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Convergence of univariate multi-level Gaussian convolution

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Abstract

In this paper we consider the approximation of smooth univariate functions f using multilevel convolution with the Gaussian basis function whose initial width parameter, denoted by h, scales by one half at each stage of the multilevel algorithm. We restrict attention to approximating entire functions (restricted to the real line) and, in this setting, we are able to provide a precise closed form series expansion for the approximation error at each stage. For entire functions of exponential type $a > 0$ that are bounded on $\mathbb{R}$ we show that the approximation error at the p^{th} level is bounded by a multiple of the product $\prod_{k=0}^{p}(\exp(a^2h^2/2^{2k+1})-1)$. Using $f(x) = \sin(ax)$ as a prototype example we derive an analytic formula for the approximation error and use this to demonstrate the accuracy of the bounds. We derive similar results for functions of exponential type $a > 0$ that are unbounded on $\mathbb{R}$. Here we use $f(x) = \exp(ax)$ as the prototype example. The bounds derived in the paper show that for an appropriately chosen starting width h the proposed algorithm delivers approximations whose errors decay like, $\mathcal{O}(1/2^{(p+1)^2})$, an exponentially fast rate with respect to the level p.

1 Introduction

To motivate the topic of this paper, we consider the situation where one wishes to approximate a univariate target function f. Suppose we have in place a method that relies on a convergence parameter h and, as such, produces an h—dependent approximation $\mathcal{A}_h(f)$. In such cases, the error profile can be viewed as (or bounded by) a function of both the location x and the convergence parameter $h > 0$, i.e., we have

$$\mathcal{E}_f(x,h) = (f - \mathcal{A}_h(f))(x), \text{ where } \mathcal{E}_f(x,h) \to 0, \text{ as } h \to 0. \tag{1}$$

A popular choice is to a form radial basis function (RBF) interpolant to the function, where h measures the spacing of the data points. The RBF interpolation method gained popularity following the publication of [16] where the invertibility of a wide class of RBF interpolation matrices was established. In addition, convergence rates for certain classes of smooth functions were provided and discussed in [24]. However, despite impressive theoretical advances, a perceived drawback of RBF interpolation stems from the observation that for large data sets, the resulting interpolant frequently struggles to maintain a good fit in a numerically stable manner. Schaback described this phenomenon as *the uncertainty principle*, meaning that, for large data sets, a trade-off exists between good reproduction quality and good numerical stability; see [18].

In an attempt to overcome this problem, a multilevel interpolation method using compactly supported RBFs was proposed in [6]. Here the algorithm begins by capturing the broad details of the target function by forming the interpolant over a coarse subset of data points using a wide support parameter for the RBF. To capture some of the finer details the idea is to choose a denser subset of the data points, reduce the support parameter according to the density, and compute the RBF interpolant of previous error profile, the so-called residual interpolant. The finer scale residual interpolant is then added to the coarse interpolant to produce a function that interpolates the data at the finer level. This procedure continues over multiple levels until some user prescribed tolerance is attained.

A hurdle in proving convergence results for the multilevel method is that, by changing support of the basis function, we also change approximation spaces; however, in relation to this, we highlight the work of Narcowich et al. [17] who analysed a related scheme but required that sequences of approximation spaces were nested. In [9] a multilevel scheme using polyharmonic splines (finitely smooth and globally supported) on uniform grids was presented and constant reduction in error per level was shown. In [12] a modified multilevel method was considered, using the thin-plate spline RBF for an initial approximation and

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with subsequent refinements performed using suitably scaled compactly supported RBFs. Wendland and coauthors have explored multilevel schemes using scaled compactly supported RBFs for solving both approximation problems and partial differential equations on spheres and compact regions in Euclidean space [5], [13],[14], [23].

Almost all theoretical results on the multilevel method (briefly reviewed above) apply only to basis functions with finite smoothness. In these cases the numerical stability is improved but one has to accept a saturation point on the accuracy. However, multilevel approximation using scaled Gaussian kernels (infinitely smooth and globally supported) has gained interest due to its key role in multilevel sparse kernel interpolation (MuSIK) and its quasi-interpolatory modification (Q-MuSIK), see [7] and [21]. These approaches have achieved successful results in different areas, see [4] and [25] for details. These successes provide the motivation for this current work. Specifically, our aim here is to present a convergence analysis of the multilevel approximation method using scaled Gaussian kernels. The approach we take differs from the standard formulation in that we replace interpolation with approximation by continuous convolution, and so, in this setting, the algorithm updates the current convolution approximation by augmenting it with the convolution of the residual error with a Gaussian kernel that has a smaller scale and hence a narrower (more peaked) profile. In an earlier paper, two of the authors presented a first convergence analysis of this approach for univariate periodic functions [11]. This work was then extended in [10] for multivariate periodic functions over sparse grids. In this paper we will demonstrate the convergence of the method when approximating a univariate real valued function.

The paper is organised as follows. In Section 2 we present the details of the univariate multilevel Gaussian convolution algorithm. In Section 3 we present a careful analysis of the early stages of the algorithm and use this to provide a strategy for extending the analysis to the general setting. Specifically, under the assumption that the target function f is the restriction to the real line of an entire function we derive a useful closed form representation for the original Gaussian convolution approximation. We then show how this can be used to develop similar closed form representations for the multilevel Gaussian convolution approximation for the first two levels of the algorithm. The careful analysis of the early stages of the algorithm allows us to present a strategy for providing a formula for the approximation at every level. The stand-out feature of this work lies in the delivery of precise closed form representations for the intermediate approximations and also for the final approximation across all levels. The work associated to formulating these results will be done in Section 4 where the key results are formulated and which culminate in the presentation of the final error bounds. The proofs of the more technical results are long and so they are deferred to an Appendix. In Section 5 we illustrate the efficiency of the derived error bounds by applying the method to $\sin(ax)$ and $\exp(ax)$, for which analytical expressions are available for comparison. We conclude in Section 6 with a discussion of possible extensions and future directions of research.

2 The multilevel Gaussian convolution algorithm

The family of scaled Gaussian kernels are given by

$$\psi_h(x) = \frac{1}{h\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{h}\right)^2} \text{ and satisfy } \int_{-\infty}^{\infty} \psi_h(x)dx = 1, \tag{2}$$

where $h > 0$, is a scale parameter controlling the width of the profile of the Gaussian. If f is a univariate integrable function then the resulting Gaussian convolution approximation is defined by

$$\mathcal{S}_f^{(0)}(x) := f * \psi_h(x) = \int_{-\infty}^{\infty} f(t)\psi_h(x-t)dt. \tag{3}$$

It is shown in [19] Chapter 2, Theorem 1 that $\lim_{h\to 0} \|f * \psi_h - f\|_1 = 0$. In the same chapter, this result is extended to the L^p norm ($p > 1$) for a uniformly continuous and bounded function f.

In this setting the multilevel convolution algorithm can be described as follows.

- Level 0. (initialisation): Let $h > 0$ denote the initial scale parameter and f the univariate integrable function to be approximated. We initialise the algorithm by setting

$$\mathcal{S}_f^{(0)}(x) := f * \psi_h(x), \quad x \in \mathbb{R}.$$

  To introduce the notation of the algorithm we define

$$\mathcal{M}_f^{(0)}(x) := \mathcal{S}_f^{(0)}(x)$$

  and call this the initial (level 0) Gaussian convolution approximation to f. The error profile of the level 0 approximation is given by

$$\mathcal{E}_f^{(0)}(x) = f(x) - \mathcal{M}_f^{(0)}(x).$$

- Level 1. Here the error profile of the previous level is approximated by convolving it with the Gaussian having half the scale of the previous level, i.e., we compute

$$\mathcal{S}_f^{(1)}(x) := \mathcal{E}_f^{(0)} * \psi_{\frac{h}{2^1}}(x) = (f - \mathcal{M}_f^{(0)}) * \psi_{\frac{h}{2^1}}(x).$$

We then use this to update the approximation from the previous level and define

$$\mathcal{M}_f^{(1)}(x) := \mathcal{S}_f^{(0)}(x) + \mathcal{S}_f^{(1)}(x),$$

and, as before, we refer to this as the level 1 Gaussian convolution approximation to f. The error profile of the level 1 approximation is given by

$$\mathcal{E}_f^{(1)}(x) = f(x) - \mathcal{M}_f^{(1)}(x) = f(x) - \mathcal{S}_f^{(0)}(x) - \mathcal{S}_f^{(1)}(x).$$

- Level 2. Continuing the pattern, the error profile from level 1 is approximated by convolving it with the Gaussian with half the scale of the previous level, i.e., we compute

$$\mathcal{S}_f^{(2)}(x) := \mathcal{E}_f^{(1)} * \psi_{\frac{h}{2^2}}(x) = (f - \mathcal{M}_f^{(1)}) * \psi_{\frac{h}{2^2}}(x).$$

We then use this to update the approximation from the previous level and define

$$\mathcal{M}_f^{(2)}(x) := \mathcal{S}_f^{(0)}(x) + \mathcal{S}_f^{(1)}(x) + \mathcal{S}_f^{(2)}(x),$$

and, as before, we refer to this as level 2 Gaussian convolution approximation to f.

- Level $p > 2$. At this stage of the algorithm all previous level approximations $\left(\mathcal{M}_f^{(k)}\right)_{k=0}^{p-1}$, have been computed recursively via

$$\mathcal{M}_f^{(k)}(x) := \sum_{j=0}^{k} \mathcal{S}_f^{(j)}(x), \quad k = 0, 1, \ldots, p-1,$$

where the components making up the approximation are given by

$$\mathcal{S}_f^{(j)}(x) = \mathcal{E}_f^{(j-1)} * \psi_{\frac{h}{2^j}}(x) = (f - \mathcal{M}_f^{(j-1)}) * \psi_{\frac{h}{2^j}}(x), \quad j = 1, \ldots, k.$$

The algorithm progresses at level p by computing

$$\mathcal{S}_f^{(p)}(x) := \mathcal{E}_f^{(p-1)} * \psi_{\frac{h}{2^p}}(x) = (f - \mathcal{M}_f^{(p-1)}) * \psi_{\frac{h}{2^p}}(x)$$

and setting

$$\mathcal{M}_f^{(p)}(x) := \mathcal{M}_f^{(p-1)}(x) + \mathcal{S}_f^{(p)}(x) = \sum_{j=0}^{p} \mathcal{S}_f^{(j)}(x),$$

to be the level p Gaussian convolution approximation to f.

3 Early stages of multilevel Gaussian convolution

We begin with the following result that will be frequently used in our investigation.

