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Recurrence Asymptotics in a Family of Matrix Semi-Classical Laguerre Polynomials

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Abstract

In this paper, we present a new three-parameter semi-classical Laguerre matrix weight function and explore the associated sequence of matrix orthogonal polynomials. We derive explicit expressions for the corresponding three-term recurrence relations, both in terms of their scalar counterparts and scalar Hankel determinants. Finally, we analyze the asymptotic behavior of the recurrence coefficients with respect to the variables t and n.

1 Introduction

One of the most important properties of orthogonal polynomials is the three-term recurrence relation [1, 2, 3]. Let W be a matrix weight function defined on the real line for which all the moments

$$w_n = \int x^n W(x)\,\mathrm{d}x$$

exist. Here and throughout, the integral is understood entrywise, so that w_n is itself a matrix given by

$$w_n = \left[\int x^n W_{ij}(x)\,\mathrm{d}x\right]_{i,j}.$$

It is well known [4, subsection 2.1] that: if $\det \eth_n \neq 0$, $n \in \mathbb{N}$, where $\eth_n = (w_{k+j})_{k,j=0}^n$ is the Hankel-block matrix

$$\eth_n = \begin{bmatrix} w_0 & w_1 & \cdots & w_n \\ w_1 & w_2 & \cdots & w_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ w_n & w_{n+1} & \cdots & w_{2n} \end{bmatrix}, \quad n \in \mathbb{N},$$

then there exist two sequences $\left\{P_n^{\mathsf{L}}(x)\right\}_{n\in\mathbb{N}}$ and $\left\{P_n^{\mathsf{R}}(x)\right\}_{n\in\mathbb{N}}$ of, respectively, left and right orthonormal polynomials such that

$$\int P_n^{\mathsf{L}}(x)W(x)x^m\mathrm{d}x = \delta_{nm}H_n^{-1}, \qquad \text{left orthogonality,}$$
$$\int x^m W(x)P_n^{\mathsf{R}}(x)\mathrm{d}x = \delta_{nm}H_n^{-1}, \qquad \text{right orthogonality,}$$

for $m = 0, 1, \ldots, n$ and $n \in \mathbb{N}$, where H_n is, for each $n \in \mathbb{N}$, a nonsingular matrix and δ_{nm} is the Kronecker delta. The left and right three-term recurrence relations then take the following forms:

$$xP_n^{\mathsf{L}} = \alpha_n^{\mathsf{L}}P_{n+1}^{\mathsf{L}} + \beta_n^{\mathsf{L}}P_n^{\mathsf{L}} + \gamma_n^{\mathsf{L}}P_{n-1}^{\mathsf{L}},$$
$$xP_n^{\mathsf{R}} = P_{n+1}^{\mathsf{R}}\alpha_n^{\mathsf{R}} + P_n^{\mathsf{R}}\beta_n^{\mathsf{R}} + P_{n-1}^{\mathsf{R}}\gamma_n^{\mathsf{R}},$$

with initial conditions $P_{-1}^{\mathsf{L}} = P_{-1}^{\mathsf{R}} = \mathbf{0}$ and $P_0^{\mathsf{L}} = P_0^{\mathsf{R}} = \mathbf{I}$. In particular, when W is symmetric, then $P_n^{\mathsf{R}} = (P_n^{\mathsf{L}})^T$ [5].

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In [6, 8, 7, 9], orthogonal polynomials associated with the semi-classical scalar Laguerre weight function

$$W^{\lambda}(x,t) = x^{\lambda} e^{-x^2+tx}, \qquad x \in \mathbb{R}^+, \tag{1}$$

on the half-line $(0,+\infty)$, with parameters $\lambda > -1$ and $t \in \mathbb{R}$, were studied. These papers investigate the recurrence coefficients of semi-classical Laguerre polynomials and reveal deep connections to integrable systems, particularly to the Painlevé equations. They establish that these recurrence coefficients satisfy fourth Painlevé and discrete Painlevé equations, linking the structure of these polynomials to rich mathematical frameworks such as Wronskian representations, Chazy systems, Dyson's Coulomb fluid for large n asymptotics and large t asymptotics. The studies collectively bridge orthogonal polynomial theory with nonlinear integrable systems, emphasizing the role of semi-classical weights in revealing hidden structures in recurrence relations.

As the theory progressed, researchers extended the scalar theory of orthogonal polynomials to the matrix one. In that way a comprehensive theory of matrix-valued orthogonal polynomials has been developed from different perspectives and found applications in several areas of mathematics and mathematical physics. One remarkable outcome of this development was the appearance of Painlevé equations. They arise naturally in the study of semi-classical weights. In [10], the authors focus on semiclassical Laguerre-type matrix weights and investigate the recurrence coefficients of the associated matrix orthogonal polynomials. They show that these coefficients satisfy integrable systems such as the Toda lattice and Painlevé III equations, highlighting the rich integrable structure inherent to the matrix setting. Building upon this foundation, [4] develops a Riemann-Hilbert framework for matrix Laguerre biorthogonal polynomials and derives nonlinear matrix equations for their recurrence coefficients, resulting in a matrix extension of the discrete Painlevé IV equation. These advances inspire us to study the recurrence coefficients from a different perspective, specifically focusing on their asymptotic behavior.

Motivated by the construction of new families of matrix orthogonal polynomials of Hermite, Laguerre, and Jacobi type [12, 11], as well as by the analysis of the three-term recurrence relations associated with semi-classical scalar weights [9, 8], we propose a new family of semi-classical matrix Laguerre weight. Building on these foundations, we investigate the corresponding three-term recurrence relations for the associated matrix orthogonal polynomials.

The applications of orthogonal polynomials are numerous and growing, spanning both the use of polynomials of fixed degree and the asymptotic analysis of the n-th orthogonal polynomial, p_n, as $n \to \infty$. In the scalar setting, a thorough overview of asymptotic results and their applications is presented by Lubinsky [13]. This theory has been powerfully extended to the matrix case. The asymptotic behavior of matrix orthogonal polynomials, which is the focus of this work, and its applications are explored in depth in the literature, see for instance [18, 14, 15, 17, 20, 19, 16, 21].

This paper is organized as follows. In Section 2, we review some key results from the literature that are used throughout the paper. In Section 3, we introduce a semi-classical Laguerre matrix weight function along with the associated sequence of matrix orthogonal polynomials. Additionally, we derive an explicit expression for the corresponding three-term recurrence relation in terms of scalar quantities. Sections 4 and 5 are dedicated to the study of the asymptotic behavior of the three-term recurrence relation, with respect to the variables t and n, respectively.

