Subdivision Schemes for Geometric Modelling

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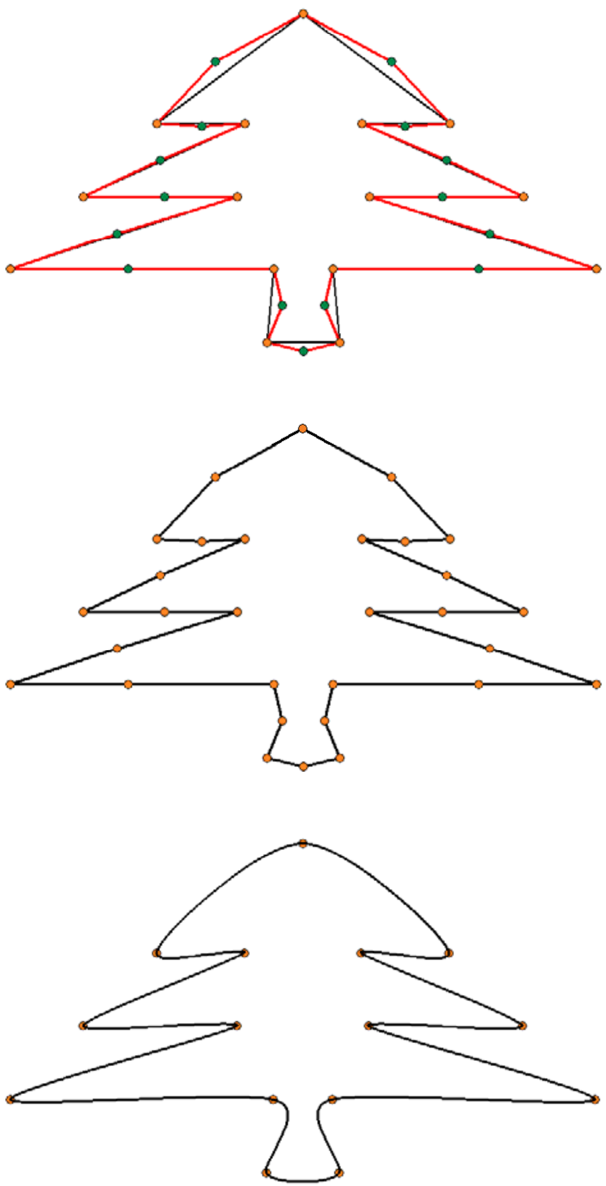
What it is all about

"Subdivision defines a smooth curve or surface as the limit of a sequence of successive refinements."

Denis Zorin & Peter Schröder
SIGGRAPH 98 course notes

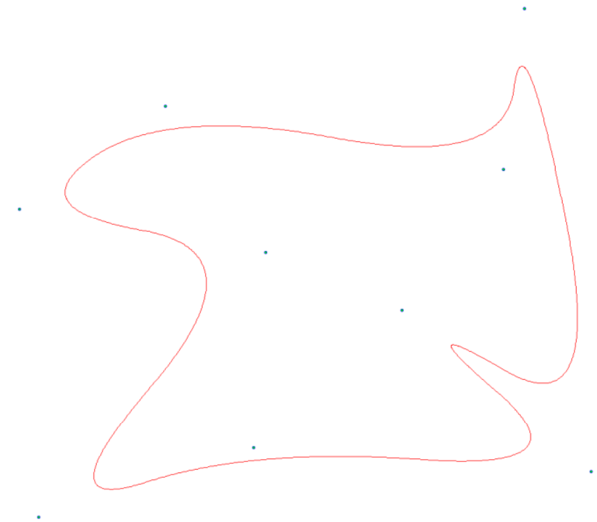
Subdivision curves

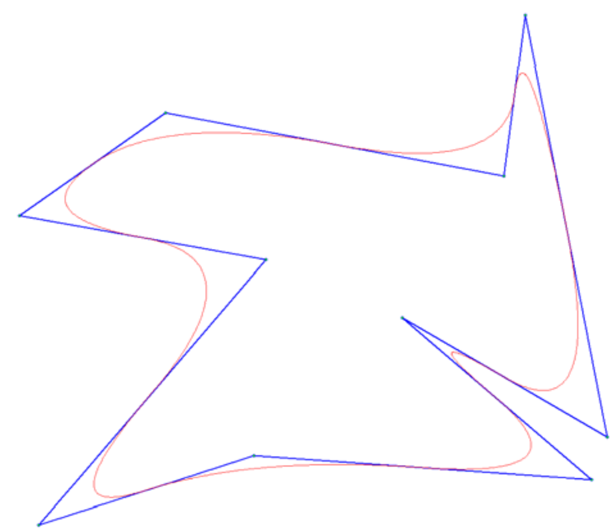
- initial control polygon
  - rough, user-sketched shape
- iterative refinement
  - adding new points
  - simple local rules
- smooth limit curve
  - finitely many steps in practice
  - up to pixel accuracy



