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Stability to perturbations of Stieltjes continued fractions on domain in the space $\mathbb{R}$ and applications

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Abstract

This paper focuses on the study and practical application of Stieltjes continued fractions as a stability to perturbations instrument for function approximation. A framework for the stability analysis of S-fractions is developed. We establish conditions under which the continued fraction is stable to perturbations of its coefficients and variable. The derived results connect the fraction's stability with the properties of its coefficients and the variable's domain. To illustrate the theoretical findings, we present an experiment modeling an asymptotic process: the reduction of consumer demand after an advertising campaign has ended. A comparative analysis is performed between a second-order S-fractional model and a polynomial model of the same order. The results show that the S-fraction model not only achieves a lower approximation error and a higher coefficient of determination but also, in contrast to the polynomial model, provides an accurate long-term forecast. The key finding is that the S-fractional model maintains its accuracy to coefficient perturbations, while similar perturbations cause the polynomial model to deviate significantly.

1 Introduction

Among the many classes of functional continued fractions, Stieltjes continued fractions (S-fractions) are of particular importance owing to their connection with the Stieltjes moment problem and the theory of orthogonal polynomials [25, 30]. The relationship between S-fractions and the moment problem, first established by Stieltjes [29], has laid the groundwork for their wide application [2, 17, 18, 28].

S-fractions find important application in the theory of differential equations, where S-fractions arise as solutions to the Riccati equation [27]. S-fractions are also an important tool in function theory and offer significant advantages over traditional power series. In many cases, their domains of convergence are wider, making them an effective tool for function approximation [3, 4, 5].

A central problem in the analytic theory of continued fractions is the study of their convergence. For S-fractions, their convergence is addressed by Stieltjes' theorem: provided that their coefficients are bounded, such a fraction converges to a holomorphic function in the entire complex plane, except for a cut along the negative real axis [29]. While classical S-fractions are used to approximate functions of a single variable, multidimensional S-fractions and multidimensional S-fractions with unequal variables serve as an effective tool for approximating functions of severalvariables [7, 8, 10, 12]. The convergence of multidimensional S-fractions and their use in approximating analytic functions of several variables have been considered in the works [1, 14, 15, 16].

However, in the era of digital computation, where continued fractions and their generalizations are used as a practical tool for function approximation, the question of convergence alone is insufficient. No less important, and often more critical, is the numerical stability of continued fractions, which characterizes the sensitivity of algorithms for computing their approximants to rounding errors in the elements and machine operation errors. If an algorithm for computing the approximants of a continued fraction is unstable, even minor errors arising from the computation of its elements can lead to a significant loss of accuracy in its approximants. Early research in this area focused on the computational stability of the BR- and FR-algorithms or their modifications [23, 24] and was continued in the works [9, 11, 13, 22].

Modern research on stability to perturbations of continued fractions involves investigating the influence of perturbations in the coefficients on the values of their approximants. In applied problems, these coefficients may be obtained from experimental

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data or as a result of computations with finite precision. In this context, results on the stability of certain classes of continued fractions have been derived. For example, in the work [6], conditions have been established and sets of stability have been constructed for continued fractions with complex partial denominators and with all partial numerators equal to one. Furthermore, this analysis has been extended to S-fractions with a complex variable. Specifically, in [20], circular and semi-circular sets of stability have been constructed, within which the boundedness of the relative errors of the approximants is guaranteed. The stability of g-fractions to perturbations of their elements has been investigated in [19, 21], where sufficient stability conditions were established and stability sets in both real and complex domains were constructed.

This paper examines the stability of S-fractions to perturbations of their coefficients and real variable. This is a critical issue in applied modeling, where system parameters are often known imprecisely, yet the model is required to be stable. The primary objective is to establish conditions that guarantee the stability of S-fractions and to showcase the practical advantages of this approach through a time-series analysis case study. The theoretical part focuses on deriving sufficient conditions for stability to perturbations. Section 2 introduces the notion of relative errors for the fraction's coefficients, its variable, and its approximants. From these definitions, recurrence formulas for the error of an S-fraction approximant are derived. The paper's main result, Theorem 3.1 (Section 3), proves that an S-fraction is stable to perturbations at a point $z_0 \in \mathbb{R}_+$ if a sequence of positive constants exists that majorizes $g_k^{(n)}(z_0)$, and the numerical series whose elements depend on these constants is divergent. This guarantees that the relative error of approximants has a continuous dependence on the errors in the S-fraction's coefficients and variable. Furthermore, stability conditions are established for the case where the coefficients and variable are bounded. In Section 4, the practical application of these findings is demonstrated by modeling the dynamics of consumer demand following an advertising campaign. Two methods are compared for data approximation: a classical 2nd order polynomial model and a 2nd order S-fractional model. The results indicate that although both models fit the training data well, the polynomial model is unsuitable for forecasting as it predicts unrealistic demand growth. The S-fractional model, however, not only provides a more accurate description of the data but also yields an adequate forecast. Upon perturbation of the coefficients, the S-fractional model maintains its accuracy. This empirically validates its theoretically proven stability, establishing it as a reliable tool for forecasting.

2 Preliminaries

We consider a functional continued fraction

$$\cfrac{a_0}{1+\cfrac{a_1 z}{1+\cfrac{a_2 z}{1+\ddots}}}, \tag{1}$$

which is called an S-fraction, where $a_k \in \mathbb{R}_+$, $k \geq 0$, $z \in \mathbb{R}$.

The expressions

$$f_n(z) = \cfrac{a_0}{1+\cfrac{a_1 z}{1+\cfrac{a_2 z}{1+\ddots+\cfrac{a_n z}{1}}}}, \quad n \geq 0,$$

are called the nth approximants of the S-fraction (1).

