



Lelong classes of plurisubharmonic functions on an affine variety

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Abstract

We study the Lelong classes $\mathcal{L}(V), \mathcal{L}^+(V)$ of psh functions on an affine variety V . We compute the Monge-Ampère mass of these functions, which we use to define the degree of a polynomial on V in terms of pluripotential theory (the *Lelong degree*). We compute the Lelong degree explicitly in a specific example. Finally, we derive an affine version of Bézout's theorem.

1 Introduction

Although they are algebraic objects, complex polynomials can also be studied as entire holomorphic functions that satisfy certain growth restrictions. The tools of complex analysis (such as the Cauchy integral formula) can be used to study their deeper properties. An example of this in one variable is the standard complex analytic proof of the fundamental theorem of algebra. In turn, the analysis of holomorphic functions involves the use of plurisubharmonic (psh) functions. In one dimension, these are the classical subharmonic functions of potential theory in the plane; in higher dimensions psh functions satisfy pluripotential theory, a nonlinear generalization based on the complex Monge-Ampère operator. The prototypical example of a psh function is $\log|F|$, where F is a holomorphic mapping.

Back in one complex variable, $z = x + iy$, a polynomial

$$p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + a_0$$

is classified by its degree $\deg(p) := d$. This gives the order of growth of p at infinity, as well as its number of zeros, counting multiplicity, by the fundamental theorem of algebra. The degree can also be given in terms of the subharmonic function $u := \log|p|$ in a couple of ways:

$$\deg(p) = \sup\left\{c > 0: \lim_{|z| \rightarrow \infty} \frac{1}{c} u(z) - \log|z| = O(1), \right\} \quad (1)$$

$$2\pi \deg(p) = \int dd^c u. \quad (2)$$

For a function φ of class C^2 , $dd^c \varphi = (\Delta \varphi) dx \wedge dy$ where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian, and the operator dd^c extends to subharmonic functions as a positive measure. Equation (1) simply reformulates the notion of growth, while (2) follows from the fundamental theorem of algebra, writing $p(z) = c \prod_i (z - a_i)$ and using the fact that $dd^c \log|z - a_i| = 2\pi \delta_{a_i}$, where δ_a denotes the discrete probability measure on $\{a\}$.

The situation is more complicated in several variables. Instead of a single polynomial, one considers a system of polynomial equations $p_1(z) = \cdots = p_k(z) = 0$, $z \in \mathbb{C}^N$. An underdetermined system has infinitely many solutions that form an algebraic set $V = \mathbf{V}(p_1, \dots, p_k)$. Consider adding another polynomial p_{k+1} ; then finding a solution to $p_1 = \cdots = p_k = p_{k+1} = 0$ is equivalent to finding a solution to

$$p_{k+1}(z) = 0, \quad z \in V.$$

Polynomials on V can be studied algebraically as elements of a ring $\mathbb{C}[V]$, or as holomorphic functions on V , or more generally, in terms of pluripotential theoretic objects: psh functions and closed positive currents. Pluripotential theory on holomorphic varieties in \mathbb{C}^N was studied by Sadullaev [12] and Zeriahi ([16], [17]). We should also mention that a general pluripotential theory on complex manifolds has been developed (by Demailly, Guedj/Zeriahi and others) that has numerous applications in complex and algebraic geometry; see [6] for a survey.

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In this paper, we study the Lelong classes $\mathcal{L}(V), \mathcal{L}^+(V)$ of psh functions on an affine variety* $V \subset \mathbb{C}^N$. These are functions of (at most) logarithmic growth as $|z| \rightarrow \infty$ on V . We obtain the formula

$$\int_V (dd^c u)^m = d(2\pi)^m \quad \text{for all } u \in \mathcal{L}^+(V), \quad (3)$$

where $m = \dim(V)$, $d = \deg(V)$, and $(dd^c)^m$ is the m -th exterior power of dd^c , or *complex Monge-Ampère operator*. Here, we use the complex analytic definition of degree and dimension (as in e.g. [8] or [13]) in terms of a branched covering projection.

In Section 2 we construct the projection explicitly for an affine variety; this is a standard construction in commutative algebra, and provides good coordinates for computation (a *Noether presentation*). In Section 3 we introduce the Lelong classes $\mathcal{L}(V), \mathcal{L}^+(V)$ of psh functions, and in Section 4, we derive formula (3). To carry out our computations, we adapt some standard convergence and comparison theorems for the complex Monge-Ampère operator in \mathbb{C}^N .

In Section 5 we introduce the Lelong degree of a polynomial $p \in \mathbb{C}[V]$ as a generalization of (2). It can also be interpreted as a Lelong number for the current $dd^c p$. We use the quantity $d(2\pi)^m$ derived in the previous section to normalize the degree. In Section 6 we compute the Lelong degree explicitly on an algebraic curve $V = \mathbf{V}(P) \subset \mathbb{C}^2$. Although we only carry out the computation for the specific polynomial $q(x, y) = y$ in this paper, we hope to expand this to a method for computing $\deg_{\mathcal{L}(V)}(q)$ for any $q \in \mathbb{C}[V]$, using a Newton polygon associated to V . Our computation shows that $\deg_{\mathcal{L}(V)}(q)$ is a rational number that gives the average growth of q along the branches of V as $|z| \rightarrow \infty$. The next step would be to generalize such a method to higher-dimensional varieties.

In the last section, we prove an affine version of Bézout's theorem. We relate it to the classical Bézout theorem in projective space via an example in \mathbb{C}^2 .

1.1 Notation

We recall some standard notation in computational algebraic geometry and several complex variables.

Write $\langle f_1, \dots, f_n \rangle$ for the ideal generated by elements f_j of a polynomial ring $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_N]$, and for $S \subset \mathbb{C}[z]$, write $\langle S \rangle$ for the ideal generated by S . Also, define the algebraic sets

$$\mathbf{V}(f_1, \dots, f_j) := \{z \in \mathbb{C}^N : f_1(z) = \dots = f_j(z) = 0\}, \quad \mathbf{V}(S) := \{z \in \mathbb{C}^N : f(z) = 0 \text{ for all } f \in S\}.$$

For $V \subset \mathbb{C}^N$, define the ideal

$$\mathbf{I}(V) := \{f \in \mathbb{C}[z] : f(z) = 0 \text{ for all } z \in V\}.$$

For $z = (z_1, \dots, z_N)$, $z_j = x_j + iy_j$, we have $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, and $dd^c = 2i\partial\bar{\partial}$, where

$$\partial(\cdot) = \sum_{j=1}^N \partial/\partial z_j(\cdot) \wedge dz_j, \quad \bar{\partial}(\cdot) = \sum_{j=1}^N \partial/\partial \bar{z}_j(\cdot) \wedge d\bar{z}_j.$$

For a psh function u , $dd^c u$ is a *positive (1, 1)-current*, i.e., a linear functional on smooth, compactly supported $(N-1, N-1)$ -forms such that $\langle dd^c u, \varphi \rangle \geq 0$ if φ is a *strongly positive form*.[†] Here we write $\langle \cdot, \cdot \rangle$ to denote the pairing of a current and a test form.

2 Noether Presentation

In this section we construct good coordinates for computation on an affine variety in \mathbb{C}^N . First, we recall the *grevlex*[‡] monomial ordering on polynomials in $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_N]$.

Definition 2.1. The *grevlex monomial ordering* is the ordering $<$ in which $z^\alpha < z^\beta$ if

1. either $|\alpha| < |\beta|$; or
2. $|\alpha| = |\beta|$ and there exists $j \in \{1, \dots, N\}$ such that $\alpha_j < \beta_j$ and $\alpha_k = \beta_k$ for all $k < j$.

Here, we are using standard multi-index notation, $\alpha = (\alpha_1, \dots, \alpha_N)$ and $|\alpha| = \alpha_1 + \dots + \alpha_N$; and similarly for β .

Denote by $\text{LT}(p)$ the leading term of a polynomial p with respect to the *grevlex* ordering. Recall that a *Gröbner basis* $\{f_1, \dots, f_N\}$ of an ideal $I \subset \mathbb{C}[z]$ is a collection of polynomials satisfying $I = \langle f_1, \dots, f_N \rangle$ and $\langle \text{LT}(I) \rangle = \langle \text{LT}(f_1), \dots, \text{LT}(f_N) \rangle$; here $\langle \text{LT}(I) \rangle$ is the ideal generated by the monomials $\text{LT}(I) = \{\text{LT}(p) : p \in I\}$.

Definition 2.2. The (*grevlex*) *normal form* of a polynomial p (with respect to I) is the unique polynomial $r(z) = \sum_\alpha r_\alpha z^\alpha$ for which the following properties hold:

1. $p = \sum_{j=1}^s q_j f_j + r$ where $\{f_1, \dots, f_s\}$ is a Gröbner basis of I ; and
2. $z^\alpha \notin \langle \text{LT}(I) \rangle$ whenever $r_\alpha z^\alpha$ is a nonzero term of r .

The existence and uniqueness of r follows from the generalized division algorithm for multivariable polynomials and the fact that the divisors are a Gröbner basis. It provides a standard polynomial representative of elements of $\mathbb{C}[z]/I$, which may be used to give a well-defined notion of degree.

*i.e., an irreducible algebraic subset of \mathbb{C}^N .

†An example of such a form is $\varphi = f\beta^{n-1}$, where f is a non-negative test function and $\beta = \sum_j dx_j \wedge dy_j$. See e.g. [9], chapter 3.

‡graded reverse lexicographic

Definition 2.3. Let $V \subset \mathbb{C}^N$ be an affine variety and $p \in \mathbb{C}[z]$. Then we define the *degree of p on V* by $\deg_V(p) := \deg(r)$, where r is the normal form of p with respect to $\mathbf{I}(V)$.

Note that the condition $|\alpha| = |\beta|$ in the definition of grevlex easily implies $\deg_V(p) \leq \deg(p)$.

Given an invertible complex linear map $L: \mathbb{C}^N \rightarrow \mathbb{C}^N$, let $L^*: \mathbb{C}[z] \rightarrow \mathbb{C}[z]$ be defined by $p(z) \mapsto p(L(z))$. We seek a change of coordinates such that the grevlex normal form becomes particularly simple. The following is a standard result in commutative algebra (Noether normalization).

Theorem 2.1. Let $V \subseteq \mathbb{C}^N$ be an (irreducible) affine variety and $I = \mathbf{I}(V)$. Then there is a non-negative integer $m \leq N$ such that for a generic complex linear change of coordinates $z \mapsto L(z)$ we have the following.

1. Write $z = (x_1, \dots, x_m, y_1, \dots, y_{N-m}) =: (x, y)$ and $J := L^*(I)$. Then the canonical map $\mathbb{C}[x] \rightarrow \mathbb{C}[x, y]/J$ induced by the inclusion $\mathbb{C}[x] \subseteq \mathbb{C}[x, y]$ is injective and finite (i.e., exhibits $\mathbb{C}[x, y]/J$ as a finite extension of $\mathbb{C}[x]$.)
2. For each $j = 1, \dots, N - m$ there is a $d_j \in \mathbb{N}$ and an irreducible polynomial $g_j \in J$ of total degree d_j of the form

$$g_j(x, y) = y_j^{d_j} - \sum_{k=0}^{d_j-1} g_{jk}(x, y_1, \dots, y_{j-1}) y_j^k \quad (4)$$

i.e., $\deg(g_{jk}) + k \leq d_j$ for each $k = 0, \dots, d_j - 1$.

Proof. The proof is by induction on N ; we follow the argument in [10]. The case $N = 1$ is almost trivial: $I = \langle p \rangle$ for some monic polynomial $p(z) = z^d + (\text{lower terms})$, and $\mathbb{C}[z]/I$ is a finite extension of \mathbb{C} .[§]

For general N , let $f \in I$ be a polynomial of degree $\deg(f) := d \geq 1$, with \widehat{f} its leading homogeneous part. Then for $L(z) = (c_1 \cdot z, \dots, c_N \cdot z)$ an invertible linear map, where $c_j \cdot z := c_{j1}z_1 + \dots + c_{jN}z_N$, we have

$$\widehat{f}(L(z)) = \widehat{f}(c_N)z_N^d + (\text{lower terms in } z_N).$$

As long as the generic condition $\widehat{f}(c_N) \neq 0$ holds, we may define

$$\tilde{g}(z) := \widehat{f}(c_N)^{-1} f(L(z)),$$

and \tilde{g} has the property (4). The natural map $\mathbb{C}[z'] \rightarrow \mathbb{C}[z', z_N]/\langle \tilde{g} \rangle$ is injective and finite, because \tilde{g} is a monic polynomial in z_N with coefficients in $\mathbb{C}[z']$. It is also easy to see that elements of $J = L^*(I)$ map to elements of J . Hence after quotienting out by J , the map $\mathbb{C}[z']/J_0 \rightarrow \mathbb{C}[z]/J$ is also injective and finite (where $J_0 = \mathbb{C}[z'] \cap J$). If $J_0 = \langle 0 \rangle$ then we are done with $m = N - 1$ and $g_1 = \tilde{g}$.