Applications

- subdivision curves in computer-aided design
  - modify initial control points
  - alternative to splines

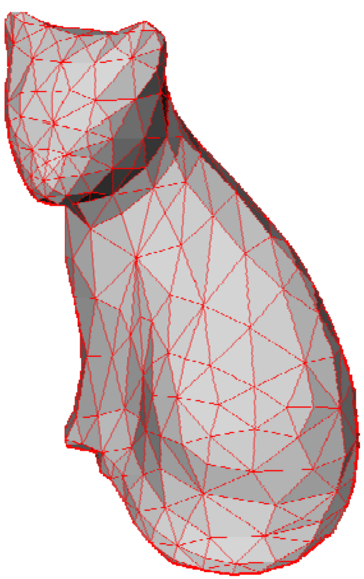


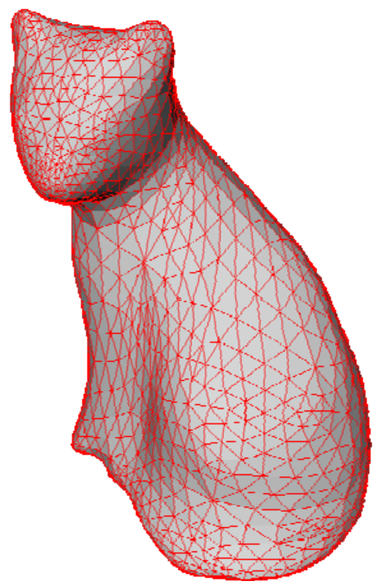


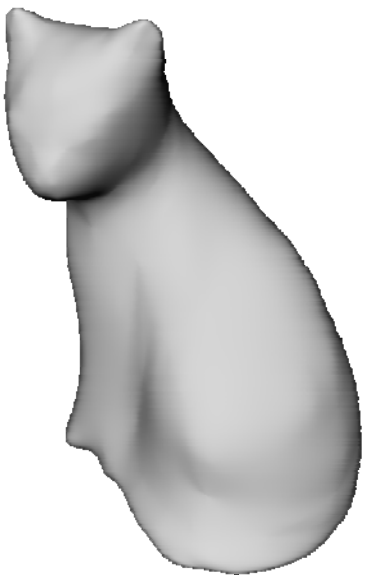
- simple and efficient curve representation

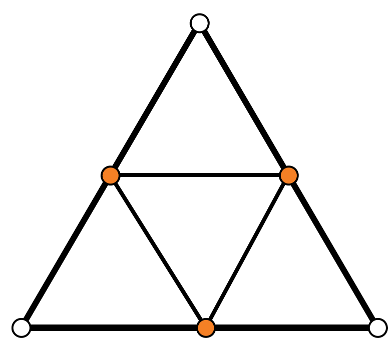
Subdivision surfaces

- iterative, regular refinement
  - simple local rules
  - efficient









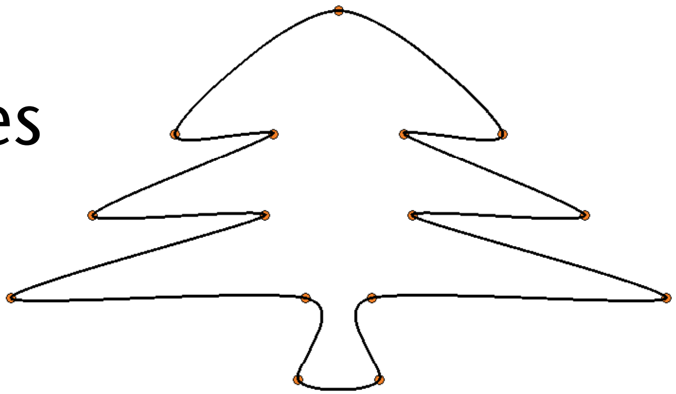
Applications

- subdivision surfaces in computer graphics

- simple and efficient surface representation

Subdivision curves and surfaces

- vast playground
  - abundance of rules and schemes
- standard goals
  - convergence
  - smoothness
- standard limitations
  - artefacts at extraordinary vertices
- new trend
  - nonlinear, geometric methods