The quantities

$$Q_k^{(n)}(z) = 1 + \cfrac{a_{k+1} z}{1+\cfrac{a_{k+2} z}{1+\cfrac{a_{k+3} z}{1+\ddots+\cfrac{a_n z}{1}}}}, \quad 0 \leq k \leq n,$$

are called the tails of the nth approximants of the S-fraction (1). The tails $Q_k^{(n)}(z)$ satisfy the following recurrence relations

$$Q_k^{(n)}(z) = 1 + \frac{a_{k+1} z}{Q_{k+1}^{(n)}(z)}, \; 0 \leq k \leq n-1, \; Q_n^{(n)}(z) = 1.$$

Let $\hat{a}_k$, $\hat{z}$ be perturbed values of the coefficients a_k and the variable z of the S-fraction (1). Then we obtain a perturbed S-fraction

$$\cfrac{\hat{a}_0}{1+\cfrac{\hat{a}_1 \hat{z}}{1+\cfrac{\hat{a}_2 \hat{z}}{1+\ddots}}} \tag{2}$$

to the S-fraction (1). We let $\hat{f}_n(\hat{z})$ denote the nth approximant of S-fraction (2) and let $\hat{Q}_k^{(n)}(\hat{z})$, $0 \leq k \leq n$, be the tails of $\hat{f}_n(\hat{z})$.

Definition 2.1. An S-fraction (1) is called stable to perturbations at a point $z_0 \in \mathbb{R}$, $z_0 \neq 0$, if for any $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that for all $\hat{a}_k \in \mathbb{R}_+$, $k \geq 0$, satisfying $|\hat{a}_k - a_k|/|a_k| < \delta_\varepsilon$, and for all $\hat{z}_0 \in \mathbb{R}$, satisfying $|\hat{z}_0 - z_0|/|z_0| < \delta_\varepsilon$, for all $n \geq 0$ the following inequalities hold $|\hat{f}_n(\hat{z}_0) - f_n(z_0)|/|f_n(z_0)| < \varepsilon$.

Let $\varepsilon_k^{(a)}$, $\varepsilon^{(z)}$, $\varepsilon_n^{(f)}$, $\varepsilon_{k,n}^{(Q)}$, be the relative errors of the coefficients a_k, the variable z, the approximants $f_n(z)$, and the tails $Q_k^{(n)}(z)$, respectively. That is, the following relations hold

$$\hat{a}_k = a_k(1+\varepsilon_k^{(a)}),\ k \geq 0,$$
$$\hat{z} = z(1+\varepsilon^{(z)}),$$
$$\hat{f}_n(\hat{z}) = f_n(z)(1+\varepsilon_n^{(f)}),\ n \geq 0,$$
$$\hat{Q}_k^{(n)}(\hat{z}) = Q_k^{(n)}(z)(1+\varepsilon_{k,n}^{(Q)}),\ 0 \leq k \leq n.$$

We will show that for the relative errors $\varepsilon_{k,n}^{(Q)}$ the following recurrent formulas hold

$$\varepsilon_{k,n}^{(Q)} = -g_{k+1}^{(n)}(z) + \frac{g_{k+1}^{(n)}(z)(1+\varepsilon_{k+1}^{(a)})(1+\varepsilon^{(z)})}{1+\varepsilon_{k+1,n}^{(Q)}},\ 0 \leq k \leq n-1, \tag{3}$$

where $\varepsilon_{n,n}^{(Q)} = 0$, and the quantities $g_k^{(n)}(z)$ are given by the formula

$$g_k^{(n)}(z) = a_k z/(Q_{k-1}^{(n)}(z) Q_k^{(n)}(z)),\ 1 \leq k \leq n. \tag{4}$$

Since $Q_n^{(n)}(z) = \hat{Q}_n^{(n)}(\hat{z}) = 1$, we have $\varepsilon_{n,n}^{(Q)} = 0$. For $0 \leq k \leq n-1$, we have

$$\begin{aligned}
\varepsilon_{k,n}^{(Q)} &= \frac{\hat{Q}_k^{(n)}(\hat{z}) - Q_k^{(n)}(z)}{Q_k^{(n)}(z)} \\
&= \frac{1+\hat{a}_{k+1}\hat{z}/\hat{Q}_{k+1}^{(n)}(\hat{z})}{Q_k^{(n)}(z)} - 1 \\
&= \frac{1}{Q_k^{(n)}(z)}\left(1 + \frac{a_{k+1}z(1+\varepsilon_{k+1}^{(a)})(1+\varepsilon^{(z)})}{Q_{k+1}^{(n)}(z)(1+\varepsilon_{k+1,n}^{(Q)})}\right) - 1 \\
&= \frac{1}{Q_k^{(n)}(z)} + \frac{a_{k+1}z(1+\varepsilon_{k+1}^{(a)})(1+\varepsilon^{(z)})}{Q_k^{(n)}(z)Q_{k+1}^{(n)}(z)(1+\varepsilon_{k+1,n}^{(Q)})} - 1.
\end{aligned}$$

Considering that

$$\begin{aligned}
\frac{1}{Q_k^{(n)}(z)} &= \frac{1}{Q_k^{(n)}(z)}\left(Q_k^{(n)}(z) - \frac{a_{k+1}z}{Q_{k+1}^{(n)}(z)}\right) \\
&= 1 - \frac{a_{k+1}z}{Q_k^{(n)}(z)Q_{k+1}^{(n)}(z)} \\
&= 1 - g_{k+1}^{(n)}(z),
\end{aligned}$$

we obtain the formulas (3).

Successively using formulas (3), we obtain a formula for the relative error of $Q_k^{(n)}(z)$ in the form of a continued fraction

$$\varepsilon_{k,n}^{(Q)} = -g_{k+1}^{(n)}(z) + \cfrac{g_{k+1}^{(n)}(z)(1+\varepsilon_{k+1}^{(a)})(1+\varepsilon^{(z)})}{1 - g_{k+2}^{(n)}(z) + \cfrac{g_{k+2}^{(n)}(z)(1+\varepsilon_{k+2}^{(a)})(1+\varepsilon^{(z)})}{1 - g_{k+3}^{(n)}(z) + \cfrac{g_{k+3}^{(n)}(z)(1+\varepsilon_{k+3}^{(a)})(1+\varepsilon^{(z)})}{1 - g_{k+4}^{(n)}(z) + \ddots + \cfrac{g_n^{(n)}(z)(1+\varepsilon_n^{(a)})(1+\varepsilon^{(z)})}{1}}}} \tag{5}$$

