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Regularity for solutions to a class of PDE's with Orlicz growth

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Abstract

We consider weak solutions $u : \Omega \to \mathbb{R}$ to partial differential equations of the form

$$\operatorname{div} a(x, Du) = 0$$

in $\Omega \subset \mathbb{R}^n$, $n > 2$, where the partial map $x \mapsto a(x, \xi)$ has a suitable Sobolev regularity and satisfies growth conditions with respect to the second variable expressed through an Orlicz function ϕ . We prove the second order regularity of the weak solutions.

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1 Introduction and statement of the result

In this paper we study the higher differentiability of solutions to weak solutions $u \in W^{1,\phi}(\Omega)$, of the equation

$$\operatorname{div} a(x, Du) = 0, \tag{1}$$

in $\Omega \subset \mathbb{R}^n$, $n > 2$. The main features of the Carathéodory function $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ considered here are the following

- the map $\xi \to a(x, \xi)$ satisfies the so called general growth conditions
- the map $x \to a(x, \xi)$ is possibly discontinuous.

To be more precise, we consider a convex function $\phi \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ that is piecewise C^2 on $\mathbb{R}_{\geq 0}$ and satisfies $\phi'(0) = \phi(0) = 0$. In what follows, we assume that $\phi'(t) > 0$ holds for any $t > 0$, as well as

$$p - 1 \leq \frac{t\phi''(t)}{\phi'(t)} \leq q - 1, \tag{2}$$

for given parameters $2 \leq p \leq q$, for every $t > 0$ for which $\phi''(t)$ exists. We note that the case $p = q$ corresponds to the model case $\phi(t) = \frac{1}{p} t^p$.

The Carathéodory function $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is such that, for given parameters $0 < \nu \leq L$ and for a function $k \in L^n_{loc}(\Omega)$ satisfies, the following conditions

$$\big\langle a(x, \xi) - a(x, \eta), \xi - \eta \big\rangle \geq \nu \phi''(|\xi| + |\eta|)|\xi - \eta|^2 \tag{3}$$

$$|a(x, \xi) - a(x, \eta)| \leq L\phi''(|\xi| + |\eta|)|\xi - \eta|, \tag{4}$$

$$|a(x, \xi)| \leq L\phi'(|\xi|), \tag{5}$$

$$|a(x, \xi) - a(y, \xi)| \leq (k(x) + k(y))|x - y|\phi'(|\xi|), \tag{6}$$

for a.e. $x, y \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$. It is worth noticing that, by virtue of the point-wise characterization of the Sobolev functions due to Hajlasz (see [32]), assumption (6) implies that the partial map $x \to a(x, \xi)$ belongs locally to $W^{1,n}$ and therefore is possibly discontinuous.

The interest for such kind of equations is based on the fact that some physical phenomena are governed by nonlinearities which are not of polynomial type and therefore the associated mathematical models involve the use of more general functions.

Many authors gave contributions on the regularity of solutions to nonlinear elliptic equations with Orlicz growth (see e.g. [12, 35]), also with L^1 or measure data as in [7, 5, 4, 6]. Moreover, results concerning the regularity of minimizers of functionals satisfying Orlicz growth conditions are also available, we refer for example to [39, 36, 16, 27, 28, 23, 15, 34]. We rely on the following notion of weak solutions to (1).

Definition 1.1. We call $u \in W^{1,\phi}(\Omega, \mathbb{R})$ a weak solution of (1) if it holds

$$\int_\Omega \langle a(x, Du), D\eta \rangle \, dx = 0 \tag{7}$$

for every $\eta \in C_0^\infty(\Omega, \mathbb{R}^n)$.

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Higher differentiability results for weak solutions of p-Laplace type equations with differentiable coefficients are commonly known from classical regularity theory (see [31] and the references therein).
In the last few years there has been an intense research activity showing that a suitable Sobolev regularity on the coefficients is still sufficient to establish the second order regularity of the solutions to p-harmonic equations either in the elliptic or in the parabolic setting, both in the scalar and in the vectorial cases ([41, 30, 24, 25, 2, 3]). We would also mention the several contributions available for the higher differentiability of local minimizers of integral functionals (see for example [40, 11, 18, 17] as well as the recent paper [21], where the second order regularity of solutions to elliptic systems has been obtained assuming that the partial map $x \to a(x,\xi)$ has derivatives in the Marcinkiewicz class $L^{n,\infty}(\Omega)$ with sufficiently small distance to $L^{\infty}(\Omega)$. For a wide discussion on such condition we refer to [19] and [20].
In case of general growth condition, despite the huge amount of papers investigating the regularity of the solutions or of the minimizers (see the recent survey [38] with the references therein), the question of the higher differentiability still needs to be explored. We refer e.g. to [22] for $p(x)$-growth, [8, 10, 9, 37] for (p,q)-growth conditions.
More specifically, for what concerns the case of ϕ-growth, we have to quote [13] where, in case of Hölder continuous coefficients, Diening and Ettwein were able to establish a fractional higher differentiability for the solutions to (1).
To our knowledge, the main result of this paper is the first higher differentiability result of integer order for equations with ϕ-growth in the gradient variable and with Sobolev coefficients.
Already for the p-Laplace equation, due to the nonlinear nature of the problem, it is not to be expected that second weak derivatives exist, but the extra differentiability quantity makes sense for the nonlinear function $V_\mu(Du) := (\mu^2 + |Du|^2)^{\frac{p-2}{4}} Du$, (see [31] and the references therein).
Therefore, our results here involve the natural corresponding auxiliary function $V : \mathbb{R}^n \to \mathbb{R}^n$ associated to ϕ defined as

$$V(\xi) := \begin{cases} \sqrt{\frac{\phi'(|\xi|)}{|\xi|}}\,\xi & \text{if } \xi \neq 0 \\ 0 & \text{if } \xi = 0 \end{cases} \tag{8}$$

and introduced for the first time in ([13]).

More precisley, our main theorem states as follows.

Theorem 1.1. *Let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function. Assume that (2)– (6) are in force and that $u \in W^{1,\phi}(\Omega,\mathbb{R})$ is a weak solution of the equation*

$$\operatorname{div} a(x,Du) = 0, \tag{9}$$

in the sense of Definition 1.1. Then $V(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ and, for every ball $B_R(x_o) \Subset \Omega$ with $R \in (0,1]$, the following local estimates hold

$$\int_{B_{\frac{R}{2}}} |D(V(Du))|^2\,dx \le \frac{c}{R^q}\int_{B_R} \phi(|Du|)\,dx + \frac{c}{R^n} \tag{10}$$

and

$$\left(\int_{B_{\frac{R}{2}}} |V(Du))|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \le \frac{c}{R^q}\int_{B_R} \phi(|Du|)\,dx + \frac{c}{R^n}. \tag{11}$$

The proof of previous Theorem is achieved combining a suitable a priori estimate with an approximation argument. The a priori estimate is established by the use of the well known difference quotient method that here presents new difficulties due to the general growth of our problem. Next we construct the approximating problem by regularizing the coefficients. In order to use the a priori estimate for the solution of such problems, we need to establish their second order regularity that doesn't seem available in literature (see Theorem 3.1). Finally, we show that such estimates are preserved in passing to the limit.

2 Preliminaries

2.1 Notation and a useful lemma

We write $B_r(x_o) \subset \mathbb{R}^n$ for the open ball of radius $r > 0$ and center $x_o \in \mathbb{R}^n$. If the center is clear from the context, we use the shorter notation B_r.

The symbol $\langle \cdot, \cdot \rangle$ will be used to denote the standard scalar product on the space $\mathbb{R}^n$.

We will write c to indicate a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts.

In what follows, we shall use the following well-known iteration lemma.

Lemma 2.1. *For $0 < R_1 < R_2$, consider a bounded function $f : [R_1, R_2] \to [0,\infty)$ with*

$$f(r_1) \le \vartheta f(r_2) + \frac{A}{(r_2-r_1)^\alpha} + \frac{B}{(r_2-r_1)^\beta} + C \qquad \text{for all } R_1 < r_1 < r_2 < R_2,$$

where $A, B, C,$ and α, β denote nonnegative constants and $\vartheta \in (0,1)$. Then we have

$$f(R_1) \le c(\alpha,\vartheta)\left(\frac{A}{(R_2-R_1)^\alpha} + \frac{B}{(R_2-R_1)^\beta} + C\right).$$

For the proof we refer to, e.g., [31, Lemma 6.1, p.191]

2.2 Preliminaries on Orlicz Functions

Here we introduce some basic properties of Orlicz functions that will be useful to frame and solve our problem (more details can be found in [42]).