Subdivision curves and surfaces

- important properties
  - convergence
    - existence of limit curve/surface
  - smoothness of the limit curve/surface
  - affine invariance
  - simple local rules
    - efficient evaluation
    - compact support of basis function
  - polynomial reproduction
    - approximation order

Outline

- ***Sep 5 – Subdivision as a linear process***
  - ***basic concepts, notation, subdivision matrix***
- Sep 6 – The Laurent polynomial formalism
  - algebraic approach, polynomial reproduction
- Sep 7 – Smoothness analysis
  - Hölder regularity of limit by spectral radius method
- Sep 8 – Subdivision surfaces
  - overview of most important schemes & properties

Basic concept and notation

- initial *control polygon*
  - sequence of *control points* $\{p_i^0\}$ (2D/3D) at *level* 0
  - open ($i \in \{i_{min}, \ldots, i_{max}\}$) or closed ($i \in \mathbb{Z}$)
- *refinement rules*
  - how to get from level j to level $j+1$
  - simplest case (*interpolatory* subdivision)

$$p_{2i}^{j+1} = p_i^j, \qquad p_{2i+1}^{j+1} = \sum_k \beta_k p_{i-k}^j$$

  - reproduce *old points* and insert *new points*

A simple interpolatory scheme

- **Example**

$$p_{2i}^{j+1} = p_i^j, \qquad p_{2i+1}^{j+1} = \underbrace{\tfrac{1}{2}}_{\beta_0} p_i^j + \underbrace{\tfrac{1}{2}}_{\beta_{-1}} p_{i+1}^j$$

  - new points at old edge midpoints
  - shape of control polygon does not change
  - points become denser with increasing level j
  - control polygon is also the *limit curve*
  - refinement rules depend on 2 old points at most, hence this is called a *2-point scheme*

A non-interpolatory scheme

- **Example**

$$p_{2i}^{j+1} = \tfrac{1}{8}\,p_{i-1}^{j} + \tfrac{6}{8}\,p_{i}^{j} + \tfrac{1}{8}\,p_{i+1}^{j}, \qquad p_{2i+1}^{j+1} = \tfrac{1}{2}\,p_{i}^{j} + \tfrac{1}{2}\,p_{i+1}^{j}$$

  - *approximating* scheme (old points are modified)
  - 3-point scheme
  - gives uniform cubic B-splines in the limit
- convenient notation for refinement rules
  - *even stencil* [1,6,1]/8 *odd stencil* [1,1]/2
  - index offsets clear by symmetry

The subdivision matrix

- write refinement rules, one below the other,

$$\begin{pmatrix} \vdots \\ p_{2i-2}^{j+1} \\ p_{2i-1}^{j+1} \\ p_{2i}^{j+1} \\ p_{2i+1}^{j+1} \\ p_{2i+2}^{j+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \ddots & & & & & & \\ & \frac{1}{8} & \frac{6}{8} & \frac{1}{8} & & & \\ & & \frac{1}{2} & \frac{1}{2} & & & \\ & & \frac{1}{8} & \frac{6}{8} & \frac{1}{8} & & \\ & & & \frac{1}{2} & \frac{1}{2} & & \\ & & & \frac{1}{8} & \frac{6}{8} & \frac{1}{8} & \\ & & & & & & \ddots \end{pmatrix} \cdot \begin{pmatrix} \vdots \\ p_{i-2}^{j} \\ p_{i-1}^{j} \\ p_{i}^{j} \\ p_{i+1}^{j} \\ p_{i+2}^{j} \\ \vdots \end{pmatrix}$$

- gives the ***subdivision matrix*** S
  - with stencils as rows (alternating and shifted)

Subdivision in the limit

- refinement from level j to level $j+1$

$$\boldsymbol{p}^{j+1} = S\boldsymbol{p}^j$$

- refinement from level 0 to level j

$$\boldsymbol{p}^j = S\boldsymbol{p}^{j-1} = S^2\boldsymbol{p}^{j-2} = \cdots = S^j\boldsymbol{p}^0$$