Otherwise, applying induction to the affine variety $V_0 = \mathbf{V}(J_0) \subseteq \mathbb{C}^{N-1}$, we obtain an integer $m \leq N$ such that for a generic linear map $L_0: \mathbb{C}^{N-1} \rightarrow \mathbb{C}^{N-1}$, the conclusion of the theorem holds, say, with polynomials g_1, \dots, g_{N-1-m} .

The theorem then holds for N with the linear change of coordinates $(z', z_N) \mapsto L(L_0(z'), z_N)$ and the polynomials g_1, \dots, g_{N-1-m} and $g_{N-m}(z) := \tilde{g}(L_0(z'), z_N)$.

Since V is irreducible, I is prime. If a g_j constructed above is reducible, it must contain an irreducible factor in I . We replace g_j by this irreducible factor. \square

Definition 2.4. Given an affine variety V , coordinates (x, y) that satisfy Theorem 2.1 will be called a *Noether presentation for V* .

A Noether presentation has a geometric interpretation. First, recall the following definition (cf., [8]).

Definition 2.5. Let X_1 and X_2 be analytic varieties. Then a surjective holomorphic map $\varphi: X_1 \rightarrow X_2$ exhibits X_1 as a *branched covering of X_2* if there is a dense open subset Ω of the regular part of X_2 such that for each $x \in \Omega$ there is some neighborhood U of x for which $\varphi^{-1}(U)$ is a union of disjoint open subsets of the regular part of X_1 . The branched covering is *locally d -sheeted* (resp. *finite*) if the number of these sets is d (resp. finite). We set $\deg_x(\varphi) := d$ to be the *degree of the covering map φ at x* . The set $X_2 \setminus \Omega$ is called the *branch locus*.

Now $x \mapsto \deg_x(\varphi)$ is a locally constant function in x (e.g. by elementary complex analysis). Hence when X_2 is connected, it is a global constant so we may write $\deg(\varphi)$ (independent of x).

Proposition 2.2. Let (x, y) be a Noether presentation for an irreducible affine variety $V \subset \mathbb{C}^N$.

1. There exists a constant $A > 0$ such that

$$\|y\| \leq A(1 + \|x\|), \text{ for all } (x, y) \in V. \quad (5)$$

2. If $P: \mathbb{C}^m \times \mathbb{C}^{N-m} \ni (x, y) \mapsto x \in \mathbb{C}^m$ is the projection, then $P(V) = \mathbb{C}^m$ and the restriction $P|_V$ exhibits V as a finite branched covering over \mathbb{C}^m . (Hence V is a complex manifold of dimension m away from the branch locus.)

Proof. We use induction on $N - m$, i.e., the number of polynomials g_1, \dots, g_{N-m} given by Theorem 2.1 that define V . (Note that the argument does not depend on N , the dimension of the ambient space.)

Step 1: Base case ($N - m = 1$).

[§]In this case, z is the 'y' variable and there is no 'x' variable.

1. We have $(x, y) = (x_1, \dots, x_{N-1}, y)$, and $V = \mathbf{V}(g_1)$ by hypothesis, where as in (4),

$$g_1(x, y) = y^{d_1} - \sum_{k=0}^{d_1-1} g_{1k}(x)y^k = 0 \text{ for all } (x, y) \in V. \quad (6)$$

We claim that the above equation implies the inequality

$$|y| \leq 2 \max_{0 \leq j \leq d_1-1} |g_{1j}(x)|^{1/(d_1-j)} \text{ for all } (x, y) \in V. \quad (7)$$

For if not,

$$\begin{aligned} |g_1(x, y)y^{-d_1}| &= |1 + g_{1(d_1-1)}(x)y^{-1} + \dots + g_{10}(x)y^{-d_1}| \\ &\geq 1 - (|g_{1(d_1-1)}(x)y^{-1}| + \dots + |g_{10}(x)y^{-d_1}|) \\ &\geq 1 - (2^{-1} + \dots + 2^{-d_1}) > 0 \end{aligned}$$

which is a contradiction. Hence (7) holds.

Fixing j , there exists a constant C_j such that $|g_{1j}(x)| \leq C_j(1 + \|x\|)^{d_1-j}$ for all $x \in \mathbb{C}^n$, since $g_{1j}(x)$ is a polynomial of degree at most $d_1 - j$. Putting this into the right-hand side of (7) and letting $A = \max_j C_j^{1/(d_1-j)}$, we obtain (5).

2. To show that P maps V onto \mathbb{C}^{N-1} , fix $x \in \mathbb{C}^{N-1}$. Then (6) is a nonzero polynomial equation of degree d_1 in y ; hence by the fundamental theorem of algebra, has d_1 solutions (counting multiplicity); write

$$g_1(x, y) = \prod_{j=1}^{d_1} (y - \alpha_j(x)). \quad (8)$$

This gives at least one point $(x, y) \in V$ that maps to x under P ; so $P(V) = \mathbb{C}^{N-1}$. The holomorphic implicit function theorem says that locally, P has a local holomorphic inverse on V at all points away from the subvariety $V \cap \{\frac{\partial}{\partial y} g_1 = 0\}$. In fact, the local inverses are given by $\alpha_j(x)$ in (8), and at such points x the values of $\alpha_j(x)$ are distinct for each j . Hence V is a holomorphic branched cover over \mathbb{C}^{N-1} , with d_1 sheets.

Step 2: Induction. Write P as the composition $P_2 \circ P_1$ given by

$$(x, y) = (x, y', y_{N-m}) \in \mathbb{C}^N \xrightarrow{P_1} (x, y') \in \mathbb{C}^{N-1} \xrightarrow{P_2} x \in \mathbb{C}^{N-m}.$$

Then $P_1(V) =: V_1$ is contained in the variety $\mathbf{V}(g_1, \dots, g_{N-m-1}) \subset \mathbb{C}^{N-1}$. Here we are using the fact that the polynomials g_1, \dots, g_{N-m-1} of (4) are independent of the last coordinate y_{N-m} .

On the other hand, if $(x, y') \in \mathbf{V}(g_1, \dots, g_{N-m-1})$ then the fundamental theorem of algebra applied to $s \mapsto g_{N-m}(x, y', s)$ gives the existence of $(x, y', y_{N-m}) \in V$, so that we have the reverse containment $\mathbf{V}(g_1, \dots, g_{N-m-1}) \subseteq V_1$. Hence $V_1 = \mathbf{V}(g_1, \dots, g_{N-m-1})$, which shows that V_1 is an affine variety in \mathbb{C}^{N-1} and (x, y') is a Noether presentation for V_1 . To prove each part of the theorem we apply induction to V_1 and $W := \mathbf{V}(g_{N-m}) \subset \mathbb{C}^N$.

1. Let $(x, y', y_{N-m}) \in V \subset W$. Then $(x, y') \in V_1$ and there exist constants $A_1, A_2 > 0$ such that

$$\|y_{N-m}\| \leq A_1(1 + \|(x, y')\|) \text{ and } \|y'\| \leq A_2(1 + \|x\|).$$

Hence

$$\begin{aligned} \|y\| \leq |y_{N-m}| + \|y'\| &\leq A_1(1 + \|(x, y')\|) + A_2(1 + \|x\|) \\ &\leq A_1(1 + \|x\| + \|y'\|) + A_2(1 + \|x\|) \\ &\leq (A_1(1 + A_2) + A_2)(1 + \|x\|) =: A(1 + \|x\|). \end{aligned}$$

2. We have $P(V) = P_2(V_1) = \mathbb{C}^m$. By induction applied to V_1 , the projection P_2 is locally biholomorphic away from a proper subvariety $V_s \subset V_1$. Also, by the implicit function theorem, $P_1: W \rightarrow \mathbb{C}^{N-1}$ is a local biholomorphism at each point of $W \setminus W_s$, where $W_s = \mathbf{V}(\frac{\partial}{\partial y_{N-m}} g_{N-m})$. Hence $P_1: V \rightarrow V_1$ is a local biholomorphism away from $P_1^{-1}(V_s) \cup W_s$. So $P = P_2 \circ P_1$ is a local biholomorphism at each point of $V \setminus (P_1^{-1}(V_s) \cup W_s)$ which gives V as a branched covering over \mathbb{C}^m . Clearly, the covering is finite, of degree $\deg(P_1) \cdot \deg(P_2)$. □

Remark 1. Clearly, V_1 is irreducible if V is, and this enters into the proof of the second part. Irreducibility of V is not necessary for Theorem 2.1; but if V were reducible, some of the g_j s will be reducible polynomials. The conclusion of the second part of the proposition may fail because:

- A component of V covers \mathbb{C}^{m+k} rather than \mathbb{C}^m . When this occurs, there are nontrivial algebraic relations among certain factors of the g_j s. Hence there is a component of V for which one can cut down (to $N - m - k$, say) the number of defining polynomials.
- A component of V is completely contained in W_s . Then there are nontrivial algebraic relations involving factors of the g_j s and $\frac{\partial}{\partial y_{N-m}} g_{N-m}$.

Now $\langle \text{LT}(I) \rangle \supseteq \langle y_1^{d_1}, \dots, y_{N-m}^{d_{N-m}} \rangle$. Hence a normal form has the structure

$$p(x, y) = \sum_{\alpha \in \mathcal{N}_y} p_\alpha(x) y^\alpha \tag{9}$$

where $p_\alpha \in \mathbb{C}[x]$ for each α , and α is taken over the finite set

$$\mathcal{N}_y := \{ \alpha = (\alpha_1, \dots, \alpha_{N-m}) \in \mathbb{Z}^{N-m} : \alpha_j < d_j \text{ for each } j \}. \tag{10}$$

Since \mathbb{C}^m is connected, the projection P has a well-defined global degree, $\text{deg}(P)$. A similar connectedness argument can be used to show the following.

Lemma 2.3. *For any two Noether presentations with projections P, \tilde{P} , we have $\text{deg}(P) = \text{deg}(\tilde{P})$.*

Proof. A complex linear perturbation of \mathbb{C}^N (i.e. a linear map $L_\epsilon = I + \epsilon T$ where $\epsilon \ll \|T\|$) takes a Noether presentation (x, y) to a Noether presentation (x_ϵ, y_ϵ) , as long as ϵ is sufficiently small: we have $(x_\epsilon, y_\epsilon) = (x, y) + \epsilon T(x, y)$. The local inverses P^{-1}, P_ϵ^{-1} of the projections $P(x, y) := x$ and $P_\epsilon(x_\epsilon, y_\epsilon) := x_\epsilon$ are holomorphic, and $\Phi_\epsilon := P_\epsilon \circ L_\epsilon \circ P^{-1}$ is a local biholomorphic map that goes to the identity as $\epsilon \rightarrow 0$. Given a local inverse P^{-1} , the composition $L \circ P^{-1} \circ \Phi_\epsilon^{-1}$ is a nearby local inverse for P_ϵ . Vice versa, given a local inverse P_ϵ^{-1} , a nearby local inverse of P is given by $L^{-1} \circ P_\epsilon^{-1} \circ \Phi_\epsilon$. Hence the number of local inverses is the same for P and P_ϵ , so $\text{deg}(P) = \text{deg}(P_\epsilon)$.

Now, fix some reference coordinate system and let L be a linear map that transforms these coordinates to a Noether presentation with projection P_L . If we identify the collection of all such linear maps L (equivalently, $N \times N$ matrices) with \mathbb{C}^{N^2} , the maps that give Noether presentations form a connected open subset of \mathbb{C}^{N^2} .⁴ By the previous paragraph, $L \mapsto \text{deg}(P_L)$ is a locally constant function on this set. Hence $\text{deg}(P_L)$ is a constant independent of L . \square

Definition 2.6. Define the *degree* of V by $\text{deg}(V) := \text{deg}(P)$, where $P : V \rightarrow \mathbb{C}^m$ is the projection in a Noether presentation of V . The *dimension* of V is $\dim(V) := m$.