Setting $k = 0$ in formula (5), we obtain a formula for the relative error of the tail $Q_0^{(n)}$

$$\varepsilon_{0,n}^{(Q)} = -g_1^{(n)}(z) + \cfrac{g_1^{(n)}(z)(1+\varepsilon_1^{(a)})(1+\varepsilon^{(z)})}{1 - g_2^{(n)}(z) + \cfrac{g_2^{(n)}(z)(1+\varepsilon_2^{(a)})(1+\varepsilon^{(z)})}{1 - g_3^{(n)}(z) + \cfrac{g_3^{(n)}(z)(1+\varepsilon_3^{(a)})(1+\varepsilon^{(z)})}{1 - g_4^{(n)}(z) + \ddots + \cfrac{g_n^{(n)}(z)(1+\varepsilon_n^{(a)})(1+\varepsilon^{(z)})}{1}}}}.$$

Since

$$\begin{aligned}\varepsilon_n^{(f)} &= \frac{\hat{f}_n(\hat{z}) - f_n(z)}{f_n(z)}\\ &= \frac{\hat{a}_0/\hat{Q}_0^{(n)}(\hat{z}) - a_0/Q_0^{(n)}(z)}{a_0/Q_0^{(n)}(z)}\\ &= -1 + \frac{1+\varepsilon_0^{(a)}}{1+\varepsilon_{0,n}^{(Q)}}\end{aligned}$$

then

$$\varepsilon_n^{(f)} = -1 + \cfrac{1+\varepsilon_0^{(a)}}{1-g_1^{(n)}(z) + \cfrac{g_1^{(n)}(z)(1+\varepsilon_1^{(a)})(1+\varepsilon^{(z)})}{1-g_2^{(n)}(z) + \cfrac{g_2^{(n)}(z)(1+\varepsilon_2^{(a)})(1+\varepsilon^{(z)})}{1-g_3^{(n)}(z) + \ddots + \cfrac{g_n^{(n)}(z)(1+\varepsilon_n^{(a)})(1+\varepsilon^{(z)})}{1}}}}. \tag{6}$$

Formula (6) represents the relative error of the nth approximant of the S-fraction in the form of a continued fraction with $n+1$ floors.

3 Sufficient conditions for stability to perturbations of S-fractions

Theorem 3.1. *Let the relative errors of the coefficients a_k and the variable z of the S-fraction (1) satisfy the inequalities*

$$|\varepsilon_k^{(a)}| \le \alpha,\ 0 \le \alpha < 1,\ k \ge 0, \tag{7}$$

$$|\varepsilon^{(z)}| \le \beta,\ 0 \le \beta < 1. \tag{8}$$

The S-fraction (1) is stable to perturbations at a point $z_0 \in \mathbb{R}_+$ if there exists a sequence $\{\eta_k\}$, $0 < \eta_k < 1$, $k \ge 1$, such that

$$0 < g_k^{(n)}(z_0) \le \eta_k,\ 1 \le k \le n, \tag{9}$$

where the quantities $g_k^{(n)}(z_0)$ are defined according to (4), and the series

$$\sum_{k=1}^{\infty} (\eta_k^{-1} - 1)(1-\eta_{k+1}) \tag{10}$$

diverges.

Proof. Assuming that conditions (7)–(9) are met, we will prove the following estimates for relative errors $\varepsilon_{k,n}^{(Q)}$ using mathematical induction for $0 \le k \le n$

$$\varepsilon_{k,n,1}^{(Q)}(\alpha,\beta) \le \varepsilon_{k,n}^{(Q)} \le \varepsilon_{k,n,2}^{(Q)}(\alpha,\beta),\ 0 \le k \le n-1, \tag{11}$$

where $\varepsilon_{k,n,j}^{(Q)}(\alpha,\beta)$, $j \in \{1,2\}$, are defined by the following recurrence relations

$$\varepsilon_{k,n,j}^{(Q)}(\alpha,\beta) = -\eta_{k+1} + \frac{\eta_{k+1}(1+(-1)^j\alpha)(1+(-1)^j\beta)}{1+\varepsilon_{k+1,n,j+(-1)^{j+1}}^{(Q)}(\alpha,\beta)},\ 0 \le k \le n-2,\ j \in \{1,2\} \tag{12}$$

with initial conditions $\varepsilon_{n-1,n,j}^{(Q)}(\alpha,\beta) = \eta_n((-1)^j(\alpha+\beta)+\alpha\beta)$, $j \in \{1,2\}$. Starting from the initial conditions at the largest index $k = n-1$ and recursively applying the relations (12) backwards down to $k = 0$, we can determine the ranges of these mappings at each step. Thus, we note that

$$\varepsilon_{k,n,1}^{(Q)} : [0,1) \times [0,1) \to (-\eta_{k+1}, 0],\ 0 \le k \le n-1, \tag{13}$$

$$\varepsilon_{k,n,2}^{(Q)} : [0,1) \times [0,1) \to \left[0, \frac{\eta_{k+1}(3+\eta_{k+2})}{1-\eta_{k+2}}\right),\ 0 \le k \le n-2, \tag{14}$$

$$\varepsilon_{n-1,n,2}^{(Q)} : [0,1) \times [0,1) \to [0, 3\eta_n). \tag{15}$$

To prove the estimates (11), we use formulas (3). For $k = n-1$, we have

$$\begin{aligned}\varepsilon_{n-1,n}^{(Q)} &= g_n^{(n)}(z_0)(\varepsilon_n^{(a)} + \varepsilon^{(z_0)} + \varepsilon_n^{(a)}\varepsilon^{(z_0)})\\ &\le \eta_n(\alpha+\beta+\alpha\beta)\\ &= \varepsilon_{n-1,n,2}^{(Q)}(\alpha,\beta)\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_{n-1,n}^{(Q)} &= g_n^{(n)}(z_0)(\varepsilon_n^{(a)} + \varepsilon^{(z_0)} + \varepsilon_n^{(a)}\varepsilon^{(z_0)}) \\
&\geq \eta_n(\alpha\beta - \alpha - \beta) \\
&= \varepsilon_{n-1,n,1}^{(Q)}(\alpha,\beta).
\end{aligned}$$

