PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 17 · 2024 · Pages 40–49

Statistical convergence of q-analogue of Aldaz-Kounchev-Render operators

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Communicated by Stefano De Marchi

Abstract

The prime purpose of this paper is to establish various approximation results of a q-analogue of Aldaz-Kounchev-Render (q-AKR) operators. Firstly, we obtain the central moments and estimate the convergence rate and Voronovskaja-type results for continuous functions via the modulus of continuity. Then, we provide the results on A-statistical convergence i.e. the Korovkin-type Theorem and the rate of A-statistical convergence of q-AKR operators. Finally, the rate of A-statistical convergence for Lipschitz-type functions is also discussed.

1 Introduction and preliminaries

In the year 2009, Aldaz et al. [4] constructed generalized Bernstein-type polynomial operators and studied their shape preserving behavior for spaces of monotone and convex functions.

Given $n \in \mathbb{N}$, $n \geq j > 1$, (where j is fixed), and for any real valued continuous function f working on $[0,1]$, the operators $B_{n,0,j}(f;x)$ are defined as

$$B_{n,0,j}(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\left(\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)}\right)^{\frac{1}{j}}\right). \tag{1}$$

These operators are positive and linear, and also preserve the test functions e_0 and e_j, where $e_i(x) = x^i$, $i = 0,1,2,\dots$ Cárdenas-Morales et al. [6] obtained the rate of convergence of the operators $B_{n,0,j}(.;x)$(when $j = 2$) in terms of classical modulus of continuity. As an application to their main result (Theorem 2.1), they also proved the asymptotic formula for $B_{n,0,j}(.;x)$ (when $j = 2,3$) and stated as a conjecture for $j \geq 4$. Birou [5] was the first who attempted this conjecture for all positive integers $j \geq 1$. Utilizing the moduli of continuity, Acu et al. [1] improved some convergence estimates given in [6, 9] for the operators $B_{n,0,j}(.;x)$. Recently, the authors in [3] conducted a comparative study of approximation results between the classical Bernstein operators and the AKR operators. Mainly they determined the class of functions for which the classical Bernstein operators approximate better than the AKR operators and vice-versa.

Among several modifications of the Bernstein operators, the modification proposed by G. M. Phillips [19] has been studied intensively in the literature. For $n \in \mathbb{N}$, an integer $q > 0$ and a function f acting on $[0,1]$, the modified Bernstein operators are defined as follows:

$$\begin{aligned} B_n^q(f;x) &:= \sum_{k=0}^{n} p_{n,k}^q(x) f\left(\frac{[k]}{[n]}\right) \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-k-1} (1-q^s x) f\left(\frac{[k]}{[n]}\right), \end{aligned} \tag{2}$$

where the q-integer $[k]_q$ and q-factorial $[k]_q!$ are defined by

$$[k]_q = [k] := \begin{cases} \dfrac{1-q^k}{1-q}, & \text{if } q \neq 1 \\ k, & \text{if } q = 1, \end{cases}$$

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and

$$[k]_q! = [k]! := \begin{cases} [k].[k-1]...[1], & \text{if } k = 1, 2, \cdots, \\ 1, & \text{if } k = 0. \end{cases}$$

For $0 \le k \le n$, the q-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]![n-k]!}.$$

The q-integers $(0 < q \le 1)$ based modification of the operators $B_{n,0,j}(.;x)$ was proposed by Finta [9]. For $x \in [0,1]$ and $f \in C[0,1]$, he defined

$$B^q_{n,0,j}(f;x) = \sum_{k=0}^{n} p^q_{n,k}(x) f\left(\left(\frac{[k][k-1]\dots[k-j+1]}{[n][n-1]\dots[n-j+1]}\right)^{\frac{1}{j}}\right), \tag{3}$$

and proved that the above mentioned operators converge to its limit operators in the space $C[0,1]$. Like the original operators (1), these operators also preserve the test functions e_0 and e_j and reduce to AKR operators for $q = 1$.

As a generalization of the notion of ordinary convergence, statistical convergence came into existence. Steinhaus [20], (also see Fast [8]) in 1951, introduced the concept of statistical convergence. The idea of A-statistical convergence (where A is non-negative regular summability matrix), which is basically a generalization of statistical convergence and ordinary convergence, was first initiated by Freedman and Sember [10]. Many researchers call it a natural generalization because if we assume A to be Cesaro matrix of order one, then A-statistical convergence becomes statistical convergence. Also, if A is an identity matrix, then A-statistical convergence converts into ordinary convergence. In [16], Kolk showed that if the non-negative and regular matrix A fulfills the condition $\lim_{n\to\infty} \max\{a_{jn}\} = 0$, then A-statistical convergence is stronger than the ordinary convergence.

Before going further, we provide some definitions related to statistical convergence.