We give some examples of Noether presentations.

Example 2.1. Let $V \subset \mathbb{C}^2$ be given by the equation $z_1 z_2 = 1$. Then $(x, y) = (z_1, z_2)$ is not a Noether presentation. Similar to the proof of Theorem 2.1, write

$$z_1 = a_1 x + b_1 y, \quad z_2 = a_2 x + b_2 y,$$

then the equation for V transforms into $b_1 b_2 y^2 + (a_1 b_2 + a_2 b_1) x y + a_1 a_2 x^2 = 1$, and (x, y) is a Noether presentation if $b_1 b_2 \neq 0$.

Example 2.2. Let $V \subset \mathbb{C}^2$ be given by the equation $z_1^2 = z_2^3$. Then $(x, y) = (z_1, z_2)$ is a Noether presentation, while $(x, y) = (z_2, z_1)$ is not. Clearly, (5) fails for the latter because $|z_1| = |z_2|^{3/2}$.

Remark 2. The existence of coordinates for which (5) holds at all points on an affine variety was already known by Zeriahi ([15]; see also Remark 4). An estimate of this form also characterizes algebraicity. Suppose W is an analytic subvariety of \mathbb{C}^N , and suppose there exist constants $A, B > 0$ such that

$$\|y\| \leq A(1 + \|x\|)^B \text{ for all } (x, y) \in W.$$

Rudin has shown [11] that W must therefore be algebraic.

3 The Lelong Class

Let V be an affine variety in \mathbb{C}^N . Then a function $u : V \rightarrow [-\infty, \infty)$ is plurisubharmonic (psh) on V if in a neighborhood of each point, u is locally the restriction of a psh function in a local embedding into \mathbb{C}^N .

Remark 3. The above notion is sufficient for this paper but a weaker notion is needed to get a class with good compactness properties ([12], [15], [7]). A function is *weakly psh* on a complex space X if it is upper semicontinuous on X and psh in local coordinates about every regular point of X . Both of these notions coincide when X is smooth.

The *Lelong class* $\mathcal{L}(V)$ is the class of psh functions on V of at most logarithmic growth:

$$\mathcal{L}(V) := \{ u \text{ psh on } V : \exists C \in \mathbb{R} \text{ such that } u(z) \leq \log^+ \|z\| + C, \forall z \in V \}.$$

We also define the class

$$\mathcal{L}^+(V) := \{ u \in \mathcal{L}(V) : \exists c \in \mathbb{R} \text{ such that } u(z) \geq \log^+ \|z\| + c, \forall z \in V \}.$$

It is easy to see that these classes are invariant under complex linear changes of coordinates. Also, for a real constant $c > 0$, write

$$c\mathcal{L}(V) := \{ cu : u \in \mathcal{L}(V) \}, \quad c\mathcal{L}^+(V) := \{ cu : u \in \mathcal{L}^+(V) \}.$$

Proposition 3.1. *Suppose that $z = (x, y)$ is a Noether presentation for an affine algebraic variety $V \subset \mathbb{C}^N$. Then*

1. $\log^+ \|x\| \in \mathcal{L}^+(V)$.
2. For any polynomial p with $\text{deg}_V(p) \geq 1$, we have $\frac{1}{\text{deg}_V(p)} \log |p| \in \mathcal{L}(V)$.

⁴Here we use the fact that the complement of a proper analytic subset of \mathbb{C}^{N^2} is connected.

Proof. We need to show that the quantity $\log^+ \|x\| - \log^+ \|z\|$ is uniformly bounded from both sides. The inequality $\log^+ \|x\| \leq \log^+ \|z\|$ is obvious, which gives an upper bound of zero. For the lower bound, we use the property $\|y\| \leq A(1 + \|x\|)$ for some $A > 0$ to estimate

$$\begin{aligned} \log^+ \|z\| = \log^+ \|(x, y)\| &\leq \log^+(\|x\| + \|y\|) \leq \log^+(\|x\| + A(1 + \|x\|)) \\ &\leq \log^+\left(\frac{A}{A+1} + \|x\|\right) + \log^+(A+1) \leq \log^+ \|x\| + C, \end{aligned}$$

where we choose $C > 0$ sufficiently large (depending on A) so that the last inequality holds for all $x \in \mathbb{C}^M$. This gives a lower bound of $-C$.

To prove the second item, let $d := \deg_v(p)$. Reduce p to its normal form of degree d , which we will also denote by p , and let \widehat{p} denote the leading homogeneous part. We have (recall (10) for the definition of \mathcal{N}_y)

$$p(x, y) = \widehat{p}(x, y) + r(x, y) = \sum_{\substack{|\beta| \leq d \\ \beta \in \mathcal{N}_y}} h_\beta(x) y^\beta + r(x, y)$$

with $h_\beta(x)$ homogeneous of degree $d - |\beta|$ for each β and $\deg r(x, y) < d$. We calculate that

$$\begin{aligned} \frac{|h_\beta(x) y^\beta|}{\|x\|^d} &\leq \frac{|h_\beta(x)|}{\|x\|^{d-|\beta|}} \left(\frac{\|y\|}{\|x\|}\right)^{|\beta|} \leq \left|h_\beta\left(\frac{x}{\|x\|}\right)\right| \cdot C^{|\beta|} \left|1 + \frac{1}{\|x\|}\right|^{|\beta|} \leq \|h_\beta\|_B (2C)^{|\beta|} \quad (\text{for } \|x\| > 1) \\ &=: C_\beta, \end{aligned}$$

where in the second inequality $\|\cdot\|_B$ denotes the sup norm on B , the closed unit ball in \mathbb{C}^m .

If $(\alpha, \beta) \in \mathbb{N}^m \times \mathbb{N}^{N-m}$ are multi-indices with $|\alpha| + |\beta| < d$, then

$$|x^\alpha y^\beta| \leq \|x\|^{|\alpha|} \|y\|^{|\beta|} \leq \|x\|^{|\alpha|} (1 + \|x\|)^{|\beta|}$$

so that

$$\frac{|x^\alpha y^\beta|}{\|x\|^d} \leq \|x\|^{d-|\alpha|} + \|x\|^{d-|\alpha|-|\beta|} \longrightarrow 0 \text{ as } \|x\| \rightarrow \infty.$$

Hence $|r(x, y)|/\|x\|^d \rightarrow 0$ as $\|x\| \rightarrow \infty$. Putting the above calculations together,

$$\frac{|p(x, y)|}{\|x\|^d} \leq \tilde{C} \text{ for sufficiently large } \|x\| > 1$$

where $\tilde{C} = 1 + \sum_\beta C_\beta$. Thus $\frac{1}{d} \log |p(x, y)| \leq \log \|x\| + \tilde{C}/d$ for sufficiently large $\|x\|$. It follows easily that $\frac{1}{d} \log |p(x, y)| \in \mathcal{L}(V)$. \square

The integer $\deg_v(p)$ is not the smallest value of d permitting an inequality of the form $\frac{1}{d} \log |p(z)| \leq \log^+ \|z\| + A$ for some $A \in \mathbb{R}$. The optimal bound may be a rational number; we will see this later, when studying the notion of *Lelong degree*.

Remark 4. In [15] a complex space X of dimension m is said to be *parabolic* if it admits a continuous psh exhaustion function $g : X \rightarrow [-\infty, \infty)$ for which $(dd^c g)^m = 0$ off some compact subset of X . The function g is called a *parabolic potential*. The Lelong class $\mathcal{L}(X, g)$ is defined by

$$\mathcal{L}(X, g) = \{u \text{ weakly psh on } X : \exists C \text{ such that } u(z) \leq g^+(z) + C, \forall z \in V\},$$

where $g^+(z) = \max\{g(z), 0\}$, and the notion of weakly psh is as in Remark 3. We define $\mathcal{L}^+(X, g)$ similarly with the added condition $u(z) \geq g^+(z) + c$.

Proposition 3.1 shows that any affine variety V is a parabolic space with parabolic potential $\log \|x\|$, where the coordinates (x, y) are a Noether presentation for V . This case was already mentioned in [15]. Our classes $\mathcal{L}(V)$, $\mathcal{L}^+(V)$ correspond to $\mathcal{L}(V, \log \|x\|)$, $\mathcal{L}^+(V, \log \|x\|)$ in that paper.

4 Comparison theorems and Monge-Ampère mass

We will establish some comparison theorems for the complex Monge-Ampère operator on an affine variety. The locally bounded theory of Bedford and Taylor [2] in \mathbb{C}^N is sufficient for this section. In particular, we use the fact that any locally bounded psh function in \mathbb{C}^N can be approximated by a decreasing sequence of smooth psh functions; and if T is a positive closed $(N - k, N - k)$ -current, u_1, \dots, u_k are psh functions, and for each j we have a monotone convergent sequence of psh functions, $u_j^{(n)} \nearrow u_j$ (or $u_j^{(n)} \searrow u_j$), then

$$dd^c u_1^{(n)} \wedge \dots \wedge dd^c u_k^{(n)} \wedge T \rightarrow dd^c u_1 \wedge \dots \wedge dd^c u_k \wedge T \text{ weak-}^* \text{ as } j \rightarrow \infty.$$

Theorem 4.1. *Let $\Omega \subset \mathbb{C}^N$ and let u, v be locally bounded psh functions on Ω . Let T be a closed positive current of bidegree $(N - j, N - j)$ on Ω for some positive integer $j \leq N$. If the set $A := \{u < v\} \cap \text{supp}(T)$ is a relatively compact subset of Ω then*

$$\int_{\{u < v\}} (dd^c v)^j \wedge T \leq \int_{\{u < v\}} (dd^c u)^j \wedge T.$$

Proof. We will verify the theorem for u, v continuous, adapting an argument of Cegrell (see [3] or [9], Section 3.7). The hypotheses can be weakened to u, v locally bounded by a standard argument using decreasing sequences of continuous psh approximants to u, v on an open neighborhood of the closure of A .

Let $\Omega_1 := \Omega \cap \{u < v\}$, and let T be a positive closed $(N-1, N-1)$ -current. For $\epsilon > 0$, define $v_\epsilon := \max\{v - \epsilon, u\}$. By continuity, $u = v$ on $\partial\Omega_1 \cap \text{supp}(T)$, and by hypothesis, the closure of this set is in Ω . Thus $\{u < v - \epsilon\} \cap \text{supp}(T)$ is a relatively compact subset of Ω_1 . We claim that

$$\int_{\Omega_1} dd^c v_\epsilon \wedge T = \int_{\Omega_1} dd^c u \wedge T. \quad (11)$$

Given $\epsilon > 0$, let φ be a smooth, compactly supported non-negative function in Ω_1 such that $\varphi \equiv 1$ in a neighborhood of the closure of the set $\{v_\epsilon > u\} \cap \text{supp}(T)$. Then $dd^c \varphi \equiv 0$ on this set, so that $(\text{supp}(dd^c \varphi) \cap \text{supp}(T)) \subset \{v_\epsilon = u\}$, and

$$\int_{\Omega_1} \varphi dd^c v_\epsilon \wedge T = \int_{\Omega_1} v_\epsilon dd^c \varphi \wedge T = \int_{\Omega_1} u dd^c \varphi \wedge T.$$

Consequently (11) holds since φ was arbitrary.

Now take a smooth, compactly supported ψ on Ω_1 , with $0 \leq \psi \leq 1$. Then $v_\epsilon \nearrow v$ on Ω_1 , so that

$$\int_{\Omega_1} \psi dd^c v \wedge T = \lim_{\epsilon \rightarrow 0} \int_{\Omega_1} \psi dd^c v_\epsilon \wedge T \leq \lim_{\epsilon \rightarrow 0} \int_{\Omega_1} dd^c v_\epsilon \wedge T = \int_{\Omega_1} dd^c u \wedge T,$$

using (11). Since ψ was arbitrary, the theorem follows when $j = 1$.