Lemma 3.1. *Let ψ_a and ψ_b denote Gaussian kernels (2) with scale factors a and b respectively. Then,*

$$\psi_a * \psi_b = \psi_{\sqrt{a^2+b^2}}. \tag{4}$$

Proof. We recall that the Fourier transform of the scaled Gaussian is given by

$$\widehat{\psi_h}(z) = \int_{\infty}^{\infty} \psi_h(x) e^{-ixz} \mathrm{d}x = e^{-\frac{1}{2}h^2 z^2}.$$

Evoking the convolution theorem we have that

$$\widehat{\psi_a * \psi_b}(z) = \widehat{\psi_a}(z)\widehat{\psi_b}(z) = e^{-\frac{1}{2}(a^2+b^2)z^2} = \widehat{\psi_{\sqrt{a^2+b^2}}}(z).$$

Applying the inverse Fourier transform to the above gives the required result. □

3.1 Level zero Gaussian convolution approximation

The following result provides an expression for the initial (level 0) approximation to the target function under investigation.

Theorem 3.2. *Let f be an entire function restricted to the real line. For $h > 0$, let ψ_h denote the h—scaled Gaussian kernel. Then, the level 0 Gaussian convolution approximation to f is given by*

$$\mathcal{M}_f^{(0)}(x) := \mathcal{S}_f^{(0)}(x) = (f * \psi_h)(x) = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^i i!} h^{2i}. \tag{5}$$

Thus, the error profile at level 0 is given by

$$f(x) - \mathcal{M}_f^{(0)}(x) := -\sum_{i=1}^{\infty} \frac{f^{(2i)}(x)}{2^i i!} h^{2i}. \tag{6}$$

Proof.

$$\begin{aligned} f * \psi_h(x) &= \int_{-\infty}^{\infty} \frac{f(t) e^{-\frac{1}{2}\left(\frac{x-t}{h}\right)^2}}{\sqrt{2\pi} h} dt \quad (\text{let } z = x - t) \\ &= \int_{-\infty}^{\infty} \frac{f(x-z) e^{-\frac{1}{2}\left(\frac{z}{h}\right)^2}}{\sqrt{2\pi} h} dz \quad (\text{let } y = -z) \\ &= \int_{-\infty}^{\infty} \frac{f(x+y) e^{-\frac{1}{2}\left(\frac{y}{h}\right)^2}}{\sqrt{2\pi} h} dy. \end{aligned}$$

Since f is an entire function we have

$$f(x+y) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} y^j.$$

Substituting, we find that

$$\begin{aligned} f * \psi_h(x) &= \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} y^j \frac{e^{-\frac{1}{2}\left(\frac{y}{h}\right)^2}}{\sqrt{2\pi} h} dy \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} \int_{-\infty}^{\infty} y^j \frac{e^{-\frac{1}{2}\left(\frac{y}{h}\right)^2}}{\sqrt{2\pi} h} dy \quad (\text{let } \lambda = y/h) \\ &= \sum_{j=0}^{\infty} \frac{h^j f^{(j)}(x)}{j!} \int_{-\infty}^{\infty} \lambda^j \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} d\lambda, \end{aligned}$$

where the exchange of summation and integration follows from the Lebesgue convergence theorem. The integral in the final expression represents the moments of the standard normal distribution for which we have

$$\int_{-\infty}^{\infty} \lambda^j \frac{e^{-\frac{\lambda^2}{2}}}{\sqrt{2\pi}} d\lambda = \begin{cases} 0 & \text{if } j = 2i+1 \text{ is odd;} \\ \frac{(2i)!}{2^i \cdot i!} & \text{if } j = 2i \text{ is even.} \end{cases}$$

Thus, we can ignore the odd terms in the above series and so conclude, as required, that

$$f * \psi_h(x) = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^i \cdot i!} h^{2i}. \tag{7}$$

□

3.2 Level one Gaussian convolution approximation

The computational step at this stage is to form the approximation $\mathcal{S}_f^{(1)}(x)$ of the previous error profile (6) by convolving it with $\psi_{\frac{h}{2^1}}(x)$, this is then added to the previous approximation $\mathcal{M}_f^{(0)}(x)$ to f to form the refined version $\mathcal{M}_f^{(1)}(x)$. The following result provides the expression for the level 1 approximation to the target function under investigation.

Theorem 3.3. *Let f be an entire function restricted to the real line. The convolution of the level 0 error profile (6) with $\psi_{\frac{h}{2^1}}(x)$ is given by*

$$\mathcal{S}_f^{(1)}(x) = (f - \mathcal{M}_f^{(0)}) * \psi_{\frac{h}{2^1}}(x) = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{3i} i!} h^{2i} \sigma_1(i), \tag{8}$$

where

$$\sigma_1(i) = 1 - (1 + 2^2)^i = 1 - 5^i. \tag{9}$$

Thus, the level 1 approximation is given by

$$\mathcal{M}_f^{(1)}(x) = \mathcal{M}_f^{(0)} + \mathcal{S}_f^{(1)}(x) = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{3i} i!} h^{2i} \mu_1(i), \tag{10}$$

where

$$\mu_1(i) = 1 + (2^2)^i - (1+2^2)^i = 1 + 4^i - 5^i. \tag{11}$$

and, consequently, the error profile at level 1 is given by

$$f(x) - \mathcal{M}_f^{(1)}(x) := -\sum_{i=2}^{\infty} \frac{f^{(2i)}(x)}{2^{3i} i!} h^{2i} \mu_1(i). \tag{12}$$

Proof.

$$\begin{aligned} \mathcal{S}_f^{(1)}(x) = (f - \mathcal{M}_f^{(0)}) * \psi_{\frac{h}{2^1}}(x) &= \left(f * \psi_{\frac{h}{2^1}} - f * \psi_h * \psi_{\frac{h}{2^1}}\right)(x) \\ &= \left(f * \psi_{h\sqrt{\frac{1}{2^2}}} - f * \psi_{h\sqrt{\frac{1+2^2}{2^2}}}\right)(x) \\ &= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^i i!} h^{2i} \left[\left(\frac{1}{2^2}\right)^i - \left(\frac{1+2^2}{2^2}\right)^i\right] \\ &= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{3i} i!} h^{2i} \big[\underbrace{1 - (1+2^2)^i}_{\sigma_1(i)}\big], \end{aligned} \tag{13}$$

where, in the first line we have used (4) and in the penultimate line we have used (6) with h replaced with $h\sqrt{\frac{1}{2^2}}$ and $h\sqrt{\frac{1+2^2}{2^2}}$ respectively. Using formula (5) for the level 0 approximation, we have that

$$\begin{aligned} \mathcal{M}_f^{(1)}(x) &= \mathcal{M}_f^{(0)} + \mathcal{S}_f^{(1)}(x) \\ &= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^i i!} h^{2i} + \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{3i} i!} h^{2i} \big[1 - 5^i\big] = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{3i} i!} h^{2i} \big[\underbrace{1 + 4^i - 5^i}_{=\mu_1(i)}\big]. \end{aligned}$$

Expression (12) for the error follows by observing that

$$\mu_1(i) = \begin{cases} 1 & \text{for } i = 0; \\ 0 & \text{for } i = 1. \end{cases} \tag{14}$$

□

3.3 Level two Gaussian convolution approximation

The computational step at this stage is to form the approximation $\mathcal{S}_f^{(2)}(x)$ of the previous error profile (12) by convolving it with $\psi_{\frac{h}{2^2}}(x)$, this is then added to previous approximation $\mathcal{M}_f^{(1)}(x)$ to f to form the refined version $\mathcal{M}_f^{(2)}(x)$. The following result provides the expression for the level 2 approximation to the target function under investigation.

Theorem 3.4. *Let f be an entire function restricted to the real line. The convolution of the level 1 error profile (12) with $\psi_{\frac{h}{2^2}}(x)$ is given by*

$$\mathcal{S}_f^{(2)}(x) = (f - \mathcal{M}_f^{(1)}) * \psi_{\frac{h}{2^2}}(x) = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{5i} i!} h^{2i} \sigma_2(i), \tag{15}$$

where

$$\sigma_2(i) = 1 - (1+2^2)^i - (1+2^4)^i + (1+2^2+2^4)^i = 1 - 5^i - 17^i + 21^i. \tag{16}$$

Thus, the level 2 approximation is given by

$$\mathcal{M}_f^{(2)}(x) = \mathcal{M}_f^{(1)} + \mathcal{S}_f^{(2)}(x) = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{5i} i!} h^{2i} \mu_2(i), \tag{17}$$

where

$$\begin{aligned} \mu_2(i) &= 1 + \big((2^2)^i - (1+2^2)^i\big) + \big((2^4)^i - (1+2^4)^i\big) - \big((2^2+2^4)^i - (1+2^2+2^4)^i\big) \\ &= 1 + (4^i - 5^i) + (16^i - 17^i) - (20^i - 21^i) \end{aligned} \tag{18}$$

and, consequently, the error profile at level 2 is given by

$$f(x) - \mathcal{M}_f^{(2)}(x) := -\sum_{i=3}^{\infty} \frac{f^{(2i)}(x)}{2^{5i} i!} h^{2i} \mu_2(i). \tag{19}$$

Proof. In the same fashion as the proof of Theorem 3.3 we can develop as follows.

$$\begin{aligned}
\mathcal{S}_f^{(2)}(x) =& (f - \mathcal{M}_f^{(1)}) * \psi_{\frac{h}{2^2}}(x) = \left(f - \mathcal{S}_f^{(0)} - \mathcal{S}_f^{(1)}\right) * \psi_{\frac{h}{2^2}}(x) \\
&= (f - f * \psi_h - f * \psi_{\frac{h}{2^1}} + f * \psi_h * \psi_{\frac{h}{2^1}}) * \psi_{\frac{h}{2^2}}(x) \\
&= f * \psi_{\frac{h}{2^2}}(x) - f * \psi_h * \psi_{\frac{h}{2^2}}(x) - f * \psi_{\frac{h}{2^1}} * \psi_{\frac{h}{2^2}}(x) + f * \psi_h * \psi_{\frac{h}{2^1}} * \psi_{\frac{h}{2^2}}(x) \\
&= f * \psi_{h\sqrt{\frac{1}{2^4}}}(x) - f * \psi_{h\sqrt{\frac{2^4+1}{2^4}}}(x) - f * \psi_{h\sqrt{\frac{2^2+1}{2^4}}}(x) + f * \psi_{h\sqrt{\frac{2^4+2^2+1}{2^4}}}(x) \\
&= f * \psi_{h\sqrt{\frac{1}{2^4}}}(x) - f * \psi_{h\sqrt{\frac{17}{2^4}}}(x) - f * \psi_{h\sqrt{\frac{5}{2^4}}}(x) + f * \psi_{h\sqrt{\frac{21}{2^4}}}(x) \\
&= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^i i!} h^{2i} \left[\left(\frac{1}{2^4}\right)^i - \left(\frac{17}{2^4}\right)^i - \left(\frac{5}{2^4}\right)^i + \left(\frac{21}{2^4}\right)^i \right] \\
&= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{5i} i!} h^{2i} \big[\underbrace{1 - 17^i - 5^i + 21^i}_{\sigma_2(i)} \big].
\end{aligned} \tag{20}$$