Definition 1.1. Let $K \subset \mathbb{N}$, then the **asymptotic density** of K is defined as

$$\delta(K) := \lim_{j\to\infty} \frac{Card\big(\{n \le j : n \in K\}\big)}{j},$$

where $Card(X)$ denotes the cardinality of the set X.

Definition 1.2. Let $x = (x_n)$ be any sequence of numbers. We say $x = (x_n)$ is **statistically convergent** to a number L, if for given $\epsilon > 0$

$$\lim_{j\to\infty} \frac{Card\big(\{k : |x_k - L| \ge \epsilon\}\big)}{j} = 0.$$

Definition 1.3. Assume $A = (a_{jn}); j, n \in \mathbb{N}$ be an infinite summability matrix, and $x = (x_n)$ be a sequence of numbers. Then A is said to be **regular** if

$$\lim x = L \implies \lim Ax = L,$$

where

$$Ax := ((Ax)_j) = \sum_{k=1}^{\infty} a_{jk} x_k < \infty, \forall j \in \mathbb{N}.$$

Definition 1.4. A sequence $x = (x_n)$ is said to be **A-statistically convergent** to L if for given any $\epsilon > 0$

$$\lim_{j\to\infty} \sum_{n:|x_n - L| \ge \epsilon} a_{jn} = 0.$$

This limit is designated by $\mathcal{S}t_A - \lim x = L$.

Over the last two decades, approximation properties of various positive linear operators have been examined in the framework of A-statistical convergence [13, 14, 12]. For instance, the authors in [18], discussed the rate of statistical convergence of a general sequence of positive linear operators acting on a subspace of $C[0,1]$. These results are obtained by the means of modulus of continuity and Lipschitz functions. In [17], Liu proposed a modification of q-Bernstein operators and proved Korovkin-type Theorem via A-statistical convergence. Further, he obtained the rate of A-statistical convergence and proved that the modified operators approximate better than the q-Bernstein operators in a subinterval of $[0,1]$. It is well known that q-type modifications of positive linear operators provide better approximation results. So, we choose q-AKR operators for our investigation.

2 Moments and Rate of Convergence

Lemma 2.1. *[19] The q-Bernstein operators $B_n^q(.;x)$ verify the following identities:*

(i) $B_n^q(1;x)=1$;

(ii) $B_n^q(t;x)=x$;

(iii) $B_n^q(t^2;x)=x^2+\dfrac{x(1-x)}{[n]}$.

Lemma 2.2. *For the operators $B_{n,0,j}^q(.;x)$, we have the following estimates:*

(i) $B_{n,0,j}^q((t-x);x)\leq \dfrac{[j-1]}{[n]}$;

(ii) $B_{n,0,j}^q((t-x)^2;x)\leq \dfrac{x(1-x)}{[n]}+\dfrac{2x[j-1]}{[n]}$;

(iii) $B_{n,0,j}^q((t-x)^4;x)\leq 8\left(\left(\dfrac{[j-1]}{[n]}\right)^4+\dfrac{Kx(1-x)}{[n]^2}\right)$.

Proof. (i) By considering the inequality proved in [2] (section 4.7), we can easily deduce that

$$0\leq \frac{[k]}{[n]}-\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{\frac{1}{j}}\leq \frac{[j-1]}{[n]};n\geq j\geq 2. \tag{4}$$

Multiply eq.(4) by the basis function $p_{n,k}^q(x)$ and summing all the terms, we get the desired result.

(ii) In view of Lemma 2.1, we can write

$$\begin{aligned}
B_{n,0,j}^q(t^2;x)-x^2 &= B_{n,0,j}^q(t^2;x)-B_n^q(t^2;x)+B_n^q(t^2;x)-x^2\\
&=\sum_{k=0}^{n}\left[\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{\frac{2}{j}}-\left(\frac{[k]}{[n]}\right)^2\right]p_{n,k}^q(x)+\frac{x(1-x)}{[n]}\\
&=\sum_{k=0}^{n}\left[\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{\frac{1}{j}}-\frac{[k]}{[n]}\right]\\
&\qquad\times\left[\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{\frac{1}{j}}+\frac{[k]}{[n]}\right]p_{n,k}^q(x)+\frac{x(1-x)}{[n]}\\
&\leq \frac{x(1-x)}{[n]}.
\end{aligned}$$