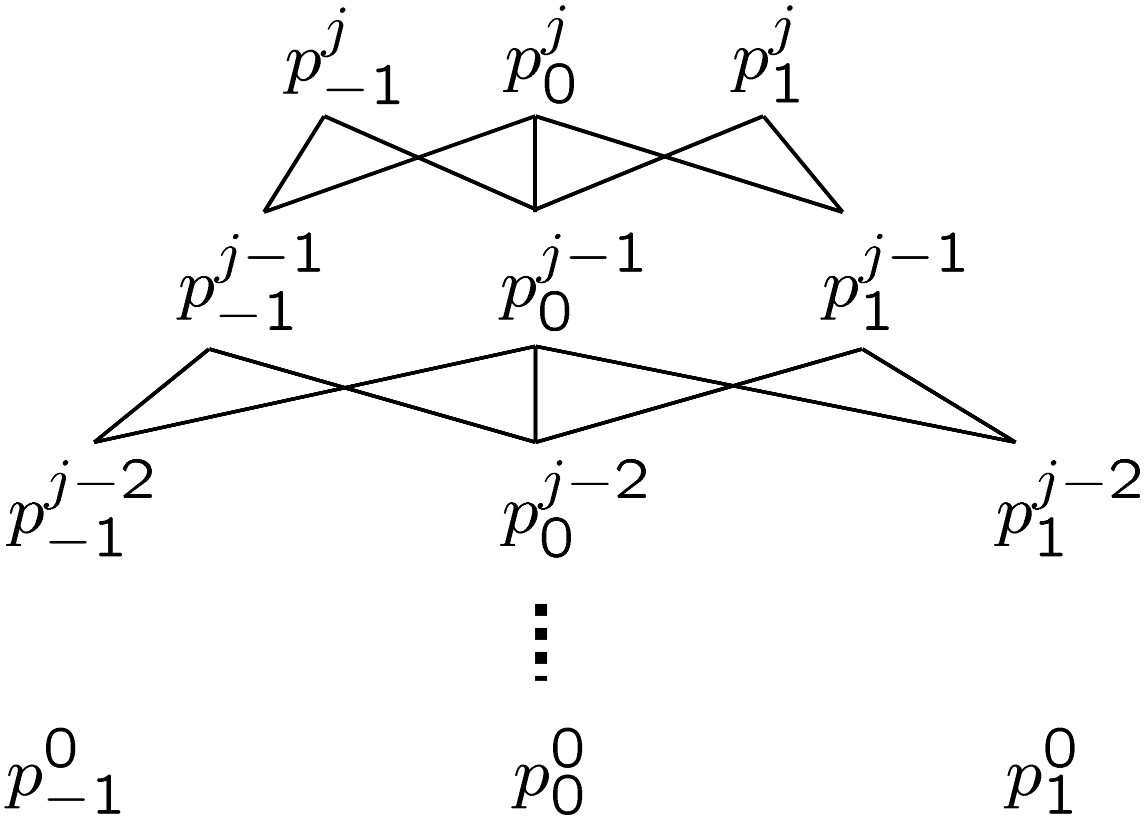
- refinement in the limit

$$\boldsymbol{p}^\infty = \lim_{j\to\infty} \boldsymbol{p}^j = \lim_{j\to\infty} S^j\boldsymbol{p}^0 = S^\infty\boldsymbol{p}^0$$

  - but S is an infinite matrix, so what is S^∞ ?

Subdivision in the limit

- local analysis with ***invariant neighbourhood***
  - which initial control points determine p_0^j?