An easy induction gives the theorem for higher powers of j . □

Recall that if $M \subset \mathbb{C}^N$ is a manifold of dimension k , then $[M]$ denotes its current of integration, i.e., the $(2N - k)$ -current that acts on a test k -form φ by

$$\langle [M], \varphi \rangle := \int_M \varphi.$$

A theorem of Lelong says that the current of integration $[V]$ over an analytic variety $V \subset \mathbb{C}^N$ of pure dimension m is a closed positive current of bidegree $(N - m, N - m)$. As a consequence, we have the following.

Corollary 4.2. *Let $V \subset \mathbb{C}^N$ be an affine variety of dimension m and let $j \in \{1, \dots, m\}$. Suppose u, v are locally bounded psh functions on V and S is a closed positive current of bidegree $(m - j, m - j)$. If $A := V \cap \{u < v\} \cap \text{supp}(S)$ is a bounded subset of V , then*

$$\int_A (dd^c v)^j \wedge S \leq \int_A (dd^c u)^j \wedge S.$$

Proof. Note that by definition, u, v are restrictions to V of locally bounded psh functions on an open set $\Omega \subset \mathbb{C}^N$ (cf. Remark 3). For these extended functions, $\{u < v\} \cap V$ is relatively compact in Ω , so we may apply the previous theorem with $T = [V] \wedge S$. □

Theorem 4.3. *Suppose that $V \subset \mathbb{C}^N$ is an affine variety of dimension m , and u, v are locally bounded psh functions on V . Suppose u, v are bounded from below outside a bounded subset of V and $v(z) = u(z) + o(u(z))$ as $\|z\| \rightarrow \infty$. Let $j \in \{1, \dots, m\}$. Then for any closed positive $(m - j, m - j)$ -current S with unbounded support,*

$$\int_V (dd^c v)^j \wedge S \leq \int_V (dd^c u)^j \wedge S.$$

Proof. By adding a positive constant, we may assume without loss of generality that $u, v > 0$ outside a bounded subset of V . Let $\epsilon, c > 0$. Then $(1 + \epsilon)u(z) - c > u(z) + o(u(z))$ as $\|z\| \rightarrow \infty$, which implies that $(1 + \epsilon)u(z) - c > v(z)$ as $\|z\| \rightarrow \infty$. Thus the set $A_{\epsilon, c} := \{(1 + \epsilon)u - c < v\} \cap V$ is bounded and we may use Corollary 4.2 to obtain the inequality

$$\int_{A_{\epsilon, c}} (dd^c v)^j \wedge S \leq (1 + \epsilon)^j \int_V (dd^c u)^j \wedge S.$$

Letting $c \rightarrow +\infty$,

$$\int_V (dd^c v)^j \wedge S \leq (1 + \epsilon)^j \int_V (dd^c u)^j \wedge S.$$

Finally, let $\epsilon \rightarrow 0$. □

We will also need the following corollary.

Corollary 4.4. *Suppose $u - v$ is $O(1)$ and S is a positive closed $(m - 1, m - 1)$ -current with the property that $\int_V dd^c w \wedge S$ is finite for some $w \in \mathcal{L}^+(V)$. Then*

$$\int_V dd^c u \wedge S = \int_V dd^c v \wedge S.$$

Proof. Take $w \in \mathcal{L}^+(V)$ as hypothesised, and let $\epsilon, c > 0$. Define $u_{\epsilon,c} := u + \epsilon w - c$. Then the set $A_{\epsilon,c} := \{u_{\epsilon,c} < v\}$ is bounded, so that

$$\int_{u_{\epsilon,c}} dd^c v \wedge S \leq \int_V dd^c u_{\epsilon,c} \wedge S = \int_V dd^c u \wedge S + \epsilon \int_V dd^c w \wedge S.$$

As before, let $c \rightarrow +\infty$ and $\epsilon \rightarrow 0$ to obtain $\int_V dd^c v \wedge S \leq \int_V dd^c u \wedge S$.

The reverse inequality is obtained by the same argument with the roles of u and v swapped. \square

Corollary 4.5. *Let u, v be psh on V , bounded from below, $u - v = O(1)$, and both functions go to infinity as $|z| \rightarrow \infty$. Let S be a positive closed $(m - j, m - j)$ -current with unbounded support. Then*

$$\int_V (dd^c u)^j \wedge S = \int_V (dd^c v)^j \wedge S. \quad (12)$$

In particular,

$$\int_V (dd^c u)^m = \int_V (dd^c v)^m. \quad (13)$$

If the value of (13) is a nonzero constant, it is equal to

$$\int (dd^c u)^j \wedge (dd^c v)^{m-j} \quad (14)$$

for each $j \in \{0, \dots, m\}$. In particular, this is true for $u, v \in \mathcal{L}^+(V)$.

Proof. Since u, v go to infinity as $|z| \rightarrow \infty$, $u - v$ is both $o(u(z))$ and $o(v(z))$. We need only show ' \geq ' in (12), (13), and (14); the opposite inequality follows by reversing the roles of u and v .

The inequality for (12) follows from Theorem 4.3, and that for (13) follows as a corollary upon taking $S \equiv 1$. For $j \in \{1, \dots, m - 1\}$ we also obtain the inequality for (14) by taking $S = (dd^c v)^{m-j}$, as long as the support of this current is unbounded; we verify this next.

Suppose on the contrary that the support is bounded, i.e.,

$$(dd^c v)^{m-1} = 0 \text{ on } V \setminus \{\|z\| \leq R\} \text{ for some } R > 0. \quad (15)$$

By hypothesis, we may choose such an R sufficiently large for which $\int_{V_R} (dd^c u)^m \neq 0$, where $V_R = \{z \in V : \|z\| \leq R\}$. Let $\varphi \geq 0$ be a smooth, compactly supported function that is identically 1 on a neighborhood of V_R . Then $\text{supp}(dd^c \varphi) \subset V \setminus V_R$, so integration by parts yields

$$0 \neq \int_{V_R} (dd^c v)^m \leq \int_V \varphi (dd^c v)^m = \int_V v dd^c \varphi \wedge (dd^c v)^{m-1} = \int_{V \setminus V_R} v dd^c \varphi \wedge (dd^c v)^{m-1} = 0,$$

using (15). This is a contradiction. \square

As we shall see, $\int_V (dd^c v)^m$ is a positive constant for $v \in \mathcal{L}^+(V)$. The common value of the integrals (13), (14) will be called the *(Monge-Ampère) mass of $\mathcal{L}^+(V)$* . Its exact value is obtained by choosing a convenient function in the class that can be integrated explicitly.

In Chapter 5 of [9], Klimek computes $\int (dd^c(\log(1 + \|z\|)))^N$ in polar coordinates to get the mass of $\mathcal{L}^+(\mathbb{C}^N)$. We give an alternative computation using a 'max' formula.

Proposition 4.6. *Let $0 \leq k \leq m$ and suppose the functions u_1, \dots, u_{k+1} are pluriharmonic (i.e. $dd^c u_j = 0$ for all j) on a domain $\Omega \subseteq \mathbb{C}^m$. Let $u(z) := \max_j u_j(z)$. Then*

$$(dd^c u)^k = [S] \wedge d^c(u_1 - u_2) \wedge d^c(u_2 - u_3) \wedge \dots \wedge d^c(u_k - u_{k+1}), \quad (16)$$

where $S = \{z \in \mathbb{C}^m : u_1(z) = u_2(z) = \dots = u_{k+1}(z)\}$, as long as S is a smooth $(2m - k)$ -dimensional manifold. \square

The formula is a direct corollary of the main theorem in [1] (i.e. without the ' dd^c ' terms, which vanish in this case). The pairing of $[S] \wedge d^c(u_1 - u_2) \wedge \dots \wedge d^c(u_k - u_{k+1})$ with a smooth $(N - k, N - k)$ -current ω is given by

$$([S] \wedge d^c(u_1 - u_2) \wedge \dots \wedge d^c(u_k - u_{k+1}), \omega) = \int_S d^c(u_1 - u_2) \wedge \dots \wedge d^c(u_k - u_{k+1}) \wedge \omega,$$

which means S must be oriented so that the integral is non-negative for $\omega \geq 0$.

Example 4.1. For $z = x + iy \in \mathbb{C}$, we have $\log^+ |z| = \max\{\log |z|, 0\}$, with $S = \{|z| = 1\}$, so $dd^c \log^+ |z| = [\{|z| = 1\}] \wedge d^c \log |z|$. We compute

$$\begin{aligned} d^c \log |z| &= d^c \left(\frac{1}{2} \log(x^2 + y^2) \right) = \frac{1}{x^2 + y^2} (x dy - y dx) \\ &= \frac{1}{r^2} (r^2 \cos^2 \theta (d\theta) + r^2 \sin^2 \theta d\theta) = d\theta, \end{aligned}$$

where $z = r e^{i\theta}$. Hence $dd^c \log^+ |z| = [\{|z| = 1\}] \wedge d\theta$, angular measure on the unit circle.

More generally, consider the function in $\mathcal{L}^+(\mathbb{C}^N)$ given by

$$u(z) = \max\{\log |z_1|, \dots, \log |z_N|, 0\}.$$

Then S is the torus $\{|z_1| = \dots = |z_N| = 1\}$, and we have

$$(dd^c u)^N = [S] \wedge d^c \log |z_1| \wedge \dots \wedge d^c \log |z_N| = [S] \wedge d\theta_1 \wedge \dots \wedge d\theta_N, \quad (17)$$

where $z_j = r_j e^{i\theta_j}$ for each j . Then $\int (dd^c u)^N = \int_S d\theta_1 \wedge \dots \wedge d\theta_N = (2\pi)^N$, showing that

$$\text{the mass of } \mathcal{L}^+(\mathbb{C}^N) \text{ is } (2\pi)^N. \quad (18)$$

Using (18) together with a Noether presentation, we can prove the following theorem.

Theorem 4.7. *For an affine variety V of dimension m , the mass of $\mathcal{L}^+(V)$ is $(\deg V)(2\pi)^m$.*

Proof. By Corollary 4.5 it suffices to compute the Monge-Ampère mass of any function in $\mathcal{L}^+(V)$. By Proposition 3.1(1), we may take $u \in \mathcal{L}^+(V)$ to be the function $u(x, y) = \log^+ \|x\|$, where (x, y) is a Noether presentation; and being independent of y , it is naturally identified as a function in $\mathcal{L}^+(\mathbb{C}^m)$. Let B be the closed unit ball in \mathbb{C}^m . We will first prove the theorem under the condition that

(*) *The projection $(x, y) \xrightarrow{P} x$ is a local biholomorphism from a neighborhood of $P^{-1}(B) \cap V$ to a neighborhood of B , and $P^{-1}(B)$ is a union of d disjoint sets.*

(In other words, B avoids the branch locus of the projection.) Then on $\mathbb{C}^m \setminus B$, the function $\log^+ \|x\|$ satisfies $(dd^c \log^+ \|x\|)^m = 0$; hence u satisfies

$$(dd^c u)^m = 0 \text{ in a neighborhood of each } (x, y) \in P^{-1}(\mathbb{C}^m \setminus B) \setminus E \quad (19)$$

where E is the algebraic set given by the union of the singular points of V and the branch points of P . Thus the (N, N) -current $T := [V] \wedge (dd^c u)^m$ is zero on the set $(\mathbb{C}^N \setminus (P^{-1}(B))) \setminus E$. We claim that it must therefore be zero on E as well. For if not, then for any test function φ ,

$$0 < \int_{\mathbb{C}^N \setminus P^{-1}(B)} \varphi T = \int_E \varphi T = \int_E \varphi (dd^c u)^m = \lim_{k \rightarrow \infty} \int_E \varphi (dd^c u_k)^m, \quad (20)$$

where $\{u_k\}$ is a decreasing sequence of smooth psh functions, $u_k \searrow u$. Since each u_k is smooth, for each k we may interpret $\int_E \varphi (dd^c u_k)^m$ classically as the integral of a smooth (m, m) -form over E , an analytic set of complex dimension $< m$, which is always zero. Hence the limit on the right-hand side of (20) is zero, a contradiction.