2 Scalar semi-classical Laguerre Orthogonal Polynomials

In this section, we recall some known results on scalar semi-classical Laguerre polynomials available in the literature.

We consider the monic orthogonal polynomials $p_n(x,t)$, for $n \in \mathbb{N}$, with respect to the semi-classical Laguerre weight (1). These polynomials satisfy the three term recurrence relation.

$$x p_n(x,t) = p_{n+1}(x,t) + a_n(t) p_n(x,t) + b_n(t) p_{n-1}(x,t). \tag{2}$$

The following properties of scalar semi-classical Laguerre polynomials, as presented in [8], will be required.

Clarkson and Jordaan [8, Theorem 4.6] proved the following result.

Theorem 2.1. *The sequence of monic polynomials $\{p_n(x,t)\}_{n\in\mathbb{N}}$ are orthogonal with respect to the semi-classical Laguerre weight (1) such that*

$$\int_0^{+\infty} p_n(x,t) p_m(x,t) W^{\lambda}(x,t)\, \mathrm{d}x = \delta_{nm}\, \mu_0^{\lambda}(t) \prod_{k=1}^{n} b_k(t), \tag{3}$$

where the zero-order moment $\mu_0^{\lambda}(t)$ is given by

$$\mu_0^{\lambda}(t) = \begin{cases} \frac{\Gamma(\lambda+1)\exp\left(\frac{1}{8}t^2\right)}{2^{(\lambda+1)/2}} D_{-\lambda-1}\left(-\frac{1}{2}\sqrt{2}t\right) & \text{if } \lambda \notin \mathbb{N} \\ \frac{1}{2}\sqrt{\pi}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\left\{\exp\left(\frac{1}{4}t^2\right)\left[1+\operatorname{erf}\left(\frac{1}{2}t\right)\right]\right\} & \text{if } \lambda = m \in \mathbb{N} \end{cases}$$

with $D_{\nu}(\zeta)$ the parabolic cylinder function and $\operatorname{erf}(z)$ *the Gauss error function [22].*

For convenience, we denote the constant of orthogonality by:

$$N_n(t) = \int_0^{+\infty} (p_n(x,t))^2 \, W^{\lambda}(x,t) \, \mathrm{d}x = \mu_0^{\lambda}(t) \prod_{k=1}^{n} b_k(t).$$

Clarkson and Jordaan [8, Theorem 4.9] further proved the following theorem.

Theorem 2.2. *The recurrence coefficients $a_n(t)$ and $b_n(t)$ in (2), associated with monic orthogonal polynomials with respect to the semi-classical Laguerre weight (1) are given by*

$$a_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\Delta_{n+1}(t)}{\Delta_n(t)}, \qquad b_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln \Delta_n(t),$$

when $\Delta_n(t)$ is the Hankel determinant given by

$$\Delta_n(t) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \quad n \in \mathbb{N},$$

where μ_n is the moment of order n. See [8, Theorem 2.1] for more details.

These explicit expressions for scalar coefficients provide a foundation for deriving recursion coefficients in the matrix case. Moreover, from [8, Lemma 5.3], we have the following asymptotic expansions As $t \longrightarrow \infty$:

$$\begin{aligned} a_n(t) &= \frac{t}{2} + \frac{\lambda}{t} + \frac{2\lambda(2n-\lambda+1)}{t^3} + \mathcal{O}\left(t^{-5}\right) \\ b_n(t) &= \frac{n}{2} - \frac{n\lambda}{t^2} - \frac{6n\lambda(n-\lambda)}{t^4} + \mathcal{O}\left(t^{-6}\right). \end{aligned} \tag{4}$$

Finally, from [9, Theorem 1.6]

$$\begin{aligned} a_n(t) &= \sqrt{\frac{2n}{3}} + \frac{t}{6} + \frac{t^2+12(1+\lambda)}{24\sqrt{6n}} - \frac{t^4+24t^2(1+\lambda)-48\left(6\lambda^2-6\lambda-5\right)}{2304\sqrt{6}n^{3/2}} \\ &+ \frac{t\left(9\lambda^2-2\right)}{144n^2} + \frac{t^6+36t^4(1+\lambda)+144t^2\left(66\lambda^2+6\lambda-13\right)-1728\left(8\lambda^3+6\lambda^2-5\lambda-3\right)}{110592\sqrt{6}n^{5/2}} \\ &+ \frac{t\left[t^2\left(27\lambda^2-7\right)-12\left(9\lambda^3+9\lambda^2-2\lambda-2\right)\right]}{1728n^3} + \mathcal{O}\left(n^{-7/2}\right), \\ b_n(t) &= \frac{n}{6} + \frac{t\sqrt{n}}{6\sqrt{6}} + \frac{t^2+6\lambda}{72} + \frac{t\left(t^2+12\lambda\right)}{288\sqrt{6n}} + \frac{2-9\lambda^2}{144n} - \frac{t\left(t^4+24\lambda t^2+3168\lambda^2-816\right)}{27648\sqrt{6}n^{3/2}} \\ &+ \frac{t^2\left(7-27\lambda^2\right)+4\lambda\left(9\lambda^2-2\right)}{1152n^2} + \frac{t\left[t^6+36\lambda t^4-144t^2\left(246\lambda^2-61\right)+1728\lambda\left(64\lambda^2-17\right)\right]}{1327104\sqrt{6}n^{5/2}} + \mathcal{O}\left(n^{-3}\right). \end{aligned} \tag{5}$$

These expansions will be instrumental in deriving the asymptotic results presented in the final sections.

3 Matrix Semi-Classical Laguerre Weight

Inspired by the work of Durán [12, 11], we introduce the following new non-scalar weight:

$$W_a^{\lambda}(x,t) = x^{\lambda} e^{-x^2+tx} \begin{bmatrix} 1+a^2x^2 & ax \\ ax & 1 \end{bmatrix}, \quad a \in \mathbb{R}. \tag{6}$$

In this section, we present a new family of matrix polynomials and establish their orthogonality with respect to the matrix weight (6) (see Theorem 3.1). We also derive explicit expressions for the recursion coefficients in terms of a scalar Hankel determinant (see Theorem 3.2), as well as for the matrix recursion coefficients in terms of the corresponding scalar ones (see Corollary 3.3).