Suppose that the estimates (11) hold for some $k = m+1$, $0 \leq m \leq n-2$. Considering formulas (3) and the ranges of the mappings (13) – (15), we prove them for $k = m$:

$$\begin{aligned}
\varepsilon_{m,n}^{(Q)} &= g_{m+1}^{(n)}(z_0)\left(\frac{(1+\varepsilon_{m+1}^{(a)})(1+\varepsilon^{(z_0)})}{1+\varepsilon_{m+1,n}^{(Q)}} - 1\right) \\
&\leq \eta_{m+1}\left(\frac{(1+\alpha)(1+\beta)}{1+\varepsilon_{m+1,n,1}^{(Q)}(\alpha,\beta)} - 1\right) \\
&= -\eta_{m+1} + \frac{\eta_{m+1}(1+\alpha)(1+\beta)}{1+\varepsilon_{m+1,n,1}^{(Q)}(\alpha,\beta)} \\
&= \varepsilon_{m,n,2}^{(Q)}(\alpha,\beta)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_{m,n}^{(Q)} &= g_{m+1}^{(n)}(z_0)\left(\frac{(1+\varepsilon_{m+1}^{(a)})(1+\varepsilon^{(z_0)})}{1+\varepsilon_{m+1,n}^{(Q)}} - 1\right) \\
&\geq \eta_{m+1}\left(\frac{(1-\alpha)(1-\beta)}{1+\varepsilon_{m+1,n,2}^{(Q)}(\alpha,\beta)} - 1\right) \\
&= -\eta_{m+1} + \frac{\eta_{m+1}(1-\alpha)(1-\beta)}{1+\varepsilon_{m+1,n,2}^{(Q)}(\alpha,\beta)} \\
&= \varepsilon_{m,n,1}^{(Q)}(\alpha,\beta).
\end{aligned}$$

Since

$$\varepsilon_n^{(f)} = -1 + \frac{1+\varepsilon_0^{(a)}}{1+\varepsilon_{0,n}^{(Q)}},$$

then setting $k = 0$ in (11) we obtain the estimates:

$$\varepsilon_{n,1}^{(f)}(\alpha,\beta) \leq \varepsilon_n^{(f)} \leq \varepsilon_{n,2}^{(f)}(\alpha,\beta),$$

where $\varepsilon_{n,j}^{(f)}(\alpha,\beta)$ are defined by

$$\varepsilon_{n,j}^{(f)}(\alpha,\beta) = -1 + \cfrac{1+(-1)^j\alpha}{1-\eta_1 + \cfrac{\eta_1(1+(-1)^{j+1}\alpha)(1+(-1)^{j+1}\beta)}{1-\eta_2 + \cfrac{\eta_2(1+(-1)^{j+2}\alpha)(1+(-1)^{j+2}\beta)}{1-\eta_3 + \ddots + \cfrac{\eta_n(1+(-1)^{j+n}\alpha)(1+(-1)^{j+n}\beta)}{1}}}}, \quad j \in \{1,2\}.$$

Note that, for $j = 1$ continued fraction $\varepsilon_{n,1}^{(f)}(\alpha,\beta)$ is a minorant and for $j = 2$ continued fraction $\varepsilon_{n,2}^{(f)}(\alpha,\beta)$ is a majorant to the relative error $\varepsilon_n^{(f)}$.

We will show that the uniform convergence of the sequences of functions $\{\varepsilon_{n,j}^{(f)}(\alpha,\beta)\}$, $j \in \{1,2\}$, on the set $[0,1) \times [0,1)$ follows from the divergence of the series (10).

Let us consider the infinite continued fractions

$$-1 + \cfrac{c_{0,j}(\alpha,\beta)}{d_0 + \cfrac{c_{1,j}(\alpha,\beta)}{d_1 + \cfrac{c_{2,j}(\alpha,\beta)}{d_2 + \ddots}}}, \quad j \in \{1,2\}, \tag{16}$$

where

$$c_{0,j} = 1 + (-1)^j\alpha, \quad c_{k,j} = \begin{cases} \eta_k(1+(-1)^{j-1}\alpha)(1+(-1)^{j-1}\beta), & k = 2m-1, \\ \eta_k(1-(-1)^{j-1}\alpha)(1-(-1)^{j-1}\beta), & k = 2m, \end{cases} \quad k \geq 1,\ j \in \{1,2\},$$

$$d_k = 1 - \eta_{k+1},\ k \geq 0.$$

Let $t_k(\omega)=a_k/(b_k+\omega)$, $k\geq 0$, be a sequence of linear fractional transformations. Let us denote by $T_k^{(n)}(\omega)=t_k\circ t_{k+1}\circ\ldots\circ t_n(\omega)$, $0\leq k\leq n$, the composition of linear fractional transformations and let $T_{n+1}^{(n)}(\omega)=\omega$. Using the recurrence relations

$$T_k^{(n)}(\omega_1)-T_k^{(m)}(\omega_2)=\frac{-a_k(T_{k+1}^{(n)}(\omega_1)-T_{k+1}^{(m)}(\omega_2))}{(b_k+T_{k+1}^{(n)}(\omega_1))(b_k+T_{k+1}^{(m)}(\omega_2))},\ 0\leq k\leq m,$$

where

$$T_{m+1}^{(n)}(\omega_1)-T_{m+1}^{(m)}(\omega_2)=\frac{a_{m+1}-\omega_2(b_{m+1}+T_{m+2}^{(n)}(\omega_1))}{b_{m+1}+T_{m+2}^{(n)}(\omega_1)},$$

for $n>m$ we obtain the formula

$$T_0^{(n)}(\omega_1)-T_0^{(m)}(\omega_2)=\frac{(-1)^{m+1}\left(a_{m+1}-\omega_2(b_{m+1}+T_{m+2}^{(n)}(\omega_1))\right)\prod\limits_{l=0}^{m}a_l}{\prod\limits_{l=1}^{m+2}(b_{l-1}+T_l^{(n)}(\omega_1))\prod\limits_{l=1}^{m+1}(b_{l-1}+T_l^{(m)}(\omega_2))}. \tag{17}$$

Let us denote by $h_{n,j}(\alpha,\beta)$, $n\geq 0$, the nth approximants of the continued fractions (16). From formula (17) for the functions $\varepsilon_{n,j}^{(f)}(\alpha,\beta)$, $j\in\{1,2\}$, follow the inequalities

$$h_{2n-1,j}(\alpha,\beta)<\varepsilon_{2n,j}^{(f)}(\alpha,\beta)<h_{2n,j}(\alpha,\beta),\ h_{2n+1,j}(\alpha,\beta)<\varepsilon_{2n+1,j}^{(f)}(\alpha,\beta)<h_{2n,j}(\alpha,\beta),\ n\geq 0,\ j\in\{1,2\}, \tag{18}$$

where $h_{-1,j}(\alpha,\beta)=0$.