Using formula (10) for the level 1 approximation we have that

$$\begin{aligned}
\mathcal{M}_f^{(2)}(x) &= \mathcal{M}_f^{(1)} + \mathcal{S}_f^{(2)}(x) \\
&= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{3i} i!} h^{2i} \left[1 + 4^i - 5^i\right] + \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{5i} i!} h^{2i} \left[1 - 5^i - 17^i + 21^i\right] \\
&= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{5i} i!} h^{2i} \big[\underbrace{1 + (4^i - 5^i) + (16^i - 17^i) - (20^i - 21^i)}_{\mu_2(i)} \big].
\end{aligned}$$

Expression (19) for the error follows by observing that

$$\mu_2(i) = \begin{cases} 1 & \text{for } i = 0; \\ 0 & \text{for } i = 1, 2. \end{cases} \tag{21}$$

□

3.4 Strategy for proving convergence

The details given for the early stage development of the multilevel Gaussian convolution algorithm provides some insight into how to approach a general proof. The work can be divided into the following parts:

- For any given level p, find the formula for the sequence $\{\sigma_p(i) : i = 0, 1, \dots\}$ such that error profile at level $(p-1)$ when convolved with $\psi_{\frac{h}{2^p}}(x)$ is given by

$$\mathcal{S}_f^{(p)}(x) = (f - \mathcal{M}_f^{(p-1)}) * \psi_{\frac{h}{2^p}}(x) = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{(2p+1)i} i!} h^{2i} \sigma_p(i). \tag{22}$$

- For any given level p, find the formula for the sequence $\{\mu_p(i) : i = 0, 1, \dots\}$ such that the level p Gaussian convolution approximation is given by

$$\mathcal{M}_f^{(p)}(x) = \mathcal{M}_f^{(p-1)}(x) + \mathcal{S}_f^{(p)}(x) = \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{(2p+1)i} i!} h^{2i} \mu_p(i). \tag{23}$$

- Investigate the precise performance of the algorithm when applied to the function $f(x) = e^{ax}$, where all convolutions can be computed analytically. This analysis, as we shall see, will shed light upon the properties of the sequence $\mu_p(i)$ that appear in the error expansion (34).

- Demonstrate that the sequence $\{\mu_p(i) : i = 0, 1, \dots\}$ satisfies

$$\mu_p(i) = \begin{cases} 1 & \text{for } i = 0; \\ 0 & \text{for } i = 1, \dots, p. \end{cases} \tag{24}$$

  In addition, provide a bound on the remainder term and use this to provide an error bound for the approximation at a general level p.

The above tasks will be achieved in the following sections of the paper. For convenience we provide the following definition that summarises the notation that will be used.

Definition 3.1. For a positive integer p we let

$$\mathcal{I}_p = \left\{ \boldsymbol{i}_p = (i_1, \dots, i_p) : i_n \in \{0,1\}, \text{ for } n = 1, \dots, p \right\} \tag{25}$$

denote the set of all $p-$dimensional binary vectors. For $p = 0$ we set $\mathcal{I}_p$ to be the empty set, i.e.,

$$\mathcal{I}_0 = \emptyset. \tag{26}$$

For a given p-dimension binary vector $\boldsymbol{i}_p = (i_1, \dots, i_p) \in \mathcal{I}_p$ we let

$$|\boldsymbol{i}_p| = \sum_{n=1}^{p} |i_n| = \sum_{n=1}^{p} i_n, \tag{27}$$

denote its sum-norm. We define the sum-norm on the empty set to be zero.

Clearly the cardinality of $\mathcal{I}_p = 2^p$ and early examples are:

$$\begin{aligned} \mathcal{I}_1 &= \left\{(0),(1)\right\}, \\ \mathcal{I}_2 &= \left\{(0,0),(1,0),(0,1),(1,1)\right\}, \\ \mathcal{I}_3 &= \left\{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,0,1),(1,0,1),(0,1,1),(1,1,1)\right\}. \end{aligned} \tag{28}$$

To capture the $2^4 = 16$ elements in $\mathcal{I}_4$ we can define the following sets:

$$\begin{aligned} \mathcal{I}_4^1 &:= \{(1, i_2, i_3, i_4)\}_{(i_2,i_3,i_4)\in\mathcal{I}_3} \\ \mathcal{I}_4^2 &:= \{(0, 1, i_3, i_4)\}_{(i_3,i_4)\in\mathcal{I}_2} \\ \mathcal{I}_4^3 &:= \{(0, 0, 1, i_4)\}_{(i_4)\in\mathcal{I}_1}, \end{aligned} \tag{29}$$

or, more generally,

$$\mathcal{I}_4^k := \{(i_1, i_2, i_3, i_4) : i_n = 0 \ (n < k), \ \ i_k = 1, \ \ i_n \in \{0,1\}, \ (k < n \le 4)\}. \tag{30}$$

Clearly $\mathcal{I}_4^1, \mathcal{I}_4^2, \mathcal{I}_4^3$ are disjoint subsets of $\mathcal{I}_4$, whose cardinalities are 8, 4 and 2 respectively. We note that the binary vectors $(0,0,0,0)$ and $(0,0,0,1)$ are not elements of $\mathcal{I}_4^k$ for any $k = 1,2,3$, and thus defining

$$\mathcal{I}_4^0 = \{(0,0,0,0)\} \quad \text{and} \quad \mathcal{I}_4^4 = \{(0,0,0,1)\}. \tag{31}$$

We can decompose $\mathcal{I}_4$ as

$$\mathcal{I}_4 = \bigcup_{k=1}^{3} \mathcal{I}_4^k \cup \mathcal{I}_4^0 \cup \mathcal{I}_4^4,$$

the equality as sets follows since $\mathcal{I}_4^k$, $k = 0, \dots, 4$ are disjoint subsets of $\mathcal{I}_4$ and

$$\left| \bigcup_{k=1}^{3} \mathcal{I}_4^k \cup \mathcal{I}_4^0 \cup \mathcal{I}_4^4 \right| = \sum_{k=1}^{3} |\mathcal{I}_4^k| + 1 + 1 = \sum_{k=1}^{3} 2^{4-k} + 1 + 1 = 2^4.$$

This approach generalises to the sets of higher dimensional binary vectors and will prove useful in the proofs section of our work. We summarise this property here.

Property 3.5. The set $\mathcal{I}_p$ (25) of $p-$dimensional binary vectors can be decomposed as follows

$$\mathcal{I}_p = \bigcup_{k=1}^{p-1} \mathcal{I}_p^k \cup \mathcal{I}_p^0 \cup \mathcal{I}_p^p,$$

where $\mathcal{I}_p^0 = \{(0,0,\dots,0)\}$, $\mathcal{I}_p^p = \{(0,\dots,0,1)\}$, and

$$\mathcal{I}_p^k = \{(i_1, \dots i_n, \dots i_p) : i_n = 0 \ (n < k), \ \ i_k = 1, \ \ i_n \in \{0,1\}, \ (k < n \le p)\}. \tag{32}$$

We remark that the above representation, which isolates the two vectors $(0,0,\dots,0)$ and $(0,\dots,0,1)$ (as elements of $\mathcal{I}_p^0$ and $\mathcal{I}_p^p$), is chosen over other possibilities as it provides the setting for the inductive proofs of the general results. In addition to this we also point out the following identity which will be used in the proofs section.

Lemma 3.6. *Let p be a non-negative integer and let $\mathcal{I}_p$ (25) denote the set of $p-$dimensional binary vectors. Then*

$$\sum_{\boldsymbol{i}_p \in \mathcal{I}_p} (-1)^{|\boldsymbol{i}_p|} = 0. \tag{33}$$

Proof. For $p = 1$ we have $\mathcal{I}_1 = \{0, 1\}$ and the identity is trivially true: $(-1)^0 + (-1)^1 = 0$. For $p > 1$ we can choose to write a typical element of $\mathcal{I}_p$ as $\boldsymbol{i}_p = (i_1, \boldsymbol{i}_{p-1})$, with $i_1 \in \{0, 1\}$ and $\boldsymbol{i}_{p-1} \in \mathcal{I}_{p-1}$. Then we have

$$\sum_{\boldsymbol{i}_p \in \mathcal{I}_p} (-1)^{|\boldsymbol{i}_p|} = \sum_{\boldsymbol{i}_{p-1} \in \mathcal{I}_{p-1}} (-1)^{0+|\boldsymbol{i}_{p-1}|} + \sum_{\boldsymbol{i}_{p-1} \in \mathcal{I}_{p-1}} (-1)^{1+|\boldsymbol{i}_{p-1}|}$$
$$= \sum_{\boldsymbol{i}_{p-1} \in \mathcal{I}_{p-1}} (-1)^{|\boldsymbol{i}_{p-1}|} - \sum_{\boldsymbol{i}_{p-1} \in \mathcal{I}_{p-1}} (-1)^{|\boldsymbol{i}_{p-1}|} = 0.$$

□

4 The main results

In view of the strategy proposed in the previous section we present here the main results concerning the form of the residual error approximations $\mathcal{S}_f^{(p)}(x)$, the series expansion for the mulilevel approximations $\mathcal{M}_f^{(p)}(x)$, and the properties of expansion coefficients. These results are verified for the early levels of the method, where we compare against the calculations given in the previous sections. The complete proofs of the more technical results are given in the Appendix.

4.1 The form of $\mathcal{S}_f^{(p)}(x)$

Theorem 4.1. *Let f be an entire function restricted to the real line and p be a non-negative integer. In reference to the multilevel Gaussian convolution algorithm, the convolution of the level $p-1$ error profile with $\psi_{\frac{h}{2^p}}(x)$ is given by*

$$\begin{aligned}\mathcal{S}_f^{(p)}(x) &= (f - \mathcal{M}_f^{(p-1)}) * \psi_{\frac{h}{2^p}}(x) \\ &= \sum_{\boldsymbol{i}_p \in \mathcal{I}_p} (-1)^{|\boldsymbol{i}_p|} f * \psi_{h\sqrt{\frac{1+2^2 i_1 + 2^4 i_2 + \cdots + 2^{2p} i_p}{2^{2p}}}}(x) \\ &= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{(2p+1)i} i!} h^{2i} \sigma_p(i),\end{aligned} \tag{34}$$

where

$$\sigma_p(i) = \sum_{\boldsymbol{i}_p \in \mathcal{I}_p} (-1)^{|\boldsymbol{i}_p|} (1 + 2^2 i_1 + 2^4 i_2 + \cdots + 2^{2p} i_p)^i. \tag{35}$$

The proof of this result is given in the appendix. Here we verify that the formula matches the expressions given in the previous section for the early levels of the method.