3.1 Matrix semi-classical Laguerre orthogonal polynomials

We propose the following family of matrix polynomials, $\{\mathbb{P}_n(x,t)\}_{n\in\mathbb{N}}$ of semi-classical Laguerre type constructed using scalar semi-classical Laguerre polynomials $\{p_n(x,t)\}_{n\in\mathbb{N}}$:

$$\mathbb{P}_n(x,t) = \begin{bmatrix} p_n(x,t) & -a a_n(t) p_n(x,t) - a b_n(t) p_{n-1}(x,t) \\ -a\, b_n(t) p_{n-1}(x,t) & (a^2 b_n(t)+1) p_n(x,t) + a^2 a_{n-1}(t) b_n(t) p_{n-1}(x,t) + a^2 b_{n-1}(t) b_n(t) p_{n-2}(x,t) \end{bmatrix}, \quad n \geq 0 \tag{7}$$

where $a_n(t)$, $b_n(t)$ are defined in Theorem 2.2 and $a \in \mathbb{R}$. Observe that $\deg(\mathbb{P}_n) = n$.

3.1.1 Orthogonality relation

Theorem 3.1. *For $n \in \mathbb{N}$, the matrix polynomials $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ already introduced in (7) are orthogonal on $(0,+\infty)$ with respect to the matrix weight function $W_a^{\lambda}(x,t)$ (6), such that:*

$$\int_0^{+\infty} \mathbb{P}_n(x,t)\, W_a^{\lambda}(x,t)\, (\mathbb{P}_m(x,t))^T \,\mathrm{d}x = \delta_{nm} H_n^{-1}(t),$$

where the matrix $H_n^{-1}(t)$ is given by:

$$H_n^{-1}(t) = \begin{bmatrix} \mu_0^{\lambda}(t)\left(1+a^2 b_{n+1}(t)\right)\prod_{k=1}^{n} b_k(t) & 0 \\ 0 & (1+a b_n(t))\,\mu_0^{\lambda}(t)\prod_{k=1}^{n} b_k(t) \end{bmatrix}.$$

Proof. The proof follows from the orthogonality properties of the scalar Laguerre polynomials and the structure of $\mathbb{P}_n$. In fact,

$$\mathbb{P}_n(x,t)\, W_a^{\lambda}(x,t)\, (\mathbb{P}_m(x,t))^T = x^{\lambda} e^{tx-x^2} \begin{bmatrix} p_n(x,t)p_m(x,t) + a^2 p_{n+1}(x,t)p_{m+1}(x,t) & -a b_m(t) p_{m-1}(x,t)p_n(x,t) + a p_m(x,t)p_{n+1}(x,t) \\ -a\, b_n(t)p_{n-1}(x,t)p_m(x,t) + a p_n(x,t)p_{m+1}(x,t) & p_n(x,t)p_m(x,t) + a^2 b_n(t) b_m(t) p_{m-1}(x,t)p_{n-1}(x,t) \end{bmatrix} \quad (8)$$

For $m \notin \{n-1,\, n,\, n+1\}$, by relation (3), it is clear that

$$\int_0^{+\infty} \mathbb{P}_n(x,t)\, W_a^{\lambda}(x,t)\, (\mathbb{P}_m(x,t))^T \,\mathrm{d}x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For $m = n$, we obtain

$$\int_0^{+\infty} \mathbb{P}_n(x,t)\, W_a^{\lambda}(x,t)\, (\mathbb{P}_n(x,t))^T \,\mathrm{d}x = \begin{bmatrix} N_n(t) + a^2 N_{n+1}(t) & 0 \\ 0 & N_n(t) + a^2 (b_n(t))^2 N_{n-1}(t) \end{bmatrix}.$$

Then, we replace $N_n(t)$ from Theorem 2.1.

For $m = n+1$, we find that the $(1,1)$, $(2,2)$ and $(1,2)$ entries of (8) are zero. The nullity of the $(2,1)$ entry follows from the identity:

$$N_n(t) = b_n(t) N_{n-1}(t).$$

In fact,

$$\begin{aligned} b_n(t)N_{n-1}(t) &= b_n(t)\int_0^{+\infty} (P_{n-1}(x,t))^2\, W_a^{\lambda}(x,t)\,\mathrm{d}x \\ &= \int_0^{+\infty} \Big(P_{n+1}(x,t) + a_n(t)P_n(x,t) + b_n(t)P_{n-1}(x,t)\Big) W_a^{\lambda}(x,t)\, P_{n-1}(x,t)\,\mathrm{d}x \\ &= \int_0^{+\infty} x P_n(x,t)\, W_a^{\lambda}(x,t)\, P_{n-1}(x,t)\,\mathrm{d}x \\ &= \int_0^{+\infty} p_n(x,t)\, W_a^{\lambda}(x,t) \big(x P_{n-1}(x,t)\big)\,\mathrm{d}x \\ &= \int_0^{+\infty} P_n(x,t)\, W_a^{\lambda}(x,t) \Big(P_n(x,t) + a_{n-1}(t)P_{n-1}(x,t) + b_{n-1}(t)P_{n-2}(x,t)\Big)\,\mathrm{d}x = N_n(t). \end{aligned}$$

Finally, for $m = n-1$, a similar argument leads to:

$$\int_0^{+\infty} \mathbb{P}_n(x,t)\, W_a^{\lambda}(x,t)\, (\mathbb{P}_{n-1}(x,t))^T \,\mathrm{d}x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

□

3.1.2 Three-term Recursion Relation

Our main objective is to obtain explicit expressions for the coefficients in the recurrence relation satisfied by the semi-classical Laguerre matrix polynomials.