If the continued fractions (16) converge uniformly on the set $[0,1)\times[0,1)$, then, taking into account the inequalities (18), the sequences of functions $\{\varepsilon_{n,j}^{(f)}(\alpha,\beta)\}$, $j\in\{1,2\}$, converge uniformly on the set $[0,1)\times[0,1)$. We will show that a sufficient condition for the uniform convergence of the continued fractions (16) on the set $[0,1)\times[0,1)$ is the divergence of the series (10).

Since the elements of the continued fractions (16) are positive, the sequence of odd approximants is strictly increasing, the sequence of even approximants is strictly decreasing, and any odd approximant is strictly less than any even approximant. Thus, for any independent integers $n\geq 0$ and $m\geq 0$, they satisfy the inequalities

$$h_{2n+1,j}(\alpha,\beta)<h_{2n+3,j}(\alpha,\beta)<h_{2m+2,j}(\alpha,\beta)<h_{2m,j}(\alpha,\beta),\ j\in\{1,2\}. \tag{19}$$

From inequalities (19) it follows that the functional sequences $\{h_{n,j}(\alpha,\beta)\}$, $j\in\{1,2\}$, converge uniformly on the set $[0,1)\times[0,1)$, if for any $\varepsilon>0$ there exists $n_\varepsilon\in\mathbb{N}$ such that for any $n>n_\varepsilon$ and for any point $(\alpha,\beta)\in[0,1)\times[0,1)$ the inequality $h_{2n,j}(\alpha,\beta)-h_{2n-1,j}(\alpha,\beta)<\varepsilon$ holds.

Using equivalent transformations, we transform the continued fractions (16) into continued fractions with partial numerators equal to one

$$-1+\cfrac{1}{r_{0,j}(\alpha,\beta)+\cfrac{1}{r_{1,j}(\alpha,\beta)+\cfrac{1}{r_{2,j}(\alpha,\beta)+\ddots}}},\ j\in\{1,2\}, \tag{20}$$

where

$$r_{k,j}(\alpha,\beta)=p_{k,j}(\alpha,\beta)q_k,\ k\geq 0,\ j\in\{1,2\},$$
$$p_{0,j}(\alpha,\beta)=(1+(-1)^j\alpha)^{-1},\ j\in\{1,2\},$$
$$p_{k,j}(\alpha,\beta)=\begin{cases}\left(\dfrac{(1+(-1)^j\alpha)(1+(-1)^j\beta)}{(1-(-1)^j\alpha)(1-(-1)^j\beta)}\right)^m, & k=2m+1,\\ \dfrac{1}{(1+(-1)^j\alpha)(1+(-1)^j\beta)}\left(\dfrac{(1-(-1)^j\alpha)(1-(-1)^j\beta)}{(1+(-1)^j\alpha)(1+(-1)^j\beta)}\right)^m, & k=2m,\end{cases}\quad k\geq 1,\ j\in\{1,2\},$$
$$q_k=\begin{cases}\prod\limits_{i=1}^{[k/2]}\eta_{2i}\left(\prod\limits_{i=0}^{[k/2]}\eta_{2i+1}\right)^{-1}(1-\eta_{k+1}), & k=2m+1,\\ \prod\limits_{i=1}^{[k/2]}\eta_{2i-1}\left(\prod\limits_{i=1}^{[k/2]}\eta_{2i}\right)^{-1}(1-\eta_{k+1}), & k=2m,\end{cases}\quad k\geq 0.$$

For the difference of the approximants $h_{2n,j}(\alpha,\beta)$ and $h_{2n-1,j}(\alpha,\beta)$ of the continued fractions (20) the formula holds [26]

$$h_{2n,j}(\alpha,\beta)-h_{2n-1,j}(\alpha,\beta)=(B_{2n-1,j}(\alpha,\beta)B_{2n,j}(\alpha,\beta))^{-1}, \tag{21}$$

where $B_{k,j}$ are their denominators, which are determined by the recurrence formulas $B_{k,j}(\alpha,\beta)=r_{k,j}B_{k-1,j}(\alpha,\beta)+B_{k-2,j}(\alpha,\beta)$, $k\ge 0$, with the initial conditions $B_{-1,j}(\alpha,\beta)=1$, $B_{-2,j}(\alpha,\beta)=0$.

From formula (21) follows the estimate

$$h_{2n,j}(\alpha,\beta)-h_{2n-1,j}(\alpha,\beta)\le\left(r_{0,j}\left(\sum_{k=0}^{s}r_{2k,j}\right)\left(\sum_{k=1}^{s}r_{2k-1,j}\right)\right)^{-1}.$$

Since

$$r_{0,j}\left(\sum_{k=0}^{s}r_{2k,j}\right)\left(\sum_{k=1}^{s}r_{2k-1,j}\right)>2^{-(j+1)}(1-\eta_1)\sum_{k=1}^{2s}(1-\eta_k)(1-\eta_{k+1})\eta_k^{-1},$$

then from the last estimate, it follows that the divergence of the series (10) ensures the uniform convergence of the continued fractions (16) on the set $[0,1)\times[0,1)$. From this follows the uniform convergence of the sequences $\{\varepsilon_{n,j}^{(f)}(\alpha,\beta)\}$, $j\in\{1,2\}$, on the set $[0,1)\times[0,1)$.