Thus T is supported on $V \cap P^{-1}(B)$, which means that

$$\int_V (dd^c u)^m = \int_{V \cap P^{-1}(\bar{B})} (dd^c u)^m = \sum_{j=1}^{\deg V} \int_{\bar{B}} (dd^c \log^+ \|x\|)^m = \deg V \int_{\bar{B}} (dd^c \log^+ \|x\|)^m$$

where condition (*) is used to get the second equality. Observe now that as a global function on \mathbb{C}^m , $\log^+ \|x\| \in \mathcal{L}^+(\mathbb{C}^m)$ and is maximal on $\mathbb{C}^m \setminus B$, so that

$$\int_{\bar{B}} (dd^c \log^+ \|x\|)^m = \int_{\mathbb{C}^m} (dd^c \log^+ \|x\|)^m = (2\pi)^m,$$

by (18). The conclusion of the theorem follows immediately.

It remains to deal with the condition (*). For each $x_0 \in \mathbb{C}^m$, observe that $P^{-1}(x_0) =: L_{x_0}$ is an affine plane of codimension m while E is an affine subvariety of dimension at most $m-1$. Hence we can find x_0 such that $L_{x_0} \cap E = \emptyset$. Then

$$\bigcup_{\|x-x_0\| < \delta} L_x \cap E = \emptyset$$

for $\delta > 0$ small enough; and in addition, $(\bigcup_{\|x-x_0\| < \delta} L_x) \cap V$ is a union of d disjoint sets. Now translate and rescale coordinates by the map

$$(x, y) \mapsto \left(\frac{1}{2\delta}(x - x_0), y\right) = (\tilde{x}, y).$$

It is easy to see that (\tilde{x}, y) is a Noether presentation for V for which (*) holds. We now consider the function $\log^+ \|\tilde{x}\|$ and proceed as above. \square

5 Lelong degree

The complex Monge-Ampère operator may be extended to certain unbounded functions. If u is a psh function on some domain Ω , we define

$$L(u) := \{z \in \Omega : u \text{ is not locally bounded in } B(z, r) \cap \Omega, \forall r > 0\}.$$

The following result (presented without proof) is a consequence of the general theory on complex manifolds developed in [5], chapter III §4.

Proposition 5.1. *Let T be a positive closed (q, q) -current in \mathbb{C}^n and let u_1, \dots, u_p be psh functions on \mathbb{C}^n with $p + q \leq n$. Suppose that for each j , $L(u_j)$ is contained in an analytic set A_j . Given $m \in \{1, \dots, p\}$, suppose that for each choice of m functions u_{j_1}, \dots, u_{j_m} the analytic set $A_{j_1} \cap \dots \cap A_{j_m}$ has codimension at least m . Then*

1. The $(p + q, p + q)$ -current $dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge T$ is well defined on \mathbb{C}^n .
2. If $\{u_j^{(k)}\}$ is a decreasing sequence of locally bounded psh functions on \mathbb{C}^n , with $u_j^{(k)} \searrow u_j$ as $k \rightarrow \infty$, then we have the weak- $*$ convergence of currents:

$$dd^c u_1^{(k)} \wedge \dots \wedge dd^c u_p^{(k)} \wedge T \rightarrow dd^c u_1 \wedge \dots \wedge dd^c u_p \wedge T. \quad (21)$$

□

As an example of the second convergence, we have

$$dd^c(\max\{\log|z_1|, -\log n\}) = [\{|z_1| = \frac{1}{n}\}] \wedge d\theta$$

by Proposition 4.6, which converges to $[\{z_1 = 0\}]$.

More generally, for an irreducible $p \in \mathbb{C}[V]$ we have $\max\{\log|p|, -\log n\} \searrow \log|p|$ as $n \rightarrow \infty$, and we have the convergence

$$dd^c \max\{\log|p(z)|, -\log n\} \wedge T = [\{|p(z)| = \frac{1}{n}\}] \wedge d^c \log|p| \wedge T \rightarrow 2\pi[\{p(z) = 0\}] \wedge T$$

for any closed positive current T . We can see this convergence near a regular point of $\{p = 0\}$ by making a local holomorphic linearization of coordinates (which we denote by H) that transforms $\log|p|$ into $\log|z_1|$, and calculating as above. We recover the *Lelong-Poincaré formula*:

$$dd^c \log|p| \wedge T = 2\pi[\{p = 0\}] \wedge T.$$

Let now $v \in \mathcal{L}^+(V)$. Suppose $u \in c\mathcal{L}(V)$ for some $c > 0$, and satisfies:

1. $L(u) \cap V$ is contained in an analytic set of dimension at most $m - 1$; and
2. $\frac{1}{c}u$ is a decreasing limit $u_j \searrow \frac{1}{c}u$ of functions $u_j \in \mathcal{L}^+(V)$.

Lemma 5.2. *Suppose $\{u_j\}_{j=1}^\infty$ and u satisfy the two conditions above. Suppose there is $v \in \mathcal{L}^+(V)$ and a compact $K \subset V$ such that the support of $dd^c u_j \wedge (dd^c v)^{m-1}$ is contained in K for each j . Then*

$$\int_V dd^c u \wedge (dd^c v)^{m-1} = \lim_{j \rightarrow \infty} c \int_V dd^c u_j \wedge (dd^c v)^{m-1} = c \deg(V)(2\pi)^m. \quad (22)$$

Proof. Let φ be a smooth, compactly supported function on V such that $\varphi|_K \equiv 1$. Then the first equality is equivalent to

$$\int_V \varphi dd^c u \wedge (dd^c v)^{m-1} = \lim_{j \rightarrow \infty} c \int_V \varphi dd^c u_j \wedge (dd^c v)^{m-1},$$

which is true by (21).

The second equality is a consequence of Corollary 4.5 and Theorem 4.7. □

We will apply the above lemma to $u = \log|p|$ where p is a polynomial. To construct the u_j s, we will need another lemma. Before stating it, we recall a standard fact: if $v \in \mathcal{L}(\mathbb{C}^m)$ is bounded from below and the Robin function

$$\rho_v(z) := \limsup_{|\lambda| \rightarrow \infty} v(\lambda z) - \log|\lambda|$$

is finite at every point of $\mathbb{C}^m \setminus \{0\}$, then $u \in \mathcal{L}^+(\mathbb{C}^m)$.

Lemma 5.3. *Let $p \in \mathbb{C}[x]$ be a polynomial and \hat{p} its leading homogeneous part. Suppose the set*

$$\{x \in \mathbb{C}^m : \hat{p}(x) = x_2 = \dots = x_m = 0\} \quad (23)$$

consists only of the origin. Let $j \in \mathbb{N}$ and define

$$u_j(x) := \max\left\{\frac{1}{\deg p} \log|p(z)|, \log \frac{|x_2|}{j}, \dots, \log \frac{|x_m|}{j}, -\log j\right\}. \quad (24)$$

Then $u_j \in \mathcal{L}^+(\mathbb{C}^m)$.

Proof. We have for each $x \in \mathbb{C}^m$ that $\log \frac{|\lambda x_k|}{j} - \log |\lambda| = \log \frac{|x_k|}{j}$ and

$$\lim_{|\lambda| \rightarrow \infty} \frac{1}{\deg p} \log |p(\lambda x)| - \log |\lambda| = \lim_{|\lambda| \rightarrow \infty} \frac{1}{\deg p} \log |\widehat{p}(x) + O(|\lambda|^{-1})| = \frac{1}{\deg p} \log |\widehat{p}(x)|,$$

so

$$\rho_{u_j}(x) = \max \left\{ \frac{1}{\deg p} \log |\widehat{p}(x)|, \log \frac{|x_2|}{j}, \dots, \log \frac{|x_m|}{j} \right\}.$$

By (23), $\rho_{u_j}(x)$ is $-\infty$ only at the origin, and finite for all other $x \in \mathbb{C}^m$. Hence $u_j \in \mathcal{L}^+(\mathbb{C}^m)$. \square

Proposition 5.4. *Let $V \subset \mathbb{C}^N$, with $\dim(V) = m$, and let (x, y) be a Noether presentation of V for which (23) holds. Suppose $p \in \mathbb{C}[x] \subseteq \mathbb{C}[V]$. Then for each $v \in \mathcal{L}^+(V)$,*

$$\deg(p) \deg(V) (2\pi)^m = \int_V dd^c \log^+ |p| \wedge (dd^c v)^{m-1} \quad (25)$$

$$= \int_V dd^c \log |p| \wedge (dd^c v)^{m-1}. \quad (26)$$

Proof. Let the functions u_j , $j \in \mathbb{N}$, be as in the previous lemma. We compute by Lemma 4.6,

$$\begin{aligned} (dd^c u_j)^m &= [S_j] \wedge d^c \left(\frac{1}{\deg p} \log |p(x)| \right) \wedge d^c \log \frac{|x_2|}{j} \wedge \dots \wedge d^c \log \frac{|x_m|}{j} \\ &= [S_j] \wedge d^c \left(\frac{1}{\deg p} \log |p(x)| \right) \wedge d^c \log |x_2| \wedge \dots \wedge d^c \log |x_m|. \end{aligned} \quad (27)$$

Let $T_j := \{x \in \mathbb{C}^m : |p(x)| \leq j^{-\deg p}\}$. Then $S_j = T_j \cap S$, where

$$S = \{|x_2| = \dots = |x_m| = 1\}.$$

Put $v := \max\{\log |x_2|, \dots, \log |x_m|, 0\}$; then we also have

$$(dd^c v)^{m-1} = [S] \wedge d^c \log |x_2| \wedge \dots \wedge d^c \log |x_m|;$$

in particular, $(dd^c v)^{m-1}$ is supported on S . Together with (27), Lemma 5.3, and Theorem 4.7, we have

$$\begin{aligned} \int_{T_j} \varphi d^c \left(\frac{1}{\deg p} \log |p(x)| \right) \wedge (dd^c v)^{m-1} &= \int_{S_j} d^c \left(\frac{1}{\deg p} \log |p(x)| \right) \wedge (dd^c v)^{m-1} \\ &= \int (dd^c u_j)^m = (2\pi)^m, \end{aligned}$$

where φ on the left is a smooth compactly supported function with $\varphi|_K \equiv 1$; here we take $K := \{|p(x)| \leq 1, |x_2| \leq 1, \dots, |x_m| \leq 1\}$ which is a fixed compact set containing S_j for all j .

For $j \geq 1$ we have $|p(z)| \leq 1$ so that $\log |p(z)| \leq 0 \leq v(z)$ on T_j . Hence $v|_{T_j} \equiv w|_{T_j}$, where $w = \max\{v, \frac{1}{\deg p} \log |p(z)|\} \in \mathcal{L}^+(\mathbb{C}^m)$, and the left-hand side is equal to

$$\int_{T_j} \varphi d^c \left(\frac{1}{\deg p} \log |p(x)| \right) \wedge (dd^c w)^{m-1}.$$

Let $\{w_\epsilon\}_{\epsilon>0}$ be smooth functions in $\mathcal{L}^+(\mathbb{C}^m)$ with $w_\epsilon \searrow w$ as $\epsilon \rightarrow 0$. Then the above integral is equal to

$$\lim_{\epsilon \rightarrow 0} \int_{T_j} \varphi d^c \left(\frac{1}{\deg p} \log |p(x)| \right) \wedge (dd^c w_\epsilon)^{m-1}. \quad (28)$$

For fixed ϵ , the integral inside the limit may be rewritten as the pairing of a current with smooth compactly supported form:

$$\langle [T_j] \wedge d^c \left(\frac{1}{\deg p} \log |p| \right), \varphi dd^c w_\epsilon \rangle = \langle dd^c \max\left\{ \frac{1}{\deg p} \log |p|, -\log j \right\}, \varphi dd^c w_\epsilon \rangle$$

where we apply Lemma 4.6 with $k = 1$; hence (28) becomes

$$\lim_{\epsilon \rightarrow 0} \int \varphi dd^c \max\left\{ \frac{1}{\deg p} \log |p|, -\log j \right\} \wedge (dd^c w_\epsilon)^{m-1} = \int \varphi dd^c \max\left\{ \frac{1}{\deg p} \log |p|, -\log j \right\} \wedge (dd^c w)^{m-1},$$

using convergence of currents (or Proposition 5.1).