A function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called an N-function if and only if there is a right-continuous, positive on the positive real line, and non-decreasing function $\psi' : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\psi'(0) = 0$ and $\lim_{t\to\infty} \psi'(t) = \infty$ such that $\psi(t) = \int_0^t \psi'(\tau)\,d\tau$. An N-function is said to satisfy the Δ_2-condition if and only if there is a constant $c > 1$ such that $\psi(2t) \leq c\,\psi(t)$ for every $t > 0$.

The conjugate of an N-function ψ is defined as

$$\psi^*(t) := \sup_{s\geq 0}(st - \psi(s)), \quad t \geq 0.$$

A direct consequence of the definition is the following Young's inequality

$$st \leq \psi(s) + \psi^*(t) \qquad \text{for any } s, t \geq 0. \tag{12}$$

We note that our assumptions on the function ϕ ensure that it is an N-function. Moreover, in view of the following lemma, condition (2) guarantees that ϕ and ϕ^* both satisfy the Δ_2-condition.

Lemma 2.2 ([1, Lemma 2.1]). *Let $\phi \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ be a function which is piecewise C^2 and satisfying (2). Then the following estimates hold true for every $\lambda > 1$ and $t > 0$:*

$$\lambda^p \phi(t) \leq \phi(\lambda t) \leq \lambda^q \phi(t), \tag{13}$$

$$\lambda^{\frac{q}{q-1}} \phi^*(t) \leq \phi^*(\lambda t) \leq \lambda^{\frac{p}{p-1}} \phi^*(t), \tag{14}$$

$$\lambda^{p-1} \phi'(t) \leq \phi'(\lambda t) \leq \lambda^{q-1} \phi'(t). \tag{15}$$

Remark 1. Under assumption (2), Lemma 2.2 implies the bounds

$$\phi(1)\min\{t^p, t^q\} \leq \phi(t) \leq \phi(1)\max\{t^p, t^q\}, \tag{16}$$

$$\phi^*(1)\min\{t^{p'}, t^{q'}\} \leq \phi^*(t) \leq \phi^*(1)\max\{t^{p'}, t^{q'}\}, \tag{17}$$

for any $t > 0$.

Moreover, we recall the notion of shifted N-functions first introduced in [13]. Here, we use the slight variant of [14, Appendix B] with even nicer properties. The shifted N-function ϕ_a, for $a \geq 0$, are defined by

$$\phi_a(t) := \int_0^t \frac{\phi'(a \vee s)}{a \vee s} s\,ds, \tag{18}$$

where $s_1 \vee s_2 := \max\{s_1, s_2\}$ for $s_1, s_2 \in \mathbb{R}$. In the model case $\phi(t) := \frac{1}{p} t^p$, the shifted N-functions satisfy

$$\begin{aligned} \phi_a(t) &\eqsim (a \vee t)^{p-2} t^2, \\ \phi_a'(t) &\eqsim (a \vee t)^{p-2} t, \end{aligned} \tag{19}$$

with constants only depending on p. The index a is called the *shift*. Obviously, $\phi_0 = \phi$. Moreover, if $a \eqsim b$, then $\phi_a(t) \eqsim \phi_b(t)$. For the shifted N-function ϕ_a we have

$$(\phi_a)^* = (\phi^*)_{\phi'(a)}, \tag{20}$$

cf. [14, Lemma 33]. A straightforward computation yields the identity

$$\frac{t\phi_a''(t)}{\phi_a'(t)} = \begin{cases} 1, & \text{if } 0 \leq t < a, \\ \frac{t\phi''(t)}{\phi'(t)}, & \text{if } t > a. \end{cases}$$

Consequently, assumption (2) implies

$$\min\{1, p-1\} \leq \frac{t\phi_a''(t)}{\phi_a'(t)} \leq \max\{1, q-1\}, \tag{21}$$

provided $t > 0$ and $\phi_a''(t)$ exists. Therefore, Lemma 2.2, implies

Lemma 2.3. *Let $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function as in (2), and $a \geq 0$. Then, for any $\lambda > 1$ and $t > 0$, the shifted N-function ϕ_a satisfies the estimates*

$$\lambda^2 \phi_a(t) \leq \phi_a(\lambda t) \leq \lambda^q \phi_a(t), \tag{22}$$

$$\lambda^{\frac{q}{q-1}} \phi_a^*(t) \leq \phi_a^*(\lambda t) \leq \lambda^2 \phi_a^*(t). \tag{23}$$

In particular, ϕ_a and ϕ_a^* satisfy the Δ_2-condition with constants respectively given by 2^q and 2^2, independently of $a \geq 0$.

For further needs we observe that if $\sigma \in (0,1)$ we have

$$\sigma^q \phi_a(t) \le \phi_a(\sigma t) \le \sigma^2 \phi_a(t), \tag{24}$$

Indeed from (22) we deduce that

$$\phi_a(t) = \phi_a\left(\frac{1}{\sigma}\sigma t\right) \le \frac{1}{\sigma^q}\phi_a(\sigma t) \implies \phi_a(\sigma t) \ge \sigma^q \phi_a(t)$$

The other inequality can be derived arguing analogously.
Since ϕ_a' is nondecreasing and ϕ_a satisfies the Δ_2-condition, we have the estimate

$$s\phi_a'(s) \le \phi_a(2s) - \phi_a(s) \le c(q)\phi_a(s) \tag{25}$$

for every $s \ge 0$ and an arbitrary shift $a \ge 0$. Moreover, as a consequence of (22) and the convexity of ϕ, we obtain

$$\phi_a(s+t) \le 2^q \phi_a\Big(\frac{s+t}{2}\Big) \le 2^{q-1}\big(\phi_a(s) + \phi_a(t)\big) \tag{26}$$

for every $s, t \ge 0$, independently of $a \ge 0$. Using the bounds (22) and (23), we observe that (12) implies the following versions of Young's inequality for the shifted N-function ϕ_a.
For every $\delta > 0$ there exists $c_\delta = c_\delta(\delta, p, q) \ge 1$ such that for all $s, t, a \ge 0$

$$\begin{aligned} s\,t &\le \delta\,\phi_a(t) + c_\delta\,\phi_a^*(s), \\ s\,t &\le c_\delta\,\phi_a(t) + \delta\,\phi_a^*(s). \end{aligned} \tag{27}$$

We use these estimates with s replaced by $\phi_a'(s)$. A direct computation and estimate (25) imply

$$\phi_a^*(\phi_a'(s)) = s\phi_a'(s) - \phi_a(s) \le c(q)\phi_a(s), \tag{28}$$

for any $s \ge 0$. Therefore, the Young type inequalities (27) yield the estimates

$$\begin{aligned} \phi_a'(s)\,t &\le \delta\,\phi_a(t) + c_\delta\,\phi_a(s), \\ \phi_a'(s)\,t &\le c_\delta\,\phi_a(t) + \delta\,\phi_a(s), \end{aligned} \tag{29}$$

for $s, t, a \ge 0$, with constants $c_\delta = c_\delta(\delta, p, q)$.

We have the following equivalent representations of the shifted N-functions.

Lemma 2.4 ([14, Lemma 40]). *Let $\phi : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be a function as in (2). Then, for every $P, Q \in \mathbb{R}^{n\times n}$, we have*

$$\phi_{|Q|}'(|P-Q|) \approx \frac{\phi'(|P| \vee |Q|)}{|P| \vee |Q|}|P-Q|,$$

$$\phi_{|Q|}(|P-Q|) \approx \frac{\phi'(|P| \vee |Q|)}{|P| \vee |Q|}|P-Q|^2,$$

where the implicit constants depend only on p and q.

The following equivalences will be useful for our aims.

Lemma 2.5 ([13, Lemma 24]). *Let ϕ be an N-function satisfying assumption (2). Then uniformly in P, Q with $|P| + |Q| > 0$ we have*

$$\phi''(|P| + |Q|)|P-Q| \sim \phi_{|P|}'(|P-Q|)$$

$$\phi''(|P| + |Q|)|P-Q|^2 \sim \phi_{|P|}(|P-Q|)$$

Next Lemma summarize the relation between $a(x,\xi)$, the auxiliary function V introduced in (8) and the shifted versions of ϕ. For the proof we remind to [14, 26].