Since the functions $\varepsilon_{n,j}^{(f)}(\alpha,\beta)$, $j\in\{1,2\}$, $n\ge 0$, for $0<\eta_k<1$, $k\ge 1$ are continuous at the point $(\alpha,\beta)=(0,0)\in[0,1)\times[0,1)$, and the sequences $\{\varepsilon_{n,j}^{(f)}(\alpha,\beta)\}$, $j\in\{1,2\}$, converge uniformly on the set $[0,1)\times[0,1)$ to the functions $\varepsilon_j(\alpha,\beta)$, $j\in\{1,2\}$, the functions $\varepsilon_j(\alpha,\beta)$, $j\in\{1,2\}$ are also continuous at the point $(0,0)$.

From the uniform convergence of the sequences of functions $\{\varepsilon_{n,j}^{(f)}(\alpha,\beta)\}$, $j\in\{1,2\}$, on the set $[0,1)\times[0,1)$ it follows that

$$\lim_{\substack{\alpha\to+0\\ \beta\to+0}}\lim_{n\to\infty}\varepsilon_{n,j}^{(f)}(\alpha,\beta)=\lim_{n\to\infty}\lim_{\substack{\alpha\to+0\\ \beta\to+0}}\varepsilon_{n,j}^{(f)}(\alpha,\beta),\ j\in\{1,2\}. \tag{22}$$

Since

$$\lim_{\substack{\alpha\to+0\\ \beta\to+0}}\varepsilon_{n,j}^{(f)}(\alpha,\beta)=\varepsilon_{n,j}^{(f)}(0,0)=0,$$

then, taking into account (22), we have $\varepsilon_j(0,0)=0$, $j\in\{1,2\}$.

In addition, from (17) we obtain the formula

$$\varepsilon_{n+1,j}^{(f)}-\varepsilon_{n,j}^{(f)}=(-1)^j\frac{(\alpha+\beta+(-1)^{n+j+1}\alpha\beta)(1+(-1)^j\alpha)\prod_{i=1}^{n+1}\eta_i\prod_{i=1}^{n}(1+(-1)^{i+j}\alpha)(1+(-1)^{i+j}\beta)}{\prod_{k=0}^{n}W_{k,j}^{(n+1)}\prod_{k=0}^{n-1}W_{k,j}^{(n)}},\ n\ge 0,\ j\in\{1,2\},$$

where

$$W_{k,j}^{(m)}=1-\eta_{k+1}+\cfrac{\eta_{k+1}(1+(-1)^{k+j+1}\alpha)(1+(-1)^{k+j+1}\beta)}{1-\eta_{k+2}+\cfrac{\eta_{k+2}(1+(-1)^{k+j+2}\alpha)(1+(-1)^{k+j+2}\beta)}{1-\eta_{k+3}+\cfrac{\eta_{k+3}(1+(-1)^{k+j+3}\alpha)(1+(-1)^{k+j+3}\beta)}{1-\eta_{k+4}+\ddots+\cfrac{\eta_m(1+(-1)^{m+j}\alpha)(1+(-1)^{m+j}\beta)}{1}}}},\ 0\le k\le m-1.$$

Then for $0\le\alpha<1$, $0\le\beta<1$, $0<\eta_k<1$, $k\ge 1$, the sequence $\{\varepsilon_{n,1}^{(f)}(\alpha,\beta)\}$ decreases monotonically, and the sequence $\{\varepsilon_{n,2}^{(f)}(\alpha,\beta)\}$ increases monotonically. Thus, for the relative error of the nth approximant of the S-fraction (1) the estimates $\varepsilon_1(\alpha,\beta)\le\varepsilon_n^{(f)}\le\varepsilon_2(\alpha,\beta)$, $n\ge 0$, hold, where $\varepsilon_j(\alpha,\beta)$, $j\in\{1,2\}$, are continuous functions at the point $(0,0)$ and $\varepsilon_j(0,0)=0$, $j\in\{1,2\}$. Therefore, for any $\varepsilon>0$, there exists $\delta_\varepsilon>0$ such that for all $(\alpha,\beta)\in[0,1)\times[0,1)$ such that $\sqrt{\alpha^2+\beta^2}<\delta_\varepsilon$, the inequalities $\varepsilon_1(\alpha,\beta)>-\varepsilon$, $\varepsilon_2(\alpha,\beta)<\varepsilon$ are satisfied. That is, for each $\hat{a}_k\in\mathbb{R}_+$, $k\ge 0$, such that $|\hat{a}_k-a_k|/|a_k|\le\alpha<\delta_\varepsilon/\sqrt{2}$, and each $\hat{z}_0\in\mathbb{R}_+$, such that $|\hat{z}_0-z_0|/|z_0|\le\beta<\delta_\varepsilon/\sqrt{2}$, for the relative errors of the approximants of the S-fraction (1) the inequalities

$$-\varepsilon<\varepsilon_1(\alpha,\beta)\le\varepsilon_n^{(f)}\le\varepsilon_2(\alpha,\beta)<\varepsilon,\ n\ge 0.$$

hold. □

Theorem 3.2. *Let the relative errors of the coefficients a_k and the variable z of an S-fraction (1) satisfy the conditions (7), (8). The set*

$$E=\{z\in\mathbb{R}:0<z\le M\},\ M>0, \tag{23}$$

is a set of stability for perturbations of the S-fraction (1), if one of the conditions is met:

1. *The series*

$$\sum_{k=0}^{\infty}a_k^{-1}(1+a_{k+1})^{-1} \tag{24}$$

diverges.

2. *There exists a positive constant A such that*

$$a_k \le A,\ k \ge 0, \tag{25}$$

and in the second case, for the relative error of the nth approximant, the following estimates hold

$$\varepsilon_1(\alpha,\beta) \le \varepsilon_n^{(f)} \le \varepsilon_2(\alpha,\beta), \tag{26}$$

where

$$\varepsilon_j(\alpha,\beta) = \frac{(-1)^j(1-2\eta(1+\alpha+\beta)+\eta^2)+\sqrt{(1-2\eta(1+\alpha+\beta)+\eta^2)^2-4\eta(1-\eta)^2(1+\alpha)(1+\beta)}}{2(1-\eta)} - 1,\ j \in \{1,2\},$$

and $\eta = AM(1+AM)^{-1}$.