Theorem 3.2. *The sequence of matrix Laguerre semi-classical orthogonal polynomials $\left\{\mathbb{P}_n(x,t)\right\}_{n\in\mathbb{N}}$ defined in*(7) *satisfies the following matrix three terms recurrence relation,*

$$x\mathbb{P}_n(x,t)=\alpha_n(t)\mathbb{P}_{n+1}(x,t)+\beta_n(t)\mathbb{P}_n(x,t)+\gamma_n(t)\mathbb{P}_{n-1}(x,t),$$

where

$$\alpha_n(t)=\begin{bmatrix} 1 & a\dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+2}(t)}{\Delta_{n+1}(t)}}{a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_{n+1}(t)+1} \\ 0 & \dfrac{a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)+1}{a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_{n+1}(t)+1} \end{bmatrix},$$

$$\beta_n(t)=\begin{bmatrix} \dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)}+a^2\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+2}(t)}{\Delta_{n+1}(t)}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_{n+1}(t)}{1+a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_{n+1}(t)} & -a\dfrac{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\frac{\Delta_n(t)}{\Delta_{n+1}(t)}}{1+a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)} \\ -a\dfrac{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\frac{\Delta_n(t)}{\Delta_{n+1}(t)}}{1+a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_{n+1}(t)} & \dfrac{a_n+a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)a_{n-1}}{1+a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)} \end{bmatrix},$$

$$\gamma_n(t)=\begin{bmatrix} \dfrac{1+a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_{n+1}(t)}{a^2+\frac{1}{\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)}} & 0 \\ \dfrac{a\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_{n+2}(t)}\right)}{1+a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)} & \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t) \end{bmatrix}.$$

when $\Delta_n(t)$ is defined in Theorem 2.2.

Proof. from Theorem 3.1, we deduce the recursion relation:

$$x\mathbb{P}_n(x,t)=\alpha_n(t)\mathbb{P}_{n+1}(x,t)+\beta_n(t)\mathbb{P}_n(x,t)+\gamma_n(t)\mathbb{P}_{n-1}(x,t),$$

where

$$\alpha_n(t)=\int_0^{+\infty} x\mathbb{P}_n(x,t)\,W_a^{\lambda}(x,t)\,\left(\mathbb{P}_{n+1}(x,t)\right)^T\,\mathrm{d}x\,H_{n+1},$$
$$\beta_n(t)=\int_0^{+\infty} x\mathbb{P}_n(x,t)\,W_a^{\lambda}(x,t)\,\left(\mathbb{P}_{n}(x,t)\right)^T\,\mathrm{d}x\,H_{n},$$
$$\gamma_n(t)=\int_0^{+\infty} x\mathbb{P}_n(x,t)\,W_a^{\lambda}(x,t)\,\left(\mathbb{P}_{n-1}(x,t)\right)^T\,\mathrm{d}x\,H_{n-1}.$$

Now, a straightforward calculation leads to

$$\alpha_n(t)=\begin{bmatrix} 1 & a\frac{a_{n+1}(t)-a_n(t)}{1+a^2b_{n+1}(t)} \\ 0 & \frac{a^2b_n(t)+1}{1+a^2b_{n+1}(t)} \end{bmatrix},\quad \beta_n(t)=\begin{bmatrix} \frac{a_n(t)+a^2a_{n+1}(t)b_{n+1}(t)}{1+a^2b_{n+1}(t)} & -a\frac{b_n(t)-b_{n+1}(t)}{1+a^2b_n(t)} \\ -a\frac{b_n(t)-b_{n+1}(t)}{1+a^2b_{n+1}(t)} & \frac{a_n(t)+a^2b_n(t)a_{n-1}(t)}{1+a^2b_n(t)} \end{bmatrix},\quad \gamma_n(t)=\begin{bmatrix} b_n(t)\frac{1+a^2b_{n+1}(t)}{1+a^2b_n(t)} & 0 \\ a\,b_n(t)\frac{a_n(t)-a_{n-1}(t)}{1+a^2b_n(t)} & b_n(t) \end{bmatrix}.$$

Considering the expressions

$$a_n(t)=\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)},\qquad b_n(t)=\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t),$$

the result follows. □

Remark 1. The sequence $\{\mathbb{P}_n(x,t)\}_{n\in\mathbb{N}}$ isn't monic, from it's expression (7) we get the leading coefficient:

$$A_n(t)=\begin{bmatrix} 1 & -a\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)} \\ 0 & a^2\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t)+1 \end{bmatrix}.$$

Hence, we can define the monic semi-classical Laguerre matrix polynomials $\{\mathcal{P}_n(x,t)\}_{n\in\mathbb{N}}$ such that:

$$\mathcal{P}_n(x,t)=\left(A_n(t)\right)^{-1}\,\mathbb{P}_n(x,t).$$

This new sequence satisfies a three terms recurence relation

$$x\mathcal{P}_n(x,t)=\mathcal{P}_{n+1}(x,t)+\tilde{\beta}_n(t)\mathcal{P}_n(x,t)+\tilde{\gamma}_n(t)\mathcal{P}_{n-1}(x,t),$$

where $\tilde{\beta}_n(t)=\left(A_n(t)\right)^{-1}\beta_n(t)A_n(t)$ and $\tilde{\gamma}_n(t)=\left(A_n(t)\right)^{-1}\gamma_n(t)A_{n-1}(t)$.

The sequence of matrix orthogonal polynomials can be expressed in terms of scalar semi-classical Laguerre polynomials, which facilitates the derivation of certain results.

3.1.3 Representation in Terms of Scalar Polynomials

The matrix orthogonal polynomials $\{\mathbb{P}_n(x,t)\}_{n\in\mathbb{N}}$ could be expressed as a linear combination of scalar ones such that:

$$\mathbb{P}_n(x,t) = A_n(t)\,p_n(x,t) + B_n(t)\,p_{n-1}(x,t) + C_n(t)\,p_{n-2}(x,t), \qquad n \in \mathbb{N},$$

where

$$A_n(t) = \begin{bmatrix} 1 & -a a_n(t) \\ 0 & a^2 b_n(t)+1 \end{bmatrix}, \qquad B_n(t) = \begin{bmatrix} 0 & -a b_n(t) \\ -a b_n(t) & a^2 b_n(t) a_{n-1}(t) \end{bmatrix}, \qquad C_n(t) = \begin{bmatrix} 0 & 0 \\ 0 & a^2 b_n(t) b_{n-1}(t) \end{bmatrix}.$$

We can easily see that the monic ones, $\mathcal{P}_n(x,t)$ satisfies

$$\mathcal{P}_n(x,t) = \mathrm{I}\,P_n(x,t) + \widetilde{B}_n(t)P_{n-1}(x,t) + \widetilde{C}_n(t)P_{n-2}(x,t), \qquad n \in \mathbb{N},$$

where I denotes the identity matrix and

$$\widetilde{B}_n(t) = \begin{bmatrix} -\frac{a^2 a_n(t) b_n(t)}{1+a^2 b_n(t)} & -a b_n(t) + \frac{a^3 a_{n-1}(t) a_n(t) b_n(t)}{1+a^2 b_n(t)} \\ -\frac{a b_n(t)}{1+a^2 b_n(t)} & \frac{a^2 a_{n-1}(t) b_n(t)}{1+a^2 b_n(t)} \end{bmatrix}, \qquad \widetilde{C}_n(t) = \begin{bmatrix} 0 & \dfrac{a^3 a_n(t) b_{n-1}(t) b_n(t)}{1+a^2 b_n(t)} \\ 0 & \dfrac{a^2 b_{n-1}(t) b_n(t)}{1+a^2 b_n(t)} \end{bmatrix}. \tag{9}$$