Lemma 2.6. *Let $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function with properties (3) and (5) for an N- function ϕ satisfying (2). Then, for all $P, Q \in \mathbb{R}^n$ and a.e. $x, y \in \Omega$ there holds*

$$\big\langle a(x,P) - a(x,Q), P - Q \big\rangle \ge c\,\phi_{|Q|}(|P-Q|) \approx |V(P) - V(Q)|^2, \tag{30}$$

$$a(x,Q)\cdot Q \approx |V(Q)|^2 \approx \phi_{|Q|}(|Q|) \approx \phi(|Q|) \tag{31}$$

with constants that depend only on p, q, ν, and L. If, additionally, assumption (4) is satisfied, then we have

$$|a(x,P) - a(x,Q)| \le c\,\phi_{|Q|}'(|P-Q|), \tag{32}$$

$$\big\langle a(x,P) - a(x,Q), P - Q \big\rangle \approx \phi_{|Q|}(|P-Q|) \approx |V(P) - V(Q)|^2. \tag{33}$$

Also of strong use is the possibility to change the shift:

Lemma 2.7 (Change of shift, [14, Corollary 44])**.** *For $\delta > 0$ there exists $c_\delta = c_\delta(\delta, p, q)$ such that for all $P, Q \in \mathbb{R}^n$ there holds*

$$\phi_{|P|}(t) \le c_\delta \phi_{|Q|}(t) + \delta\,|V(P) - V(Q)|^2,$$
$$(\phi_{|P|})^*(t) \le c_\delta (\phi_{|Q|})^*(t) + \delta\,|V(P) - V(Q)|^2.$$

In particular, the choice $P = 0$ allows to add a shift in the form

$$\phi(t) \le c\phi_{|Q|}(t) + c|V(Q)|^2 \le c\phi_{|Q|}(t) + c\phi(|Q|) \tag{34}$$

for every $Q \in \mathbb{R}^n$, where we used (31) for the last estimate. Moreover, the choice of $Q = 0$ in the preceding lemma allows to remove the shift from the N-function. For the proof we refer to [14].

Lemma 2.8 (Removal of shift)**.** *For all $P \in \mathbb{R}^n$, all $t \ge 0$ and all $\delta \in (0, 1]$ there holds*

$$\phi'_{|P|}(t) \le \phi'\left(\frac{t}{\delta}\right) \vee \left(\delta\phi'(|P|)\right), \tag{35}$$

$$\phi_{|P|}(t) \le \delta\,\phi(|P|) + c\,\delta\,\phi\left(\frac{t}{\delta}\right), \tag{36}$$

$$(\phi_{|P|})^*(t) \le \delta\,\phi(|P|) + c\,\delta\,\phi^*\left(\frac{t}{\delta}\right), \tag{37}$$

where c depends only on p and q.

For an N-function ϕ satisfying the Δ_2-condition, the space $L^\phi(\Omega)$ consists of those functions $u \in L^1(\Omega)$ that satisfy

$$\int_\Omega \phi(|u|)\,dx < \infty.$$

The Orlicz space $L^\phi(\Omega)$ becomes a Banach space with the norm

$$\|u\|_{L^\phi(\Omega)} := \inf\left\{\lambda > 0 \colon \int_\Omega \phi\left(\frac{|u(x)|}{\lambda}\right) dx \le 1\right\}.$$

The Orlicz-Sobolev space $W^{1,\phi}(\Omega, \mathbb{R}^n)$ is defined as the space of functions $u \in L^\phi(\Omega)$ that are weakly differentiable with $Du \in L^\phi(\Omega)$, and is equipped with the norm $\|u\|_{W^{1,\phi}} := \|u\|_{L^\phi} + \|Du\|_{L^\phi}$. Finally, we define the subspace $W_0^{1,\phi}(\Omega, \mathbb{R}^n) \subset W^{1,\phi}(\Omega, \mathbb{R}^n)$ as the completion of $C_0^\infty(\Omega, \mathbb{R}^n)$ with respect to the $W^{1,\phi}$-norm.

2.3 Difference quotients

In the sequel we recall some properties of the finite difference operator. We use the customary notation

$$\tau_h F(x) \equiv \tau_{h,i} F(x) := F(x + he_i) - F(x), \tag{38}$$

for any $F \in L^1_{\mathrm{loc}}(\Omega)$, $i = 1, \dots, n$, $x \in \Omega$, and $h \neq 0$ with $x + he_i \in \Omega$.
We start with some elementary properties, cf. [31].

Proposition 2.9. *Consider two functions $F, G \in W^{1,\phi}(\Omega, \mathbb{R}^n)$. For $h \neq 0$ and the inner parallel sets*

$$\Omega_{|h|} := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > |h|\},$$

we have the following properties.

(*i*) $\tau_h F \in W^{1,\phi}(\Omega_{|h|}, \mathbb{R}^n)$ *and* $D_i(\tau_h F) = \tau_h(D_i F)$.

(*ii*) *If the support of at least one of the functions F and G is contained in $\Omega_{|h|}$, then we have the discrete integration by parts formula*

$$\int_\Omega F\,\tau_h G\,dx = \int_\Omega G\,\tau_{-h} F\,dx.$$

(*iii*) *We have the following product rule for the finite differences:*

$$\tau_h(FG)(x) = F(x + he_i)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result about the finite difference operator is a kind of integral version of the Lagrange Theorem.

Lemma 2.10. *Let ϕ be an N-function satisfying the Δ_2-condition and $F \in W^{1,\phi}(B_R, \mathbb{R}^n)$. Then, for any $\rho > 0$ and $h \in \mathbb{R}_{\neq 0}$ with $\rho + |h| \le R$, we have the estimates*

$$\int_{B_\rho} \phi\left(\frac{|\tau_h F(x)|}{|h|}\right) dx \le \int_{B_{\rho+|h|}} \phi(|DF(x)|)\,dx. \tag{39}$$

Moreover,

$$\int_{B_\rho} \phi(|F(x + he_i)|)\,dx \le \int_{B_{\rho+|h|}} \phi(|F(x)|)\,dx. \tag{40}$$

For the proof we refer to [26].

3 Higher differentiability

The first result of this section is the second order regularity of solutions to (1) in case the coefficients are Lipschitz continuous. It will be fundamental to legitimate the construction of approximating problems whose solutions are sufficiently regular. Next, we shall establish an a priori estimate for the solutions that possess the same second order regularity proven before.

3.1 The case of smooth coefficients

Consider a Carathéodory function $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and the equation

$$\operatorname{div} A(x, Dv) = 0. \tag{41}$$

Assume that there exist an N-function $\phi(t)$ satisfying (2) and positive constants ν, L, K such that

$$\big\langle A(x,\xi) - A(x,\eta), \xi - \eta \big\rangle \geq \nu \phi''(|\xi| + |\eta|)|\xi - \eta|^2 \tag{42}$$

$$|A(x,\xi) - A(x,\eta)| \leq L\phi''(|\xi| + |\eta|)|\xi - \eta|, \tag{43}$$

$$|A(x,\xi)| \leq L\phi'(|\xi|), \tag{44}$$

$$|A(x,\xi) - A(y,\xi)| \leq K|x - y|\phi'(|\xi|), \tag{45}$$

for a.e. $x, y \in \Omega$ and every $\xi, \eta \in \mathbb{R}^n$. The following higher differentiability result holds true.

Theorem 3.1. *Let $v \in W^{1,\phi}(\Omega)$ be a weak solution to (41) under the assumptions (42)– (45). Then*

$$V(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega) \qquad \textit{and} \qquad \phi(|Dv|) \in L^{\frac{n}{n-2}}_{\mathrm{loc}}(\Omega).$$

Proof. Since $v \in W^{1,\phi}(\Omega)$ is a weak solution to (41), we have

$$\int_\Omega \langle A(x, Dv), D\eta\rangle\, dx = 0 \qquad \text{for every } \eta \in W^{1,\phi}_0(\Omega, \mathbb{R}^n). \tag{46}$$

We fix a ball $B_R \Subset \Omega$ with $R \in (0,1]$. For radii $0 < \frac{R}{2} < R_1 < R_2 \leq R$, we consider the maximal step size $h_o := \frac{1}{2}(R_2 - R_1)$. Then, we fix radii r_1, r_2 with $R_1 \leq r_1 < r_2 \leq R_2 - h_o = \frac{1}{2}(R_1 + R_2)$ and choose a cut-off function $\zeta \in C^\infty_0(B_{r_2}, [0,1])$ with $\zeta = 1$ on B_{r_1} and such that $|\nabla\zeta| \leq \frac{c}{r_2 - r_1}$.