Proof. If the set E is defined according to (23) then the following estimates hold

$$\begin{aligned} g_k^{(n)}(z) &= \frac{a_k z}{Q_{k-1}^{(n)}(z)Q_k^{(n)}(z)} \\ &= \frac{a_k z/Q_k^{(n)}(z)}{1+a_k z/Q_k^{(n)}(z)} \\ &\le \frac{a_k M}{1+a_k M} < 1. \end{aligned}$$

Let us set $\eta_k = a_k M(1+a_k M)^{-1}$, then condition (24) is equivalent to the condition (10) and according to Theorem 3.1, the set (23) is a set of stability for perturbations of the S-fraction (1).

If the coefficients of the S-fraction (1) satisfy condition (25), then $g_k^{(n)}(z) \le \eta$, where $\eta = AM(1+AM)^{-1}$. Consequently, the conditions of Theorem 3.1 are satisfied, and the set (23) is a set of stability for perturbations of the S-fraction (1).

Let us establish estimates for the relative error of the nth approximant of the S-fraction (1) under condition (25). Let $h_{n,j}(\alpha,\beta)$ be the nth approximants of the periodic continued fractions

$$-1+\frac{1}{\eta(1+(-1)^{j+1}\beta)}\,\cfrac{c_{0,j}(\alpha,\beta)}{d_0+\cfrac{c_{1,j}(\alpha,\beta)}{d_1+\cfrac{c_{2,j}(\alpha,\beta)}{d_2+\ddots}}},\ j\in\{1,2\}, \tag{27}$$

where

$$c_{k,j} = \begin{cases} \eta(1+(-1)^{j-1}\alpha)(1+(-1)^{j-1}\beta),\ k=2m-1, \\ \eta(1-(-1)^{j-1}\alpha)(1-(-1)^{j-1}\beta),\ k=2m, \end{cases} \qquad d_k = 1-\eta,\ k\ge 0.$$

Repeating the proof of Theorem 3.1, we obtain

$$h_{2n+1,1}(\alpha,\beta) < \varepsilon_{2n+1,1}^{(f)}(\alpha,\beta) < \varepsilon_{2n,1}^{(f)}(\alpha,\beta) < \varepsilon_{2n-1,1}^{(f)}(\alpha,\beta) < \ldots < \varepsilon_{0,1}^{(f)}(\alpha,\beta),$$

and

$$\varepsilon_{0,2}^{(f)}(\alpha,\beta) < \ldots < \varepsilon_{2n-1,2}^{(f)}(\alpha,\beta) < \varepsilon_{2n,2}^{(f)}(\alpha,\beta) < \varepsilon_{2n+1,2}^{(f)}(\alpha,\beta) < h_{2n,2}(\alpha,\beta),$$

where the continued fractions

$$\varepsilon_{n,j}^{(f)}(\alpha,\beta) = -1+\cfrac{1+(-1)^j\alpha}{1-\eta+\cfrac{\eta(1+(-1)^{j+1}\alpha)(1+(-1)^{j+1}\beta)}{1-\eta+\cfrac{\eta(1+(-1)^{j+2}\alpha)(1+(-1)^{j+2}\beta)}{1-\eta+\ddots+\cfrac{\eta(1+(-1)^{j+n}\alpha)(1+(-1)^{j+n}\beta)}{1}}}},\ j\in\{1,2\},$$

with $j=1$ is the minorant, and with $j=2$ is the majorant of the relative error $\varepsilon_n^{(f)}$. Thus,

$$\lim_{n\to\infty} h_{2n+1,1}(\alpha,\beta) \le \varepsilon_n^{(f)} \le \lim_{n\to\infty} h_{2n,2}(\alpha,\beta). \tag{28}$$

The points $\omega_{1,j}$, $\omega_{2,j}$, where

$$\omega_{i,j} = \frac{(-1)^j(1-2\eta(1+\alpha+\beta)+\eta^2)+(-1)^{i+1}\sqrt{(1-2\eta(1+\alpha+\beta)+\eta^2)^2-4\eta(1-\eta)^2(1+\alpha)(1+\beta)}}{2(1-\eta)},$$

Table 1: Metrics of models on training data

Model	R^2 (training)	$RMSE$ (training)
polynomial model of the 2nd order	0.9383	3.7782
S-fractional model of the 2nd order	0.9854	1.8361

are fixed points of the linear fractional transformations

$$\frac{\eta(1+(-1)^j\alpha)(1+(-1)^j\beta)}{1-\eta+\dfrac{\eta(1+(-1)^{j-1}\alpha)(1+(-1)^{j-1}\beta)}{1-\eta+\omega}},\ j\in\{1,2\}.$$

Since $\omega_{1,j}\neq\omega_{2,j}$,

$$\left|\frac{B_{0,j}\omega_{2,j}+B_{1,j}}{B_{0,j}\omega_{1,j}+B_{1,j}}\right|<1,$$

and $T_{0,j}^{(k)}(0)\neq\omega_{2,j}$, $k\in\{0,1\}$, where $B_{k,j}$ is the denominator of the kth approximant, $T_{0,j}^{(k)}(\omega)=T_{0,j}^{(k-1)}(t_{k,j}(\omega))$, $T_{0,j}^{(0)}(\omega)=t_{0,j}(\omega)$, $t_{k,j}(\omega)=c_{k,j}(\alpha,\beta)/(d_k+\omega)$, $k\in\{0,1\}$, then, according to the theorem on the convergence of periodic continued fractions [24], the continued fractions (27) converge to $\eta^{-1}(1+(-1)^j\beta)^{-1}\omega_1^{(j)}$, $j\in\{1,2\}$. Then, taking into account inequalities (28), we obtain the estimates

$$(1-\beta)\eta^{-1}\omega_1^{(1)}-1\le\varepsilon_n^{(f)}\le(1+\beta)^{-1}\eta^{-1}\omega_1^{(2)}-1,$$

which are equivalent to estimates (26). □

By setting in Theorem 3.2 $a_k=\sqrt{k}$, $k\ge1$, we obtain

Example 3.1. Let the relative errors of the coefficients a_k and the variable z of the S-fraction

$$\frac{a_0}{1+\dfrac{z}{1+\dfrac{\sqrt{2}z}{1+\dfrac{\sqrt{3}z}{1+\ddots}}}},\ a_0\in\mathbb{R}_+, \tag{29}$$

satisfy the conditions (7),(8), then the set (23) is a set of its stability to perturbations.