Which allows us to express the three terms recurrence relation $\tilde{\beta}_n(t)$ and $\tilde{\gamma}_n(t)$ in terms of these coefficients $\widetilde{B}_n(t)$ and $\widetilde{C}_n(t)$ such that

$$\tilde{\beta}_n(t) = a_n(t)\mathrm{I} + \widetilde{B}_n(t) - \widetilde{B}_{n+1}(t), \qquad \tilde{\gamma}_n(t) = b_n(t)\mathrm{I} + \widetilde{B}_n(t)a_{n-1}(t) + \widetilde{C}_n(t) - \widetilde{C}_{n+1}(t) - \tilde{\beta}_n(t)\widetilde{B}_n(t). \tag{10}$$

These relations facilitate the computation of the asymptotic expansion in the next sections.

Corollary 3.3. *The monic semi-classical Laguerre matrix polynomials $\{\mathcal{P}_n(x,t)\}_{n\in\mathbb{N}}$ satisfies*

$$x\mathcal{P}_n(x,t) = \mathcal{P}_{n+1}(x,t) + \tilde{\beta}_n(t)\mathcal{P}_n(x,t) + \tilde{\gamma}_n(t)\mathcal{P}_{n-1}(x,t), \qquad n \in \{1,2,\ldots\},$$

with

$$\tilde{\beta}_n(t) = \begin{bmatrix} \dfrac{a_n(t)}{1+a^2 b_n(t)} + \dfrac{a^2 a_{n+1}(t) b_{n+1}(t)}{1+a^2 b_{n+1}(t)} & a\left(b_{n+1}(t) - b_n(t)\right) + a^3 a_n(t)\left(\frac{a_{n-1}(t)b_n(t)}{1+a^2 b_n(t)} - \frac{a_{n+1}(t)b_{n+1}(t)}{1+a^2 b_{n+1}(t)}\right) \\ \dfrac{a(b_{n+1}(t) - b_n(t))}{(1+a^2 b_n(t))(1+a^2 b_{n+1}(t))} & \dfrac{a^2 a_{n-1}(t) b_n(t)}{1+a^2 b_n(t)} + \dfrac{a_n(t)}{1+a^2 b_{n+1}(t)} \end{bmatrix},$$

$$\tilde{\gamma}_n(t) = \begin{bmatrix} \tilde{\gamma}_n^{11}(t) & \tilde{\gamma}_n^{12}(t) \\ \tilde{\gamma}_n^{21}(t) & \tilde{\gamma}_n^{22}(t) \end{bmatrix},$$

where

$$\tilde{\gamma}_n^{11}(t) = \frac{b_n(t)\left((1+a^2 b_n(t))(1+a^2 b_{n+1}(t)) + a^2(a_n(t)^2 - a_{n-1}(t)a_n(t))\right)}{(1+a^2 b_n(t))^2},$$

$$\tilde{\gamma}_n^{12}(t) = \frac{a b_n(t)\left((1+a^2 b_n(t))\left(a_n(t)(1+a^2 b_{n-1}(t)) - a_{n-1}(t)a_n^2(t) - a_{n-1}(t)a^2 b_{n+1}(t)\right) + a^2 a_{n-1}(t)\left(a_{n-1}(t)a_n(t) - a_n^2(t) - b_n(t)\right)\right)}{(1+a^2 b_n(t))^2},$$

$$\tilde{\gamma}_n^{21}(t) = \frac{a(a_n(t) - a_{n-1}(t))b_n(t)}{(1+a^2 b_n(t))^2},$$

$$\tilde{\gamma}_n^{22}(t) = \frac{b_n(t)\left((1+a^2 b_n(t))(1+a^2 b_{n-1}(t)) + a^2 a_{n-1}(t)(a_{n-1}(t) - a_n(t))\right)}{(1+a^2 b_n(t))^2}.$$

4 Large-t Asymptotics

Theorem 4.1. *As $t \longrightarrow \infty$, the coefficients (9), $\tilde{B}_n(t)$ and $\tilde{C}_n(t)$ have the asymptotic expansions*

$$\tilde{B}_n(t) = {}_2b_n^a\,t^2 + {}_1b_n^a\,t + {}_0b_n^a + \frac{{}_{-1}b_n^a}{t} + \frac{{}_{-2}b_n^a}{t^2} + \frac{{}_{-3}b_n^a}{t^3} + \mathcal{O}\left(\frac{1}{t^4}\right),$$

$$\tilde{C}_n(t) = {}_1c_n^a\,t + {}_0c_n^a + \frac{{}_{-1}c_n^a}{t} + \frac{{}_{-2}c_n^a}{t^2} + \frac{{}_{-3}c_n^a}{t^3} + \frac{{}_{-4}c_n^a}{t^4} + \mathcal{O}\left(\frac{1}{t^5}\right),$$

where

$$
{}_{-3}b_n^a = \frac{2a^2\lambda n}{(2+a^2n)^3}\begin{bmatrix} -4\lambda-4+n\big(4+2a^2(\lambda-2)\big)+n^2\big(a^4(\lambda-1)-2a^2\big)-2a^4n^3 & 0 \\ 0 & 4\lambda-4-n(4+4a^2+2a^2\lambda)+n^2(2a^2-a^4(1+\lambda))+2a^4n^3 \end{bmatrix},
$$

$$
{}_{-2}b_n^a = \frac{a\lambda n}{(2+a^2n)^2}\begin{bmatrix} 0 & \frac{8+a^2n(16-4a^2\lambda)+a^4n^2(16-a^2\lambda)+5a^6n^3}{2+a^2n} \\ 4 & 0 \end{bmatrix},
$$

$$
{}_{-1}b_n^a = \frac{a^4\lambda n^2}{(2+a^2n)^2}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
{}_{0}b_n^a = -\frac{an}{2+a^2n}\begin{bmatrix} 0 & \frac{1}{2}\,\frac{4-2a^2\lambda+4a^2n-2a^4\lambda n+a^4n^2}{2+a^2n} \\ 1 & 0 \end{bmatrix},
$$