We choose $\eta = \tau_{-h}(\zeta^2 \tau_h u) \in W^{1,\phi}_0(B_{R_2})$ as an admissible test function in (46), obtaining the identity

$$\int_{B_R} \big\langle A(x, Du), D(\tau_{-h}(\zeta^2\tau_h u))\big\rangle\, dx = 0, \tag{47}$$

Exploiting the properties of the difference quotient, we rewrite the integral on the left-hand side as

$$\begin{aligned} &\int_{B_R} \big\langle A(x, Dv), D(\tau_{-h}(\zeta^2\tau_h v))\big\rangle\, dx \\ &= \int_{B_R} \zeta^2 \big\langle \tau_h[A(x, Dv)], \tau_h Dv\big\rangle\, dx + 2\int_{B_R} \big\langle \tau_h[A(x, Dv)], \zeta\nabla\zeta\tau_h v\big\rangle\, dx. \end{aligned} \tag{48}$$

Now, we decompose

$$\begin{aligned} \tau_h[A(x, Dv)] &= \Big(A(x + he_i, Dv(x + he_i)) - A(x + he_i, Dv(x))\Big) \\ &\quad + \Big(A(x + he_i, Dv(x)) - A(x, Dv(x))\Big) \\ &=: \mathcal{A}'_h + \mathcal{B}'_h. \end{aligned}$$

Joining (47) and (48), keeping in mind the abbreviations above and the definition of η , we arrive at the identity

$$\begin{aligned} \int_{B_{r_2}} \zeta^2 \big\langle \mathcal{A}'_h, \tau_h Dv\big\rangle\, dx &= -\int_{B_{r_2}} \zeta^2 \big\langle \mathcal{B}'_h, \tau_h Dv\big\rangle\, dx - 2\int_{B_{r_2}} \big\langle \mathcal{A}'_h, \zeta\nabla\zeta\tau_h v\big\rangle\, dx \\ &\quad - 2\int_{B_{r_2}} \big\langle \mathcal{B}'_h, \zeta\nabla\zeta\tau_h v\big\rangle\, dx \\ &=: \mathrm{J}_1 + \mathrm{J}_2 + \mathrm{J}_3. \end{aligned} \tag{49}$$

According to (42), by Lemmas 2.5 and 2.6, the left-hand side is bounded from below by

$$\int_{B_{r_2}} \zeta^2 \big\langle \mathcal{A}'_h, \tau_h Dv\big\rangle\, dx \geq c\int_{B_{r_2}} \zeta^2 |\tau_h V(Dv)|^2\, dx, \tag{50}$$

for a constant $c = c(\nu, p, q)$. For the estimate of J_1, we introduce the abbreviation

$$\mathcal{D}(h) := |Dv(x)| \vee |Dv(x + he_i)|. \tag{51}$$

Using in turn assumption (45), the monotonicity of ϕ', and Young's inequality with a parameter $\varepsilon \in (0,1]$, we get

$$\begin{aligned}
|\mathrm{J}_1| &\le \int_{B_{r_2}} \zeta^2 |\mathcal{B}'_h| \, |\tau_h Dv| \, dx \le cK|h| \int_{B_{r_2}} \zeta^2 \phi'(|Dv|) |\tau_h Dv| \, dx \\
&\le cK|h| \int_{B_{r_2}} \zeta^2 \frac{\phi'(\mathcal{D}(h))}{\mathcal{D}(h)} \mathcal{D}(h) \, |\tau_h Dv| \, dx \\
&\le \varepsilon \int_{B_R} \zeta^2 \frac{\phi'(\mathcal{D}(h))}{\mathcal{D}(h)} |\tau_h Dv|^2 \, dx + c_\varepsilon K^2 |h|^2 \int_{B_{r_2}} \zeta^2 \phi'(\mathcal{D}(h)) \, \mathcal{D}(h) \, dx.
\end{aligned}$$

In order to bound the first integral on the right-hand side, we apply the second assertion of Lemma 2.4 and $(30)_2$, while the last integral is bounded by (25), estimate (26) with $a = 0$, and (40). In this way, we deduce

$$\begin{aligned}
|\mathrm{J}_1| &\le c\varepsilon \int_{B_R} \zeta^2 \phi_{|Dv|}(|\tau_h Dv|) \, dx + c_\varepsilon K^2 |h|^2 \int_{B_{r_2}} \phi(\mathcal{D}(h)) \, dx \\
&\le c\varepsilon \int_{B_{r_2}} \zeta^2 |\tau_h V(Dv)|^2 \, dx + c_\varepsilon K^2 |h|^2 \int_{B_{R_2}} \phi(|Dv|) \, dx.
\end{aligned} \tag{52}$$

Next, we exploit assumption (43) and the properties of ζ to obtain

$$\begin{aligned}
|\mathrm{J}_2| &\le 2 \int_{B_{r_2}} |\mathcal{A}'_h| \, |\zeta \, \nabla \zeta \, \tau_h v| \, dx \le \frac{c}{r_2 - r_1} \int_{B_{r_2}} \phi''(|\mathcal{D}(h)|) \, |\tau_h Dv| |\tau_h v| \, dx \\
&\le \frac{c}{r_2 - r_1} \int_{B_{r_2}} \phi'_{|Dv|} \left(|\tau_h Dv|\right) |\tau_h v| \, dx \\
&\le \varepsilon \int_{B_{r_2}} (\phi_{|Dv|})^* \left(\phi'_{|Dv|} \left(|\tau_h Dv|\right)\right) \, dx + c_\varepsilon \int_{B_{r_2}} \phi_{|Dv|} \left(\frac{|h|}{r_2 - r_1} \frac{|\tau_h v|}{|h|} \right) dx \\
&\le \varepsilon \, c \int_{B_{r_2}} \phi_{|Dv|} \left(|\tau_h Dv|\right) \, dx + \frac{c_\varepsilon |h|^2}{(r_2 - r_1)^q} \int_{B_{r_2}} \phi_{|Dv|} \left(\frac{|\tau_h v|}{|h|} \right) dx \\
&\le \varepsilon \, c \int_{B_{r_2}} \phi_{|Dv|} \left(|\tau_h Dv|\right) \, dx + \frac{c_\varepsilon |h|^2}{(r_2 - r_1)^q} \int_{B_{R_2}} \phi_{|Dv|} \left(|Dv|\right) \, dx \\
&\le \varepsilon \, c \int_{B_{r_2}} |\tau_h V(Dv)|^2 \, dx + \frac{c_\varepsilon |h|^2}{(r_2 - r_1)^q} \int_{B_{R_2}} \phi \left(|Dv|\right) \, dx,
\end{aligned} \tag{53}$$

where we used in turn the first assertion in Lemma 2.5, Young's inequality at (27), estimate (28), the homogeneity inequalities at (22) and (24), the first assertion in Lemma 2.10, the equivalence in (30), and Lemma 2.7 with one shift equal to zero. For the estimate of J_3, we use (45) and Young's inequality as follows

$$\begin{aligned}
|\mathrm{J}_3| &\le 2 \int_{B_{r_2}} |\mathcal{B}'_h| \, |\zeta \nabla \zeta \tau_h v| \, dx \le cK \frac{|h|}{r_2 - r_1} \int_{B_{r_2}} \phi'(|\mathcal{D}(h)|) \, |\tau_h v| \, dx \\
&= cK \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \phi'(|\mathcal{D}(h)|) \, \frac{|\tau_h v|}{|h|} \, dx \\
&\le cK \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \left(\phi^*(\phi'(|\mathcal{D}(h)|)) + \phi \left(\frac{|\tau_h v|}{|h|} \right) \right) dx \\
&\le cK \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \phi(|\mathcal{D}(h)|) \, dx + cK \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \phi \left(\frac{|\tau_h v|}{|h|} \right) dx \\
&\le cK \frac{|h|^2}{r_2 - r_1} \int_{B_{R_2}} \phi(|Dv|) \, dx,
\end{aligned} \tag{54}$$

where we also used (28) and the first assertion in Lemma 2.10. Collecting the estimates (50), (52), (53) and (54) , recalling identity (49) and using the properties of ζ, we deduce the bound

$$\int_{B_{r_1}} |\tau_h V(Dv)|^2\,dx \le c\varepsilon \int_{B_{r_2}} |\tau_h V(Dv)|^2\,dx + c_\varepsilon(K)\frac{|h|^2}{(r_2-r_1)^q}\int_{B_{R_2}} \phi(|Dv|)\,dx$$

for every $\varepsilon \in (0,1]$ and all radii r_1, r_2 with $R_1 \le r_1 < r_2 \le R_2$. Choosing $\varepsilon = \frac{1}{2c}$, we can use Lemma 2.1, thus obtaining

$$\int_{B_{R_1}} |\tau_h V(Dv)|^2\,dx \le c(K)\frac{|h|^2}{(R_2-R_1)^q}\int_{B_{R_2}} \phi(|Dv|)\,dx.$$

Previous inequality implies that $V(Dv) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ with the following estimate

$$\int_{B_{R_1}} |DV(Dv)|^2\,dx \le \frac{c}{(R_2-R_1)^q}\int_{B_{R_2}} \phi(|Dv|)\,dx,$$

with a constant c depending on n, ν, L, p, q and K. The other assertion easily comes from the Sobolev imbedding Theorem. □

3.2 The a priori estimate

This subsection contains the a priori estimate suitable for establishing the main result. More precisely, we are going to establish the following Theorem.

