



A Singular Sturm-Liouville Framework for Heun-Type Orthogonality and Eigenfunction Expansions

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Abstract

Heun functions arise as local solutions of the general second-order Fuchsian differential equation with four regular singular points, but such local solutions do not, in general, form an orthogonal basis. In this paper, we develop a singular Sturm–Liouville framework in which Heun-type solutions can be realized as orthogonal eigenfunctions suitable for spectral expansions. The accessory parameter is interpreted as a spectral parameter, and the general Heun equation is transformed into a formally self-adjoint weighted Sturm–Liouville problem.

The singular endpoints are analyzed through Weyl’s limit-point/limit-circle classification, which determines the admissible boundary data for self-adjoint realizations. Within an explicit parameter regime, the resulting operator has compact resolvent, leading to a discrete spectrum and a complete orthogonal system in the associated weighted Hilbert space. The corresponding normalization and eigenfunction expansion formulas are then established within this spectral setting.

The proposed framework provides a parameter-specific mathematical foundation for differential equations reducible to Heun form and is intended to support future analytical and spectral applications in wave propagation, quantum models, plasma-equilibrium theory, and tokamak-related configurations.

1 Introduction

Special functions play a central role in the spectral analysis of differential equations arising in mathematical physics. Classical families such as Bessel functions, Legendre functions, Jacobi polynomials, and hypergeometric functions are important not only as explicit solutions of differential equations, but also as eigenfunctions of suitable boundary-value problems. When the associated differential operator admits an appropriate self-adjoint realization, these functions may form orthogonal systems and provide expansion bases in suitable weighted function spaces [29, 30]. The general Heun equation occupies a distinguished position among second-order linear differential equations. It is a Fuchsian equation with four regular singular points and may be regarded as a natural extension of the Gauss hypergeometric equation, which has three regular singularities [22, 27, 28]. The presence of the additional singular point gives Heun functions a richer local and global structure and introduces an accessory parameter that is not determined solely by the local characteristic exponents. This feature explains their appearance in several problems of mathematical physics, including wave propagation, quantum spectral problems, perturbation theory, and boundary-value problems involving nontrivial singular behavior [16, 24, 23]. Recent investigations have further emphasized both the analytical and computational aspects of Heun-type equations. In particular, numerical evaluation of the general Heun function and the performance of symbolic algorithms have been examined in detail [7], while Heun-based analytical formulations have also appeared in plasma-equilibrium models [6, 18]. Related mathematical-physics applications include spectral and nonlinear problems arising in magnetic quantum models [25]. The existence of local Heun solutions, however, does not by itself imply orthogonality. A Frobenius solution constructed near a regular singular point is a local analytic object, whereas orthogonality is a global spectral property that depends on the differential expression, the underlying weighted space, the operator domain, and the admissible boundary conditions. Earlier studies of orthogonal Heun functions have shown that suitable eigenvalue conditions lead to weighted orthogonality relations and normalization formulas on the interval $0, 1$ [5, 3]. To place this structure in an operator framework, the accessory parameter may be interpreted as a spectral parameter and the Heun equation may be transformed into a formally self-adjoint differential expression through an appropriate weight function. The resulting problem is naturally connected with singular Sturm–Liouville theory, since the endpoints of the interval may coincide with regular singular

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points of the differential equation. In this setting, formal self-adjointness alone is not sufficient: the admissible operator domain also depends on the limiting behavior of the associated Lagrange boundary form [29, 30, 8, 17]. Modern Sturm–Liouville theory has developed in several complementary directions. Non-self-adjoint and indefinite spectral problems have received considerable attention in recent studies [11, 9]. Related investigations have addressed discontinuous and impulsive Sturm–Liouville operators [15, 14]. Almost-periodic spectral problems and operator pencils with impulse have also been considered [10, 1, 4]. Inverse spectral problems involving discontinuous coefficients and related Dirac-type systems have also been investigated [13, 12]. Further developments concern delayed and noncanonical dynamic equations [19, 20]. Recent inverse spectral questions for related first-order systems further illustrate the continuing development of this area [26]. These works demonstrate the variety of operator structures that arise in modern spectral problems and provide a broader context for the singular self-adjoint setting considered here. Boundary conditions also play a central role in related nonlinear boundary-value problems. Recent studies of Robin-type problems, for example, illustrate how the analytical structure of the boundary conditions may substantially influence the corresponding solution framework [21]. The present work differs from these nonlinear formulations by focusing on a linear singular spectral realization of the general Heun equation, but the underlying emphasis on admissible boundary data is closely related. The present paper develops a unified spectral formulation of the general Heun equation on the interval $(0, 1)$. Starting from the differential equation, we identify the accessory parameter as the spectral parameter, derive the corresponding weight and formally self-adjoint Sturm–Liouville expression, and relate the local Frobenius behavior at the singular endpoints to the admissible spectral realization. In this way, the passage from local Heun solutions to orthogonal spectral eigenfunctions is made explicit within a single Heun-specific framework. A central part of the analysis concerns the singular endpoints $z = 0$ and $z = 1$. While simple restrictions on the exponent parameters provide useful sufficient conditions for the vanishing of the boundary contribution for regular local solutions, they do not constitute a complete classification of the singular endpoints. We therefore refine the endpoint analysis by means of Frobenius asymptotics and Weyl’s limit-point/limit-circle theory. This classification determines the boundary data required for self-adjoint realizations and provides the appropriate setting for the subsequent analysis of the resolvent and spectral structure. A second objective is to clarify the distinction between orthogonality and completeness. Orthogonality of eigenfunctions associated with distinct spectral values follows once an admissible self-adjoint realization has been specified. A complete eigenfunction expansion, however, requires additional spectral information. In particular, the discreteness of the spectrum and the completeness of the eigenfunctions are connected with compactness properties of the resolvent. The present treatment therefore separates the endpoint classification from the compact-resolvent analysis instead of taking compactness solely as an abstract assumption. The framework developed here provides a mathematical basis for future applications in which differential equations reduce to Heun form. Such applications include wave models, quantum systems, and plasma-equilibrium problems, where singular points may encode boundary behavior, transition regions, or geometrical constraints. In particular, the present analysis may support later use of Heun-type spectral expansions in analytical models relevant to shaped tokamak and fusion configurations [2, 6, 18], while the present paper remains focused on the underlying mathematical structure.