$$
{}_{1}b_n^a = \frac{a^2n}{2(2+a^2n)}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},
$$

$$
{}_{2}b_n^a = \begin{bmatrix} 0 & \frac{a^3n}{8+4a^2n} \\ 0 & 0 \end{bmatrix},
$$

$$
{}_{1}c_n^a = \begin{bmatrix} 0 & \frac{a^3n(n-1)}{4(2+a^2n)} \\ 0 & 0 \end{bmatrix},
$$

$$
{}_{0}c_n^a = \begin{bmatrix} 0 & 0 \\ 0 & \frac{a^2n(n-1)}{2(2+a^2n)} \end{bmatrix},
$$

$$
{}_{-1}c_n^a = \begin{bmatrix} 0 & -\frac{a^3\lambda n(n-1)}{(2+a^2n)^2} \\ 0 & 0 \end{bmatrix},
$$

$$
{}_{-2}c_n^a = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{a^2\lambda(n-1)(4n+a^2n^2)}{(2+a^2n)^2} \end{bmatrix},
$$

$$
{}_{-3}c_n^a = \begin{bmatrix} 0 & \frac{a^3\lambda(n-1)n\big(16\lambda+8a^2\lambda n+a^4\lambda n^2-(2+a^2n)\big[8(n-1)-4a^2n+a^2n^2\big]\big)}{(2+a^2n)^3} \\ 0 & 0 \end{bmatrix}.
$$

Proof. Taking into account the asymptotic expensions (4) of $a_n(t)$ and $b_n(t)$, we have:

$$
\begin{aligned}
-\frac{a^2a_n(t)b_n(t)}{1+a^2b_n(t)} &= -\frac{a^2n}{2(2+a^2n)}\,t-\frac{a^4\lambda n^2}{(2+a^2n)^2}\,\frac{1}{t} \\
&\quad+\frac{2a^2\lambda n\big[\lambda(-4+a^2n(2+a^2n))-(2+a^2n)\big(2+n(-2+a^2(1+2n))\big)\big]}{(2+a^2n)^3}\,\frac{1}{t^3}+\mathcal{O}\Big(\frac{1}{t^5}\Big), \\
-ab_n(t)+\frac{a^3a_{n-1}a_n(t)b_n(t)}{1+a^2b_n(t)} &= \frac{a^3n}{8+4a^2n}\,t^2-\Big(\frac{an}{2}-\frac{a^3\lambda n(1+a^2n)}{(2+a^2n)^2}\Big) \\
&\quad+\frac{a\lambda n\big[8+a^2n\big(16-4a^2(\lambda-4n)+a^4n(-\lambda+5n)\big)\big]}{(2+a^2n)^3}\,\frac{1}{t^2}+\mathcal{O}\Big(\frac{1}{t^4}\Big), \\
\frac{a^2a_{n-1}(t)b_n(t)}{1+a^2b_n(t)} &= \frac{a^2n}{2(2+a^2n)}\,t+\frac{a^4\lambda n^2}{(2+a^2n)^2}\,\frac{1}{t} \\
&\quad-\frac{2a^2\lambda n\big[4-4\lambda+2(2+a^2(2+\lambda))n+a^2(-2+a^2(1+\lambda))n^2-2a^4n^3\big]}{(2+a^2n)^3}\,\frac{1}{t^3}+\mathcal{O}\Big(\frac{1}{t^5}\Big), \\
\frac{a^3a_n(t)b_{n-1}(t)b_n(t)}{1+a^2b_n(t)} &= \frac{a^3(n-1)n}{4(2+a^2n)}\,t-\frac{a^3\lambda(n-1)n}{(2+a^2n)^2}\,\frac{1}{t} \\
&\quad+\frac{a^3\lambda(n-1)n\big[\lambda(4+a^2n)^2-(2+a^2n)\big(-8+(8+a^2(n-4))n\big)\big]}{(2+a^2n)^3t^3}+\mathcal{O}\Big(\frac{1}{t^5}\Big),
\end{aligned}
$$

$$-\frac{ab_n(t)}{1+a^2b_n(t)} = -\frac{an}{2+a^2n} + \frac{4a\lambda n}{(2+a^2n)^2}\frac{1}{t^2} + \frac{8a\lambda n\left[3n(2+a^2n)-2\lambda(3+a^2n)\right]}{(2+a^2n)^3}\frac{1}{t^4} + \mathcal{O}\left(\frac{1}{t^6}\right),$$

$$\frac{a^2b_{n-1}(t)b_n(t)}{1+a^2b_n(t)} = \frac{a^2(n-1)n}{2(2+a^2n)} - \frac{a^2\lambda(n-1)(4n+a^2n^2)}{(2+a^2n)^2}\frac{1}{t^2} + \frac{2a^2\lambda(n-1)n\left[-3(2+a^2n)\left(-2+(4+a^2(n-1))n\right)+\lambda(28+3a^2n(6+a^2n))\right]}{(2+a^2n)^3}\frac{1}{t^4} + \mathcal{O}\left(\frac{1}{t^6}\right).$$

From the expressions of $\tilde{B}_n(t)$ and $\tilde{C}_n(t)$ (9), truncating the previous expansions at the appropriate order yields the desired result. □

Taking these expressions into account, we deduce the large-t asymptotic expansions of the recurrence coefficients $\tilde{\beta}_n(t)$ and $\tilde{\gamma}_n(t)$ associated with the monic polynomials. For computational purposes, the expansions will be truncated at third order.

