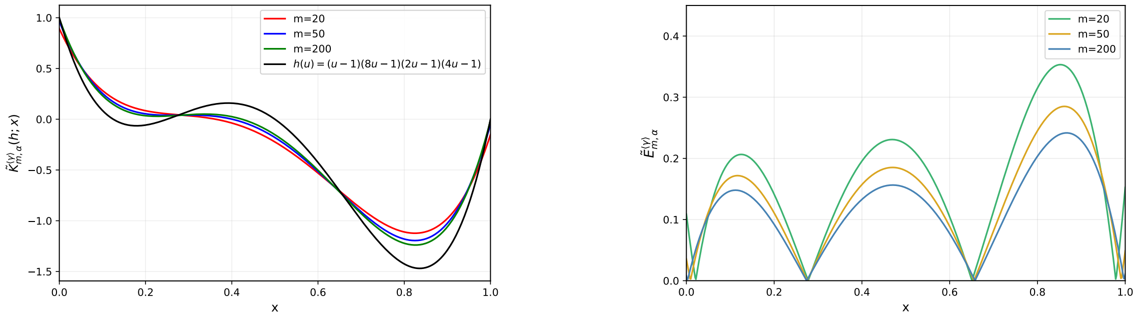
Illustration 2. Now, for $h(u) = (u-1)(8u-1)(2u-1)(4u-1)$ let $\alpha = 0.95$ and $\gamma = 0.05$. Let the error functions be the same as above. We will then look at four different graphs like illustrated above, for different values of m. (See Figures 4 and 5)



(a) Approximation Process

(b) Error Function

Figure 4: Kantorovich Operators, $K^{\langle\gamma\rangle}_{m,\alpha}(h;x)$

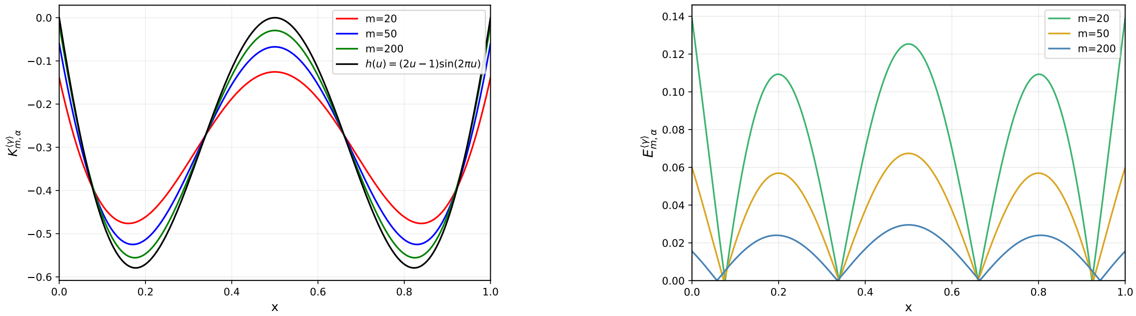


(a) Approximation Process

(b) Error Function

Figure 5: Genuine-type Modification, $\tilde{K}^{\langle\gamma\rangle}_{m,\alpha}(h;x)$

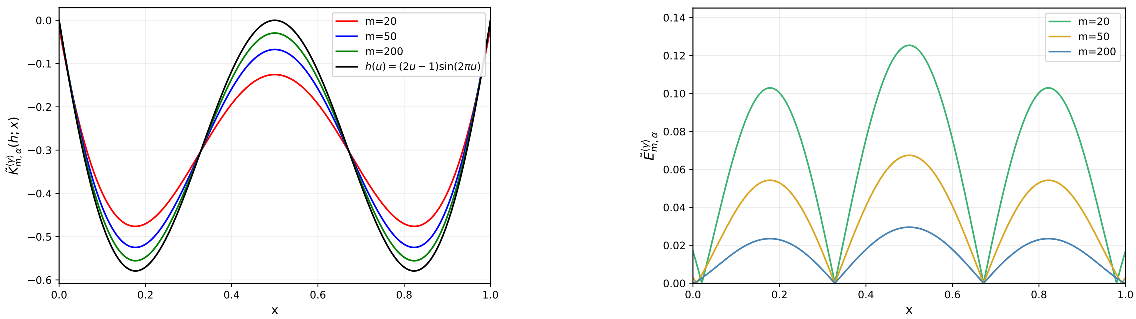
Illustration 3. Let $\alpha = 0.9$ and $\gamma = 0.005$. Figures 6 and 7 represent the approximation and error of the function $h(u) = (2u-1)\sin 2\pi u$ for the operators $K^{\langle\gamma\rangle}_{m,\alpha}(h;u)$ and its genuine-type modification $\tilde{K}^{\langle\gamma\rangle}_{m,\alpha}(h;u)$, respectively.