2 The General Heun Equation in Operator Form

We begin with the general Heun equation in its canonical Fuchsian form,

$$\frac{d^2 H}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{dH}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-a)} H = 0, \quad (1)$$

where $a \neq 0, 1$. The parameters $\alpha, \beta, \gamma, \delta, \epsilon$ determine the local exponent structure of the equation, while q is the accessory parameter. The four regular singular points are

$$z = 0, \quad z = 1, \quad z = a, \quad z = \infty,$$

and the parameters satisfy the Fuchsian relation

$$\alpha + \beta + 1 = \gamma + \delta + \epsilon. \quad (2)$$

At the two finite singular points that will later form the endpoints of the spectral interval, the characteristic exponents are

$$z = 0: \quad 0, 1 - \gamma, \quad z = 1: \quad 0, 1 - \delta.$$

These local exponents will play a central role in the singular endpoint classification developed below [27, 28].

The presence of the fourth singular point distinguishes the general Heun equation from the Gauss hypergeometric equation. In particular, the accessory parameter q is not determined solely by the local characteristic exponents. This makes it natural to regard q , after an appropriate operator realization has been specified, as a spectral parameter.

To expose this structure, let

$$\Pi(z) = z(z-1)(z-a).$$

Multiplying (1) by $\Pi(z)$ gives

$$z(z-1)(z-a)H'' + \left[\gamma z(z-1)(z-a) + \delta z(z-a) + \epsilon z(z-1) \right] H' + \alpha\beta z H = qH.$$

We therefore introduce the differential expression

$$\mathcal{L} := z(z-1)(z-a) \frac{d^2}{dz^2} + [\gamma(z-1)(z-a) + \delta z(z-a) + \epsilon z(z-1)] \frac{d}{dz} + \alpha\beta z. \quad (3)$$

The Heun equation may then be written in the eigenvalue-type form

$$\mathcal{L}H = qH. \quad (4)$$

Equation (4) identifies the accessory parameter q as a formal spectral parameter. Its interpretation within a self-adjoint spectral problem requires an appropriate weighted space, operator domain, and endpoint conditions. An integrating weight will be used to rewrite the differential expression in formally self-adjoint Sturm–Liouville form.

3 Self-Adjoint Transformation and Singular Sturm–Liouville Structure

Define

$$\varrho(z) = z^{\gamma-1}(z-1)^{\delta-1}(z-a)^{\epsilon-1}, \quad \mathcal{P}(z) = z^{\gamma}(z-1)^{\delta}(z-a)^{\epsilon}. \quad (5)$$

Then

$$\mathcal{P}(z) = z(z-1)(z-a)\varrho(z),$$

and multiplication of (4) by $\varrho(z)$ yields

$$\frac{d}{dz} (\mathcal{P}(z)H'(z)) + \alpha\beta z \varrho(z)H(z) = q \varrho(z)H(z). \quad (6)$$

Equation (6) is the weighted divergence form associated with the Heun equation. Related weighted formulations occur in the spectral and orthogonality theory of Heun functions [5, 3].

For sufficiently smooth functions u and v , introduce

$$\mathfrak{S}H = \frac{d}{dz} (\mathcal{P}(z)H'(z)) + \alpha\beta z \varrho(z)H(z).$$

The associated Lagrange identity is

$$u \mathfrak{S}v - v \mathfrak{S}u = \frac{d}{dz} [\mathcal{P}(z)(u(z)v'(z) - u'(z)v(z))], \quad (7)$$

with boundary form

$$\mathcal{B}[u, v](z) = \mathcal{P}(z)(u(z)v'(z) - u'(z)v(z)) = \mathcal{P}(z)W[u, v](z). \quad (8)$$

We restrict attention to

$$0 < z < 1, \quad a \notin [0, 1].$$

Thus $z = a$ lies outside the spectral interval, while $z = 0$ and $z = 1$ remain singular endpoints.

Admissible endpoint conditions must eliminate the limiting boundary contribution

$$[\mathcal{B}[u, v](z)]_0^1 = \lim_{z \rightarrow 1^-} \mathcal{B}[u, v](z) - \lim_{z \rightarrow 0^+} \mathcal{B}[u, v](z) = 0. \quad (9)$$

For singular endpoints, this condition alone does not characterize a self-adjoint realization; the required endpoint data are determined by the limit-point/limit-circle classification [8, 17].

For regular local Heun solutions,

$$\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \delta > 0$$

are sufficient for the corresponding boundary contributions to vanish when the functions and their first derivatives remain finite at the endpoints. These conditions concern only the regular Frobenius branches and do not classify the full local solution space.