Altogether, we have

$$(2\pi)^m = \int \varphi dd^c \max\left\{ \frac{1}{\deg p} \log |p|, -\log j \right\} \wedge (dd^c w)^{m-1}.$$

We may drop φ from the above integral, as the support of $dd^c \max\{\log |p|, -\log j\} \wedge (dd^c w)^{m-1}$ is the set $K \cap \{|p| = \frac{1}{j}\} \subset K$. Finally, by Theorem 4.3, we may replace w with any other $v \in \mathcal{L}^+(\mathbb{C}^m)$ since the support of $dd^c \max\{\log |p|, -\log j\}$ is the (unbounded)

set $\{|p| = \frac{1}{j}\}$. When $j = 1$, this proves (25) when $V = \mathbb{C}^m$. Proposition 5.1 may be applied to take the limit as $j \rightarrow \infty$ on the right-hand side to yield

$$(2\pi)^m = \frac{1}{\deg p} \int \varphi dd^c \log |p| \wedge (dd^c w)^{m-1},$$

which proves (26) when $V = \mathbb{C}^m$.

For a general variety V , the same argument as in the proof of Theorem 4.7 gives

$$\int_V dd^c \log |p| \wedge (dd^c v)^{m-1} = \sum_{k=1}^{\deg V} \int_{\mathbb{C}^m} dd^c \log |p| \wedge (dd^c v)^{m-1}$$

if $v = \log \|x\|$ and the support of the current $[V] \wedge dd^c \log |p| \wedge (dd^c v)^{m-1}$ is away from the branch locus of the projection. Otherwise, as in that proof, we modify v by an affine map so that the support of $(dd^c v)^{m-1}$ is away from the branch locus, and the above equation holds. Applying the \mathbb{C}^m case to each integral in the right-hand sum yields formula (26). Similarly, we also get (25). Finally, by Theorem 4.3, these formulas hold for all functions in $\mathcal{L}^+(V)$. \square

Proposition 5.4 motivates the following definition.

Definition 5.1. Fix $v \in \mathcal{L}^+(V)$. Given $p \in \mathbb{C}[V]$, the *Lelong degree of p on V* is defined by

$$\deg_{\mathcal{L}(V)}(p) := \frac{1}{\deg(V)(2\pi)^m} \int_V dd^c \log |p| \wedge (dd^c v)^{m-1}.$$

Remark 5. 1. The Lelong degree is independent of v by Corollary 4.5, and coincides with the usual degree when $p \in \mathbb{C}[x] \subseteq \mathbb{C}[V]$ by Proposition 5.4.

2. The definition in terms of a Monge-Ampère formula is similar to that of a Lelong number: $\deg_{\mathcal{L}(V)}(p)$ may be interpreted as a Lelong number for the current $dd^c \log |p|$.
3. The Lelong degree is independent of coordinates in \mathbb{C}^m since it is defined in terms of the dd^c -operator which is invariant under biholomorphic maps. In particular, the definition makes sense without reference to a Noether presentation. A Noether presentation is convenient for computation.

Example 5.1. Let $V \subset \mathbb{C}^2$ be the quadratic curve with equation $x = y^2$. Then (x, y) is a Noether presentation. Since $\log |y| = \frac{1}{2} \log |x|$, we have

$$\deg_{\mathcal{L}(V)}(y) = \frac{1}{2} \deg(x) = \frac{1}{2}$$

by Proposition 5.4.

For a general polynomial $p \in \mathbb{C}[V]$, its normal form is $p_1(x) + yp_2(x)$. Using the estimates

$$\max\{|p_1|, |yp_2|\} \leq |p_1 + yp_2| \leq 2 \max\{|p_1|, |yp_2|\}$$

we see that $\log |p| - \max\{\log |p_1|, \log |yp_2|\} = O(1)$. Thus using Theorem 4.3, we may replace $\log |p|$ by $u = \max\{\log |p_1|, \log |yp_2|\}$ in the computation of Lelong degree. If $\deg(p_1) > \deg(p_2)$ then for sufficiently large values of x , we have

$$\log |p_1(x)| > \log |p_2(x)| + \frac{1}{2} \log |x| = \log |yp_2(x)|,$$

so that $u = \log |p_1(x)|$. On the other hand, if $\deg(p_1) \leq \deg(p_2)$ then $u = \log |yp_2(x)| = \frac{1}{2} \log |x| + \log |p_2(x)|$ for sufficiently large values of x . It follows that

$$\deg_{\mathcal{L}(V)}(p) = \max\{\deg(p_1), \deg(p_2) + \frac{1}{2}\}.$$

It may or may not be possible to replace $\log |p|$ by $\log^+ |p|$ in the computation of Lelong degree. We need an additional condition.

Proposition 5.5. Suppose $z = (x, y)$ is a Noether presentation of V where $x = (x_1, \dots, x_m)$. Suppose $p \in \mathbb{C}[V]$ satisfies one of the following conditions: for each $j = 1, \dots, m$, either the set

$$K_j := \{z \in V : |p(x, y)| \leq 1, |x_k| \leq 1 \text{ for all } k \neq j\}$$

is compact, or if K_j is unbounded, then $\log |p|$ is $O(1)$ on K_j as $|x| \rightarrow \infty$.

Then

$$\deg_{\mathcal{L}(V)}(p) = \frac{1}{\deg(V)(2\pi)^m} \int_V dd^c \log^+ |p| \wedge (dd^c v)^{m-1}.$$

More generally, we may replace $\log^+ |p|$ in the integral by $\max\{\log |p|, c\}$ for any $c \in \mathbb{R}$.

Proof. Let us take the function $v := \max\{\log |x_1|, \dots, \log |x_m|, 0\} \in \mathcal{L}^+(V)$. Suppose the first condition holds. Let $K := \bigcup_{j=1}^m K_j$. By Proposition 4.6, the support of the current $dd^c \log^+ |p| \wedge (dd^c v)^{m-1}$ is contained in K .

Let φ be a smooth, compactly supported function on V such that $\varphi|_K \equiv 1$. Then $dd^c \varphi|_K \equiv 0$, so that

$$\int_V \log^+ |p| dd^c \varphi \wedge (dd^c v)^{m-1} = \int_V \log |p| dd^c \varphi \wedge (dd^c v)^{m-1}.$$

This is the same as $\int_V \varphi dd^c \log^+ |p| \wedge (dd^c \nu)^{m-1} = \int_V \varphi dd^c \log |p| \wedge (dd^c \nu)^{m-1}$, and the result follows since we may replace φ by 1.

If $\log |p(z)|$ is $O(1)$ on K_j , then

$$\liminf_{\substack{|z| \rightarrow \infty \\ z \in K_j}} \log |p| \geq M + 1$$

for some $M \in (-\infty, -1]$, so the set

$$K_{j,M} := \{z \in V : |p(x, y)| \leq e^M, |x_k| \leq 1 \text{ for all } k \neq j\}$$

is compact. Hence $e^{-M}p$ satisfies the previous condition, so

$$\int_V dd^c \log |e^{-M}p| \wedge (dd^c \nu)^{m-1} = \int_V dd^c \log^+ |e^{-M}p| \wedge (dd^c \nu)^{m-1}.$$

We want to replace $e^{-M}p$ with p on both sides. We may do this on the left-hand side because

$$dd^c \log |e^{-M}p| = dd^c (\log |p| - M) = dd^c \log |p|.$$

On the right-hand side, $\log^+ |e^{-M}p| - \log^+ |p| = O(1)$, so we may apply Corollary 4.4. \square

Example 5.2. Let $p(x, y) = x - y$ on the quadratic curve in \mathbb{C}^2 given by $x^2 - y^2 = 1$. Then $p(x, y) \rightarrow 0$ as (x, y) tends to infinity along the curve in the direction of the asymptote $y = x$. The set K in the above proof is $\{|x - y| \leq 1\}$, which is unbounded, and $\log |p| \rightarrow -\infty$ on K as $|x| \rightarrow \infty$. So neither condition holds for p .

The conclusion of the proposition also fails, by a calculation. Write $w = x - y$; then

$$\int_V dd^c \log^+ |p| = \int_{\mathbb{C} \setminus \{0\}} dd^c \log^+ |w| = \int_{\{|w|=1\}} d^c \log |w| = \int d\theta = 2\pi.$$

On the other hand, $p(x, y) = x - y \neq 0$ for all $(x, y) \in V$. Hence $\log |p|$ is harmonic and $\int_V dd^c \log |p| = 0$.

We close this section with the following result, used in the last section.

Proposition 5.6. Let $V = \mathbf{V}(p_1, p_2, \dots, p_k)$ be an irreducible affine variety in \mathbb{C}^N of dimension $m = N - k$. Define $W \subset \mathbb{C}^{N+1}$ by

$$W := \{(t, z) \in \mathbb{C}^{N+1} : p_1(z) - t = p_2(z) = \dots = p_k(z) = 0\} = \mathbf{V}(p_1 - t, p_2, \dots, p_k),$$

and define $V_t := \mathbf{V}(p_1 - t, p_2, \dots, p_k)$ when t is fixed. If t is sufficiently small then V_t is irreducible. Moreover, $\deg(W) = \deg(V_t) = \deg(V)$, and

$$\deg_{\mathcal{L}(W)}(p) = \deg_{\mathcal{L}(V)}(p) = \deg_{\mathcal{L}(V_t)}(p)$$

for any polynomial p that does not depend on t .

Proof. Restricting to a hyperplane (corresponding to fixing t in the original coordinates), we obtain a Noether presentation for V_t (or V if $t = 0$), and clearly $\deg(V_t) = \deg(W)$. If V is irreducible then so is V_t for t sufficiently small.

Consider now, for $\epsilon > 0$, the functions in $\mathcal{L}^+(W)$ given by

$$w_\epsilon(t, z) := \max\{\log |t|, \log |x_1|, \dots, \log |x_m|, \log \epsilon\},$$

where x are the independent variables in a Noether presentation of V , and let

$$v_\epsilon(z) := \max\{\log |x_1|, \dots, \log |x_m|, \log \epsilon\}.$$

Then a calculation as in (17) yields

$$(dd^c w_\epsilon)^m = [\{t = \epsilon\}] \wedge d\theta(t) \wedge (dd^c v_\epsilon)^{m-1} = dd^c(\max\{\log |t|, \log \epsilon\}) \wedge (dd^c v_\epsilon)^{m-1}.$$

Considering $p \in \mathbb{C}[V]$ as a polynomial in $\mathbb{C}[W]$ which is independent of t , and writing $\tau_\epsilon = \max\{\log |t|, \log \epsilon\}$, we have

$$(2\pi)^{m+1} \deg(W) \deg_{\mathcal{L}(W)}(p) = \int_W dd^c \log |p| \wedge (dd^c w_\epsilon)^m = \int_W dd^c \log |p| \wedge (dd^c v_\epsilon)^{m-1} \wedge dd^c \tau_\epsilon. \quad (29)$$

By Corollary 4.4, we can replace $\tau_\epsilon(z)$ on the right-hand side of (29) by $\tau_\delta(z)$ for any $\delta > 0$, since $\tau_\epsilon - \tau_\delta = O(1)$. Then

$$\begin{aligned} (2\pi)^{m+1} \deg(W) \deg_{\mathcal{L}(W)}(p) &= \int_W dd^c \log |p| \wedge (dd^c v_\epsilon)^{m-1} \wedge dd^c \tau_\delta = 2\pi \int_V dd^c \log |p| \wedge (dd^c v_\epsilon)^{m-1} \\ &= (2\pi)^{m+1} \deg(V) \deg_{\mathcal{L}(V)}(p) \end{aligned}$$

upon letting $\delta \rightarrow 0$ and using Monge-Ampère convergence. We showed above that $\deg(W) = \deg(V)$, hence $\deg_{\mathcal{L}(W)}(p) = \deg_{\mathcal{L}(V)}(p)$.

By the same type of argument, $\deg_{\mathcal{L}(W)}(p) = \deg_{\mathcal{L}(V_t)}(p)$ for any fixed $t \in \mathbb{C}$. \square

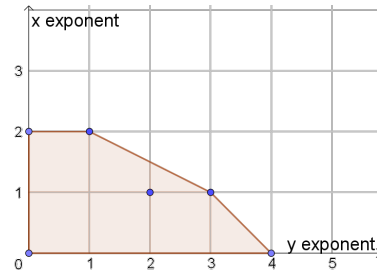
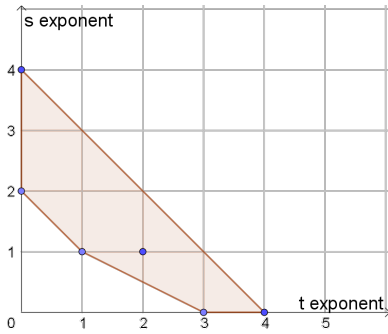


Figure 1: Newton polygons of P (above) and \tilde{P} (left) in Example 6.1.