(a) Approximation Process

(b) Error Function

Figure 6: Kantorovich Operators, $K^{\langle\gamma\rangle}_{m,\alpha}(h;x)$

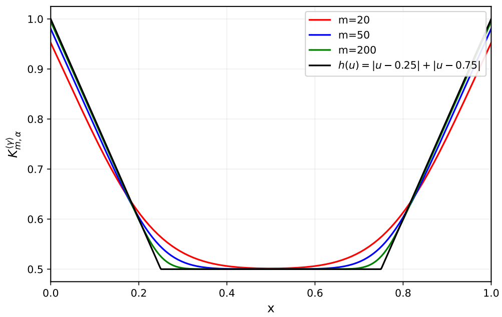


(a) Approximation Process

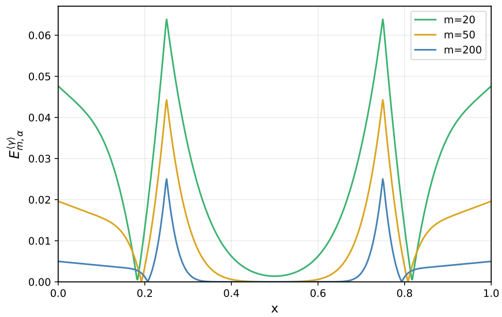
(b) Error Function

Figure 7: Genuine-type Modification, $\tilde{K}^{\langle\gamma\rangle}_{m,\alpha}(h;x)$

Illustration 4. Let $\alpha = 0.85$ and $\gamma = 0.001$. We approximate the non-differentiable function $h(u) = |u-0.25| + |u-0.75|$ on $u \in [0,1]$, represented graphically by Figures 8 and 9.

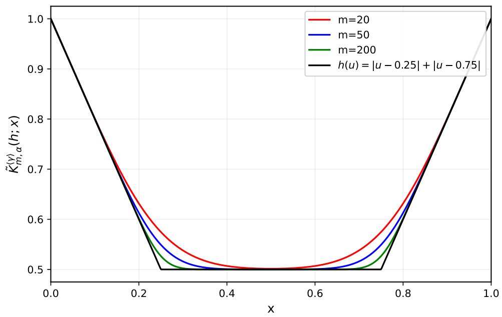


(a) Approximation Process

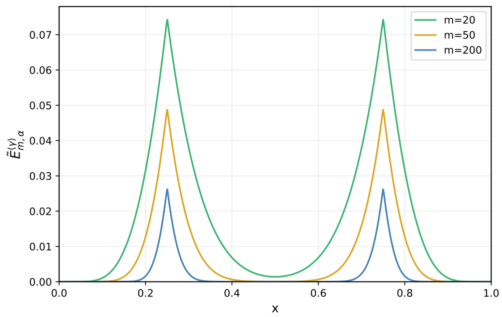


(b) Error Function

Figure 8: Kantorovich Operators, $K_{m,\alpha}^{\langle\gamma\rangle}(h;x)$



(a) Approximation Process



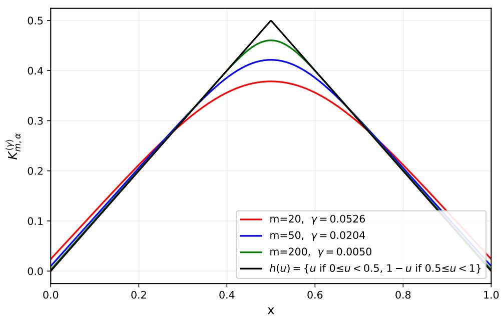
(b) Error Function

Figure 9: Genuine-type Modification, $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$

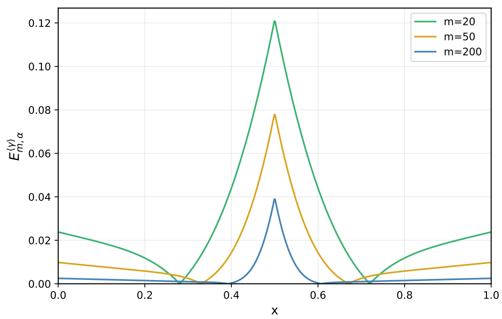
Illustration 5. Consider the piece-wise function

$$h(u) = \begin{cases} u, & 0 \le u < 0.5 \\ 1-u, & 0.5 \le u < 1, \end{cases}$$

for $\alpha = 0.75$ and $\gamma = 1/(m-1)$. Figures 10 and 11 represent the approximation and error of the function $h(u)$ for the operators $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ and its genuine-type modification $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$, respectively.