3.1 Weighted operator realization and Weyl classification

For the operator-theoretic analysis, we specialize to the real parameter regime

$$a > 1, \quad \gamma > 0, \quad \delta > 0, \quad \epsilon \in \mathbb{R}, \quad \alpha\beta \in \mathbb{R}. \quad (10)$$

On the interval $(0, 1)$, the branch-dependent coefficients $\mathcal{P}$ and ϱ differ from the corresponding positive representatives only by a common nonzero constant factor. We therefore set

$$p(z) = z^{\gamma}(1-z)^{\delta}(a-z)^{\epsilon}, \quad \rho(z) = z^{\gamma-1}(1-z)^{\delta-1}(a-z)^{\epsilon-1}. \quad (11)$$

Thus

$$p(z) > 0, \quad \rho(z) > 0, \quad 0 < z < 1,$$

and (6) is equivalently written as

$$(pH')' + \alpha\beta z \rho H = q\rho H.$$

Define the Sturm–Liouville differential expression

$$\tau H = \frac{1}{\rho(z)} \left[-(p(z)H'(z))' - \alpha\beta z \rho(z)H(z) \right], \quad (12)$$

so that

$$\tau H = \lambda H, \quad \lambda = -q.$$

The corresponding Hilbert space is

$$\mathcal{H} = L^2_\rho(0, 1), \quad \langle u, v \rangle_\rho = \int_0^1 u(z) \overline{v(z)} \rho(z) dz.$$

The maximal realization associated with τ is

$$T_{\max} H = \tau H$$

on the domain

$$\mathcal{D}(T_{\max}) = \left\{ H \in L^2_\rho(0, 1) : H, pH' \in AC_{\text{loc}}(0, 1), \tau H \in L^2_\rho(0, 1) \right\}. \quad (13)$$

The closure of the compactly supported restriction of $T_{\max}$ defines the minimal operator $T_{\min}$. In the present real Sturm–Liouville setting,

$$T_{\min}^* = T_{\max},$$

and self-adjoint realizations are restrictions of $T_{\max}$ extending $T_{\min}$, with endpoint conditions determined by the Weyl classification [8].

For $u, v \in \mathcal{D}(T_{\max})$, the corresponding sesquilinear Lagrange bracket is

$$[u, v](z) = p(z) \left(u(z) \overline{v'(z)} - u'(z) \overline{v(z)} \right),$$

and Green's identity is

$$\langle T_{\max} u, v \rangle_\rho - \langle u, T_{\max} v \rangle_\rho = [u, v](1^-) - [u, v](0^+). \quad (14)$$

An endpoint is limit-circle if all solutions of

$$(\tau - \lambda_0)H = 0, \quad \lambda_0 \in \mathbb{C} \setminus \mathbb{R},$$

belong locally to L^2_ρ near that endpoint; otherwise it is limit-point [8].

Proposition 3.1 (Weyl classification of the singular endpoints). *Under the assumptions (10), the endpoint $z = 0$ is limit-circle for*

$$0 < \gamma < 2,$$

and limit-point for

$$\gamma \geq 2.$$

Similarly, $z = 1$ is limit-circle for

$$0 < \delta < 2,$$

and limit-point for

$$\delta \geq 2.$$

Proof. We first consider $z = 0$. Since $\alpha > 1$,

$$p(z) \sim a^\epsilon z^\gamma, \quad \rho(z) \sim a^{\epsilon-1} z^{\gamma-1}, \quad z \rightarrow 0^+.$$

The characteristic equation at $z = 0$ is

$$r(r + \gamma - 1) = 0,$$

with exponents

$$r_1 = 0, \quad r_2 = 1 - \gamma.$$

A regular local solution may be chosen so that

$$H_1(z) = 1 + O(z).$$

A second independent solution can be represented by reduction of order as

$$H_2(z) = H_1(z) \int^z \frac{dt}{p(t)H_1(t)^2}.$$

Since $p(t) \asymp t^\gamma$,

$$H_2(z) \sim \begin{cases} z^{1-\gamma}, & \gamma \neq 1, \\ \log z, & \gamma = 1, \end{cases} \quad z \rightarrow 0^+.$$

For the regular branch,

$$\int_0^c |H_1(z)|^2 \rho(z) dz \asymp \int_0^c z^{\gamma-1} dz,$$

which is finite because $\gamma > 0$. For the second branch, when $\gamma \neq 1$,

$$\int_0^c |H_2(z)|^2 \rho(z) dz \asymp \int_0^c z^{2(1-\gamma)} z^{\gamma-1} dz = \int_0^c z^{1-\gamma} dz.$$

This integral converges precisely when $\gamma < 2$. At $\gamma = 1$,

$$\int_0^c |\log z|^2 dz < \infty.$$

Hence $z = 0$ is limit-circle for $0 < \gamma < 2$ and limit-point for $\gamma \geq 2$. At $\gamma = 2$, the second weighted square behaves as z^{-1} and is non-integrable.

At $z = 1$, set $s = 1 - z$. Then

$$p(z) \sim (a-1)^\epsilon s^\delta, \quad \rho(z) \sim (a-1)^{\epsilon-1} s^{\delta-1}, \quad s \rightarrow 0^+.$$

The characteristic exponents are

$$0, \quad 1 - \delta.$$

The corresponding local behaviors are

$$H_1(z) \sim 1, \quad H_2(z) \sim \begin{cases} (1-z)^{1-\delta}, & \delta \neq 1, \\ \log(1-z), & \delta = 1. \end{cases}$$

The regular branch belongs locally to L_ρ^2 because $\delta > 0$, whereas

$$\int_{1-c}^1 |H_2(z)|^2 \rho(z) dz \asymp \int_0^c s^{1-\delta} ds,$$

which converges precisely when $\delta < 2$. Therefore $z = 1$ is limit-circle for $0 < \delta < 2$ and limit-point for $\delta \geq 2$. $\square$

For separated self-adjoint realizations, each limit-circle endpoint requires one independent boundary condition, whereas no independent boundary condition is imposed at a limit-point endpoint [8, 17]. Hence

$$0 < \gamma < 2, \quad 0 < \delta < 2$$

defines the limit-circle/limit-circle regime, with one separated condition at each endpoint.

The earlier sufficient conditions

$$\operatorname{Re} \gamma > 0, \quad \operatorname{Re} \delta > 0$$

apply only to the regular Frobenius branches, whereas Proposition 3.1 classifies the endpoints using the weighted square-integrability of the full local solution space.

The Weyl classification determines the admissible endpoint data but does not, by itself, imply compactness of the resolvent. Compactness is established separately for the double limit-circle regime.