Theorem 4.2. *As $t \longrightarrow \infty$, the recurrence coefficients* (10) *$\tilde{\beta}_n(t)$ and $\tilde{\gamma}_n(t)$ have the asymptotic expansions*

$$\tilde{\beta}_n(t) = {}_2\beta_n^a\, t^2 + {}_1\beta_n^a\, t + {}_0\beta_n^a + \frac{{}_{-1}\beta_n^a}{t} + \frac{{}_{-2}\beta_n^a}{t^2} + \mathcal{O}\left(\frac{1}{t^3}\right),$$

$$\tilde{\gamma}_n(t) = {}_1\gamma_n^a\, t + {}_0\gamma_n^a + \frac{{}_{-2}\gamma_n^a}{t^2} + \frac{{}_{-3}\gamma_n^a}{t^3} + \mathcal{O}\left(\frac{1}{t^4}\right),$$

where

$${}_{-2}\beta_n^a = \frac{a\lambda}{(2+a^2n)^2(2+a^2(n+1))^2}\begin{bmatrix} 0 & -\left(16+a^2\left(24+48n+5a^2(2+n(4+a^2(n+1)))^2\right)\right)+2a^4\frac{(16+32n+a^2(4+n(n+1)(6+a^2n)(6+a^2(n+1))))\lambda}{(2+a^2n)(2+a^2(n+1))} \\ 4(a^4n(n+1)-4) & 0 \end{bmatrix},$$

$${}_{-1}\beta_n^a = \frac{\lambda}{(2+a^2n)^2(2+a^2(n+1))^2}\begin{bmatrix} 16+a^2(16+32n+a^2(n+1)(8+n(24+a^2(n+1)(8+a^2n)))) & 0 \\ 0 & 16+a^2(16+n(32+a^2(16+n(24+a^2(n+1)(8+a^2(n+1)))))) \end{bmatrix},$$

$${}_0\beta_n^a = \frac{a}{(2+a^2n)(2+a^2(n+1))}\begin{bmatrix} 0 & \frac{1}{2} - \frac{a^2(4+a^2(4+8n)+3a^4n(n+1))\lambda}{(2+a^2n)(2+a^2(n+1))} \\ 2 & 0 \end{bmatrix},$$

$${}_1\beta_n^a = \begin{bmatrix} \frac{1}{2}+\frac{1}{2+a^2n}-\frac{1}{2+a^2(n+1)} & 0 \\ 0 & \frac{1}{2}-\frac{1}{2+a^2n}+\frac{1}{2+a^2(n+1)} \end{bmatrix},$$

$${}_2\beta_n^a = \begin{bmatrix} 0 & -\frac{a^3}{2(2+a^2n)(2+a^2(n+1))} \\ 0 & 0 \end{bmatrix},$$

$${}_1\gamma_n^a = \begin{bmatrix} 0 & -\frac{a^3n}{4+2a^2n} \\ 0 & 0 \end{bmatrix},$$

$${}_0\gamma_n^a = \begin{bmatrix} \frac{n(2+a^2(n+1))}{4+2a^2n} & 0 \\ 0 & -\frac{1}{2}+\frac{n}{2}+\frac{1}{2+a^2n} \end{bmatrix},$$

$${}_{-2}\gamma_n^a = -\frac{\lambda n}{(2+a^2n)^2}\begin{bmatrix} 4+4a^2n+a^4n(n+1) & 0 \\ 0 & 4+4a^2n+a^4(n-1)n \end{bmatrix},$$

$${}_{-3}\gamma_n^a = \frac{a\lambda n}{(2+a^2n)^2}\begin{bmatrix} 0 & \frac{2(8+a^2(-20\lambda+n(40+14a^2(2n-\lambda)+a^4n(5n-\lambda))))}{(2+a^2n)} \\ 8 & 0 \end{bmatrix}.$$

Proof. It suffices to apply the previous theorem 4.1 in combination with the relation (10). The remainder of the proof follows by straightforward computation. □

As a consequence,

$$\tilde{\beta}_n(t) \underset{t\to\infty}{\sim} = {}_2\beta_n^a\, t^2 + {}_1\beta_n^a\, t + {}_0\beta_n^a, \qquad \tilde{\gamma}_n(t) \underset{t\to\infty}{\sim} = {}_1\gamma_n^a\, t + {}_0\gamma_n^a.$$

In particular, for $a = 0$ we recover the scalar case [8, Lemma 5.3]:

$$\tilde{\beta}_n(t) \underset{t\to\infty}{\sim} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} t, \qquad \tilde{\gamma}_n(t) \underset{t\to\infty}{\sim} = \begin{bmatrix} \frac{n}{2} & 0 \\ 0 & \frac{n}{2} \end{bmatrix}.$$

5 Large-n Asymptotics

Theorem 5.1. *Suppose $a \neq 0$. As $n \longrightarrow \infty$, the coefficients (9), $\tilde{B}_n(t)$ and $\tilde{C}_n(t)$ have the asymptotic expansions*

$$\tilde{B}_n(t) = {}_1b_t^a\, n + {}_{\frac{1}{2}}b_t^a \sqrt{n} + {}_0b_t^a + \frac{{}_{-\frac{1}{2}}b_t^a}{\sqrt{n}} + \frac{{}_{-1}b_t^a}{n} + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right),$$

$$\tilde{C}_n(t) = {}_{\frac{3}{2}}c_t^a\, n^{\frac{3}{2}} + {}_1c_t^a\, n + {}_{\frac{1}{2}}c_t^a \sqrt{n} + {}_0c_t^a + \frac{{}_{-\frac{1}{2}}c_t^a}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right),$$

where

$$\begin{aligned}
{}_1b_t^a &= \begin{bmatrix} 0 & \frac{a}{2} \\ 0 & 0 \end{bmatrix}, \\
{}_{\frac{1}{2}}b_t^a &= \begin{bmatrix} -\sqrt{\frac{2}{3}} & \frac{at}{2\sqrt{6}} \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix}, \\
{}_0b_t^a &= \begin{bmatrix} -\frac{t}{6} & -\frac{4}{a} - \frac{a}{72}t^2 - \frac{a}{12}\lambda \\ -\frac{1}{a} & \frac{t}{6} \end{bmatrix}, \\
{}_{-\frac{1}{2}}b_t^a &= \begin{bmatrix} \frac{a^2(4\lambda+t^2-4)+96}{8a^2\sqrt{6}} & \frac{at\left(t^2+12\lambda\right)}{96\sqrt{6}} \\ 0 & -\frac{a^2(4\lambda+t^2+4)+96}{8a^2\sqrt{6}} \end{bmatrix}, \\
{}_{-1}b_t^a &= \begin{bmatrix} -\frac{1}{72}t\left(\frac{72}{a^2} + t^2 + 6\lambda\right) & \frac{24}{a^3} + \frac{2t^2+12\lambda}{3a} + \frac{1}{432}a\left(2t^4 + 24t^2\lambda + 153\lambda^2 - 54\right) \\ \frac{a^2(t^2+6\lambda)+72}{12a^3} & \frac{1}{72}t\left(\frac{72}{a^2} + t^2 + 6\lambda\right) \end{bmatrix}, \\
{}_{\frac{3}{2}}c_t^a &= \begin{bmatrix} 0 & \frac{a}{3\sqrt{6}} \\ 0 & 0 \end{bmatrix}, \\
{}_1c_t^a &= \begin{bmatrix} 0 & \frac{at}{12} \\ 0 & \frac{1}{6} \end{bmatrix}, \\
{}_{\frac{1}{2}}c_t^a &= \begin{bmatrix} 0 & \frac{a^2\left(5t^2+12\lambda-36\right)-288}{144\sqrt{6}a} \\ 0 & \frac{t}{6\sqrt{6}} \end{bmatrix}, \\
{}_0c_t^a &= \begin{bmatrix} 0 & \frac{a^2t\left(t^2+12\lambda-18\right)-72t}{432a} \\ 0 & -\frac{1}{6} - \frac{1}{a^2} \end{bmatrix}, \\
{}_{-\frac{1}{2}}c_t^a &= \begin{bmatrix} 0 & \frac{55296+192a^2\left(7t^2+36\lambda+36\right)+a^4\left(13t^4+24t^2+152t^2\lambda-336+288\lambda-96\lambda^2\right)}{4608\sqrt{6}a^3} \\ 0 & \frac{t^3+12t\lambda-24t}{288\sqrt{6}} \end{bmatrix}.
\end{aligned}$$