Theorem 3.2. *Let ϕ be an N-function with the property (2), and consider a weak solution $u \in W^{1,\phi}(\Omega)$ of (1) under the assumptions (3)– (6). Then, if*

$$V(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega), \tag{55}$$

there exists a radius $\bar{R} = \bar{R}(c, n, p, q, \nu, L, ||k||_{L^n})$ such that the following estimate

$$\int_{B_{\frac{R}{2}}} |D(V(Du))|^2\,dx \le \frac{c}{R^q}\int_{B_R} \phi(|Du|)\,dx + \frac{c}{R^n}, \tag{56}$$

holds for every ball $B_R(x_o) \Subset \Omega$ with $R < \bar{R}$ and with a constant $c = c(n, \nu, L, p, q)$.

Proof. As before, we fix a ball $B_R \Subset \Omega$ with $R \in (0,1]$. For radii $0 < \frac{R}{2} < R_1 < R_2 \le R$, we consider the maximal step size $h_o := \frac{1}{2}(R_2 - R_1)$. Then, we fix radii r_1, r_2 with $R_1 \le r_1 < r_2 \le R_2 - h_o = \frac{1}{2}(R_1 + R_2)$ and choose a cut-off function $\zeta \in C_0^\infty(B_{r_2}, [0,1])$ with $\zeta = 1$ on B_{r_1} and $|\nabla\zeta| \le \frac{c}{r_2 - r_1}$.
We choose $\eta = \tau_{-h}(\zeta^2\tau_h u) \in W_0^{1,\phi}(B_{R_2})$ as an admissible test function in (7), obtaining the identity

$$\begin{aligned}\int_{B_R} \zeta^2\langle \mathcal{A}_h, \tau_h Du\rangle\,dx &= -\int_{B_R} \zeta^2\langle \mathcal{B}_h, \tau_h Du\rangle\,dx - 2\int_{B_R} \langle \mathcal{A}_h, \zeta\nabla\zeta\tau_h u\rangle\,dx \\ &\quad -2\int_{B_R} \langle \mathcal{B}_h, \zeta\nabla\zeta\tau_h u\rangle\,dx \\ &=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,\end{aligned} \tag{57}$$

where now, we used the decomposition

$$\begin{aligned}\tau_h[a(x, Du)] &= \Big(a(x+he_i, Du(x+he_i)) - a(x+he_i, Du(x))\Big) \\ &\quad + \Big(a(x+he_i, Du(x)) - a(x, Du(x))\Big) \\ &=: \mathcal{A}_h + \mathcal{B}_h.\end{aligned}$$

According to (3), by Lemmas 2.5 and 2.6, the left-hand side of (57) is bounded from below by

$$\int_{B_R} \zeta^2\langle \mathcal{A}_h, \tau_h Du\rangle\,dx \ge c\int_{B_{r_2}} \zeta^2|\tau_h V(Du)|^2\,dx, \tag{58}$$

for a constant $c = c(\nu, p, q)$. For the estimate of I_1, we use the abbreviations (51) and

$$\mathcal{K}(h) := |k(x)| \vee |k(x+he_i)|.$$

Using in turn assumption (6), the monotonicity of ϕ', and Young's inequality with a parameter $\kappa \in (0,1]$, we estimate

$$
\begin{aligned}
|\mathrm{I}_1| &\le \int_{B_{r_2}} \zeta^2 |\mathcal{B}_h|\,|\tau_h Du|\,dx \le c|h| \int_{B_{r_2}} \zeta^2 \mathcal{K}(h)\phi'(|Du|)|\tau_h Du|\,dx \\
&= c|h| \int_{B_{r_2}} \zeta^2 \mathcal{K}(h) \frac{\phi'(\mathcal{D}(h))}{\mathcal{D}(h)} \mathcal{D}(h)\,|\tau_h Du|\,dx \\
&\le \kappa \int_{B_{r_2}} \zeta^2 \frac{\phi'(\mathcal{D}(h))}{\mathcal{D}(h)} |\tau_h Du|^2\,dx + c_\kappa |h|^2 \int_{B_{r_2}} \zeta^2 \mathcal{K}^2(h)\phi'(\mathcal{D}(h))\,\mathcal{D}(h)\,dx.
\end{aligned} \tag{59}
$$

In order to bound the first integral on the right-hand side, we apply Lemma 2.4 and $(30)_2$, while the last integral is bounded by (25), the assumption on $k(x)$, Hölder's inequality, estimate at (26) with $a = 0$, and (40). In this way, we deduce

$$
\begin{aligned}
|\mathrm{I}_1| &\le c\kappa \int_{B_{r_2}} \zeta^2 \phi_{|Du|}(|\tau_h Du|)\,dx + c_\kappa |h|^2 \int_{B_{r_2}} \zeta^2 \mathcal{K}^2(h)\phi(\mathcal{D}(h))\,dx \\
&\le c\kappa \int_{B_{r_2}} \zeta^2 |\tau_h V(Du)|^2\,dx + c_\kappa |h|^2 \left(\int_{B_{r_2}} \mathcal{K}^n(h)\,dx \right)^{\frac{2}{n}} \left(\int_{B_{r_2}} \zeta^{\frac{2n}{n-2}} \phi(\mathcal{D}(h))^{\frac{n}{n-2}}\,dx \right)^{\frac{n-2}{n}} \\
&\le c\kappa \int_{B_{r_2}} \zeta^2 |\tau_h V(Du)|^2\,dx + c_\kappa |h|^2 \left(\int_{B_{R_2}} k^n(x)\,dx \right)^{\frac{2}{n}} \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx \right)^{\frac{n-2}{n}}.
\end{aligned} \tag{60}
$$

Arguing exactly as we did for the estimation of J_2 in Theorem 3.1, we obtain

$$
\begin{aligned}
|\mathrm{I}_2| &\le \phi'_{|Dv|}\,(|\tau_h Dv|)\,|\tau_h v|\,dx \\
&\le \varepsilon\, c \int_{B_{r_2}} |\tau_h V(Du)|^2\,dx + \frac{c_\varepsilon |h|^2}{(r_2 - r_1)^q} \int_{B_{r_2}} \phi\,(|Du|)\,dx.
\end{aligned} \tag{61}
$$