4 Single-Orthogonality Relation

We continue under the real parameter assumptions of Section 3, in particular $a > 1$, and work in the weighted Hilbert space

$$\mathcal{H} = L_\rho^2(0, 1).$$

Let T be a separated self-adjoint realization of the Sturm–Liouville differential expression τ , with endpoint conditions determined by the Weyl classification. Let $H_m, H_n \in \mathcal{D}(T)$ satisfy

$$TH_m = \lambda_m H_m, \quad TH_n = \lambda_n H_n,$$

where

$$\lambda_m = -q_m, \quad \lambda_n = -q_n.$$

Since T is self-adjoint, $\lambda_m, \lambda_n \in \mathbb{R}$. Green's identity (14) gives

$$(\lambda_m - \lambda_n) \langle H_m, H_n \rangle_\rho = [H_m, H_n](1^-) - [H_m, H_n](0^+). \quad (15)$$

The boundary contribution vanishes for functions in the domain of the chosen self-adjoint realization. Hence

$$(\lambda_m - \lambda_n) \langle H_m, H_n \rangle_\rho = 0.$$

For distinct spectral values $q_m \neq q_n$, equivalently $\lambda_m \neq \lambda_n$, we obtain

$$\int_0^1 H_m(z) \overline{H_n(z)} \rho(z) dz = 0, \quad q_m \neq q_n. \quad (16)$$

Thus Heun-type eigenfunctions corresponding to distinct accessory values are orthogonal in $L_\rho^2(0, 1)$. For real-valued eigenfunctions, (16) reduces to

$$\int_0^1 \rho(z) H_m(z) H_n(z) dz = 0, \quad q_m \neq q_n.$$

Orthogonality alone is insufficient to ensure completeness; additional spectral properties of the self-adjoint realization are required.

5 Self-Adjoint Hilbert Realization and Spectral Consequences

The weighted Hilbert space

$$\mathcal{H} = L_\rho^2(0, 1),$$

the maximal operator $T_{\max}$, its domain $\mathcal{D}(T_{\max})$, and the corresponding Green identity were introduced in Section 3. We continue throughout this section under the real parameter assumptions

$$a > 1, \quad \gamma > 0, \quad \delta > 0, \quad \epsilon \in \mathbb{R}, \quad \alpha\beta \in \mathbb{R},$$

and use the positive Sturm–Liouville coefficients p and ρ defined in (11).

The Sturm–Liouville spectral parameter λ and the Heun accessory parameter q are related by

$$\lambda = -q.$$

5.1 Self-adjoint realization

The Weyl classification obtained in Proposition 3.1 determines the endpoint data required for a self-adjoint realization. In the limit-circle/limit-circle regime

$$0 < \gamma < 2, \quad 0 < \delta < 2,$$

one independent boundary condition is required at each endpoint.

Theorem 5.1 (Self-adjoint realization in the LC–LC regime). *Assume*

$$0 < \gamma < 2, \quad 0 < \delta < 2.$$

Let $v_0, v_1 \in \mathcal{D}(T_{\max})$ be real-valued boundary reference functions associated with the endpoints 0 and 1, respectively, such that the endpoint boundary functionals

$$H \mapsto [H, v_0](0^+), \quad H \mapsto [H, v_1](1^-)$$

are nontrivial. Define

$$T = T_{\max}|_{\mathcal{D}(T)},$$

where

$$\mathcal{D}(T) = \left\{ H \in \mathcal{D}(T_{\max}) : \begin{array}{l} \lim_{z \rightarrow 0^+} p(z)W[H, v_0](z) = 0, \\ \lim_{z \rightarrow 1^-} p(z)W[H, v_1](z) = 0 \end{array} \right\}. \quad (17)$$

Then T is a separated self-adjoint realization of the Heun Sturm–Liouville operator in $L_\rho^2(0, 1)$.

Proof. By Proposition 3.1, both endpoints are limit-circle. Consequently, one independent boundary condition is required at each endpoint for a separated self-adjoint realization.

For $u, v \in \mathcal{D}(T)$, Green's identity (14) gives

$$\langle Tu, v \rangle_\rho - \langle u, Tv \rangle_\rho = [u, v](1^-) - [u, v](0^+).$$

The two boundary conditions in (17) eliminate the corresponding endpoint boundary forms, so T is symmetric. Since these two independent separated conditions provide precisely the boundary data required at the two limit-circle endpoints, the resulting restriction of $T_{\max}$ is self-adjoint [8, 17]. $\square$

At a limit-point endpoint no independent boundary condition is imposed. Thus, in the mixed LC–LP cases only one separated boundary condition is required, whereas if both endpoints are limit-point the minimal operator is essentially self-adjoint.

Corollary 5.2 (Reality of the spectral values). *Every Heun accessory value q associated with an eigenfunction of the self-adjoint realization T is real.*

Proof. Since T is self-adjoint, every eigenvalue λ of T is real. The relation

$$q = -\lambda$$

therefore implies $q \in \mathbb{R}$. □

The orthogonality of eigenfunctions corresponding to distinct accessory values has already been established in Section 4.

5.2 Compact resolvent in the double limit-circle regime

In the double limit-circle regime, compactness follows from the endpoint classification together with the general singular Sturm–Liouville resolvent theorem.

Theorem 5.3 (Hilbert–Schmidt resolvent). *Assume*

$$a > 1, \quad 0 < \gamma < 2, \quad 0 < \delta < 2, \quad \epsilon \in \mathbb{R}, \quad \alpha\beta \in \mathbb{R},$$

and let T be a self-adjoint realization of the Heun Sturm–Liouville operator in $L^2_\rho(0, 1)$. Then, for every ζ in the resolvent set of T ,

$$R_\zeta(T) = (T - \zeta I)^{-1}$$

is a Hilbert–Schmidt operator:

$$R_\zeta(T) \in \mathcal{B}_2(L^2_\rho(0, 1)), \tag{18}$$

where $\mathcal{B}_2$ denotes the Hilbert–Schmidt class. In particular, T has compact resolvent.