Using the identity $(a-b)^2=a^2-b^2-2b(a-b)$ and part (i), we get

$$B_{n,0,j}^q((t-x)^2;x)\leq \frac{x(1-x)}{[n]}+\frac{2x[j-1]}{[n]}.$$

(iii) Taking into account the inequality $(t+x)^4\leq 8(t^4+x^4)$ and the estimation (see [21]) $B_n^q((t-x)^4;x)=\dfrac{Kx(1-x)}{[n]^2}$, we conclude that

$$\begin{aligned}
B_{n,0,j}^q((t-x)^4;x) &= \sum_{k=0}^{n}p_{n,k}^q(x)\left(\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{\frac{1}{j}}-x\right)^4\\
&\leq 8\sum_{k=0}^{n}p_{n,k}^q(x)\left(\left(\frac{[k][k-1]\ldots[k-j+1]}{[n][n-1]\ldots[n-j+1]}\right)^{\frac{1}{j}}-\frac{k}{n}\right)^4\\
&\quad+8\sum_{k=0}^{n}p_{n,k}^q(x)\left(\frac{k}{n}-x\right)^4\\
&\leq 8\left(\frac{[j-1]}{[n]}\right)^4\sum_{k=0}^{n}p_{n,k}^q(x)+8\sum_{k=0}^{n}p_{n,k}^q(x)\left(\frac{k}{n}-x\right)^4\\
&=8\left(\left(\frac{[j-1]}{[n]}\right)^4+\frac{Kx(1-x)}{[n]^2}\right).
\end{aligned}$$

□

Lemma 2.3. *The operators $B^q_{n,0,j}(.;x)$ satisfy the following inequalities:*

(i) $B^q_{n,0,j}\left(|(t-x)^2_{q_n}|;x\right) \leq \dfrac{x(1-x)}{[n]} + \dfrac{2x[j-1]}{[n]}$
$$+\frac{x(1-q^n)}{[n]}\sqrt{\frac{x(1-x)}{[n]}+\frac{2x[j-1]}{[n]}};$$

(ii) $B^q_{n,0,j}\left(|t-x||(t-x)^2_{q_n}|;x\right) \leq \sqrt{\dfrac{x(1-x)}{[n]}+\dfrac{2x[j-1]}{[n]}}\sqrt{8\left(\left(\dfrac{[j-1]}{[n]}\right)^4+\dfrac{Kx(1-x)}{[n]^2}\right)}$
$$+\frac{x(1-q^n)}{[n]}\left(\frac{x(1-x)}{[n]}+\frac{x[j-1]}{[n]}\right).$$

Proof. (i) For the proof of part (i), we use the following algebraic inequality:

$$|(t-x)^2_q| \leq (t-x)^2 + \frac{x(1-q^n)}{[n]}|t-x|. \tag{5}$$

Since the operators $B^q_{n,0,j}(.;x)$ are positive and linear, therefore they are monotone also. Using Cauchy-Schwarz inequality and Lemma 2.2 in eq.(5), we get

$$\begin{aligned}
B^q_{n,0,j}\left(|(t-x)^2_q|;x\right) &\leq B^q_{n,0,j}\left((t-x)^2;x\right) + \frac{x(1-q^n)}{[n]}B^q_{n,0,j}\left(|t-x|;x\right)\\
&\leq B^q_{n,0,j}\left((t-x)^2;x\right) + \frac{x(1-q^n)}{[n]}\sqrt{B^q_{n,0,j}\left((t-x)^2;x\right)}\\
&\leq \frac{x(1-x)}{[n]}+\frac{2x[j-1]}{[n]}+\frac{x(1-q^n)}{[n]}\sqrt{\frac{x(1-x)}{[n]}+\frac{2x[j-1]}{[n]}}.
\end{aligned}$$

(ii) In order to prove part (ii), multiply eq.(5) by $|t-x|$, we have

$$|t-x||(t-x)^2_q| \leq |t-x|(t-x)^2 + \frac{x(1-q^n)}{[n]}|t-x|^2.$$

Using the monotone property of the operators $B^q_{n,0,j}(.;x)$ with Cauchy-Schwarz inequality, we have

$$\begin{aligned}
B^q_{n,0,j}\left(|t-x||(t-x)^2_q|;x\right) &\leq B^q_{n,0,j}\left(|t-x|(t-x)^2;x\right) + \frac{x(1-q^n)}{[n]}B^q_{n,0,j}\left((t-x)^2;x\right)\\
&\leq \sqrt{B^q_{n,0,j}((t-x)^2;x)}\sqrt{B^q_{n,0,j}((t-x)^4;x)}\\
&\quad + \frac{x(1-q^n)}{[n]}B^q_{n,0,j}\left((t-x)^2;x\right)\\
&\leq \sqrt{\frac{x(1-x)}{[n]}+\frac{2x[j-1]}{[n]}}.\sqrt{8\left(\left(\frac{[j-1]}{[n]}\right)^4+\frac{Kx(1-x)}{[n]^2}\right)}\\
&\quad + \frac{x(1-q^n)}{[n]}\left(\frac{x(1-x)}{[n]}+\frac{2x[j-1]}{[n]}\right).
\end{aligned}$$

□

Next, we compute the rate of convergence of the operators $B^q_{n,0,j}(.;x)$ in terms of the modulus of continuity. For $f \in C[0,1]$ and $\delta > 0$, the modulus of continuity of f, designated by $\omega(f,\delta)$, and is given by

$$\omega(f,\delta) = \sup\left\{|f(t)-f(x)| : |t-x| \leq \delta;\ t,x \in [0,1]\right\}. \tag{6}$$