- Level $p = 0$. Here formula (34) gives

$$\mathcal{S}_f^{(0)}(x) = \sum_{\boldsymbol{i}_0 \in \emptyset} f * \psi_{h\sqrt{\frac{1+2^2\cdot 0 + 2^4 \cdot 0 + \cdots + 2^{2p}\cdot 0}{2^0}}}(x) = f * \psi_h$$

  and this coincides with (5).

- Level $p = 1$. Here formula (34) gives

$$\begin{aligned}\mathcal{S}_f^{(1)}(x) &= \sum_{\boldsymbol{i}_1 \in \mathcal{I}_1} (-1)^{|\boldsymbol{i}_p|} f * \psi_{h\sqrt{\frac{1+2^2 i_1}{2^2}}}(x) = \\ &= (-1)^0 f * \psi_{h\sqrt{\frac{1}{2^2}}}(x) + (-1)^1 f * \psi_{h\sqrt{\frac{1+2^2}{2^2}}}(x) \\ &= \left(f * \psi_{h\sqrt{\frac{1}{2^2}}} - f * \psi_{h\sqrt{\frac{1+2^2}{2^2}}} \right)(x)\end{aligned}$$

  and this coincides with (13).

- Level $p = 2$. Here formula (34) gives

$$\begin{aligned}\mathcal{S}_f^{(2)}(x) &= \sum_{\boldsymbol{i}_2 \in \mathcal{I}_2} (-1)^{|\boldsymbol{i}_2|} f * \psi_{h\sqrt{\frac{1+2^2 i_1 + 2^4 i_2}{2^4}}}(x) \\ &= (-1)^{0+0} f * \psi_{h\sqrt{\frac{1}{2^4}}}(x) + (-1)^{1+0} f * \psi_{h\sqrt{\frac{1+2^2}{2^4}}}(x) \\ &+ (-1)^{0+1} f * \psi_{h\sqrt{\frac{1+2^4}{2^4}}}(x) + (-1)^{1+1} f * \psi_{h\sqrt{\frac{1+2^2+2^4}{2^4}}}(x) \\ &= f * \psi_{h\sqrt{\frac{1}{2^4}}}(x) - f * \psi_{h\sqrt{\frac{1+2^4}{2^4}}}(x) - f * \psi_{h\sqrt{\frac{1+2^2}{2^4}}}(x) + f * \psi_{h\sqrt{\frac{1+2^2+2^4}{2^4}}}(x)\end{aligned}$$

  and this coincides with (20).

4.2 The series representation of $\mathcal{M}_f^{(p)}(x)$

Theorem 4.2. *Let f be an entire function restricted to the real line and p be a non-negative integer. In reference to the multilevel Gaussian convolution algorithm, the level p approximation to f is given by*

$$\mathcal{M}_f^{(p)}(x) \quad = f(x) + \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)}{2^{(2p+1)i} i!} h^{2i} \mu_p(i), \tag{36}$$

where

$$\mu_p(i) = \sum_{\boldsymbol{i}_p \in \mathcal{I}_p} (-1)^{1+|\boldsymbol{i}_p|} \Big[(2^2 i_1 + 2^4 i_2 + \cdots + 2^{2p} i_p)^i - (1 + 2^2 i_1 + 2^4 i_2 + \cdots + 2^{2p} i_p)^i \Big]. \tag{37}$$

The proof of this result is given in the appendix. Here we verify that the formula matches the expressions given in the previous section for the early levels of the method.

- Level $p = 0$. Here formula (37) gives

$$\sum_{\boldsymbol{i}_0 \in \emptyset} (-1)^{1+0} \Big[(0)^i - (1)^i \Big] = 1 \quad \text{for all } i \geq 1.$$

  In this case the RHS of (36) is

$$f(x) + \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)}{2^i i!} h^{2i}$$

  and this coincides with (5).

- Level $p = 1$. Here formula (37) gives

$$\begin{aligned} \mu_1(i) &= \sum_{\boldsymbol{i}_1 \in \mathcal{I}_1} (-1)^{1+|\boldsymbol{i}_1|} \Big[(2^2 i_1)^i - (1 + 2^2 i_1)^i \Big] \\ &= (-1)^{1+0} \Big[0^i - (1)^i \Big] + (-1)^{1+1} \Big[(2^2)^i - (1 + 2^2)^i \Big] \\ &= 1 + (2^2)^i - (1 + 2^2)^i \end{aligned}$$

  and this coincides with (11).

- Level $p = 2$. Here formula (37) gives

$$\begin{aligned} \mu_2(i) &= \sum_{\boldsymbol{i}_2 \in \mathcal{I}_2} (-1)^{1+|\boldsymbol{i}_2|} \Big[(2^2 i_1 + 2^4 i_2)^i - (1 + 2^2 i_1 + 2^4 i_2)^i \Big] \\ &= (-1)^{1+0+0} \Big[(0)^i - (1)^i \Big] + (-1)^{1+1+0} \Big[(2^2)^i - (1 + 2^2)^i \Big] \\ &+ (-1)^{1+0+1} \Big[(2^4)^i - (1 + 2^4)^i \Big] + (-1)^{1+1+1} \Big[(2^2 + 2^4)^i - (1 + 2^2 + 2^4)^i \Big] \\ &= 1 + \big((2^2)^i - (1 + 2^2)^i \big) + \big((2^4)^i - (1 + 2^4)^i \big) - \big((2^2 + 2^4)^i - (1 + 2^2 + 2^4)^i \big) \end{aligned}$$

  and this coincides with (18).

4.3 Performance of the algorithm when applied to $f(x) = e^{ax}$ for $a > 0$.

Here we examine the performance of the multilevel algorithm when applied to the scaled exponential function $f(x) = \exp(ax)$. The results of this will, as we shall see in the next section, allow us to understand the behaviour of the error expansion and enable us to formulate error bounds.

The Gaussian convolution of the scaled exponential function is given by

$$\exp_a * \psi_h(x) = \int_{-\infty}^{\infty} \exp(at) \frac{\exp\left(-\frac{1}{2}\left(\frac{x-t}{h}\right)^2\right)}{\sqrt{2\pi} h} dt. \tag{38}$$

Using the substitution $z = (x - t)/h$ and the fact that $\frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$ is the standard normal density function, we can compute this analytically as follows

$$\begin{aligned} \exp_a * \psi_h(x) &= \int_{-\infty}^{\infty} \exp(a(x - zh)) \frac{\exp\left(-\frac{1}{2}\left(z^2 + 2hz\right)^2\right)}{\sqrt{2\pi}} dz \\ &= \exp(ax) \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{1}{2}\left((z + ah)^2 - a^2 h^2\right)\right)}{\sqrt{2\pi}} dz = \exp\left(\frac{a^2 h^2}{2}\right) \exp(ax). \end{aligned} \tag{39}$$

In the notation of the multilevel algorithm we have:

$$\mathcal{E}^{(0)}_{exp_a}(x) = (1 - e^{\frac{a^2h^2}{2}})\exp(ax).$$

Using (39) we have that the level 1 approximation is given by

$$\mathcal{M}^{(1)}_{exp_a}(x) = e^{\frac{a^2h^2}{2}}\exp(ax) + (1 - e^{\frac{a^2h^2}{2}})\exp_a * \psi_{\frac{h}{2}}(x) = \left[e^{-\frac{a^2h^2}{2}} + (1 - e^{\frac{a^2h^2}{2}})e^{\frac{a^2h^2}{8}}\right]\exp(ax)$$

and

$$\mathcal{E}^{(1)}_{exp_a}(x) = \left[1 - e^{\frac{a^2h^2}{2}} - (1 - e^{\frac{a^2h^2}{2}})e^{\frac{a^2h^2}{8}}\right]\exp(ax) = (1 - e^{\frac{a^2h^2}{2}})(1 - e^{\frac{a^2h^2}{8}})\exp(ax).$$

Continuing one can show inductively that the error profile at level p is given precisely by

$$\mathcal{E}^{(p)}_{exp_a}(x) = \prod_{k=0}^{p}(1 - e^{\frac{a^2h^2}{2^{2k+1}}})\exp(ax). \tag{40}$$

Appealing to formula (36) with $f(x) = \exp(ax)$ we have that

$$\begin{aligned}\mathcal{E}^{(p)}_{f=exp_a}(x) &= \exp(ax) - \mathcal{M}^{(p)}_{f=exp_a}(x)\\ &= -\left(\sum_{i=1}^{\infty}\frac{a^{2i}h^{2i}}{2^{(2p+1)i}i!}\mu_p(i)\right)\exp(ax) = \prod_{k=0}^{p}(1 - e^{\frac{a^2h^2}{2^{2k+1}}})\exp(ax),\end{aligned} \tag{41}$$

and this leads to the identity

$$\sum_{i=1}^{\infty}\frac{a^{2i}h^{2i}\mu_p(i)}{2^{(2p+1)i}i!} = -\prod_{k=0}^{p}(1 - e^{\frac{a^2h^2}{2^{2k+1}}}) = (-1)^p\prod_{k=0}^{p}(e^{\frac{a^2h^2}{2^{2k+1}}} - 1), \tag{42}$$

where, for a given p, the sequence $(\mu_p(i))_{i\geq 1}$ is given by (37).