6 Curves in \mathbb{C}^2

The Lelong degree on an algebraic curve in \mathbb{C}^2 may be computed from Puiseux series. Let

$$V = \{(x, y) \in \mathbb{C}^2 : P(x, y) = 0\},$$

and suppose $(x_0, y_0) \in V$. Then by the theory of Puiseux series, there is a neighborhood Ω of (x_0, y_0) in \mathbb{C}^2 such that any point $(x, y) \in \Omega \cap V$ is given by a Puiseux series about x_0 :

$$y = \sum_{j=j_0}^{\infty} a_j(x-x_0)^{j/d}, \text{ where } j_0 \in \mathbb{Z} \text{ and } d \in \mathbb{N}.$$

We apply Puiseux series (more precisely, the lowest term) as follows.

Proposition 6.1. *Suppose $V \subset \mathbb{C}^2$ is an algebraic curve, with Noether presentation (x, y) . There exists a $\gamma \in (-\infty, 1] \cap \mathbb{Q}$ such that if $R > 0$ is sufficiently large, then*

$$y = x^\gamma(c + o(|x|^\gamma)) \text{ as } |x| \rightarrow \infty, \tag{30}$$

and we take the limit along a continuous path in which $(x, y) \in V$ and $|x| > R$. As a consequence, $\log |y| = \gamma \log |x| + O(1)$ along such a path.

Proof. Let $V = \{P(x, y) = 0\}$. Under the change of coordinates

$$(x, y) \mapsto (1/x, y/x) =: (s, t),$$

for each (x, y) with $x \neq 0$ we have $P(x, y) = 0$ if and only if $\tilde{P}(s, t) = 0$, where \tilde{P} is the polynomial obtained by replacing each term $a_{jk}x^jy^k$ with $a_{jk}s^{j'}t^k$, where $j' = \deg(P) - j - k$.

Consider a Puiseux series at the origin that gives t in terms of s when $\tilde{P}(s, t) = 0$:

$$t = \sum_{j=j_0}^{\infty} a_j s^{j/d} = a_{j_0} s^{j_0/d} + o(|s|^{j_0/d}) \text{ as } |s| \rightarrow 0;$$

changing back to affine coordinates and multiplying through by x yields

$$y = a_{j_0} x^{1-j_0/d} + o(|x|^{1-j_0/d}) \text{ as } |x| \rightarrow \infty.$$

For sufficiently large $|x|$, this says that $|y| > \frac{1}{2}|a_{j_0}||x|^{1-j_0/d}$. Since (x, y) is a Noether presentation, the estimate in (5) must hold, so $1 - j_0/d \leq 1$. We set $\gamma := 1 - j_0/d$. The Puiseux series in s is valid on an open set for which the origin is a limit point. A continuous path in this open set with $s \rightarrow 0$ corresponds to a continuous path in x with $|x| \rightarrow \infty$. \square

The value of γ in the above proposition is easy to read off from the Newton polygon associated to $\tilde{P}(s, t)$.^{||} We illustrate with the following example.

Example 6.1. Let $V \subset \mathbb{C}^2$ have defining polynomial

$$P(x, y) = y^4 + xy^3 + xy^2 + x^2y + x^2 + 1.$$

The associated polynomial is $\tilde{P}(s, t) = t^4 + t^3 + st^2 + st + s^2 + s^4$.

Write a Puiseux series for t in terms of s as $t = \sum_{k=1}^{\infty} c_k s^{\gamma_k}$. Then the lowest power γ_1 is the negative of a slope of a so-called lower segment of the boundary of the Newton polygon (i.e. any downward translate of a lower segment gives a line segment that does not intersect the polygon).

In our case, the polygon N has three lower segments (see Figure 1, left).

^{||}It is the first step in an iterative algorithm for computing the terms of a Puiseux series. See e.g. [14] for a description.

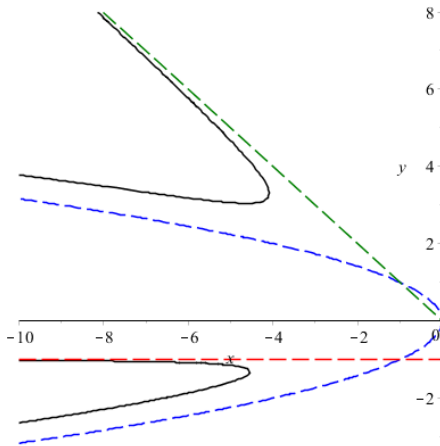


Figure 2: The real points of V are given by the two solid curves. The three dashed curves approximate $V_0, V_{\frac{1}{2}}, V_1$. The next terms in the series for $\gamma = 0$ and $\gamma = 1$ are $o(1)$, so the dashed lines approximating V_0, V_1 actually converge to these branches. For $\gamma = \frac{1}{2}$, the next term in the Puiseux series for each branch of $V_{\frac{1}{2}}$ is $O(|x|^{1/4})$. In this case, the branches of $W_{\frac{1}{2}}$ follow $V_{\frac{1}{2}}$ but separate from it very slowly. Plotted with Maple 18.

1. Line segment joining $(3, 0)$ and $(4, 0)$, with slope 0. Here $\gamma_1 = 0$, and we write the Puiseux series as

$$t = c_1 + t_1(s) = c_1 + c_2s^{\gamma_2} + \dots$$

Put this into the equation for the curve (write $c := c_1$ for convenience):

$$\begin{aligned} 0 = P(s, c + t_1(s)) &= (c + t_1(s))^4 + (c + t_1(s))^3 + s(c + t_1(s))^2 + s^2 + s^4 \\ &= (c^4 + c^3) + (\text{higher terms in } s). \end{aligned}$$

Equating coefficients in the constant term, we obtain $0 = c^4 + c^3$ which is a nonzero coefficient when $c = -1$. Hence $t = -1 + o(1)$ as $s \rightarrow 0$, which becomes

$$y = -x + o(|x|) \quad \text{as } |x| \rightarrow \infty.$$

2. Line segment joining $(3, 0)$ and $(1, 1)$. Here $\gamma_1 = \frac{1}{2}$, so that $t = cs^{1/2} + t_1(s)$. Equating the lowest nonzero coefficient in the Puiseux series equation for $P(s, t) = 0$, we obtain $0 = t^{3/2}(c^3 + c)$ so that $c = \pm i$. Hence we obtain two more series: $t = is^{1/2} + o(|s|^{1/2})$, and $t = -is^{1/2} + o(|s|^{1/2})$, giving

$$y = ix^{1/2} + o(|x|^{1/2}) \text{ or } y = -ix^{1/2} + o(|x|^{1/2}) \quad \text{as } |x| \rightarrow \infty.$$

3. Line segment joining $(1, 1)$ and $(0, 2)$. Here $\gamma_1 = 1$, so that $t = cs + t_1(s)$. Solving for the lowest nonzero coefficient in $P(s, t) = 0$, we obtain $t^2(c + 1) = 0$, so that $c = -1$. At the end we obtain

$$y = -1 + o(1) \quad \text{as } |x| \rightarrow \infty.$$

For this example, (x, y) is a Noether presentation because $P(x, y)$ is of the form (4), and $\deg(V) = 4$ since generically there are 4 solutions in y for fixed x . The four branches of V over \mathbb{C} correspond to the four Puiseux series. Write this as

$$V = V_1 \cup V_{\frac{1}{2},1} \cup V_{\frac{1}{2},2} \cup V_0,$$

where the (first) subscript corresponds to the exponent γ in (30).

For points x where $|x| > R$ and R is sufficiently large, the branches associated to Puiseux series with different values of γ are disjoint, because the corresponding y values are of a different order of magnitude. In particular, V_1 and V_0 are isolated branches for large $|x|$. The two branches $V_{\frac{1}{2},1}, V_{\frac{1}{2},2}$ have y values of the same order of magnitude for large $|x|$. Write $V_{\frac{1}{2}} := V_{\frac{1}{2},1} \cup V_{\frac{1}{2},2}$; then for large $|x|$, $V_{\frac{1}{2}}$ is approximated up to $o(|x|^{1/2})$ by the curve $y^2 + x = 0$, obtained by discarding the lower order terms in both series. We will denote this curve by $W_{\frac{1}{2}}$. See Figure 2.

Let $V(R) := V \cap \{|x| > R\}$, and similarly for other sets. We evaluate $\log |y|$ on these three pieces of $V(R)$:

$$\log |y| = \begin{cases} \log |-x + o(x)| & \text{on } V_1(R) & = \log |x| + O(1) & \text{on } \mathbb{C}(R) \\ \log |-1 + o(1)| & \text{on } V_0(R) & = O(1) & \text{on } \mathbb{C}(R) \\ \log |y| & \text{on } V_{\frac{1}{2}}(R) & = \log |\tilde{y}| + O(1) & \text{on } W_{\frac{1}{2}}(R) \end{cases}.$$

Here, \tilde{y} is just given by the closest value on the approximating curve: if $(x, y) \in V_{\frac{1}{2}}$ then $(x, \tilde{y}) \in W_{\frac{1}{2}}$.

We now compute the Lelong degree of y . Clearly $V \setminus V(R)$ is compact, by (5). Set $C > \sup\{|y| : (x, y) \in V \setminus V(R)\}$, so that $dd^c \max\{\log |y|, \log C\}$ is supported on the set $\{|y| = C\} \subset V(R)$. Then

$$\begin{aligned} \int_V dd^c \max\{\log |y|, C\} &= \int_{V(R)} dd^c \max\{\log |y|, C\} \\ &= \int_{V_1(R)} dd^c \max\{\log |y|, C\} + \int_{V_0(R)} dd^c \max\{\log |y|, C\} + \int_{V_{\frac{1}{2}}(R)} dd^c \max\{\log |y|, C\} \\ &= \int_{\mathbb{C}(R)} dd^c \max\{\log |x| + O(1), \log C\} + \int_{\mathbb{C}(R)} dd^c \log C + \int_{W_{\frac{1}{2}}(R)} dd^c \max\{\log |\tilde{y}| + O(1), \log C\} \\ &= 2\pi + 0 + 2\pi = 4\pi, \end{aligned}$$

where we use the fact that $\log |\tilde{y}| = \frac{1}{2} \log |x|$ on $W_{\frac{1}{2}}$. Hence $\deg_{\mathcal{L}(V)}(y) = \frac{4\pi}{4(2\pi)} = \frac{1}{2}$.

Remark 6. 1. The end result of the computation is to take the average growth of $\log |y|$ over all four branches:

$$\deg_{\mathcal{L}(V)}(y) = \frac{1}{4}(1 + \frac{1}{2} + \frac{1}{2} + 0) = \frac{1}{2}. \tag{31}$$

2. The additional condition of Proposition 5.5 was needed to simplify the computation of the integral, replacing $\log |y|$ by $\max\{\log |y|, C\}$ and projecting to \mathbb{C} . The condition is easily seen to hold if all Puiseux series exponents are nonnegative. If one of the exponents is negative then the Monge-Ampère computation fails; however, the averaging formula (31) should still hold with the appropriate sign. For example, $y^3 - x^2y + x + 1 = 0$ is a degree 3 curve with the 3 series

$$y = x + \dots, \quad y = -x + \dots, \quad y = x^{-1} + \dots,$$

and we should have $\deg_{\mathcal{L}(V)}(y) = \frac{1}{3}(1 + 1 - 1) = \frac{1}{3}$.

3. Clearly, the minimum nonnegative constant d permitting the inequality $\frac{1}{d} \log |p(z)| \leq \log^+ |z| + A$ for some $A \in \mathbb{R}$ and all $z \in V$ is given by the maximum value of γ . (For Example 6.1, $d = 1$.)

6.1 Lelong degree formula for y

Let $V = \mathbf{V}(P) \subset \mathbb{C}^2$, where P is an irreducible polynomial of the form

$$P(x, y) = y^{\deg P} + q(x, y). \tag{32}$$

This condition ensures that (x, y) is a Noether presentation. Also, since p is irreducible, q contains a nonzero term in x^k , for some $0 \leq k \leq \deg(P)$. Let d be the maximum such value. Then

$$\deg_{\mathcal{L}(V)}(y) = \frac{d}{\deg(P)}. \tag{33}$$

In Example 6.1, the highest power of P in x alone is x^2 . Hence by the above formula, $\deg_{\mathcal{L}(V)}(y) = \frac{2}{4} = \frac{1}{2}$.