Remark 1. If $f \in C[0,1]$ and $\delta > 0$. Then for any $t,x \in [0,1]$, we have

$$|f(t)-f(x)| \leq \omega(f,\delta)\left(1+\frac{|t-x|}{\delta}\right).$$

Theorem 2.4. *Suppose f is a continuous function on $[0,1]$ and $q := q_n$, $q_n \in (0,1)$ be any sequence such that $q_n \to 1$ as $n \to \infty$. Then*

$$\left|B^{q_n}_{n,0,j}(f;x)-f(x)\right| \leq 2\omega\left(f,\sqrt{\frac{x(1-x)}{[n]_{q_n}}+\frac{2x[j-1]}{[n]_{q_n}}}\right).$$

Proof. Since $f \in C[0,1]$ and $x \in [0,1]$. Therefore by the virtue of linear and monotone properties of the operators $B^{q_n}_{n,0,j}(.;x)$, we can write

$$|B^{q_n}_{n,0,j}(f;x)-f(x)| \leq B^{q_n}_{n,0,j}(|f(t)-f(x)|);x).$$

Applying Remark 1 for any arbitrary chosen $\delta > 0$

$$\begin{aligned} |B^{q_n}_{n,0,j}(f;x)-f(x)| &\leq B^{q_n}_{n,0,j}(|f(t)-f(x)|);x) \\ &\leq \omega(f,\delta)\Big(1+\frac{1}{\delta}B^{q_n}_{n,0,j}(|t-x|;x)\Big). \end{aligned} \tag{7}$$

Using Cauchy-Schwarz inequality, we can write

$$B^{q_n}_{n,0,j}((|t-x|);x) \leq \sqrt{B^{q_n}_{n,0,j}\,((t-x)^2;x)}. \tag{8}$$

From eq.(7) and by Lemma 2.2, we get

$$\begin{aligned} |B^{q_n}_{n,0,j}(f;x)-f(x)| &\leq \omega(f,\delta)\Big(1+\frac{1}{\delta}\sqrt{B^{q_n}_{n,0,j}((t-x)^2);x)}\Big) \\ &\leq \omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{x(1-x)}{[n]_{q_n}}+\frac{2x[j-1]}{[n]_{q_n}}}\right) \\ &\leq 2\omega(f,\delta_{n,x}), \end{aligned}$$

where

$$\delta := \delta_{n,x} = \sqrt{\frac{x(1-x)}{[n]_{q_n}}+\frac{2x[j-1]}{[n]_{q_n}}}.$$

Hence, the desired result is obtained. □

Theorem 2.5. *Suppose that $q := q_n$, $q_n \in (0,1)$ and $q_n \to 1$ as $n \to \infty$. Then, for every $x \in [0,1]$ and $f \in C[0,1]$, the following inequality holds:*

$$\begin{aligned} &\left|[n]_{q_n}\Big(B^{q_n}_{n,0,j}(f;x)-f(x)-D_{q_n}f(x)B^{q_n}_{n,0,j}(t-x;x)+\frac{D^2_{q_n}f(x)}{[2]_{q_n}!}B^{q_n}_{n,0,j}\big((t-x)^2_{q_n};x\big)\Big)\right| \\ &\leq O(1)\omega\left(D^2_{q_n}f;\frac{1}{\sqrt{[n]_{q_n}}}\right), \end{aligned}$$

where $D_{q_n}f$ is the q-derivative of f, for $x \neq 0$, defined by

$$D_{q_n}f(x) := \frac{f(q_nx)-f(x)}{x(q_n-1)}.$$

Proof. Using Taylor's formula in q-calculus [15] for the function f, we have

$$f(t) = f(x)+D_{q_n}f(x)(t-x)+\frac{D^2_{q_n}f(x)}{[2]_{q_n}!}(t-x)^2_{q_n}+h_2(t,q_n,x), \tag{9}$$

where the remainder term $h_2(t,q_n,x)$ is given by

$$h_2(t,q_n,x) = \frac{D^2_{q_n}f(\eta)-D^2_{q_n}f(x)}{[2]_{q_n}!}(t-x)^2_{q_n},$$

and η lies between x and t.
Operating $B^{q_n}_{n,0,j}(.;x)$ on both sides of eq.(9), we get

$$\left|B^{q_n}_{n,0,j}(f;x)-f(x)-D_{q_n}fB^{q_n}_{n,0,j}(t-x;x)+\frac{D^2_{q_n}f}{[2]_{q_n}!}B^{q_n}_{n,0,j}\Big((t-x)^2_{q_n};x\Big)\right| \leq B^{q_n}_{n,0,j}\big(|h_2(t,q_n,x)|;x\big). \tag{10}$$