For the estimate of I_3, we use (6) and Young's inequality with exponents n and $\frac{n}{n-1}$ as follows

$$
\begin{aligned}
|\mathrm{I}_3| &\le 2 \int_{B_{r_2}} |\mathcal{B}_h|\,|\zeta \nabla \zeta \tau_h u|\,dx \le c \frac{|h|}{r_2 - r_1} \int_{B_{r_2}} \mathcal{K}(h)\phi'(|\mathcal{D}(h)|)\,|\tau_h u|\,dx \\
&\le c \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \mathcal{K}^n(h)\,dx + c \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \left(\phi'(|\mathcal{D}(h)|)\, \frac{|\tau_h u|}{|h|} \right)^{\frac{n}{n-1}} dx \\
&\le c \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \mathcal{K}^n(h)\,dx + c \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \left(\phi^*(\phi'(|\mathcal{D}(h)|)) + \phi\left(\frac{|\tau_h u|}{|h|} \right) \right)^{\frac{n}{n-1}} dx \\
&\le c \frac{|h|^2}{r_2 - r_1} \int_{B_{R_2}} k^n(x)\,dx + c \frac{|h|^2}{r_2 - r_1} \int_{B_{R_2}} \phi^{\frac{n}{n-1}}(|Du|)\,dx \\
&+ c \frac{|h|^2}{r_2 - r_1} \int_{B_{r_2}} \left(\phi\left(\frac{|\tau_h u|}{|h|} \right) \right)^{\frac{n}{n-1}} dx
\end{aligned} \tag{62}
$$

where we also used Young's inequality at (27), (28) and the second assertion in Lemma 2.10.
Note that the a priori assumption $V(Du) \in W^{1,2}_{\mathrm{loc}}(\Omega)$ in particular implies, by Sobolev imbedding Theorem, $\phi^{\frac{n}{n-2}}(|Du|) \in L^1_{\mathrm{loc}}(\Omega)$ and so also $\phi^{\frac{n}{n-1}}(|Du|) \in L^1_{\mathrm{loc}}(\Omega)$. Therefore we are legitimate to use the first assertion of Lemma 2.10 with the Orlicz function $\phi(t)^{\frac{n}{n-1}}$ to bound last integral in previous estimate as follows

$$
\int_{B_{r_2}} \left(\phi\left(\frac{|\tau_h u|}{|h|} \right) \right)^{\frac{n}{n-1}} dx \le c \int_{B_{R_2}} (\phi(|Du|))^{\frac{n}{n-1}}\,dx.
$$

Inserting the above inequality in estimate (62), we get

$$
\begin{aligned}
|\mathrm{I}_3| &\le c \frac{|h|^2}{r_2 - r_1} \int_{B_{R_2}} k^n(x)\,dx + c \frac{|h|^2}{r_2 - r_1} \int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-1}}\,dx \\
&\le c \frac{|h|^2}{r_2 - r_1} \int_{B_{R_2}} k^n(x)\,dx + c \frac{|h|^2}{r_2 - r_1} \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx \right)^{\frac{n-2}{n-1}} |B_{R_2}|^{\frac{1}{n-1}} \\
&\le c \frac{|h|^2}{r_2 - r_1} \int_{B_{R_2}} k^n(x)\,dx + c_\sigma |h|^2 \frac{|B_{R_2}|^{\frac{n}{n-1}}}{(r_2 - r_1)^n} + \sigma |h|^2 \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx \right)^{\frac{n-2}{n}},
\end{aligned} \tag{63}
$$

where we used Hölder's and Young's inequality and $\sigma \in (0,1)$ will be chosen later. Collecting the estimates (58), (60), (61), (63) and recalling identity (57), we deduce the bound

$$\begin{aligned}\int_{B_{r_2}} \zeta^2 |\tau_h V(Du)|^2\,dx \le c\kappa \int_{B_{r_2}} \zeta^2 |\tau_h V(Du)|^2\,dx + \varepsilon c \int_{B_{r_2}} |\tau_h V(Du)|^2\,dx \\ + c_\kappa |h|^2 \left(\int_{B_{R_2}} k^n(x)\,dx\right)^{\frac{2}{n}} \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \\ + \frac{c_{\varepsilon,\kappa}|h|^2}{(r_2-r_1)^q} \int_{B_{R_2}} \phi(|Du|)\,dx + c\frac{|h|^2}{r_2-r_1}\int_{B_{R_2}} k^n(x)\,dx \\ + c_\sigma |h|^2 \frac{|B_{R_2}|^{\frac{n}{n-1}}}{(r_2-r_1)^n} + \sigma |h|^2 \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}},\end{aligned}$$

for every $\kappa, \varepsilon, \sigma \in (0,1)$ and all radii r_1, r_2 with $R_1 \le r_1 < r_2 \le R_2$. Choosing $\kappa = \frac{1}{2c}$, we can reabsorb the first integral in the right hand side of the previous inequality by the left hand side thus getting

$$\begin{aligned}\int_{B_{r_1}} |\tau_h V(Du)|^2\,dx \le \int_{B_{r_2}} \zeta^2 |\tau_h V(Du)|^2\,dx \le c\varepsilon \int_{B_{r_2}} |\tau_h V(Du)|^2\,dx \\ + c|h|^2 \left(\int_{B_{R_2}} k^n(x)\,dx\right)^{\frac{2}{n}} \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \\ + \frac{c_\varepsilon|h|^2}{(r_2-r_1)^2} \int_{B_{R_2}} \phi(|Du|)\,dx + c\frac{|h|^2}{r_2-r_1}\int_{B_{R_2}} k^n(x)\,dx \\ + c_\sigma |h|^2 \frac{|B_{R_2}|^{\frac{n}{n-1}}}{(r_2-r_1)^n} + \sigma |h|^2 \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}}.\end{aligned}$$

Choosing now $\varepsilon = \frac{1}{2c}$, we can use Lemma 2.1 to obtain

$$\begin{aligned}\int_{B_{R_1}} |\tau_h V(Du)|^2\,dx \le c|h|^2 \left(\int_{B_{R_2}} k^n(x)\,dx\right)^{\frac{2}{n}} \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \\ + c\sigma|h|^2 \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} + \frac{c|h|^2}{(R_2-R_1)^q} \int_{B_{R_2}} \phi(|Du|)\,dx \\ + c\frac{|h|^2}{R_2-R_1}\int_{B_{R_2}} k^n(x)\,dx + c_\sigma |h|^2 \frac{|B_{R_2}|^{\frac{n}{n-1}}}{(R_2-R_1)^n}.\end{aligned}$$

Dividing both sides of previous inequality by $|h|^2$ and taking the limit as $h \to 0$, by virtue of the a priori assumption on $V(Du)$, we get

$$\begin{aligned}\int_{B_{R_1}} |DV(Du)|^2\,dx \le c\left[\sigma + \left(\int_{B_{R_2}} k^n(x)\,dx\right)^{\frac{2}{n}}\right] \left(\int_{B_{R_2}} \phi(|Du|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \\ + \frac{c}{(R_2-R_1)^q} \int_{B_{R_2}} \phi(|Du|)\,dx \\ + \frac{c}{R_2-R_1}\int_{B_{R_2}} k^n(x)\,dx + c_\sigma \frac{|B_{R_2}|^{\frac{n}{n-1}}}{(R_2-R_1)^n}\end{aligned} \tag{64}$$

for all radii with $\frac{R}{2} \le R_1 < R_2 \le R$ where the dependencies of the constant are given by $c = c(n, \nu, L, p, q)$.
Let $\lambda > 1$ and $\frac{R}{2} \le r < \lambda r \le R$ and select

$$R_1 = r + \frac{\lambda - 1}{4} r \qquad R_2 = \lambda r - \frac{\lambda-1}{4} r$$

so that

$$\frac{R}{2} \le r < R_1 < R_2 < \lambda r \le R.$$

Consider a cut off function $\eta \in C_0^\infty(B_{\lambda r})$ such that $0 \le \eta \le 1$, $\eta = 1$ on B_{R_2} such that $|\nabla \eta| \le \frac{C}{(\lambda-1)r}$. By the Sobolev imbedding theorem, we have

$$\begin{aligned}\left(\int_{B_{R_2}} |V(Du)|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} &\le \left(\int_{B_{\lambda r}} |\eta V(Du)|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \le c(n)\int_{B_{\lambda r}} |D(\eta V(Du))|^2\,dx \\ &\le c(n)\left(\int_{B_{\lambda r}} \eta^2 |D(V(Du))|^2\,dx + \int_{B_{\lambda r}} |\nabla\eta|^2 |V(Du)|^2\,dx\right).\end{aligned}$$

Recalling that $|V(t)|^2 \sim \phi(t)$ and using previous inequality in (64) we get

$$\begin{aligned}\int_{B_{R_1}} |DV(Du)|^2\,dx \le{}& c\left[\sigma + \left(\int_{B_{R_2}} k^n(x)\,dx\right)^{\frac{2}{n}}\right]\int_{B_{\lambda r}} |D(V(Du))|^2\,dx \\ &+ \frac{c}{(\lambda r - R_2)^2}\left[\sigma + \left(\int_{B_{R_2}} k^n(x)\,dx\right)^{\frac{2}{n}}\right]\int_{B_{\lambda r}} \phi(|Du|)\,dx \\ &+ \frac{c}{(R_2-R_1)^q}\int_{B_{R_2}} \phi(|Du|)\,dx \\ &+ \frac{c}{R_2-R_1}\int_{B_{R_2}} k^n(x)\,dx + c_\sigma \frac{|B_{R_2}|^{\frac{n}{n-1}}}{(R_2-R_1)^n}.\end{aligned} \tag{65}$$