Proof. By Proposition 3.1, the assumptions

$$0 < \gamma < 2, \quad 0 < \delta < 2$$

place both endpoints in the limit-circle case. The operator T is, by assumption, a self-adjoint restriction of the corresponding maximal Sturm–Liouville operator.

The general limit-circle resolvent theorem therefore applies and yields

$$(T - \zeta I)^{-1} \in \mathcal{B}_2(L^2_\rho(0, 1)),$$

for every ζ in the resolvent set of T ; see [8, Theorem 7.1]. Since every Hilbert–Schmidt operator is compact, the resolvent of T is compact. □

Corollary 5.4 (Discreteness and completeness). *Under the assumptions of Theorem 5.3, the spectrum of T is real and purely discrete. Its eigenvalues have finite multiplicity and no finite accumulation point. Accordingly, the eigenfunctions may be indexed as*

$$H_0, H_1, H_2, \dots,$$

with corresponding real accessory values

$$q_0, q_1, q_2, \dots,$$

where spectral values are repeated according to multiplicity when necessary. These eigenfunctions may be chosen to form a complete orthogonal system in $L^2_\rho(0, 1)$. In particular,

$$\overline{\text{span}\{H_0, H_1, H_2, \dots\}} = L^2_\rho(0, 1). \tag{19}$$

Proof. By assumption, T is self-adjoint, and Theorem 5.3 shows that it has compact resolvent. Hence the spectral theorem for self-adjoint operators with compact resolvent implies that the spectrum consists entirely of isolated real eigenvalues of finite multiplicity, with no finite accumulation point, and that the associated eigenfunctions form a complete orthogonal system in $L^2_\rho(0, 1)$.

Since

$$q_n = -\lambda_n,$$

the corresponding accessory values q_n are real and discrete, and the same eigenfunctions give the complete Heun spectral system. The purely discrete nature of the spectrum is also consistent with [8, Corollary 7.2]. □

Remark 1. The parameter conditions

$$0 < \gamma < 2, \quad 0 < \delta < 2$$

therefore provide an explicit regime in which resolvent compactness, spectral discreteness, and completeness follow from the endpoint classification together with the singular Sturm–Liouville resolvent theorem, rather than being imposed as independent assumptions. The Weyl classification itself does not imply loss of compactness when one or both endpoints are limit-point.

6 Normalization, Formal Expansions, and the Heun-Type Basis Property

Let H_n be a nonzero eigenfunction of the self-adjoint realization T ,

$$TH_n = \lambda_n H_n, \quad q_n = -\lambda_n.$$

Its squared norm in $L^2_\rho(0, 1)$ is

$$N_n = \|H_n\|_\rho^2 = \int_0^1 |H_n(z)|^2 \rho(z) dz. \quad (20)$$

Since $H_n \in L^2_\rho(0, 1)$, $H_n \neq 0$, and $\rho(z) > 0$ on $(0, 1)$,

$$0 < N_n < \infty.$$

Quadratic normalization integrals of this type occur in the theory of orthogonal Heun functions; explicit formulas in terms of local Heun solutions and their Wronskians were obtained by Becker [5].

Define

$$\widehat{H}_n(z) = \frac{H_n(z)}{\sqrt{N_n}}.$$

After choosing an orthonormal basis within each eigenspace when necessary, the normalized eigenfunctions satisfy

$$\langle \widehat{H}_m, \widehat{H}_n \rangle_\rho = \int_0^1 \widehat{H}_m(z) \overline{\widehat{H}_n(z)} \rho(z) dz = \delta_{mn}.$$

The associated formal expansion is

$$F(z) \sim \sum_{n=0}^{\infty} c_n H_n(z). \quad (21)$$

Whenever the expansion converges in $L^2_\rho(0, 1)$, orthogonality gives

$$c_n = \frac{\int_0^1 F(z) \overline{H_n(z)} \rho(z) dz}{\int_0^1 |H_n(z)|^2 \rho(z) dz}. \quad (22)$$

Orthogonality alone is insufficient to ensure completeness; additional spectral properties of the self-adjoint realization are required. In the double limit-circle regime, completeness follows from the compact-resolvent result of Section 5.

Theorem 6.1 (Heun-type expansion basis). *Assume the hypotheses of Corollary 5.4. Let*

$$\{H_n\}_{n=0}^{\infty}$$

be a complete orthogonal system of Heun eigenfunctions in $L^2_\rho(0, 1)$, and let N_n be defined by (20). Then every

$$F \in L^2_\rho(0, 1)$$

admits the expansion

$$F(z) = \sum_{n=0}^{\infty} c_n H_n(z) \quad \text{in } L^2_\rho(0, 1), \quad (23)$$

where

$$c_n = \frac{\int_0^1 F(z) \overline{H_n(z)} \rho(z) dz}{N_n}.$$

Equivalently,

$$F(z) = \sum_{n=0}^{\infty} \langle F, \widehat{H}_n \rangle_\rho \widehat{H}_n(z) \quad \text{in } L^2_\rho(0, 1).$$

Proof. By Corollary 5.4, the eigenfunctions of the self-adjoint realization form a complete orthogonal system in $L^2_\rho(0, 1)$. After normalization and, when necessary, the choice of an orthonormal basis within each eigenspace, $\{\widehat{H}_n\}$ is a complete orthonormal system.

Hence the Hilbert-space expansion theorem gives

$$F = \sum_{n=0}^{\infty} \langle F, \widehat{H}_n \rangle_\rho \widehat{H}_n$$

with convergence in the L^2_ρ -norm. Since

$$\widehat{H}_n = \frac{H_n}{\sqrt{N_n}},$$

the coefficient of H_n is

$$c_n = \frac{\langle F, H_n \rangle_\rho}{N_n},$$

which yields (23). □

In particular,

$$\lim_{N \rightarrow \infty} \left\| F - \sum_{n=0}^N c_n H_n \right\|_\rho = 0,$$

and Parseval's identity for the normalized system is

$$\|F\|_\rho^2 = \sum_{n=0}^{\infty} |\langle F, \widehat{H}_n \rangle_\rho|^2.$$

These statements concern convergence in the weighted L^2 -norm; no pointwise or uniform convergence is asserted.