4.4 Properties of the expansion sequence $\mu_p(i)$

Theorem 4.3. *The first p terms of the sequence $\mu_p(i)$ given by (37) are zero and the $(p+1)^{th}$ term is given by*

$$\frac{\mu_p(p+1)}{2^{(2p+1)(p+1)}(p+1)!} = \frac{(-1)^p}{2^{(p+1)^2}}. \tag{43}$$

Consequently, the error committed at level p of the multilevel Gaussian convolution approximation to f is given by

$$\mathcal{E}^{(p)}_f(x) = f(x) - \mathcal{M}^{(p)}_f(x) = \frac{(-1)^{p+1}h^{2(p+1)}}{2^{(p+1)^2}}f^{(2(p+1))}(x) - \sum_{i=p+2}^{\infty}\frac{h^{2i}\mu_p(i)}{2^{(2p+1)i}i!}f^{(2i)}(x). \tag{44}$$

Proof. The results of the above theorem regarding the early coefficients have been verified in Section 2 for $p = 1$ and 2. The formula for the $(p+1)^{th}$ coefficient in these case is established as follows. For $p = 1$ we observe from (11) that

$$\frac{\mu_1(2)}{2^{(3)\cdot 2)}(2)!} = \frac{(1 + 4^2 - 5^2)}{2^7} = \frac{-8}{2^7} = -\frac{1}{2^4} = -\frac{1}{2^{2^2}}.$$

Similarly, for $p = 2$ we observe from (18) that

$$\frac{\mu_2(3)}{2^{(5)\cdot 3)}(3)!} = \frac{1 + (4^3 - 5^3) + (16^3 - 17^3) - (20^3 - 21^3)}{2^{15}\cdot 6} = \frac{284}{2^{15}\cdot 6} = \frac{64}{2^{15}} = \frac{1}{2^9} = \frac{1}{2^{2^3}}.$$

Here we provide a general proof by developing identity (42) with $a = 1$, as follows:

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{h^{2i}\mu_p(i)}{2^{(2p+1)i}i!} &= (-1)^p\prod_{k=0}^{p}(e^{\frac{h^2}{2^{2k+1}}} - 1)\\ &= (-1)^p\prod_{k=0}^{p}\left(\sum_{j=1}^{\infty}\frac{\left(\frac{h^2}{2^{2k+1}}\right)^j}{j!}\right)\\ &= (-1)^p\frac{h^2}{2}\cdot\frac{h^2}{2^3}\cdots\frac{h^2}{2^{2p+1}}\prod_{k=0}^{p}\left(\sum_{j=1}^{\infty}\frac{\left(\frac{h^2}{2^{2k+1}}\right)^{j-1}}{j!}\right)\\ &= \frac{(-1)^p h^{2(p+1)}}{2^{p+1}2^{2(1+2+\cdots+p)}}\prod_{k=0}^{p}\left(\sum_{j=0}^{\infty}\frac{\left(\frac{h^2}{2^{2k+1}}\right)^j}{(j+1)!}\right)\\ &= \frac{(-1)^p h^{2(p+1)}}{2^{(p+1)^2}}\prod_{k=0}^{p}\left(\sum_{j=0}^{\infty}\frac{\left(\frac{h^2}{2^{2k+1}}\right)^j}{(j+1)!}\right)\end{aligned} \tag{45}$$

The leading term of this expression is given by

$$\frac{(-1)^p h^{2(p+1)}}{2^{(p+1)^2}}$$

and thus, by matching coefficients, it follows that $\mu_p(i) = 0$ for $i = 1, \ldots, p$. □

Remark 1. We note that the expansion coefficients of

$$\prod_{k=0}^{p}(e^{\frac{h^2}{2^{2k+1}}} - 1) = \left(\sum_{j=1}^{\infty}\frac{\left(\frac{h^2}{2}\right)^j}{j!}\right)\left(\sum_{j=1}^{\infty}\frac{\left(\frac{h^2}{2^3}\right)^j}{j!}\right)\cdots\left(\sum_{j=1}^{\infty}\frac{\left(\frac{h^2}{2^{2p+1}}\right)^j}{j!}\right)$$

are all positive and so, in view of (42), we can conclude that if p is even then $(\mu_p(i))_{i\geq p+1}$ is a sequence of positive values, and if p is odd it is a sequence of negative values. In particular, this allows us to conclude that

$$\sum_{i=p+1}^{\infty}\frac{h^{2i}|\mu_p(i)|}{2^{(2p+1)i}i!} = \prod_{k=0}^{p}(e^{\frac{h^2}{2^{2k+1}}} - 1) \quad \text{and} \quad \sum_{i=p+1}^{\infty}\frac{a^{2i}h^{2i}|\mu_p(i)|}{2^{(2p+1)i}i!} = \prod_{k=0}^{p}(e^{\frac{a^2h^2}{2^{2k+1}}} - 1). \tag{46}$$

The following Lemma shows that the sequence $(\frac{|\mu_p(i)|}{2^{(2p+1)i}i!})_{i\geq p+1}$ is monotonically decreasing.

Lemma 4.4. *For a positive integer p define*

$$\prod_{k=0}^{p}(e^{\frac{h^2}{2^{2k+1}}} - 1) = \frac{h^{2(p+1)}}{2^{(p+1)^2}}F_p(h) \quad \text{where} \quad F_p(h) = \prod_{k=0}^{p}\left(\sum_{j=0}^{\infty}\frac{\left(\frac{h^2}{2^{2k+1}}\right)^j}{(j+1)!}\right) = \sum_{i=0}^{\infty}a_i h^{2i}. \tag{47}$$

Then the sequence $(a_n)_{n\geq 0}$ is monotonically decreasing and consequently the sequence $(\frac{|\mu_p(i)|}{2^{(2p+1)i}i!})_{i\geq p+1}$ is also monotonically decreasing.

Proof. We have observed that $a_1 = 1$ and matching the coefficients of h^2 we have

$$a_2 = \frac{1}{2}\sum_{k=0}^{p}\frac{1}{2^{2k+1}} = \frac{1}{3}\left(1 - \frac{1}{2^{2(p+1)}}\right) < a_1.$$

Consider the product of two power series

$$\left(\sum_{n=0}^{\infty}u_i x^i\right)\left(\sum_{i=0}^{\infty}v_i x^i\right) = \sum_{i=0}^{\infty}w_i x^i.$$

In ([3]) it is shown that if both $(u_i)_{i\geq 0}$ and $(v_i)_{i\geq 0}$ are monotonically decreasing then so is the sequence $(w_i)_{i\geq 0}$. Setting $x = h^2$, we observe that all sequences in the power series product (47) are monotonically decreasing and thus we can apply this to conclude that $(a_i)_{i\geq 0}$ is monotonically decreasing. The proof is complete by appealing to (46) to observe that

$$\frac{|\mu_p(p+1+i)|}{2^{(2p+1)(p+1+i)}(p+1+i)!} = \frac{h^{2(p+1)}}{2^{(p+1)^2}}a_i, \quad i \geq 0.$$

□

4.5 General error bounds

Theorem 4.2 establishes that

$$|f(x) - \mathcal{M}_f^{(p)}(x)| \leq \sum_{i=p+1}^{\infty}\frac{h^{2i}|\mu_p(i)|\cdot|f^{(2i}(x)|}{2^{(2p+1)i}i!} \tag{48}$$

and so, given that we know (from (46) and Lemma 4.4), that

$$\sum_{i=p+1}^{\infty}\frac{h^{2i}|\mu_p(i)|}{2^{(2p+1)i}i!} = \prod_{k=0}^{p}(e^{\frac{h^2}{2^{2k+1}}} - 1) \quad \text{and} \quad \frac{|\mu_p(i)|}{2^{(2p+1)i}i!} \text{ is monotonic decreasing,} \tag{49}$$

then we can convert the above into meaningful error bounds for functions whose derivative growth can be suitably bounded. The growth of an entire function is determined by its order and its type as demonstrated in the following definition.

Definition 4.1. An entire function is said to be of order ρ and type $a > 0$ if, for every $\epsilon > 0$ there exists a positive constant C_ϵ say such that

$$|f(z)| \leq C_\epsilon \exp((a+\epsilon)|z|^\rho), \quad z \in \mathbb{C},$$

where the value a is assumed to be the greatest lower bound of positive values such that the above inequality holds. In the case where the order $\rho = 1$ it is common to denote f as a function of exponential type $a > 0$.

The textbooks [15] and [2] are excellent references for the theory of entire functions. In Chapter 1 of [15] examples are constructed of entire functions of various orders and types, in particular $\exp(az)$ and $\sin(az)$ are highlighted as examples of functions exponential type $a > 0$. Furthermore, the connection between the growth of the function and the size of its derivatives is explored. Following [20] we let $E[a > 0]$ denote the class of function of exponential type $a > 0$ and $B[a > 0]$ denote the class of functions in $E[a > 0]$ which are bounded on the real line. For functions in $B[a > 0]$ we can evoke Bernstein's inequality to deduce that

$$\sup_{x\in\mathbb{R}} |f^{(n)}(x)| \le a^n \sup_{x\in\mathbb{R}} |f(x)|, \quad \text{for all } n \ge 0. \tag{50}$$

For functions in $E[a > 0]$ that are not bounded on $\mathbb{R}$ there is not, in general, a bound similar to (50). In view of this we define $E^*[a > 0]$ to be the sub-class of $E[a > 0]$ whose elements satisfy a uniform bound, rather than the ϵ—dependent one given in Definition 4.1. This is captured in the following definition.

Definition 4.2. An entire function is said to be of strict exponential type $a > 0$, if there exists a positive constant C, such that

$$|f(z)| \le C \exp(a|z|^\rho), \quad z \in \mathbb{C}.$$

We let $E^*[a > 0]$ denote the class of functions of strict exponential type.

The growth of the derivatives of functions in $E^*[a > 0]$ can be controlled using Cauchy's integral formula, which provides:

$$|f^{(n)}(x)| \le \frac{n!}{r^n} \max_{|z-x|=r} |f(z)|, \quad \text{for all } r > 0.$$

Using the reverse triangle inequality $|z| \le |x| + r$, together with the growth condition, we have

$$|f^{(n)}(x)| \le C e^{a|x|} n! \frac{e^{ar}}{r^n}, \quad \text{for all } r > 0.$$

Following [15] we can optimize the above bound in r by observing that the minimum of the function $B(r) = \frac{e^{ar}}{r^n}$ occurs when $r = \frac{n}{a}$. Taking this value for the radius r we have that

$$|f^{(n)}(x)| \le C a^n e^{a|x|} \frac{n!}{n^n} e^n.$$

Using the crude upper bound $n! \le n^n$ we arrive at

$$|f^{(n)}(x)| \le C e^{a|x|} (ae)^n, \quad n \ge 1. \tag{51}$$

Using the bounds (50) and (51) we can propose the following result.