In order to find an estimation of $B^{q_n}_{n,0,j}\big(|h_2(t,q_n,x)|;x\big)$, we observe from Remark 1 that

$$\left|D^2_{q_n}f(\eta)-D^2_{q_n}f(x)\right| \leq \omega\big(D^2_{q_n}f,\delta\big)\left(1+\frac{|t-x|}{\delta}\right),$$

where η lies between x and t.
Hence, we may write

$$B^{q_n}_{n,0,j}\big(|h_2(t,q_n,x)|;x\big) \leq \frac{\omega(D^2_{q_n}f,\delta)}{[2]_{q_n}!}\Big[B^{q_n}_{n,0,j}\big(|(t-x)^2_{q_n}!|;x\big)+\frac{1}{\delta}B^{q_n}_{n,0,j}\big(|t-x||(t-x)^2_{q_n}|;x\big)\Big]. \tag{11}$$

From the estimations obtained in Lemma 2.3, eq. (11) gives

$$B^{q_n}_{n,0,j}\Big(|h_2(t,q_n,x)|;x\Big) \le \frac{\omega(D^2_{q_n}f,\delta)}{[2]_{q_n}!}\Bigg[\frac{x(1-x)+2x[j-1]}{[n]_{q_n}} + \frac{x(1-q_n^n)}{[n]_{q_n}}$$
$$\sqrt{\frac{x(1-x)+2x[j-1]}{[n]_{q_n}}} + \frac{1}{\delta}\Bigg[\Bigg(\sqrt{\frac{x(1-x)+2x[j-1]}{[n]_{q_n}}}\Bigg)$$
$$\Bigg(\sqrt{\frac{8}{[n]^2_{q_n}}\left(\frac{[j-1]^4}{[n]^2_{q_n}} + Kx(1-x)\right)}\Bigg)$$
$$+\frac{x(1-q_n^n)}{[n]_{q_n}}\left(\frac{x(1-x)+2x[j-1]}{[n]_{q_n}}\right)\Bigg]\Bigg].$$

By selecting $\delta = \dfrac{1}{\sqrt{[n]_{q_n}}}$, and assuming the fact that $0 < q_n^n \le 1$ and $x \in [0,1]$, we get

$$B^{q_n}_{n,0,j}\Big(|h_2(t,q_n,x)|;x\Big) \le \frac{\omega\left(D^2_{q_n}f,[n]^{\frac{-1}{2}}_{q_n}\right)}{[2]_{q_n}!}\Bigg[\frac{1}{[n]_{q_n}}\left(\frac{1}{4}+2[j-1]\right) + \frac{1}{[n]^{\frac{3}{2}}_{q_n}}\left(\sqrt{\frac{1}{4}+2[j-1]}\right)$$
$$+\frac{1}{[n]_{q_n}}\left(\sqrt{\frac{1}{4}+2[j-1]}\right) \times \left(\sqrt{8\left(\frac{[j-1]^4}{[n]^2_{q_n}}+\frac{K}{4}\right)}\right)$$
$$+\frac{1}{[n]^{\frac{3}{2}}_{q_n}}\left(\frac{1}{4}+2[j-1]\right)\Bigg]$$
$$= \frac{\omega\left(D^2_{q_n}f,[n]^{\frac{-1}{2}}_{q_n}\right)}{[2]_{q_n}!}\left[\frac{1}{[n]_{q_n}}(A_j + B_j\sqrt{A_j}) + \frac{1}{[n]^{\frac{3}{2}}_{q_n}}(A_j+\sqrt{A_j})\right]$$
$$= \frac{\omega\left(D^2_{q_n}f,[n]^{\frac{-1}{2}}_{q_n}\right)}{[n]_{q_n}}\left[\frac{(A_j + B_j\sqrt{A_j})}{[2]_{q_n}!} + \frac{(A_j+\sqrt{A_j})}{\sqrt{[n]_{q_n}}[2]_{q_n}!}\right],$$

where

$$A_j = \frac{1}{4} + 2[j-1],$$

and

$$B_j = \sqrt{8\left(\frac{[j-1]^4}{[n]^2_{q_n}}+\frac{K}{4}\right)}.$$

This implies

$$B^{q_n}_{n,0,j}\Big(|h_2(t,q_n,x)|;x\Big) \le \frac{\omega\left(D^2_{q_n}f,[n]^{\frac{-1}{2}}_{q_n}\right)}{[n]_{q_n}}\left[\frac{(A_j + B_j\sqrt{A_j})}{[2]_{q_n}!} + \frac{(A_j+\sqrt{A_j})}{\sqrt{[n]_{q_n}}[2]_{q_n}!}\right].$$