Remark 2. For real-valued eigenfunctions and F , the Hermitian formulas reduce to

$$N_n = \int_0^1 \rho(z) H_n^2(z) dz, \quad c_n = \frac{\int_0^1 \rho(z) F(z) H_n(z) dz}{\int_0^1 \rho(z) H_n^2(z) dz}.$$

Thus the bilinear formulas are the real-valued specialization of the Hilbert-space formulation.

7 Numerical Illustration: A Singular Weighted HeunG Expansion

We now illustrate the Heun eigenfunction expansion in a parameter regime that retains the singular endpoint structure developed in the preceding sections. The purpose of the computation is to provide a numerical consistency check of the weighted spectral formulation, rather than an independent proof of the self-adjointness or completeness results established above.

We use the Mathematica convention

$$H(z; q) = \text{HeunG}(a, q, \alpha, \beta, \gamma, \delta, z)$$

and choose

$$a = 2, \quad \alpha = 1, \quad \beta = 2, \quad \gamma = \delta = \frac{3}{2}.$$

The Fuchsian relation gives

$$\epsilon = \alpha + \beta + 1 - \gamma - \delta = 1,$$

so that

$$\alpha + \beta + 1 = \gamma + \delta + \epsilon = 4.$$

Since

$$0 < \gamma < 2, \quad 0 < \delta < 2,$$

both endpoints $z = 0$ and $z = 1$ belong to the limit-circle class of Proposition 3.1. The positive Sturm–Liouville coefficients introduced in Section 3 become

$$p(z) = z^{3/2}(1-z)^{3/2}(2-z)$$

and

$$\rho(z) = z^{1/2}(1-z)^{1/2}.$$

Thus the numerical model involves a genuinely nonconstant weight and retains the nontrivial endpoint behavior of the singular Sturm–Liouville problem, rather than reducing to the constant-weight case.

The local Heun solution used in the computation is

$$H(z; q) = \text{HeunG}\left(2, q, 1, 2, \frac{3}{2}, \frac{3}{2}, z\right).$$

The Mathematica HeunG branch selects the regular Frobenius solution at $z = 0$.

For numerical evaluation, the singular endpoints are approached through the truncated interval

$$[\eta, 1 - \eta].$$

The truncation is used solely as a numerical regularization and is not regarded as a replacement of the singular boundary-value problem on $(0, 1)$. To assess the sensitivity of the computed spectrum to this regularization, the calculation is repeated for several decreasing values of η .

At the right endpoint, we impose the approximate zero-flux condition

$$p(1 - \eta)H'(1 - \eta; q) \simeq 0, \quad (24)$$

where the derivative is evaluated using the one-sided finite difference

$$H'(1 - \eta; q) \approx \frac{H(1 - \eta; q) - H(1 - \eta - h; q)}{h}, \quad h = 10^{-5}.$$

Solving (24) gives the first four numerical accessory values reported in Table 1.

Table 1: First four numerical accessory values for decreasing endpoint truncation η .

η	q_0	q_1	q_2	q_3
1.0×10^{-2}	1.05477	-3.30254	-10.6858	-21.2033
5.0×10^{-3}	1.05638	-3.26818	-10.5270	-20.7744
2.0×10^{-3}	1.05705	-3.25321	-10.4530	-20.5596
1.0×10^{-3}	1.05720	-3.24981	-10.4355	-20.5058
5.0×10^{-4}	1.05726	-3.24858	-10.4290	-20.4857

The values in Table 1 show a clear numerical stabilization as the endpoint truncation is reduced. In particular, the variation of the first four accessory values decreases substantially as η approaches zero. This indicates that the computed values are not artifacts of a single fixed truncation.

For the subsequent normalization and expansion calculations, we use

$$\eta = 5 \times 10^{-4},$$

for which

$$\begin{aligned} q_0 &\approx 1.05726, & q_1 &\approx -3.24858, \\ q_2 &\approx -10.4290, & q_3 &\approx -20.4857. \end{aligned}$$

Let

$$H_n(z) = H(z; q_n).$$

The weighted normalization integrals are

$$N_n = \int_{\eta}^{1-\eta} |H_n(z)|^2 \rho(z) dz.$$

Numerically,

$$\begin{aligned} N_0 &\approx 0.609104, & N_1 &\approx 0.159447, \\ N_2 &\approx 0.0713241, & N_3 &\approx 0.0401872. \end{aligned}$$

Defining the normalized modes

$$\phi_n(z) = \frac{H_n(z)}{\sqrt{N_n}}, \quad n = 0, 1, 2, 3,$$

we compute the weighted Gram matrix

$$G_{mn} = \int_{\eta}^{1-\eta} \phi_m(z) \overline{\phi_n(z)} \rho(z) dz.$$

The resulting matrix is

$$G \approx \begin{pmatrix} 1 & -2.48476 \times 10^{-5} & -3.43177 \times 10^{-5} & -4.93964 \times 10^{-5} \\ -2.48476 \times 10^{-5} & 1 & -7.26474 \times 10^{-5} & -8.90974 \times 10^{-5} \\ -3.43177 \times 10^{-5} & -7.26474 \times 10^{-5} & 1 & -1.44442 \times 10^{-4} \\ -4.93964 \times 10^{-5} & -8.90974 \times 10^{-5} & -1.44442 \times 10^{-4} & 1 \end{pmatrix}. \quad (25)$$

The diagonal entries are equal to one by normalization, while the largest off-diagonal entry has magnitude approximately

$$1.44 \times 10^{-4}.$$

Thus the computed modes are numerically orthogonal to high accuracy with respect to the nonconstant weight ρ .