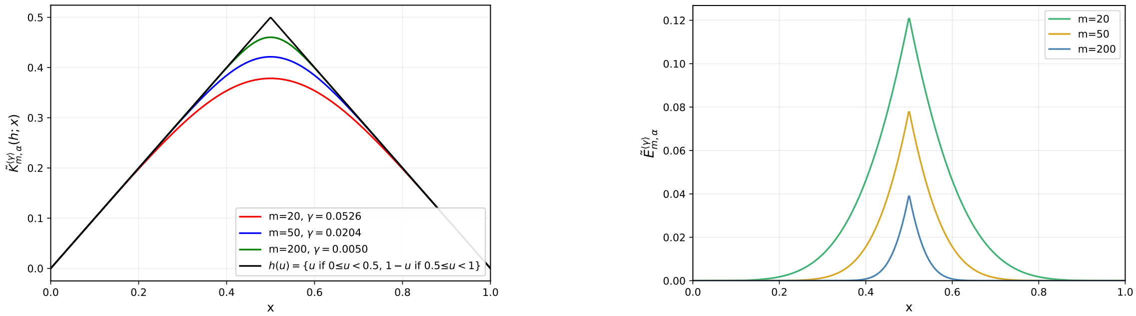


(a) Approximation Process



(b) Error Function

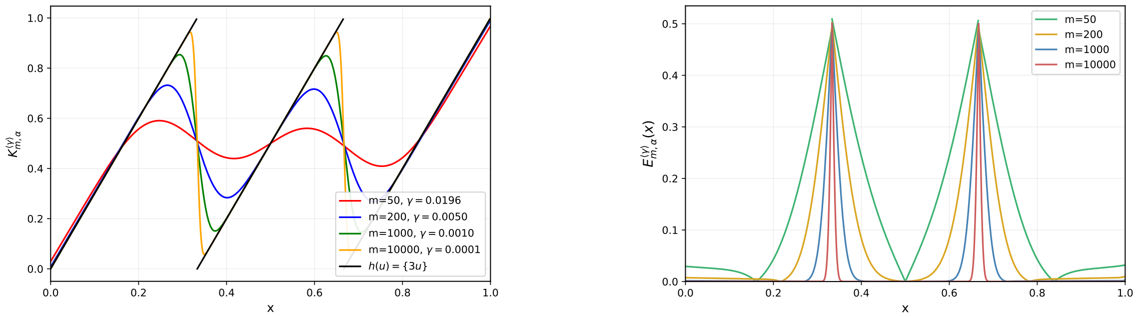
Figure 10: Kantorovich Operators, $K_{m,\alpha}^{\langle\gamma\rangle}(h;x)$



(a) Approximation Process **(b)** Error Function

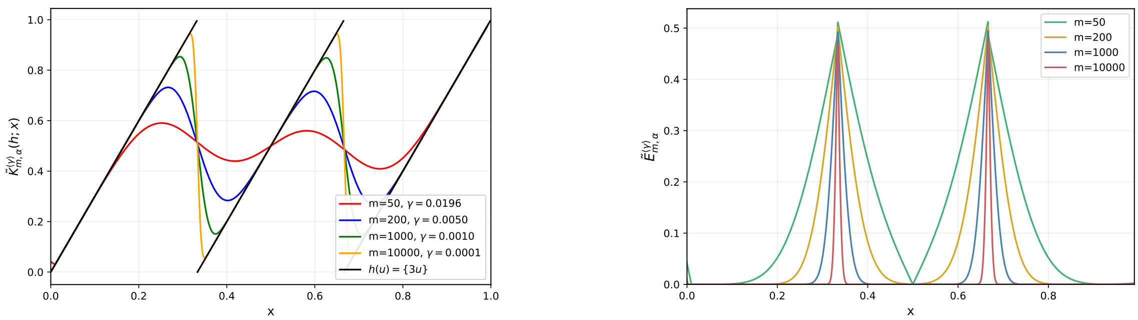
Figure 11: Genuine-type Modification, $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$

Illustration 6. Now, we take the non-continuous but integrable function $h(u) = \{3u\}$ for $u \in [0,1]$, where $\{.\}$ denotes the fractional-part function. Let $\alpha = 0.95$ and $\gamma = 1/(m+1)$. Then, Figures 12 and 13 represent the approximation and error of the function $h(u)$ for the operators $K_{m,\alpha}^{\langle\gamma\rangle}(h;u)$ and its genuine-type modification $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;u)$, respectively.



(a) Approximation Process **(b)** Error Function

Figure 12: Kantorovich Operators, $K_{m,\alpha}^{\langle\gamma\rangle}(h;x)$



(a) Approximation Process **(b)** Error Function

Figure 13: Genuine-type Modification, $\tilde{K}_{m,\alpha}^{\langle\gamma\rangle}(h;x)$

5 Conclusion

Throughout this paper, we have assumed that the parameter γ is of order $1/m$. The importance of this parameter in the approximation process can be seen from Lemma 2.2 and the Bohman-Korovkin Theorem, which is evident from the graphical illustration presented in Figure 1. The smaller the value of γ, the better the approximation. The asymptotic behaviour of the proposed operator is also established using a Voronovskaya type theorem and the modulus of continuity. These results are derived for the spaces of Lebesgue integrable functions and continuous functions, providing an explicit upper bound for the approximation error of the proposed operators.

To improve approximation, we have also introduced a modification to the Kantorovich operators, called the genuine-type modification. This modification has the advantage of giving the first central moment an absolute zero value, rather than approximating it to zero. As a result, the genuine-type modification provides better approximations for continuous functions near the end points $\{0,1\}$ than the basic Kantorovich operators (5). This difference is demonstrated through various graphical examples presented in this paper.

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