4 Application: modeling the time series of demand reduction after advertising

In marketing practice, it is important to evaluate the dynamics of demand for a product after the completion of an advertising campaign. As a rule, demand decreases rapidly immediately after the cessation of advertising, and over time stabilizes at a certain level. To analyze this dependence, we will consider a time series with a duration of 20 weeks, to which random noise has been added, which imitates the fluctuations of the real market. The task is to build models that adequately describe the data and provide a reliable forecast.

For data approximation, we will use two approaches: polynomial model of the 2nd order

$$c_0+c_1z+c_2z^2$$

and a S-fractional model of the 2nd order

$$\frac{a_0}{1+\dfrac{a_1z}{1+\dfrac{a_2z}{1}}}.$$

Polynomials are a classic method of approximation, but for time series that have asymptotic behavior, they can give an incorrect forecast. The S-fractional model of the 2nd order is a rational function with positive coefficients. This model allows you to effectively approximate time series with asymptotic behavior and demonstrates stability to parameter perturbations. Optimization of the coefficients of the polynomial and S-fractional models (with the restriction on the positivity of the parameters) was performed by the least squares method.

The accuracy of both models on the training data was evaluated by the R^2 and $RMSE$ criteria. From Table 1 it is clear that the fractional model reproduces the data better (closer to one R^2 value and a small $RMSE$ value).

An important aspect is the forecast of demand in the interval of 21–30 weeks, where the models do not have training points. From Figure 1 we see that the S-fractional model predicts the demand behavior much more accurately ($RMSE=4.2754$)

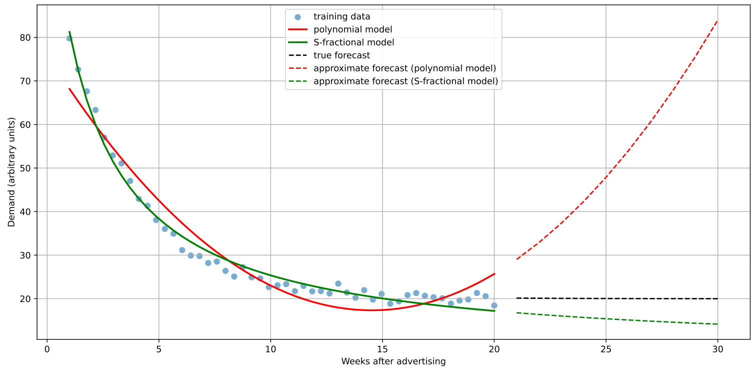

Figure 1: Modeling demand after advertising.

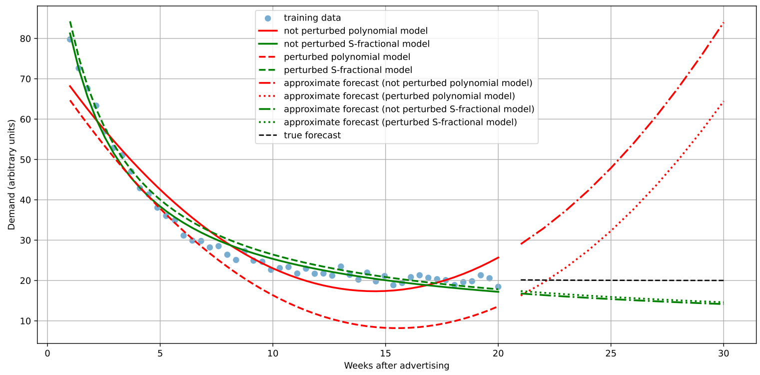

Figure 2: Stability to perturbations of models, including forecast.

compared to the polynomial model ($RMSE = 23.0880$). The forecast of the polynomial model demonstrates an increase in demand, which contradicts the real dynamics. Instead, the S-fraction retains its proximity to the true forecast.

An important property of the S-fractional model is its stability to parameter perturbations: when the coefficients change by $\pm 5\%$, the shape of the curve almost does not change, while the polynomial loses its adequacy (see Figure 2). The S-fractional model reliably describes both the training data and the forecast, and the perturbation of the parameters does not lead to significant deviations. After the perturbation of the coefficients, the polynomial model significantly deviates from the true data, both in the training and in the forecast zone.

Thus, the polynomial model of the 2nd order gives a good approximation on the training data, but is unsuitable for forecasting asymptotic processes. The S-fractional model of the 2nd order with the restriction of the positivity of the coefficients provides a higher accuracy on the training data and a good forecast. The obtained results confirm the expediency of using fractional-rational models in the tasks of time series analysis.

5 Conclusions

This paper introduces an approach to the stability analysis of S-fractions of a real variable. We have established sufficient conditions for stability to perturbations, which impose specific constraints on the S-fraction's coefficients and define a set of stability. This framework provides a means to assess whether an S-fraction is a reliable approximation tool when dealing with imprecise data. A case study modeling consumer demand after an advertising campaign demonstrated that the S-fractional model outperforms the classical polynomial model. It not only achieves superior approximation accuracy on the training data but also delivers an adequate and realistic forecast for asymptotic processes. Conversely, the polynomial model was found to be unsuitable for forecasting, as its predictions conflict with the process's actual dynamics. The core theoretical findings were empirically illustrated through an experiment with coefficient perturbations. The S-fractional model exhibited stability: its forecast remained close to the true trajectory even when its parameters were perturbed. This is a crucial property for real-world systems characterized by parameter inaccuracy.

Thus, these results justify the use of fractional-rational models, and S-fractions in particular, as powerful approximation tools. Due to their structure and stability to perturbations, they represent a reliable approach, especially in time-series forecasting applications that involve long-term prediction and must account for parameter inaccuracy.

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