We retain the smooth non-polynomial test function

$$F(z) = z(1-z)e^z$$

used in the original numerical illustration. Using the first four normalized Heun modes, we form

$$F_4(z) = \sum_{n=0}^3 c_n \phi_n(z),$$

where the weighted projection coefficients are

$$c_n = \int_{\eta}^{1-\eta} F(z) \phi_n(z) \rho(z) dz.$$

The computed coefficients are

$$\begin{aligned} c_0 &\approx 0.19927, & c_1 &\approx 0.00389846, \\ c_2 &\approx -0.0680782, & c_3 &\approx -0.00343338. \end{aligned}$$

The resulting approximation is shown in Figure 1. The weighted squared L^2 -error is

$$\int_{\eta}^{1-\eta} |F(z) - F_4(z)|^2 \rho(z) dz \approx 9.18 \times 10^{-7}. \quad (26)$$

Equivalently,

$$\|F - F_4\|_{L^2_{\rho}([\eta, 1-\eta])} \approx 9.58 \times 10^{-4}.$$

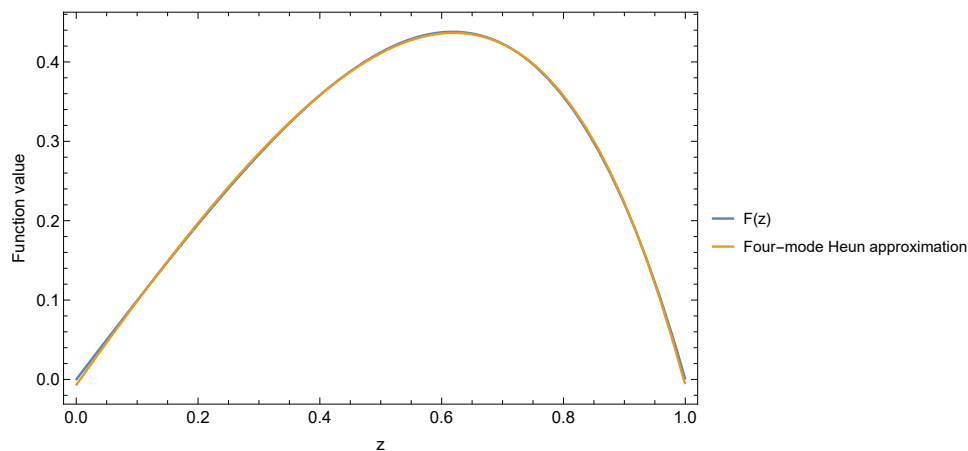


Figure 1: Four-mode weighted HeunG approximation of the test function $F(z) = z(1-z)e^z$ for $\eta = 5 \times 10^{-4}$. The target function and the truncated Heun expansion are nearly indistinguishable throughout the computational interval.

This revised numerical experiment retains the essential singular structure that is absent from the constant-weight example. In particular, the weight

$$\rho(z) = \sqrt{z(1-z)}$$

is nonconstant and reflects the endpoint behavior of the associated Sturm–Liouville problem. Moreover, rather than relying on a single fixed truncation, the dependence of the accessory values on η is explicitly examined.

The stabilization of the computed accessory values, the near-identity weighted Gram matrix, and the small weighted approximation error together provide a numerical consistency check of the singular spectral framework. The truncation $[\eta, 1-\eta]$ remains a numerical regularization; the self-adjoint realization, discreteness of the spectrum, and completeness of the Heun eigenfunctions follow from the analytical results of the preceding sections and from Theorem 6.1.



8 Discussion

The preceding results identify the mechanism by which Heun-type eigenfunctions may serve as expansion functions. The central point is that the basis property is not a consequence of the local analytic definition of the Heun function alone. Rather, it is produced by the spectral structure of a self-adjoint singular Sturm–Liouville realization of the Heun equation.

This distinction is essential. A local Heun function obtained from a Frobenius expansion near a regular singular point is primarily a local solution of a differential equation. Such a solution does not automatically belong to an orthogonal system, nor does it automatically generate an expansion basis. Orthogonality appears only after the accessory parameter is interpreted as a spectral parameter and admissible endpoint conditions select a self-adjoint family of eigenfunctions. Thus, the endpoint data play a structural role: they convert local Heun solutions into spectral objects.

In the present formulation, the interval $(0, 1)$ is chosen so that the endpoints coincide with two regular singular points of the Heun equation. For the operator-theoretic analysis we specialize to

$$a > 1,$$

so that the third finite singularity lies outside the spectral interval. The resulting problem is therefore singular at both endpoints.

The endpoint conditions are governed by Weyl's limit-point/limit-circle classification rather than by regular endpoint theory. Under the real positive-weight assumptions introduced in Section 3, the endpoint $z = 0$ is limit-circle for

$$0 < \gamma < 2$$

and limit-point for $\gamma \geq 2$, while $z = 1$ is limit-circle for

$$0 < \delta < 2$$

and limit-point for $\delta \geq 2$. Consequently, each limit-circle endpoint requires one independent boundary condition for a separated self-adjoint realization, whereas no independent boundary condition is imposed at a limit-point endpoint.

This classification clarifies the role of the Lagrange boundary form. The vanishing of the limiting boundary contribution is the mechanism that produces symmetry on the chosen domain, while self-adjointness additionally requires the appropriate endpoint data dictated by the Weyl classification. In the weighted Hilbert space $L^2_\rho(0, 1)$, the relevant sesquilinear boundary form is

$$[u, v](z) = p(z) \left(u(z) \overline{v'(z)} - u'(z) \overline{v(z)} \right).$$

Thus the passage from formal symmetry to a genuine self-adjoint operator is a domain-dependent statement and not merely a consequence of rewriting the equation in divergence form [8, 17].