Since

$$\left(\frac{(A_j + B_j\sqrt{A_j})}{[2]_{q_n}!} + \frac{(A_j+\sqrt{A_j})}{\sqrt{[n]_{q_n}}[2]_{q_n}!}\right) = O(1),$$

therefore by using eq.(10), the proof is completed. □

3 Rate of A-Statistical Convergence

Korovkin-type Theorem concerning the statistical convergence of positive linear operators was proved by Gadjiev and Orhan [11]. The credit of proving Korovkin-type Theorem via A-statistical convergence also goes to Gadjiev and Orhan, which is given as follows:

Theorem 3.1. *[11] Assume that $L_n : C[a,b] \to C[a,b]$ be a sequence of positive linear operators and $A = (a_{jn})$ be a non-negative and regular summability matrix. If for $i = 0,1,2$*

$$\mathcal{S}t_A - \lim_{n\to\infty} \|L_n e_i - e_i\| = 0.$$

Then, for every function $f \in C[a,b]$

$$\mathcal{S}t_A - \lim_{n\to\infty} \|L_n f - f\| = 0,$$

where $\|f\| = \sup\{|f(x)| \, : x \in [a,b]\}$.

Theorem 3.2. *Suppose $A = (a_{jn})$ be a non-negative and regular summability matrix, then for every $f \in C[0,1]$*

$$\mathcal{S}t_A - \lim_{n\to\infty} \|B^{q_n}_{n,0,j} f - f\| = 0. \tag{12}$$

Proof. For the proof of this Theorem, it is enough, if we validate the hypothesis of Theorem 3.1. A simple calculation shows that, for $i = 0, 1$

$$\mathcal{S}t_A - \lim_{n\to\infty} \|B^{q_n}_{n,0,j} e_i - e_i\| = 0,$$

and for $i = 2$, consider

$$\begin{aligned} \|B^{q_n}_n e_2 - e_2\| &= \sup_{x\in[0,1]} |B^{q_n}_n(e_2; x) - e_2(x)| \\ &= \sup_{x\in[0,1]} \left| x^2 + \frac{x(1-x)}{[n]_{q_n}} - x^2 \right| \le \frac{1}{4[n]_{q_n}}. \end{aligned}$$

This implies

$$\|B^{q_n}_{n,0,j} e_2 - e_2\| \le \|B^{q_n}_n e_2 - e_2\| \le \frac{1}{4[n]_{q_n}}.$$

For given any $\epsilon > 0$, we define the following sets:

$$S^* = \left\{ n : \|B^{q_n}_{n,0,j} e_2 - e_2\| \ge \epsilon \right\}$$

$$S = \left\{ n : \frac{1}{4[n]_{q_n}} \ge \epsilon \right\}.$$

Then, obviously $S^* \subseteq S$ and $0 \le \sum_{n\in S^*} a_{jn} \le \sum_{n\in S} a_{jn}, \quad \forall j \in \mathbb{N}$.

By using Definition 1.4, we get $\lim_{j\to\infty} \sum_{n\in S} a_{jn} = 0$, which implies $\lim_{j\to\infty} \sum_{n\in S^*} a_{jn} = 0$.

Hence

$$\mathcal{S}t_A - \lim_{n\to\infty} \|B^{q_n}_{n,0,j} e_2 - e_2\| = 0.$$

Therefore, the hypothesis of the Theorem 3.1 is satisfied, so the conclusion implies the required proof. □

For comparing the rates of summation in the framework of classical summability, no single definition appears to have emerged as the standard definition [16]. Although several definitions were available in the literature for this task. Duman et al. [7] presented some more useful definitions to estimate the rate of convergence. One of them is defined with rates of a positive and non-increasing sequence given in Definition 3.1, and another one is based on the concept of convergence in measure spaces in Measure theory given by Definition 3.2. Below we list these definitions, and then we compute the rate of A-statistical convergence for a sequence of operators $B^q_{n,0,j}(.;x)$. In the following definitions, $A = (a_{jn})$; $j, n \in \mathbb{N}$ is a non-negative and regular summability matrix and $< p_n >$ is a non-increasing sequence in $[0, \infty)$.

Definition 3.1. A sequence $x = (x_n)$ is said to be **A-statistically convergent** to L with rate $O(p_n)$, designated by $x_n - L = \mathcal{S}t_A - O(p_n)$, as $n \to \infty$,
if for given any $\epsilon > 0$,

$$\lim_{j\to\infty} \frac{1}{p_j} \sum_{n:|x_n - L|\ge\epsilon} a_{jn} = 0.$$

Definition 3.2. A sequence $x = (x_n)$ is said to be **A-statistically convergent** to L with rate $O_m(p_n)$, designated by $x_n - L = \mathcal{S}t_A - O_m(p_n)$, as $n \to \infty$,
if for given any $\epsilon > 0$

$$\lim_{j\to\infty} \sum_{n:|x_n - L|\ge\epsilon p_n} a_{jn} = 0.$$

Theorem 3.3. *Consider $A = (a_{jn})$ as a non-negative and regular summability matrix and $< p_n >$ be any sequence in $[0, \infty)$ which is non-increasing. Further assume that*

$$\omega(f, \delta_{n,j}) = \mathcal{S}t_A - O(p_n), \text{ as } n \to \infty, \tag{13}$$

where

$$\delta_{n,j} = \sqrt{\frac{1}{4[n]_{q_n}} + \frac{2[j-1]}{[n]_{q_n}}}. \tag{14}$$