Theorem 4.5. *Let f be an entire function of exponential type $a > 0$.*

- *If f is bounded on the real line, then with $M = \sup_{x\in\mathbb{R}} |f(x)|$, the error profile of the multilevel Gaussian convolution algorithm at level p satisfies*

$$|f(x) - \mathcal{M}_f^{(p)}(x)| \le M \prod_{k=0}^{p} (e^{\frac{a^2h^2}{2^{2k+1}}} - 1), \quad \text{for all } x \in \mathbb{R}. \tag{52}$$

Furthermore, if $h < \frac{1}{a}$, then

$$\sup_{x\in\mathbb{R}} |f(x) - \mathcal{M}_f^{(p)}(x)| = \mathcal{O}\left(\frac{(ah)^{2(p+1)}}{2^{(p+1)^2}}\right). \tag{53}$$

- *If $f \in E^*[a > 0]$ is unbounded on the real line then, restricting attention to the interval $I = [\alpha, \beta]$ and setting $M_I = \max_{x\in I} e^{a|x|}$, the error profile of the multilevel Gaussian convolution algorithm at level p satisfies*

$$|f(x) - \mathcal{M}_f^{(p)}(x)| \le C \cdot M_I \prod_{k=0}^{p} (e^{\frac{a^2e^2h^2}{2^{2k+1}}} - 1), \quad \text{for all } x \in [\alpha, \beta]. \tag{54}$$

Proof. We prove each of the two statements in turn.

- For the first case we can use formula (48) and the derivative bound (50) it follows that

$$\begin{aligned} |f(x) - \mathcal{M}_f^{(p)}(x)| &\le \sum_{i=p+1}^{\infty} |f^{(2i)}(x)| h^{2i} \frac{|\mu_p(i)|}{2^{(2p+1)i} i!} \\ &\le M \sum_{i=p+1}^{\infty} (ah)^{2i} \frac{|\mu_p(i)|}{2^{(2p+1)i} i!} \\ &= M \prod_{k=0}^{p} (e^{\frac{a^2h^2}{2^{2k+1}}} - 1). \end{aligned}$$

The final line follows from (49) with h replaced by ah. The bound (53) holds since, by Lemma 4.4 the error expansion coefficients are monotonically decreasing when $h < \frac{1}{a}$ with the first term being the dominant term.

- For the second case we can use formula (48) and the derivative bound (51) to show that if $x \in [\alpha, \beta]$ it follows that

$$
\begin{aligned}
|f(x) - \mathcal{M}_f^{(p)}(x)| &\leq \sum_{i=p+1}^{\infty} |f^{(2i)}(x) h^{2i}| \frac{|\mu_p(i)|}{2^{(2p+1)i} i!} \\
&\leq C e^{a|x|} \sum_{i=p+1}^{\infty} \frac{(a \cdot e \cdot h)^{2i} |\mu_p(i)|}{2^{(2p+1)i} i!} \\
&\leq C \cdot M_I \prod_{k=0}^{p} (e^{\frac{a^2 e^2 h^2}{2^{2k+1}}} - 1).
\end{aligned}
$$

The final line follows from (49) with h replaced by aeh.

□

5 Illustrative examples

Let us first investigate the performance of the algorithm with the target function $f(x) = \sin(ax)$. It is well-known that $f(z) = \sin(az)$ is an example of function of exponential type a whose restriction to the real line is bounded. In this case the Gaussian convolution can be computed analytically and we have

$$
\sin_a * \psi_h(x) = \int_{-\infty}^{\infty} \sin(t) \frac{\exp\left(-\frac{1}{2}\left(\frac{x-t}{h}\right)^2\right)}{\sqrt{2\pi} h} dt.
$$

Using the change of variable $z = \frac{x-t}{h}$ we have

$$
\begin{aligned}
\sin_a * \psi_h(x) &= \int_{-\infty}^{\infty} \sin(a(x+hz)) \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} dz \\
&= \int_{-\infty}^{\infty} \sin(a(x+hz)) \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} dz \\
&= \sin(ax) \int_{-\infty}^{\infty} \cos(ahz) \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} dz + \cos(ax) \int_{-\infty}^{\infty} \frac{\sin(ahz) \exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} dz \\
&= \sin(ax) \int_{-\infty}^{\infty} \frac{\cos(ahz) \exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} dz.
\end{aligned}
$$

The integral involving $\sin(ahz)$ in the penultimate line is zero as the integrand is an odd function. The remaining integral involving $\cos(ahz)$ is given in [8] Formula 3.896.1 and so we have

$$
\sin_a * \psi_h(x) = e^{-\frac{a^2 h^2}{2}} \sin(ax). \tag{55}
$$

Using the same approach as for $f(x) = \exp(ax)$ in section 4.3 one can show that the analytical error of the multilevel algorithm at level p for $f(x) = \sin(ax)$ is given by

$$
\mathcal{E}_{sin_a}^{(p)}(x) = \prod_{k=0}^{p} (1 - e^{-\frac{a^2 h^2}{2^{2k+1}}}) \sin(ax). \tag{56}
$$

Appealing to formula (44) with $f(x) = \sin(x)$ we have that

$$
\mathcal{E}_{f=sin_a}^{(p)}(x) = \left[\sum_{i=p+1}^{\infty} \frac{(-1)^{i+1} a^{2i} h^{2i} \mu_p(i)}{2^{(2p+1)i} i!} \right] \sin(ax). \tag{57}
$$

We have that

$$
\left| \frac{d^{2i}}{dx^{2i}} \sin(ax) \right| \leq \begin{cases} 1 & \text{for } a = 1; \\ a^{2i} & \text{for } a > 1. \end{cases}
$$

Since $f(z) = \sin(az)$ satisfies the conditions for the first part of Theorem 4.5 we can conclude that

$$
|\mathcal{E}_{f=sin_a}^{(p)}(x)| \leq \prod_{k=0}^{p} (e^{\frac{a^2 h^2}{2^{2k+1}}} - 1). \tag{58}
$$

In this particular case we can observe that the upper bound can be expressed as

$$\prod_{k=0}^{p}(e^{\frac{a^2h^2}{2^{2k+1}}}-1) = \exp\left(a^2h^2\left(1+\frac{1}{2^2}+\cdots+\frac{1}{2^{2p}}\right)\right)\prod_{k=0}^{p}(1-e^{-\frac{a^2h^2}{2^{2k+1}}})$$
$$= \exp\left(\frac{2a^2h^2}{3}\left(1-\frac{1}{2^{2(p+1)}}\right)\right)\prod_{k=0}^{p}(1-e^{-\frac{a^2h^2}{2^{2k+1}}})$$
$$= \exp\left(\frac{2a^2h^2}{3}\left(1-\frac{1}{2^{2(p+1)}}\right)\right)\sup_{x\in\mathbb{R}}|\mathcal{E}^{(p)}_{f=sin_a}(x)|.$$

This suggests that an initial choice of $h = 1/a$ ensures that the upper bound is no greater than $\exp(2/3) \approx 1.94773...$ times the true value of $\sup_{x\in\mathbb{R}} |\mathcal{E}^{(p)}_{f=sin_a}(x)|$, and that the error bound improves for smaller values of $h < 1/a$.

In Table 1 we consider $f(x) = \sin(x)$ and we compare the precise analytical error term (56) against the leading order multiple term of the series representation (57) and the upper bound (58). Here we see that the leading order term is a very accurate approximation to the true error, this being due to the fact that the remaining terms in the series expansion decay at a slower rate for this case, as was observed in Lemma 4.4.

In Table 2 we consider $f(x) = \sin(4x)$ and we compare the precise analytical error term (56) against the leading order multiple term of the series representation (57) and the upper bound (58). In this case we set $h = 1/4$ which ensures that leading order term is dominant in the error expansion. The results for this case are similar to those observed for $\sin(x)$.

level p	$\prod_{k=0}^{p}(1-e^{-\frac{h^2}{2^{2k+1}}})$	$\frac{h^{2(p+1)}}{2^{(p+1)^2}}$	$\prod_{k=0}^{p}(e^{\frac{h^2}{2^{2k+1}}}-1)$
0	1.175e-01	1.25e-01	1.331e-01
1	3.61519e-03	3.90625e-03	4.226e-03
2	2.81336e-05	3.05176e-05	3.31495e-05
3	5.48949e-08	5.96046e-08	6.4808e-08
4	2.67976e-11	2.91038e-11	3.1652e-11
5	3.27099e-15	3.55271e-15	3.9968e-15
6	9.98212e-20	1.0842e-19	1.21975e-19
7	7.61572e-25	8.27181e-25	9.30596e-25

Table 1: Comparison of analytic multilevel error with leading order series approximation and general upper bound for $f(x) = \sin(x)$ with $h = \frac{1}{2}$.

level p	$\prod_{k=0}^{p}(1-e^{-\frac{4^2h^2}{2^{2k+1}}})$	$\frac{(4h)^{2(p+1)}}{2^{(p+1)^2}}$	$\prod_{k=0}^{p}(e^{\frac{4^2h^2}{2^{2k+1}}}-1)$
0	3.9347e-01	5.0 e-01	6.4872e-01
1	4.6233e-02	6.25e-02	8.6376e-02
2	1.4224e-03	1.9531e-03	2.7419e-03
3	1.1070e-05	1.5258e-05	2.1504e-05
4	2.1599e-08	2.9802e-08	4.2043e-08
5	1.0544e-11	1.4552e-11	2.0534e-11
6	1.2870e-15	1.7764e-15	2.5067e-15
7	3.9276e-20	5.42101e-20	7.64996e-20

Table 2: Comparison of analytic multilevel error and leading order series approximation and general upper bound for $f(x) = \sin(4x)$ with $h = \frac{1}{4}$.

For an example of a function that is of exponential type a yet unbounded on the real line we revisit $f(z) = \exp(az)$. The analytic Multilevel error at level p for the scaled exponential was computed in Section 4.3 (40) and given that this function trivially satisfies the growth condition $|f(z)| \le e^{a|z|}$ we can use the second part of Theorem 4.5 to bound this error as

$$|\mathcal{E}^{(p)}_{f=exp_a}(x)| \le \max_{x\in[\alpha,\beta]} e^{a|x|} |\prod_{k=0}^{p}(e^{\frac{a^2e^2h^2}{2^{2k+1}}}-1). \tag{59}$$

In Table 3 we consider $f(x) = \exp(3x)$ on the interval $[1, 2]$ and we compare the precise analytical error term (40) against the upper bound (59). Here we see that the upper bound predicts that the error decays at an exponentially fast rate as the algorithm progresses. The upper bounds in this case are not as tight as those for $\sin(ax)$ this is due to the fact that bounds on the higher derivatives derived via Cauchy's integral formula for functions that are unbounded on the real line are not as tight as those derived using Bernstein's inequality for functions bounded on the real line.

level p	$e^{3\cdot 2}\left\|\prod_{k=0}^{p}(1-e^{\frac{3^2h^2}{2^{2k+1}}})\right\|$	$e^{3\cdot 2}\prod_{k=0}^{p}(e^{\frac{(3e)^2h^2}{2^{2k+1}}}-1)$
0	28.2439	261.713
1	4.8186e-01	34.8467
2	2.0422e-03	1.10615
3	2.1604e-06	8.6757e-03
4	5.7112e-10	1.6961e-05
5	3.77423e-14	8.28386e-09
6	6.2352e-19	1.0113e-12
7	2.5752e-27	3.086216e-17

Table 3: Comparison of analytic multilevel error and the general upper bound (59) for $f(x)=\exp(3x)$ on $[1,2]$ with $h=\frac{1}{3e}$.