Our choice of R_1, R_2 implies that

$$R_2 - R_1 = \frac{1}{2}(\lambda-1)r \ \text{ and } \ \lambda r - R_2 = \frac{1}{4}(\lambda-1)r$$

and so (65) gives

$$\begin{aligned}\int_{B_r} |DV(Du)|^2\,dx \le{}& c\left[\sigma + \left(\int_{B_R} k^n(x)\,dx\right)^{\frac{2}{n}}\right]\int_{B_{\lambda r}} |D(V(Du))|^2\,dx \\ &+ \frac{c}{(\lambda-1)^2 r^2}\left[\sigma + \left(\int_{B_R} k^n(x)\,dx\right)^{\frac{2}{n}}\right]\int_{B_{\lambda r}} \phi(|Du|)\,dx \\ &+ \frac{c}{(\lambda-1)^q r^q}\int_{B_R} \phi(|Du|)\,dx \\ &+ \frac{c}{(\lambda-1)r}\int_{B_R} k^n(x)\,dx + c_\sigma \frac{|B_{R_2}|^{\frac{n}{n-1}}}{(\lambda-1)^n r^n}.\end{aligned} \tag{66}$$

By the absolute continuity of the integral, we may choose R small enough to have

$$c\left(\int_{B_R} k^n(x)\,dx\right)^{\frac{2}{n}} \le \frac{1}{4}$$

and so, choosing $\sigma \in (0,1)$ such that $c\sigma \le \frac{1}{4}$ we have

$$c\left[\sigma + \left(\int_{B_{R_2}} k^n(x)\,dx\right)^{\frac{2}{n}}\right] \le \frac{1}{2}.$$

Therefore (66) yields

$$\begin{aligned}\int_{B_r} |DV(Du)|^2\,dx \le{}& \frac{1}{2}\int_{B_{\lambda r}} |D(V(Du))|^2\,dx + \frac{c}{(\lambda-1)^n r^n} \\ &+ \frac{c}{(\lambda-1)^2 r^2}\int_{B_R} \phi(|Du|)\,dx + \frac{c}{(\lambda-1)^q r^q}\int_{B_R} \phi(|Du|)\,dx.\end{aligned}$$

Since previous estimate holds true for every $\frac{R}{2} \le r \le \lambda r \le R$, and for every $\lambda \in (1,2)$ we can use again Lemma 2.1 to conclude

$$\int_{B_{\frac{R}{2}}} |D(V(Du))|^2\,dx \le \frac{c}{R^q}\int_{B_R} \phi(|Du|)\,dx + \frac{c}{R^n},$$

where we also used that $R \le 1$. □

4 Proof of Theorem 1.1

For a function $\rho \in C_c^\infty(B_1(0))$, $\rho \geq 0$, such that $\int_{B_1(0)} \rho \, dx = 1$, let ρ_ε be the corresponding family of mollifiers and set

$$a_\varepsilon(x,\xi) = \int_{B_1(0)} \rho(\omega) a(x+\varepsilon\omega, \xi)\, d\omega.$$

Let $u \in W^{1,\phi}(\Omega, \mathbb{R}^n)$ be a weak solution to equation (7), fix a ball $B_R \Subset \Omega$ and let $v_\varepsilon \in u + W_0^{1,\phi}(B_R, \mathbb{R}^n)$ be the unique solution to the Dirichlet problem

$$\begin{cases} \operatorname{div} a_\varepsilon(x, Dv_\varepsilon) = 0 & \text{in } B_R \\ v_\varepsilon = u & \text{on } \partial B_R. \end{cases} \tag{67}$$

Note that assumptions (3)–(5) imply the corresponding conditions

$$\langle a_\varepsilon(x,\xi) - a_\varepsilon(x,\eta), \xi - \eta \rangle \geq \nu \phi''(|\xi| + |\eta|)|\xi - \eta|^2 \tag{68}$$

$$|a_\varepsilon(x,\xi) - a_\varepsilon(x,\eta)| \leq L\phi''(|\xi| + |\eta|)|\xi - \eta| \tag{69}$$

$$|a_\varepsilon(x,\xi)| \leq L\phi'(|\xi|), \tag{70}$$

while (6) yields

$$|a_\varepsilon(x,\xi) - a_\varepsilon(x,\eta)| \leq |x-y|(k_\varepsilon(x) + k_\varepsilon(y))\phi'(|\xi|) \tag{71}$$

where

$$k_\varepsilon = k * \rho_\varepsilon.$$

Theorem 3.1 implies that $V(Dv_\varepsilon) \in W^{1,2}_{\mathrm{loc}}(B_R)$ and so we can use estimate in Theorem 3.2 to deduce that

$$\int_{B_{\frac{r}{2}}} |D(V(Dv_\varepsilon))|^2\, dx \leq \frac{c}{r^q} \int_{B_r} \phi(|Dv_\varepsilon|)\, dx + \frac{c}{r^n} \tag{72}$$

$$\left(\int_{B_{\frac{r}{2}}} \phi(|Dv_\varepsilon|)^{\frac{n}{n-2}}\, dx \right)^{\frac{n-2}{n}} \leq \frac{c}{r^q} \int_{B_r} \phi(|Dv_\varepsilon|)\, dx + \frac{c}{r^n} \tag{73}$$

for every $r < R$ and where, since $k_\varepsilon \to k$ strongly in L^n, both the constant and the radius r can be chosen independently of ε. By assumption (69) and since v_ε solves problem (67) we get

$$\begin{aligned}
&\nu \int_{B_R} |Dv_\varepsilon - Du|^2 \phi''(|Dv_\varepsilon| + |Du|)\, dx \leq \int_\Omega \langle a_\varepsilon(x, Dv_\varepsilon) - a_\varepsilon(x, Du), Dv_\varepsilon - Du \rangle\, dx \\
= &\int_{B_R} \langle a_\varepsilon(x, Dv_\varepsilon), Dv_\varepsilon - Du \rangle\, dx - \int_\Omega \langle a_\varepsilon(x, Du), Dv_\varepsilon - Du \rangle\, dx \\
= & -\int_{B_R} \langle a_\varepsilon(x, Du), Dv_\varepsilon - Du \rangle\, dx \\
\leq &\int_{B_R} |a_\varepsilon(x, Du)||Dv_\varepsilon - Du|\, dx \leq \int_{B_R} \phi'(|Du|)|Dv_\varepsilon - Du|\, dx \\
\leq &\ \varepsilon \int_{B_R} \phi_{|Du|}(|Dv_\varepsilon - Du|)\, dx + c_\varepsilon \int_{B_R} (\phi_{|Du|})^*(\phi'(|Du|))\, dx \\
\leq &\ \varepsilon \int_{B_R} \phi_{|Du|}(|Dv_\varepsilon - Du|)\, dx + c_\varepsilon \int_{B_R} \phi(|Du|)\, dx,
\end{aligned}$$

where we used (70), Young's inequality (27) and (37). By the second equivalence of Lemma 2.4, previous estimate implies

$$\begin{aligned}
\nu \int_{B_R} \phi_{|Du|}(|Dv_\varepsilon - Du|)\, dx &\leq \nu \int_{B_R} |Dv_\varepsilon - Du|^2 \phi''(|Dv_\varepsilon| + |Du|)\, dx \\
&\leq \varepsilon \int_{B_R} \phi_{|Du|}(|Dv_\varepsilon - Du|)\, dx + c_\varepsilon \int_{B_R} \phi(|Du|)\, dx
\end{aligned}$$

and so, choosing ε sufficiciently small, also

$$\int_{B_R} \phi_{|Du|}(|Dv_\varepsilon - Du|)\, dx \leq c \int_{B_R} \phi(|Du|)\, dx$$

that, by virtue of (34), implies

$$\int_{B_R} \phi(|Dv_\varepsilon|)\,dx \quad \le \quad c\int_{B_R} \phi(|Du|)\,dx. \tag{74}$$

Since assumption (2) guaratees the reflexivity of the space $W^{1,\phi}(B_R)$, from previous estimate we deduce that there exists $v \in W^{1,\phi}(B_R)$ such that

$$v_\varepsilon \rightharpoonup v \quad \text{weakly in} \quad W^{1,\phi}(B_R).$$