Then, for every function $f \in C[0,1]$

$$\|B^{q_n}_{n,0,j} f - f\| = \mathcal{S}t_A - O(p_n), \text{ as } n \to \infty.$$

Proof. Since the operators $B^{q_n}_{n,0,j}(.;x)$ are linear and monotone, therefore, for a continuous function f defined on $[0,1]$ and any $\delta>0$, Remark 1 yields

$$\begin{aligned}|B^{q_n}_{n,0,j}(f;x)-f(x)| &\le B^{q_n}_{n,0,j}(|f(t)-f(x)|;x)\\ &\le \omega(f,\delta)B^{q_n}_{n,0,j}\left(1+\frac{|t-x|}{\delta};x\right)\\ &= \omega(f,\delta)\Big(1+\frac{1}{\delta}B^{q_n}_{n,0,j}(|t-x|;x)\Big).\end{aligned} \tag{15}$$

Using Cauchy-Schwarz inequality, we can write

$$B^{q_n}_{n,0,j}(|t-x|;x)\le\sqrt{B^{q_n}_{n,0,j}((t-x)^2;x)}. \tag{16}$$

Combining eq.(15) and eq.(16), and by Lemma 2.2, we get

$$\begin{aligned}|B^{q_n}_{n,0,j}(f;x)-f(x)| &\le \omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{B^{q_n}_{n,0,j}((t-x)^2;x)}\right)\\ &\le \omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{x(1-x)}{[n]_{q_n}}+\frac{2x[j-1]}{[n]_{q_n}}}\right).\end{aligned} \tag{17}$$

By taking supremum on both sides of eq.(17) for all $x\in[0,1]$, we obtain

$$\|B^{q_n}_{n,0,j}f-f\|\le\omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{1}{4[n]_{q_n}}+\frac{2[j-1]}{[n]_{q_n}}}\right).$$

If we take

$$\delta:=\delta_{n,j}=\sqrt{\frac{1}{4[n]_{q_n}}+\frac{2[j-1]}{[n]_{q_n}}},$$

then

$$\|B^{q_n}_{n,0,j}f-f\|\le 2\omega(f,\delta_{n,j}).$$

For given $\epsilon>0$, let us define the following sets:

$$S_1=\left\{n:\left\|B^{q_n}_{n,0,j}f-f\right\|\ge\epsilon\right\}$$
$$S_2=\left\{n:\omega(f,\delta_{n,j})\ge\frac{\epsilon}{2}\right\}.$$

Then, it is clear that $S_1\subset S_2$,
and

$$0\le\frac{1}{p_j}\sum_{n\in S_1}a_{jn}\le\frac{1}{p_j}\sum_{n\in S_2}a_{jn}\ ,\forall j\in\mathbb{N}. \tag{18}$$

Hence, the proof can be easily obtained by taking $j\to\infty$ in eq.(18), using eq.(13) and Definition 3.1. □

Using the argument of Theorem 3.3 and Definition 3.2, the following result hold:

Corollary 3.4. *Consider $A=(a_{jn})$ as a non-negative and regular summability matrix and $<p_n>$ be any sequence in $[0,\infty)$ which is non-increasing. Further assume that*

$$\omega(f,\delta_{n,j})=\mathcal{S}t_A-O_m(p_n),\ \textit{as}\ n\to\infty, \tag{19}$$

where

$$\delta_{n,j}=\sqrt{\frac{1}{4[n]_{q_n}}+\frac{2[j-1]}{[n]_{q_n}}}. \tag{20}$$

Then, for every function $f\in C[0,1]$

$$\|B^{q_n}_{n,0,j}f-f\|=\mathcal{S}t_A-O_m(p_n),\ \textit{as}\ n\to\infty.$$

Let $f\in C[0,1]$ and $\rho\in(0,1]$. We say that f is a Lipschitz function of order ρ $(0<\rho\le1)$, if there exists a real number $\lambda>0$ such that, for all $t_1,t_2\in[0,1]$

$$|f(t_1)-f(t_2)|\le\lambda|t_1-t_2|^{\rho}. \tag{21}$$

The class of all such functions is denoted by $Lip_\lambda(\rho)$.

In the next Theorem, we derive the rate of A-statistical convergence of the operators $B^{q_n}_{n,0,j}(.;x)$ for functions belonging to the class $Lip_\lambda(\rho)$.