Let us describe how formula (33) arises, using Example 6.1 as an illustration. First, the Newton polygon N of $\tilde{P}(s, t)$ has 3 lower segments, the negative of whose slopes are $\gamma_1 = 0$, $\gamma_2 = \frac{1}{2}$, and $\gamma_3 = 1$. The corresponding growths of y in terms of x as $|x| \rightarrow \infty$ are then calculated to be

$$c_1 := 1 - \gamma_1 = 1, \quad c_2 := 1 - \gamma_2 = \frac{1}{2}, \quad c_3 := 1 - \gamma_3 = 0.$$

Let us relate this to the Newton polygon of $P(x, y)$, which we denote by N_0 . The lower segments of N correspond to upper segments of N_0 (i.e., that separate from N_0 when translated up). The values of c_j for each j are precisely the negatives of the slopes of the segments in N_0 . (Compare the two polygons in Figure 1.)

In general, if $c_j = \frac{m_j}{n_j}$ (in simplest form) as $|x| \rightarrow \infty$, we get an approximation to $y^{n_j} = ax^{m_j}$ (for some a), a curve which has n_j branches over x . By an argument using continuity, the curve ought to be approximating n_j such branches of V (in Example 6.1, two branches for $c_2 = \frac{1}{2}$). As every branch for c_j is associated to a collection of n_j branches given by an approximating curve, the total number of branches must be a integer multiple, say $q_j n_j$. Altogether, $\deg(V) = \sum q_j n_j$.

In view of Remark 6(1) and the previous paragraph, we obtain

$$\deg_{\mathcal{L}(V)}(y) = \frac{1}{\deg(V)} \sum q_j n_j \frac{m_j}{n_j} = \frac{1}{\deg(V)} \sum q_j m_j = \frac{\sum q_j m_j}{\sum q_j n_j}.$$

Now $\sum q_j n_j = \deg(V) = \deg(P)$ is the horizontal length of the Newton polygon, and m_j/n_j are the negatives of the slopes of the upper segments (indexed by j), which we can consider as forming the graph of a piecewise linear function on the interval $[0, \deg(P)]$. Integrating the slopes over this interval gives $\sum q_j m_j$, which gives the total decrease in height of the function. Since the graph starts at $(0, d)$ and ends at $(\deg(P), 0)$, we get d .

7 A Bézout theorem for affine varieties

Let $V \subset \mathbb{C}^N$ be an irreducible affine variety of dimension $m \geq 1$ given by $N - m$ polynomials $p_1, p_2, \dots, p_{N-m} \in \mathbb{C}[z]$. Then

$$V =: V_m \subset V_{m+1} \subset \dots \subset V_{N-1} \subset V_N := \mathbb{C}^N \quad (34)$$

where V_j is given by the polynomials p_1, \dots, p_{N-j} for $j = m, \dots, N - 1$.

Theorem 7.1. *Suppose V is irreducible. Then*

$$\deg(V) = \prod_{j=m}^N \deg_{\mathcal{L}(V_{N-j})}(p_j).$$

Proof. Irreducibility of V_{m+1}, \dots, V_N follows from irreducibility of V by induction: if $V_{j+1} = V_{j+1,1} \cup V_{j+1,2}$, then $V_j = W_1 \cup W_2$, where $W_k = \mathbf{V}(p_{N-j}) \cap V_{j+1,k}$, contradicting the inductive hypothesis that V_j is irreducible. We also have $\dim(V_{j+1}) - \dim(V_j) \in \{0, 1\}$; since $\dim(V_m) = m$ and $\dim(V_N) = N$, the difference in dimension must be 1 throughout, showing that $\dim(V_j) = j$.

Fix j , and write $V_j = V_{j+1} \cap \mathbf{V}(p_{N-j})$. Let ν be a smooth function in $\mathcal{L}^+(V_{j+1})$. Then

$$\begin{aligned} \deg_{\mathcal{L}(V_{j+1})}(p_{N-j}) &= \frac{1}{(2\pi)^{j+1} \deg(V_{j+1})} \int_{V_{j+1}} dd^c \log |p_{N-j}| \wedge (dd^c \nu)^j \\ &= \frac{1}{(2\pi)^{j+1} \deg(V_{j+1})} \int_{V_j} 2\pi (dd^c \nu)^j \\ &= \frac{\deg(V_j)}{\deg(V_{j+1})}. \end{aligned}$$

Hence

$$\prod_{j=m}^N \deg_{\mathcal{L}(V_{j+1})}(p_{N-j}) = \prod_{j=m}^N \frac{\deg(V_j)}{\deg(V_{j+1})} = \frac{\deg(V)}{\deg(\mathbb{C}^N)} = \deg(V).$$

□

Corollary 7.2 (Affine Bézout theorem). *Let $p_1, \dots, p_N \in \mathbb{C}[z]$ be polynomials, each of degree ≥ 1 . Suppose $V := \mathbf{V}(p_1, \dots, p_N)$ is finite and $V_1 := \mathbf{V}(p_2, \dots, p_N)$ is irreducible. Then the number of points of V is at most*

$$\prod_{j=1}^N \deg_{\mathcal{L}(V_{N-j})}(p_j).$$

Proof. Let $W := \mathbf{V}(p_1(z) - t, p_2(z), \dots, p_N(z)) \subset \mathbb{C}^{N+1}$. Then W is an irreducible curve for generic values of t . By the previous result,

$$\deg(W) = \prod_{j=1}^N \deg_{\mathcal{L}(W_{j+1})}(p_{N+1-j}) = \deg(p_1) \prod_{j=1}^{N-1} \deg_{\mathcal{L}(V_j)}(p_{N-j}),$$

where we use $\deg_{\mathcal{L}(W_{N+1})}(p_1) = \deg(p_1)$ (since $W_{N+1} = \mathbb{C}^{N+1}$ by our convention), and $\deg_{\mathcal{L}(W_{j+1})}(p_{N-j}) = \deg_{\mathcal{L}(V_j)}(p_{N-j})$ by Proposition 5.6.

For $\epsilon > 0$, let $L_\epsilon : \mathbb{C}^{N+1} \rightarrow \mathbb{C}^{N+1}$ be a linear map close to the identity ($\|L_\epsilon - I\| \rightarrow 0$ as $\epsilon \rightarrow 0$), such that $(t_\epsilon, z_\epsilon) = L_\epsilon(t, z)$ is a Noether presentation for W . For any value of t_ϵ away from the branch locus of the projection π to the first coordinate, we have local inverses ζ_1, \dots, ζ_D of π , where $D = \deg(W)$. Pick one of these (say ζ_1) and define $\varphi_\epsilon := \pi \circ L_\epsilon^{-1} \circ \zeta_1$. Then φ_ϵ is holomorphic and locally invertible for ϵ sufficiently small (since $\varphi'_\epsilon \rightarrow 1$ locally uniformly as $\epsilon \rightarrow 0$).

For each $j = 1, \dots, D$, the map $L_\epsilon^{-1} \circ \zeta_j \circ \varphi_\epsilon^{-1}$ gives a local inverse for π in a neighborhood of t in the original coordinates, $(t, x) \xrightarrow{\pi} t$. For fixed x these local inverses give D points (t, x) , for generic values of t , i.e., $\deg(W)$ points of $\mathbf{V}(p_1 - t, p_2, \dots, p_N)$. Letting $t \rightarrow 0$, a continuity argument gives $\deg(W)$ as an upper bound for the number of points in $\mathbf{V}(p_1, \dots, p_N)$. □

Remark 7. 1. To remove the condition that V_1 is irreducible, we can extend the definition of Lelong degree to unions of affine varieties of the same dimension. We then treat each component in the above proof separately, and sum over all components at the end.

2. We can introduce the notion of multiplicity of a point to get a formula with equality. The point b is of multiplicity m if there is $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ there exists $\delta > 0$ such that for a generic choice of $t = (t_1, \dots, t_N) \in \mathbb{C}^N$ with $\|t\| < \delta$, the set

$$\mathbf{V}(p_1 - t_1, \dots, p_N - t_N) \cap \{|z| < \epsilon\}$$

consists of exactly m points. In the above proof, b is of multiplicity m if there are m distinct points $(t, b_{1,t}), \dots, (t, b_{m,t})$ of W that coalesce into $(0, b)$ as $t \rightarrow 0$ along a generic path.

Example 7.1. We illustrate with a simple example in \mathbb{C}^2 : compute the number of points of $V = \mathbf{V}(p_1, p_2)$, where

$$p_1(x, y) := y^4 + xy^3 + xy^2 + x^2y + x^2 + 1, \quad p_2(x, y) := y - 1. \quad (35)$$

1. **Bézout's theorem in \mathbb{P}^2 .** Homogenizing in the variable t gives

$$y^4 + xy^3 + xy^2t + x^2yt + x^2t^2 + t^4 = 0, \quad y - t = 0. \quad (36)$$

The intersection of a degree 4 curve with a degree 1 curve is $4 \cdot 1 = 4$ points (counting multiplicity), by Bézout's theorem. To get the affine points, we must discard points at infinity: putting $t = 0$ in (36) gives the equations $y^4 + xy^3 = 0$ and $y = 0$, yielding the point $[t : x : y] = [0 : 1 : 0]$. This is a point of multiplicity 2 as can be seen as follows. First, dehomogenize in the variable x to get local coordinates at infinity: setting $x = 1$ in (36) gives

$$y^4 + y^3 + y^2t + yt + t^2 + t^4 = 0, \quad y = t.$$

Substituting the second equation into the first yields

$$0 = 2t^2 + 2t^3 + 2t^2 = 2t^2(t^2 + t + 1),$$

so $t = 0$ is a root of multiplicity 2.

Discarding the double point at infinity leaves 2 affine points of V .

2. **Affine Bézout theorem with $V_2 = \mathbf{V}(p_2)$.** We have $\deg(p_2) = 1$. Next, (x, y) is a Noether presentation of V_2 ; plugging $y - 1 = 0$ into p_1 gives $2x^2 + 2x + 2$ as its normal form, so $\deg_{\mathcal{L}(V_2)}(p_1) = 2$ by Proposition 5.4. We obtain $1 \cdot 2 = 2$ points.
3. **Affine Bézout theorem with $V_2 = \mathbf{V}(p_1)$.** We have $\deg(p_1) = 4$. Now $\deg_{\mathcal{L}(V_2)}(p_2) = \deg_{\mathcal{L}(V_2)}(y - 1)$. The change of coordinates $(x, y) \mapsto (x, y + 1)$ takes $y - 1$ to y and takes P to a polynomial whose highest exponent in x alone is still x^2 , so applying (33) in these new coordinates gives the same result. Thus $\deg_{\mathcal{L}(V_2)}(y - 1) = \frac{1}{2}$, and we obtain $4 \cdot \frac{1}{2} = 2$ points, as before.

Example 7.2. We close the paper with an example in \mathbb{C}^3 . We want to compute the number of points in $\mathbf{V}(p_1, p_2, p_3)$ where p_1, p_2, p_3 are given by

$$p_1 = z^3 - x, \quad p_2 = y^2 - xz, \quad p_3 = z^4 - y + 1.$$

First, $\deg(p_1) = 3$. Let $V_2 := \mathbf{V}(p_1)$; then (x, y) are independent variables in a Noether presentation of V_2 , and $|z| = |x|^{1/3}$, so $\deg_{\mathcal{L}(V_2)}(p_2) = \deg(y^2) = 2$. Next, x is an independent variable in a Noether presentation of $V_1 := \mathbf{V}(p_1, p_2)$, with $|y| = |x|^{2/3}$ and $|z| = |x|^{1/3}$. These are the affine points of the well-known twisted cubic, with parametrization $t \mapsto (t^3, t^2, t)$. We have $\deg_{\mathcal{L}(V_1)}(p_3) = \deg(z^4) = 4/3$. Finally, the number of points of $\mathbf{V}(p_1, p_2, p_3)$ is $3 \cdot 2 \cdot (4/3) = 8$. (The solutions can be found numerically.)

Remark 8. The twisted cubic is a standard example of a curve that is not a complete intersection in \mathbb{P}^3 . However, it is a complete intersection in \mathbb{C}^3 , and (34) holds for $V = \mathbf{V}(p_1, p_2)$.

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