Inserting (74) in (72), we get

$$\int_{B_{\frac{r}{2}}} |D(V(Dv_\varepsilon))|^2\,dx \le \frac{c}{r^q}\int_{B_R} \phi(|Du|)\,dx + \frac{c}{r^n} \tag{75}$$

i.e. the sequence $V(Dv_\varepsilon)$ is bounded in $W^{1,2}_{\text{loc}}(B_R)$. Therefore there exists $w \in W^{1,2}_{\text{loc}}(B_R)$ such that

$$V(Dv_\varepsilon) \rightharpoonup w \quad \text{weakly in} \quad W^{1,2}_{\text{loc}}(B_R)$$

and

$$V(Dv_\varepsilon) \to w \quad \text{strongly in} \quad L^2_{\text{loc}}(B_R).$$

The continuity of the operator $V(\xi)$ together with the uniqueness of the weak limit implies that $w = V(Dv)$ and so that

$$V(Dv_\varepsilon) \rightharpoonup V(Dv) \quad \text{weakly in} \quad W^{1,2}_{\text{loc}}(B_R)$$

and

$$V(Dv_\varepsilon) \to V(Dv) \quad \text{strongly in} \quad L^2_{\text{loc}}(B_R)$$

and, up to a subsequence, also

$$V(Dv_\varepsilon) \to V(Dv) \quad \text{a.e. in} \quad B_r, \qquad r < R.$$

Using Fatou's Lemma in the left hand side of (75), we obtain

$$\int_{B_{\frac{r}{2}}} |D(V(Dv))|^2\,dx \le \frac{c}{r^q}\int_{B_R} \phi(|Du|)\,dx + \frac{c}{r^n}. \tag{76}$$

We are left to prove that $v \equiv u$. To this aim, we first observe that v solves (1). Indeed, for every $\varphi \in C^1_c(B_r)$ where $B_r \subset B_R$, we have

$$\begin{aligned}
&\int_{B_R} \langle a(x,Dv), D\varphi\rangle\,dx \\
= &\int_{B_R} \langle a(x,Dv) - a_\varepsilon(x,Dv), D\varphi\rangle\,dx + \int_{B_R} \langle a_\varepsilon(x,Dv) - a_\varepsilon(x,Dv_\varepsilon), D\varphi\rangle\,dx \\
&+ \int_{B_R} \langle a_\varepsilon(x,Dv_\varepsilon), D\varphi\rangle\,dx \\
= &\int_{B_R} \langle a(x,Dv) - a_\varepsilon(x,Dv), D\varphi\rangle\,dx + \int_{B_R} \langle a_\varepsilon(x,Dv) - a_\varepsilon(x,Dv_\varepsilon), D\varphi\rangle\,dx,
\end{aligned}$$

where we used that v_ε solves problem (67). Therefore, we are left to prove that the right hand side of previous equality vanishes as $\varepsilon \to 0$. This will come if we show that

$$\lim_{\varepsilon\to 0} I_1^\varepsilon := \lim_{\varepsilon\to 0}\left|\int_{B_R} \langle a(x,Dv) - a_\varepsilon(x,Dv), D\varphi\rangle\,dx\right| = 0$$

$$\lim_{\varepsilon\to 0} I_2^\varepsilon := \lim_{\varepsilon\to 0}\left|\int_{B_R} \langle a_\varepsilon(x,Dv) - a_\varepsilon(x,Dv_\varepsilon), D\varphi\rangle\,dx\right| = 0.$$

For what concerns I_1^ε we have by the definition of a_ε and by (71) that

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_1^\varepsilon \quad &\le \quad \lim_{\varepsilon\to 0} ||D\varphi||_{L^\infty(B_r)} \int_{B_r} |a(x,Dv) - a_\varepsilon(x,Dv)|\,dx \\
&\le \quad \lim_{\varepsilon\to 0} ||D\varphi||_{L^\infty(B_r)} \int_{B_r} \left| a(x,Dv) - \int_{B_1(0)} \rho(\omega) a(x+\varepsilon\omega, Dv)\,d\omega \right| dx \\
&\le \quad \lim_{\varepsilon\to 0} ||D\varphi||_{L^\infty(B_r)} \int_{B_r}\int_{B_1(0)} \rho(\omega)\left| a(x,Dv) - a(x+\varepsilon\omega, Dv)\right| d\omega\,dx
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{\varepsilon\to 0}\varepsilon\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}\int_{B_1(0)}\rho(\omega)\Big|(k(x)+k(x+\varepsilon\omega))\phi'(|Dv|)\Big|\,d\omega\,dx\\
&\le \lim_{\varepsilon\to 0}\varepsilon\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}(k(x)+k_\varepsilon(x))\phi'(|Dv|)\,dx\\
&\le \lim_{\varepsilon\to 0}\varepsilon\|D\varphi\|_{L^\infty(B_r)}\left(\int_{B_r}(k(x)+k_\varepsilon(x))^n\right)^{\frac{1}{n}}\left(\int_{B_r}\phi'(1+|Dv|)^{\frac{n}{n-1}}\,dx\right)^{\frac{n-1}{n}}\\
&\le c\lim_{\varepsilon\to 0}\varepsilon\|D\varphi\|_{L^\infty(B_r)}\left(\int_{B_r}(k(x)+k_\varepsilon(x))^n\right)^{\frac{1}{n}}\left(\int_{B_r}\phi(1+|Dv|)^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \qquad (77)
\end{aligned}
$$

where we used that $\phi'(t)$ is increasing and that

$$\phi'(1+t)\sim\frac{\phi(1+t)}{1+t}\le\phi(1+t).$$

Now observing that, by the Sobolev imbedding Theorem, (76) implies that

$$\phi(|Dv|)^{\frac{n}{n-2}}\sim V(Dv)^{\frac{2n}{n-2}}\in L^1_{\mathrm{loc}}(B_R)$$

and since $k_\varepsilon\to k$ strongly in $L^n(\Omega)$, we deduce that the right hand side of (77) goes to zero. For I_2^ε, using the definition of a_ε and (69), we observe that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}I_2^\varepsilon &\le \lim_{\varepsilon\to 0}\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}|a_\varepsilon(x,Dv)-a_\varepsilon(x,Dv_\varepsilon)|\,dx\\
&\le \lim_{\varepsilon\to 0}\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}\int_{B_1(0)}\rho(\omega)\Big|a(x+\varepsilon\omega,Dv)-a(x+\varepsilon\omega,Dv_\varepsilon)\Big|\,d\omega\,dx\\
&\le \lim_{\varepsilon\to 0}\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}|Dv-Dv_\varepsilon|\phi''(|Dv|+|Dv_\varepsilon|)\,dx\\
&\le \lim_{\varepsilon\to 0}\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}\phi'_{|Dv|}(|Dv-Dv_\varepsilon|)\,dx\\
&\le \lim_{\varepsilon\to 0}c_\delta\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}(\phi_{|Dv|})^*\left(\phi'_{|Dv|}(|Dv-Dv_\varepsilon|)\right)dx+\delta\int_{B_r}\phi_{|Dv|}(1)\,dx\\
&\le \lim_{\varepsilon\to 0}c_\delta\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}\phi_{|Dv|}(|Dv-Dv_\varepsilon|)\,dx+\delta\int_{B_r}\phi_{|Dv|}(1)\,dx\\
&\le \lim_{\varepsilon\to 0}c_\delta\|D\varphi\|_{L^\infty(B_r)}\int_{B_r}|V(Dv)-V(Dv_\varepsilon)|^2\,dx+\delta\int_{B_r}\phi(|Dv|)\,dx,
\end{aligned}
$$

where we used Lemma 2.5 , Young's inequality at (27) and Lemma 2.6. Since $V(Dv_\varepsilon)\to V(Dv)$ strongly in $L^2_{\mathrm{loc}}(B_R)$, we get

$$\lim_{\varepsilon\to 0}I_2^\varepsilon\le\delta\int_{B_r}\phi(|Dv|)\,dx$$

and then $\lim_{\varepsilon\to 0}I_2^\varepsilon=0$ follows letting $\delta\to 0$. Therefore we obtain

$$\begin{cases}\operatorname{div}a(x,Dv)=0 & \text{in } B_R\\[2ex] v=u & \text{on } \partial B_R\end{cases}$$

and then the conclusion follows observing that $v=u$ a.e. in B_R, since the solution of previous problem is unique.

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