6 Future work

There are several directions in which this work can be taken. A natural follow up would be to extend the method to higher dimensions where it would offer the most potential. In this case the Gaussian kernel would be replaced by the d−dimensional ($d>1$) version and the convolution of $f:\mathbb{R}^d\to\mathbb{R}$ is given by

$$f*\Psi_h(\mathbf{x})=\int_{\mathbb{R}^d}f(\mathbf{y})\Psi_h(\mathbf{x}-\mathbf{y})d\mathbf{y}\quad \Psi_h(\mathbf{x})=\frac{1}{h^d(2\pi)^{\frac{d}{2}}}\exp\left(-\frac{1}{2}\frac{\|\mathbf{x}\|^2}{h^2}\right).$$

If we impose the condition that f be the restriction to $\mathbb{R}^d$ of an entire function in $\mathbb{C}^d$ then same arguments in Theorem 3.2 can be applied to show that

$$f*\Psi_h(\mathbf{x})-f(\mathbf{x})=-\sum_{|\boldsymbol{\alpha}|\geq 1}\frac{h^{2|\boldsymbol{\alpha}|}}{2^{|\boldsymbol{\alpha}|}\boldsymbol{\alpha}!}D^{2\boldsymbol{\alpha}}f(\mathbf{x}) \tag{60}$$

where $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d)$ is a d−dimensional multi-index, $|\boldsymbol{\alpha}|=\alpha_1+\cdots+\alpha_d$, $D^{2\boldsymbol{\alpha}}=\partial_{x_1}^{2\alpha_1}\cdots\partial_{x_d}^{2\alpha_d}$ and $\boldsymbol{\alpha}!=\alpha_1!\cdots\alpha_d!$. We note that (60) above is the higher-dimensional analogue of (6). Furthermore, it is well-known that the Fourier transform of $\Psi_a(\mathbf{x})$ ($a>0$) is given by

$$\widehat{\Psi_a}(\mathbf{z})=\int_{\mathbb{R}^d}\Psi_a(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{z}}d\mathbf{x}=e^{-\frac{a^2\|\mathbf{z}\|^2}{2}}$$

and, using the same proof technique as Lemma 3.1 one can establish that

$$\Psi_a*\Psi_b=\Psi_{\sqrt{a^2+b^2}}. \tag{61}$$

The two ingredients (60) and (61) are enough to allow an investigation of the early performance of the higher dimensional multilevel Gaussian convolution algorithm and from there it ought to be possible to generalise, as in the univariate case, to provide analagous error bounds in the higher dimensional setting; this is a topic for future research. We remark that once the higher-dimensional analogue is established one could explore the multilevel algorithm over a sequence of sparse grids as was initiated in [10] for the periodic setting.

An alternative project would be to consider the discrete version, where we replace convolution with quasi-interpolation; here it would be interesting to compare the convergence in both cases and understand the connection, given that the quasi-interpolant will be a good approximation to the convolution. Another direction would be to restrict attention to approximation on the finite interval $[-1,1]$ say, and explore how the two sided truncations explain the expected deterioration in the convergence rate near the end points; the quasi-interpolation version of this could be tested on both equally spaced points and also on Chebychev points. Work on such problems, albeit using the multi-quadric basis function, can be found in [1]. We will leave these issues for future work.

A Appendix: Proofs

A.1 Proof of Theorem 4.1

Proof. The proof will be established in two parts. First, we will use induction to prove that:

$$\mathcal{S}_f^{(p)}(x)=\sum_{\mathbf{i}_p\in\mathcal{I}_p}(-1)^{|\mathbf{i}_p|}f*\psi_{h\sqrt{\frac{1+2^2i_1+2^4i_2+\cdots+2^{2p}i_p}{2^{2p}}}}(x). \tag{62}$$

Then, secondly, we will use this formula, to show that

$$\mathcal{S}_f^{(p)}(x)=\sum_{i=1}^{\infty}\frac{f^{(2i)}(x)h^{2i}\sigma_p(i)}{2^{(2p+1)i}i!},\ \text{ where }\ \sigma_p(i)=\sum_{\mathbf{i}_p\in\mathcal{I}_p}(-1)^{|\mathbf{i}_p|}(1+2^2i_1+2^4i_2+\cdots+2^{2p}i_p)^i. \tag{63}$$

Part 1

We have shown in (3.2), (13), and (20), that (62) holds for $p = \{0,1,2\}$. By induction we assume that (62) is valid for $j = \{0,1,2,\dots,p\}$ and consider $\mathcal{S}_f^{(p+1)}(x)$ which, by formula (34), is given by

$$\mathcal{S}_f^{(p+1)}(x) = (f - \mathcal{M}_f^{(p)}) * \psi_{\frac{h}{2^{p+1}}}(x), \text{ where } \mathcal{M}_f^{(p)}(x) = \sum_{j=0}^{p} \mathcal{S}_f^{(j)}(x).$$

Using the inductive hypothesis, keeping in mind that $\mathcal{S}_f^{(0)}(x) = f * \psi_h(x)$, and employing Lemma 3.1, we have

$$\begin{aligned}\mathcal{S}_f^{(p+1)}(x) &= \Big(f - \sum_{j=0}^{p} \mathcal{S}_f^{(j)}(x)\Big) * \psi_{\frac{h}{\sqrt{2^{2(p+1)}}}}(x) = \Big(f - f*\psi_h - \sum_{j=1}^{p} \mathcal{S}_f^{(j)}(x)\Big) * \psi_{\frac{h}{\sqrt{2^{2(p+1)}}}}(x)\\ &= f * \psi_{\frac{h}{\sqrt{2^{2(p+1)}}}}(x) - f * \psi_{h\sqrt{\frac{1+2^{2(p+1)}}{2^{2(p+1)}}}}(x)\\ &- \sum_{j=1}^{p} \sum_{\boldsymbol{i}_j \in \mathcal{I}_j} (-1)^{|\boldsymbol{i}_j|} f * \psi_{h\sqrt{\frac{1}{2^{2(p+1)}} + \frac{1+2^2 i_1 + 2^4 i_2 + \cdots + 2^{2j} i_j}{2^{2j}}}}(x).\end{aligned} \tag{64}$$

Simplifying and re-indexing the sum (setting $k = p+1-j$) we have that

$$\mathcal{S}_f^{(p+1)}(x) = f * \psi_{\frac{h}{\sqrt{2^{2(p+1)}}}}(x) - f * \psi_{h\sqrt{\frac{1+2^{2(p+1)}}{2^{2(p+1)}}}}(x) - \sum_{k=1}^{p} \mathcal{A}_f^{(k,p+1)}(x) \tag{65}$$

where,

$$\mathcal{A}_f^{(k,p+1)}(x) = \sum_{\boldsymbol{i}_{p+1-k} \in \mathcal{I}_{p+1-k}} (-1)^{|\boldsymbol{i}_{p+1-k}|} f * \psi_{h\sqrt{\frac{1+2^{2k}+2^{2(k+1)} i_1 + \cdots + 2^{2(p+1)} i_{p+1-k}}{2^{2(p+1)}}}}(x), \quad 1 \le k \le p.$$

Let α_k denote the numerator sum under the square root in the above expression and observe that

$$\begin{aligned}\alpha_1 &= 1 + 2^2 \cdot 1 + 2^4 \cdot i_1 + 2^6 \cdot i_2 + \cdots + 2^{2p} \cdot i_{p-1} + 2^{2(p+1)} \cdot i_p\\ \alpha_2 &= 1 + 2^2 \cdot 0 + 2^4 \cdot 1 + 2^6 \cdot i_1 + \cdots + 2^{2p} \cdot i_{p-2} + 2^{2(p+1)} \cdot i_{p-1}\\ &\vdots\\ \alpha_p &= 1 + 2^2 \cdot 0 + 2^4 \cdot 0 + 2^6 \cdot 0 + \cdots + 2^{2p} \cdot 1 + 2^{2(p+1)} \cdot i_1\end{aligned}$$

where in each line $i_j \in \{0,1\}$, $j = 1,\dots,p$. Without placing any concern on the indexing of the binary terms, we can express the general term as

$$\alpha_k = 1 + 2^2 \cdot 0 \cdots 2^{2(k-1)} \cdot 0 + 2^{2k} \cdot 1 + 2^{2(k+1)} i_{k+1} + \cdots + 2^{2(p+1)} \cdot i_{p+1},$$

which can be written as 1 plus the dot product of $(2^2, 2^4, \cdots, 2^{2(p+1)})$ and a vector $\boldsymbol{i}_{p+1}^k = (0,\cdots,1,i_{k+1},\cdots,i_{p+1})$, i.e.,

$$\alpha_k = 1 + (2^2, 2^4, \cdots, 2^{2(p+1)}) \underbrace{(0,\cdots,1,i_{k+1},\cdots,i_{p+1})^T}_{=\boldsymbol{i}_{p+1}^k}$$

We recognise that vectors of the form $\boldsymbol{i}_{p+1}^k$ belong to the disjoint subspaces $\mathcal{I}_{p+1}^k$ (see (32)) of $\mathcal{I}_{p+1}$, where $k = 1,\dots,p$. Thus we can write

$$\begin{aligned}\mathcal{A}_f^{(k,p+1)}(x) &= \sum_{\boldsymbol{i}_{p+1}^k \in \mathcal{I}_{p+1}^k} (-1)^{|\boldsymbol{i}_{p+1}^k|} f * \psi_{h\sqrt{\frac{1+2^2 i_1 + \cdots + 2^{2k} i_k + 2^{2(k+1)} i_{k+1} + \cdots + 2^{2(p+1)} i_{p+1}}{2^{2(p+1)}}}}(x)\\ &= -\sum_{\boldsymbol{i}_{p+1}^k \in \mathcal{I}_{p+1}^k} (-1)^{i_{k+1}+\cdots+i_{p+1}} f * \psi_{h\sqrt{\frac{1+2^2 i_1 + \cdots + 2^{2k} i_k + 2^{2(k+1)} i_{k+1} + \cdots + 2^{2(p+1)} i_{p+1}}{2^{2(p+1)}}}}(x)\end{aligned}$$

We recall from Property 3.5 that

$$\mathcal{I}_{p+1} = \bigcup_{k=1}^{p} \mathcal{I}_{p+1}^k \cup \mathcal{I}_{p+1}^0 \cup \mathcal{I}_{p+1}^{p+1},$$

where $\mathcal{I}_{p+1}^0 = \{(0,0,\dots,0)\}$, $\mathcal{I}_{p+1}^{p+1} = \{(0,\dots,0,1)\}$.