Theorem 3.5. *Consider $A=(a_{jn})$ as a non-negative and regular summability matrix and $<p_n>$ be any sequence in $[0,\infty)$ which is non-increasing. If*

$$\delta_{n,j} = \mathcal{S}t_A - O(p_n), \text{ as } n \to \infty, \tag{22}$$

where $\delta_{n,j}$ is given by eq.(14). Then, for every function $f \in Lip_\lambda(\rho)$

$$\|B^{q_n}_{n,0,j} f - f\| = \mathcal{S}t_A - O(p_n), \text{as } n \to \infty.$$

Proof. Suppose $f \in \mathrm{Lip}_\lambda(\rho)$. Then in view of the eq.(21) and linear and monotone properties of the operators of $B^{q_n}_{n,0,j}(.;x)$, we have

$$\begin{aligned}|B^{q_n}_{n,0,j}(f;x) - f(x)| &\le B^{q_n}_{n,0,j}(|f(t)-f(x)|;x)\\ &\le \lambda B^{q_n}_{n,0,j}\big(|t-x|^\rho;x\big).\end{aligned} \tag{23}$$

Applying Hölder's inequality by taking $p=\dfrac{2}{\rho}, q=\dfrac{2}{2-\rho}$, and using Lemma 2.2, we obtain

$$\begin{aligned}B^{q_n}_{n,0,j}\big(|t-x|^\rho;x\big) &\le \Big(B^{q_n}_{n,0,j}\big((t-x)^2;x\big)\Big)^{\frac{\rho}{2}}\\ &\le \left(\frac{x(1-x)}{[n]_{q_n}} + \frac{2x[j-1]}{[n]_{q_n}}\right)^{\frac{\rho}{2}}.\end{aligned} \tag{24}$$

Combining eq.(23) and eq.(24), we get

$$|B^{q_n}_{n,0,j}(f;x) - f(x)| \le \lambda\left(\frac{x(1-x)}{[n]_{q_n}} + \frac{2x[j-1]}{[n]_{q_n}}\right)^{\frac{\rho}{2}}.$$

This implies

$$\|B^{q_n}_{n,0,j} f - f\| \le \lambda(\delta_{n,j})^\rho.$$

For given $\epsilon > 0$, let us define the following sets:

$$\begin{aligned}S_1 &= \Big\{n : \|B^{q_n}_{n,0,j} f - f\| \ge \epsilon\Big\},\\ S_2 &= \Big\{n : (\delta_{n,j})^\rho \ge \frac{\epsilon}{\lambda}\Big\}.\end{aligned}$$

Then, it is obvious that $S_1 \subset S_2$, and for the given sequence $<p_n>$, we have

$$0 \le \frac{1}{p_j}\sum_{n\in S_1} a_{jn} \le \frac{1}{p_j}\sum_{n\in S_2} a_{jn}, \forall j \in \mathbb{N}. \tag{25}$$

We conclude that the proof is over by taking $j \to \infty$ in eq.(25), using eq.(22) and Definition 3.1. □

Using the concept of convergence in measure from Measure theory and following the argument of Theorem 3.5, we can easily prove the next result.

Corollary 3.6. *Consider $A=(a_{jn})$ as a non-negative and regular summability matrix and $<p_n>$ be any sequence in $[0,\infty)$ which is non-increasing. If*

$$\delta_{n,j} = \mathcal{S}t_A - O_m(p_n), \text{ as } n \to \infty, \tag{26}$$

where $\delta_{n,j}$ is given by eq.(14). Then for every function $f \in Lip_\lambda(\rho)$

$$\|B^{q_n}_{n,0,j} f - f\| = \mathcal{S}t_A - O_m(p_n) \text{ as } n \to \infty.$$

4 Conclusion

Recently, Aldaz et al. constructed an intriguing Bernstein-type polynomial operators known as AKR operators. Due to their construction, it is not easy to find the exact values of moments and central moments of these operators. As a consequence, it is difficult to derive the fundamental approximation results of these operators.

As it is known that the q-positive linear operators possess better approximation results. Therefore we investigated the approximation properties of q-analogue of these operators in the setting of A-statistical convergence.

We determined an upper bound for the central moments and then established the Korovkin-type Theorem and statistical rate of convergence of these operators using A-statistical convergence. Also, we have included rate of A-statistical convergence of Lipschitz-type function spaces in our investigation.

The results of this article can be helpful for various approximation techniques like difference approximation, simultaneous approximation, and Bézier approximation related to Bernstein-type operators.

Acknowledgement: The authors express sincere gratitude to the reviewers for offering valuable comments and suggestions, which have greatly enhanced the presentation of the paper. Additionally, the authors thank the editor for promptly completing the review process.

Funding: The first author is thankful to the University Grants Commission-India(UGC-In), UGC-Ref. N0.: 4242/(CSIR-UGC NET JUNE 2019) to provide fellowship during his Ph.D. tenure (August 2020-August 2025).

Conflict of interest: The authors declare that they have no conflict of interest.

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