Subdivision in the limit

- local analysis for p_0^j in the limit (as $j \to \infty$)

$$\begin{pmatrix} p_{-1}^j \\ p_0^j \\ p_1^j \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{8} & \frac{6}{8} & \frac{1}{8} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}}_{=S} \cdot \begin{pmatrix} p_{-1}^{j-1} \\ p_0^{j-1} \\ p_1^{j-1} \end{pmatrix}$$

  - compute eigendecomposition of S

$$S = Q \cdot \Lambda \cdot Q^{-1}, \quad Q = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & & \\ & 1/2 & \\ & & 1/4 \end{pmatrix}$$

  - and so $S^j = (Q \cdot \Lambda \cdot Q^{-1})^j = Q \cdot \Lambda^j \cdot Q^{-1}$

Subdivision in the limit

- now letting $j \to \infty$

$$S^{\infty} = \lim_{j\to\infty} S^j = \lim_{j\to\infty} Q \cdot \Lambda^j \cdot Q^{-1} = Q(\lim_{j\to\infty} \Lambda^j) Q^{-1}$$

$$\begin{pmatrix} p_{-1}^{\infty} \\ p_0^{\infty} \\ p_1^{\infty} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} p_{-1}^{0} \\ p_0^{0} \\ p_1^{0} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} p_{-1}^{0} \\ p_0^{0} \\ p_1^{0} \end{pmatrix}$$

Subdivision in the limit

- observations
  - convergence requires that 1 is an eigenvalue of S and that all other eigenvalues are smaller
  - first row of Q^{-1} gives the ***limit stencil*** $[1,4,1]/6$
  - applying it to three consecutive control points gives the limit position of the central one
  - works on any level
  - can be used to "snap" the points to the limit curve
  - can be used to estimate distance to the limit curve

The 4-point scheme

- **Example**

$$p_{2i}^{j+1} = p_i^j$$
$$p_{2i+1}^{j+1} = \frac{1}{16}(-p_{i-1}^j + 9p_i^j + 9p_{i+1}^j - p_{i+2}^j)$$

  - classical *interpolatory 4-point scheme* [Dubuc 1986]
  - based on local cubic interpolation
  - even stencil [1] odd stencil [–1,9,9,–1]/16
  - invariant neighbourhood size: 5
  - eigenvalues of S: 1, ½, ¼, ¼, ⅛
  - limit stencil [1] (true for all interpolatory schemes)

The general 3-point scheme

- **Example**

$$p_{2i}^{j+1} = \alpha_1 p_{i-1}^j + \alpha_0 p_i^j + \alpha_{-1} p_{i+1}^j$$
$$p_{2i+1}^{j+1} = \beta_0 p_i^j + \beta_{-1} p_{i+1}^j$$

- constraints: symmetry and summation to 1
  - even stencil $[w, 1-2w, w]$ odd stencil $[1,1]/2$
- invariant neighbourhood size: 3
- eigenvalues of S: $1, 1/2, 1/2-2w$
- certainly not convergent for $w \notin (-1/4, 3/4)$
- limit stencil $[2w, 1, 2w]/(1+4w)$

Summary

- initial control points $\{p_i^0\}$ and refined data $\{p_i^j\}$
- refinement rules
  - even stencil $[\ldots, \alpha_2, \alpha_1, \alpha_0, \alpha_{-1}, \alpha_{-2}, \ldots]$
  - odd stencil $[\ldots, \beta_2, \beta_1, \beta_0, \beta_{-1}, \beta_{-2}, \ldots]$
  - rules $p_{2i}^{j+1} = \sum_k \alpha_k p_{i-k}^j, \qquad p_{2i+1}^{j+1} = \sum_k \beta_k p_{i-k}^j$
- subdivision matrix
  - stencils as rows (alternating, shifted)

$$S = \begin{pmatrix} \ddots & & & & & & \\ & \alpha_1 & \alpha_0 & \alpha_{-1} & \alpha_{-2} & \alpha_{-3} & \\ & \beta_1 & \beta_0 & \beta_{-1} & \beta_{-2} & \beta_{-3} & \\ & \alpha_2 & \alpha_1 & \alpha_0 & \alpha_{-1} & \alpha_{-2} & \\ & \beta_2 & \beta_1 & \beta_0 & \beta_{-1} & \beta_{-2} & \\ & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \alpha_{-1} & \\ & & & & & & \ddots \end{pmatrix}$$

Summary

- local limit analysis
  - determine size n of invariant neighbourhood
  - consider local $n \times n$ subdivision matrix S
  - ***necessary condition*** for convergence

    1 is the unique largest eigenvalue of S

  - if coefficients of even/odd stencil sum to 1, then
    - 1 is an eigenvalue of S with eigenvector (1,…,1)
    - subdivision scheme is affine invariant
  - limit stencil given by normalized ***left eigenvector*** of S with eigenvalue 1 (usual (right) eigenvector of S^T)