Once a self-adjoint realization has been selected, eigenfunctions associated with distinct accessory values are orthogonal in the weighted Hilbert space:

$$\int_0^1 H_m(z) \overline{H_n(z)} \rho(z) dz = 0, \quad q_m \neq q_n.$$

For real-valued eigenfunctions, this reduces to the corresponding bilinear orthogonality relation. The bilinear formulation is therefore best viewed as the real-valued specialization of the Hilbert-space theory rather than as a separate parallel construction.

The analogy with classical systems such as Fourier–Bessel, Legendre, or Jacobi expansions becomes fully valid only after the spectral structure has been specified. Orthogonality alone does not imply a basis property. Discreteness and completeness require additional spectral information about the self-adjoint realization.

In the double limit-circle regime

$$0 < \gamma < 2, \quad 0 < \delta < 2,$$

this additional step can be made explicit. The self-adjoint Heun Sturm–Liouville realization has Hilbert–Schmidt, and hence compact, resolvent. It follows that its spectrum is purely discrete, with real eigenvalues of finite multiplicity and no finite accumulation point, and that the corresponding eigenfunctions form a complete orthogonal system in $L^2_\rho(0, 1)$. Thus, in this parameter regime, the basis property is a consequence of the singular endpoint classification together with the compact-resolvent spectral theory, rather than an independent assumption.

Recent spectral studies have considered a considerably broader range of Sturm–Liouville-type operators, including non-self-adjoint, indefinite, impulsive, discontinuous, almost-periodic, and inverse spectral problems [11, 15, 14, 9, 10, 1, 4]. Related developments include oscillation theory for delay and noncanonical Sturm–Liouville equations and inverse spectral recovery for Dirac-type systems [19, 20, 26]. Compared with these directions, the present paper deliberately remains within a self-adjoint singular Heun framework. Its purpose is not to develop a new general extension theory, but to connect Frobenius endpoint behavior, Weyl classification, admissible self-adjoint boundary data, compact resolvent, orthogonality, and completeness within a single Heun-specific spectral construction.

The analysis therefore separates three levels of structure. The first is the local Heun solution, determined by the singularity structure of the differential equation. The second is the Heun eigenfunction, obtained after the differential expression is realized

as a self-adjoint singular Sturm–Liouville operator with admissible endpoint data. The third is the Heun expansion basis, obtained when the resulting self-adjoint realization has discrete spectrum and a complete eigenfunction system. These three levels should not be identified with one another.

This perspective also distinguishes the present approach from recent works focused primarily on numerical Heun evaluation or on application-specific analytical expansions. Computational aspects of the general Heun function have been investigated in [7], while Heun and related spectral expansion techniques have appeared in plasma-equilibrium and magnetic mathematical-physics models [6, 18, 25]. The present contribution is complementary to these works: it concentrates on the operator-theoretic conditions under which the Heun functions themselves acquire a genuine orthogonal-basis interpretation.

The revised numerical illustration provides a computational consistency check of this interpretation. In contrast with the earlier constant-weight example, the new parameter choice yields the nonconstant weight

$$\rho(z) = \sqrt{z(1-z)},$$

and the sensitivity of the accessory values to the endpoint truncation is examined explicitly. The resulting weighted Gram matrix is close to the identity, and the four-mode expansion of $F(z) = z(1-z)e^z$ achieves a small weighted L^2 -error. These numerical results illustrate the spectral construction but do not replace the analytical arguments establishing self-adjointness, discreteness, and completeness.

The framework is intentionally formulated at the level of the general Heun equation. This makes it adaptable to later applications in which differential equations reduce to Heun form. In such problems, the geometry, physical boundary conditions, and regularity requirements determine the relevant self-adjoint realization and therefore the corresponding spectral family. In this sense, Heun functions can play a role analogous to Bessel, Legendre, or Jacobi systems: not as arbitrary local special functions, but as eigenfunctions selected by a well-posed singular spectral problem.

9 Conclusion

In this paper, we developed a singular Sturm–Liouville framework for interpreting Heun functions as spectral expansion functions on the interval $(0, 1)$. The accessory parameter q was identified as a spectral parameter after transforming the general Heun equation into a formally self-adjoint weighted form.

A central point of the analysis is that local Heun solutions do not acquire orthogonality or basis properties solely from their differential-equation structure. These properties depend on the operator realization and the admissible endpoint data. Using Weyl’s limit-point/limit-circle theory, we obtained the explicit endpoint classification

$$z = 0 : \quad 0 < \gamma < 2 \text{ (limit-circle)}, \quad \gamma \geq 2 \text{ (limit-point)},$$

and

$$z = 1 : \quad 0 < \delta < 2 \text{ (limit-circle)}, \quad \delta \geq 2 \text{ (limit-point)}.$$

This determines the boundary data required for separated self-adjoint realizations of the Heun Sturm–Liouville operator.

In the double limit-circle regime

$$0 < \gamma < 2, \quad 0 < \delta < 2,$$

the resolvent of the self-adjoint realization is Hilbert–Schmidt and hence compact. Consequently, the spectrum is real and purely discrete, and the corresponding Heun eigenfunctions form a complete orthogonal system in $L^2_\rho(0, 1)$. The formal Heun expansion therefore becomes a genuine weighted L^2 -basis expansion in this parameter regime.

The revised numerical example retains the singular structure through the nonconstant weight

$$\rho(z) = \sqrt{z(1-z)}.$$

The computed accessory values stabilize as the endpoint truncation is reduced, the weighted Gram matrix is close to the identity, and a four-mode HeunG expansion approximates the test function $F(z) = z(1-z)e^z$ with weighted squared L^2 -error of approximately

$$9.18 \times 10^{-7}.$$

The results clarify the distinction between local Heun solutions, self-adjoint Heun eigenfunctions, and complete Heun expansion bases. This framework provides a parameter-specific spectral foundation for future applications in problems that reduce to Heun form, including wave, quantum-mechanical, and plasma-equilibrium models.



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