We observe that

$$f * \psi_{\frac{h}{\sqrt{2^{2(p+1)}}}}(x) = (-1)^{0+\cdots+0} f * \psi_{h\sqrt{\frac{1+2^2\cdot 0+\cdots+2^{2(p+1)}\cdot 0}{2^{2(p+1)}}}}(x)$$

and

$$-f * \psi_{h\sqrt{\frac{1+2^{2(p+1)}}{2^{2(p+1)}}}}(x) = (-1)^{0+\cdots+0+1} f * \psi_{h\sqrt{\frac{1+2^2\cdot 0+\cdots+2^{2p}\cdot 0+2^{2(p+1)}\cdot 1}{2^{2(p+1)}}}}(x).$$

Thus, in view of (65), we can conclude that

$$\begin{aligned}
\mathcal{S}_f^{(p+1)}(x) &= f * \psi_{\frac{h}{\sqrt{2^{2(p+1)}}}}(x) - f * \psi_{h\sqrt{\frac{1+2^{2(p+1)}}{2^{2(p+1)}}}}(x) - \sum_{k=1}^{p} \mathcal{A}_f^{(k,p+1)}(x) \\
&= (-1)^{0+\cdots+0} f * \psi_{h\sqrt{\frac{1+2^2\cdot 0+\cdots+2^{2(p+1)}\cdot 0}{2^{2(p+1)}}}}(x) \\
&+ (-1)^{0+\cdots+0+1} f * \psi_{h\sqrt{\frac{1+2^2\cdot 0+\cdots+2^{2p}\cdot 0+2^{2(p+1)}\cdot 1}{2^{2(p+1)}}}}(x). \\
&+ \sum_{k=1}^{p} \sum_{i_{p+1}^k \in \mathcal{I}_{p+1}^k} (-1)^{i_{k+1}+\cdots+i_{p+1}} f * \psi_{h\sqrt{\frac{1+2^2 i_1+\cdots+2^{2k} i_k + 2^{2(k+1)} i_{k+1}+\cdots+2^{2(p+1)} i_{p+1}}{2^{2(p+1)}}}}(x) \\
&= \sum_{i_{p+1}\in\mathcal{I}_{p+1}} (-1)^{|i_{p+1}|} f * \psi_{h\sqrt{\frac{1+2^2 i_1+2^4 i_2+\cdots+2^{2(p+1)} i_{p+1}}{2^{2(p+1)}}}}(x).
\end{aligned}$$

Hence, formula (62) is valid.

Part 2

For the second part, we have established that

$$\mathcal{S}_f^{(p)}(x) = \sum_{i_p\in\mathcal{I}_p} (-1)^{|i_p|} f * \psi_{h\sqrt{\frac{1+2^2 i_1+2^4 i_2+\cdots+2^{2p} i_p}{2^{2p}}}}(x)$$

and from here we can use (7) to deal with the convolution to give

$$\begin{aligned}
\mathcal{S}_f^{(p)}(x) &= \sum_{i_p\in\mathcal{I}_p} (-1)^{|i_p|} \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^i i!} \left(h \sqrt{\frac{1+2^2 i_1+2^4 i_2+\cdots+2^{2p} i_p}{2^{2p}}} \right)^{2i} \\
&= \sum_{i_p\in\mathcal{I}_p} (-1)^{|i_p|} \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^i i!} h^{2i} \frac{(1+2^2 i_1+2^4 i_2+\cdots+2^{2p} i_p)^i}{2^{2pi}} \\
&= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{(2p+1)i} i!} h^{2i} \sum_{i_p\in\mathcal{I}_p} (-1)^{|i_p|} (1+2^2 i_1+2^4 i_2+\cdots+2^{2p} i_p)^i \\
&= \sum_{i=0}^{\infty} \frac{f^{(2i)}(x)}{2^{(2p+1)i} i!} h^{2i} \sigma_p(i),
\end{aligned}$$

where

$$\sigma_p(i) = \sum_{i_p\in\mathcal{I}_p} (-1)^{|i_p|} (1+2^2 i_1+2^4 i_2+\cdots+2^{2p} i_p)^i. \tag{66}$$

Now, for the term corresponding to $i=0$ in this formula, we have:

$$\begin{aligned}
&\frac{f^{(2*0)}(x)}{2^{(2p+1)*0} 0!} h^{2*0} \sum_{i_p\in\mathcal{I}_p} (-1)^{|i_p|} (1+2^2 i_1+2^4 i_2+\cdots+2^{2p} i_p)^0 \\
&= f(x) \sum_{i_p\in\mathcal{I}_p} (-1)^{|i_p|} = 0,
\end{aligned}$$

where the final line follows from Lemma 3.6. This concludes the proof of Theorem 4.1.

□

A.2 Proof of Theorem 4.2

Proof. Using induction, we prove that the multilevel Gaussian convolution approximation at level p for an entire function f as presented in equations (36, 37) is as follows:

$$\mathcal{M}_f^{(p)}(x) = \; = f(x) + \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)}{2^{(2p+1)i} i!} h^{2i} \mu_p(i), \tag{A.2.1}$$

where

$$\mu_p(i) = \sum_{i_p\in\mathcal{I}_p} (-1)^{1+|i_p|} \Big[(2^2 i_1+\cdots+2^{2p} i_p)^i - (1+2^2 i_1+\cdots+2^{2p} i_p)^i \Big]. \tag{A.2.2}$$

Equations (5), (10), and (17) show that that the above expression is valid for $p = 0, 1, 2$. By induction we assume that it is valid for level p and we consider the level $(p + 1)$ case.

Referencing the Multilevel convolution algorithm as defined in the introduction, we know that $\mathcal{M}_f^{(p+1)}(x) = \mathcal{M}_f^{(p)}(x) + \mathcal{S}_f^{(p+1)}(x)$. Hence, using the inductive hypothesis and Theorem 4.1, we substitute to obtain:

$$\begin{aligned}\mathcal{M}_f^{(p+1)}(x) &= f(x)\\ &+ \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+1)i}i!} \sum_{i_p \in \mathcal{I}_p} (-1)^{1+|i_p|} \Big[(2^2 i_1 + \cdots + 2^{2p} i_p)^i - (1 + 2^2 i_1 + \cdots + 2^{2p} i_p)^i \Big] \\ &+ \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+3)i}i!} \sum_{i_{p+1} \in \mathcal{I}_{p+1}} (-1)^{|i_{p+1}|} (1 + 2^2 i_1 + \cdots + 2^{2(p+1)} i_{p+1})^i.\end{aligned} \tag{67}$$

In the first series of the above expression we multiply the terms in its outer sum by $\frac{2^{2i}}{2^{2i}}$ to give

$$\begin{aligned}&\sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+1)i}i!} \sum_{i_p \in \mathcal{I}_p} (-1)^{1+|i_p|} \Big[(2^2 i_1 + \cdots + 2^{2p} i_p)^i - (1 + 2^2 i_1 + \cdots + 2^{2p} i_p)^i \Big] \\ &= \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+3)i}i!} \sum_{i_p \in \mathcal{I}_p} (-1)^{1+|i_p|} \Big[(2^4 i_1 + \cdots + 2^{2p+2} i_p)^i - (2^2 + 2^4 i_1 + \cdots + 2^{2p+2} i_p)^i \Big] \\ &= \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+3)i}i!} \sum_{i_p \in \mathcal{I}_p} (-1)^{1+|i_p|} \Big[(2^2 \cdot 0 + 2^4 i_1 + \cdots + 2^{2p+2} i_p)^i - (2^2 \cdot 1 + 2^4 i_1 + \cdots + 2^{2p+2} i_p)^i \Big].\end{aligned}$$

We now extend the p-dimensional binary vector to a $(p + 1)$-dimensional binary vector such that $i_k \mapsto i_{k+1}$ for $k = 1, \ldots, p$ and introducing a new $i_1 \in \{0, 1\}$ to obtain:

$$\begin{aligned}&\sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+3)i}i!} \sum_{i_p \in \mathcal{I}_p} (-1)^{1+|i_p|} \Big[(2^2 \cdot 0 + 2^4 i_1 + \cdots + 2^{2p+2} i_p)^i - (2^2 \cdot 1 + 2^4 i_1 + \cdots + 2^{2p+2} i_p)^i \Big] \\ &= \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+3)i}i!} \sum_{i_p \in \mathcal{I}_p} (-1)^{1+|i_p|} \Big[(-1)^{i_1} (2^2 \cdot i_1 + 2^4 \cdot i_2 + \cdots + 2^{2p+2} \cdot i_{p+1})^i \Big] \\ &= \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+3)i}i!} \sum_{i_{p+1} \in \mathcal{I}_{p+1}} (-1)^{1+|i_{p+1}|} \Big[(2^2 \cdot i_1 + 2^4 i_2 + \cdots + 2^{2p+2} i_{p+1})^i \Big].\end{aligned}$$

Using the conclusion above in (67) we find that

$$\begin{aligned}\mathcal{M}_f^{(p+1)}(x) &= f(x)\\ &+ \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+3)i}i!} \sum_{i_{p+1} \in \mathcal{I}_{p+1}} (-1)^{1+|i_{p+1}|} \Big[(2^2 \cdot i_1 + 2^4 i_2 + \cdots + 2^{2p+2} i_{p+1})^i \Big] \\ &+ \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)h^{2i}}{2^{(2p+3)i}i!} \sum_{i_{p+1} \in \mathcal{I}_{p+1}} (-1)^{|i_{p+1}|} (1 + 2^2 i_1 + \cdots + 2^{2p+2} i_{p+1})^i \\ &= f(x) + \sum_{i=1}^{\infty} \frac{f^{(2i)}(x)}{2^{(2p+3)i}i!} h^{2i} \sum_{i_{p+1} \in \mathcal{I}_{p+1}} (-1)^{1+|i_{p+1}|} \mu_{p+1}(i),\end{aligned} \tag{68}$$

where

$$\mu_{p+1}(i) = \sum_{i_{p+1} \in \mathcal{I}_{p+1}} (-1)^{1+|i_{p+1}|} \Big[(2^2 i_1 + 2^4 i_2 + \cdots + 2^{2p+2} i_{p+1})^i - (1 + 2^2 i_1 + \cdots + 2^{2p+2} i_{p+1})^i \Big].$$

This concludes the proof of Theorem 4.2. □

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