Proof. Taking into account the asymptotic expensions (5) of $a_n(t)$ and $b_n(t)$, we derive those of $a_{n-1}(t)$ and $b_{n-1}(t)$:

$$\begin{aligned}
a_{n-1}(t) = {} & \sqrt{\frac{2n}{3}} + \frac{t}{6} + \frac{t^2 + 12(-1+\lambda)}{24\sqrt{6n}} + \frac{-t^4 - 24t^2(-1+\lambda) + 48(-5+6\lambda(1+\lambda))}{2304\sqrt{6}n^{3/2}} \\
& + \frac{t(-2+9\lambda^2)}{144n^2} + \frac{t^6 + 36t^4(-1+\lambda) - 1728(-1+\lambda)(-1+2\lambda)(3+4\lambda) + 144t^2(-13-6\lambda+66\lambda^2)}{110592\sqrt{6}n^{5/2}} \\
& + \frac{-12t(-1+\lambda)(-2+9\lambda^2) + t^3(-7+27\lambda^2)}{1728n^3} + \mathcal{O}(n^{-7/2}), \\
b_{n-1}(t) = {} & \frac{n}{6} + \frac{t}{6\sqrt{6}}\sqrt{n} + \frac{t^2+6\lambda-12}{72} + \frac{t(t^2+12\lambda-24)}{288\sqrt{6}\sqrt{n}} + \frac{2-9\lambda^2}{144n} \\
& - \frac{t\left(t^4 + 24t^2(\lambda-2) + 48(66\lambda^2 - 12\lambda - 5)\right)}{27648\sqrt{6}n^{3/2}} + \frac{t^2(7-27\lambda^2) + 4(\lambda-2)(9\lambda^2-2)}{1152n^2} \\
& + \frac{t^7 + 36t^5(\lambda-2) - 144t^3(6\lambda(41\lambda+2)-73) + 1728t(\lambda-2)(64\lambda^2-4\lambda-13)}{1327104\sqrt{6}n^{5/2}} + \mathcal{O}(n^{-3}).
\end{aligned}$$

Next, we establish the asymptotic expansions of each component of $\tilde{B}_n(t)$ and $\tilde{C}_n(t)$, following the approach used in the proof of Theorem 4.1 (albeit with more complex expressions). Finally, by truncating these expansions at the appropriate order, we obtain the desired result. □

Taking these expressions into account, we obtain the asymptotic expansions of the recurrence coefficients $\tilde{\beta}_n(t)$ and $\tilde{\gamma}_n(t)$.

Theorem 5.2. *Suppose $a \neq 0$. As $n \longrightarrow \infty$, the recurrence coefficients (10) $\tilde{\beta}_n(t)$ and $\tilde{\gamma}_n(t)$ have the asymptotic expansions*

$$\tilde{\beta}_n(t) = {}_1\beta_t^a\, n + {}_{\frac{1}{2}}\beta_t^a \sqrt{n} + {}_0\beta_t^a + \frac{{}_{-\frac{1}{2}}\beta_t^a}{\sqrt{n}} + \frac{{}_{-1}\beta_t^a}{n} + \mathcal{O}\left(\frac{1}{n^{\frac{3}{2}}}\right),$$

$$\tilde{\gamma}_n(t) = {}_1\gamma_t^a\, n + {}_{\frac{1}{2}}\gamma_t^a \sqrt{n} + {}_0\gamma_t^a + \frac{{}_{-\frac{1}{2}}\gamma_t^a}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n}\right),$$

where

$${}_{-1}\beta_t^a = \begin{bmatrix} 0 & -\frac{a}{2} \\ 0 & 0 \end{bmatrix},$$

$${}_{-\frac{1}{2}}\beta_t^a = \begin{bmatrix} \frac{t^2+12(-1+\lambda)}{24\sqrt{6}} & \frac{at}{4\sqrt{6}} \\ 0 & \frac{t^2+12(3+\lambda)}{24\sqrt{6}} \end{bmatrix},$$

$${}_0\beta_t^a = \begin{bmatrix} \frac{t}{6} & 0 \\ 0 & \frac{t}{6} \end{bmatrix},$$

$${}_{\frac{1}{2}}\beta_t^a = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ 0 & \sqrt{\frac{2}{3}} \end{bmatrix},$$

$${}_1\gamma_t^a = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{6} \end{bmatrix},$$

$${}_{\frac{1}{2}}\gamma_t^a = \begin{bmatrix} \frac{t}{6\sqrt{6}} & -\frac{a}{2\sqrt{6}} \\ 0 & \frac{t}{6\sqrt{6}} \end{bmatrix},$$

$${}_0\gamma_t^a = \begin{bmatrix} \frac{1}{72}(t^2+12+6\lambda) & -\frac{at}{12} \\ 0 & \frac{1}{72}(t^2-12+6\lambda) \end{bmatrix},$$

$${}_{-\frac{1}{2}}\gamma_t^a = \begin{bmatrix} \frac{t(t^2+12+6\lambda)}{288\sqrt{6}} & \frac{12-a^2}{4\sqrt{6}a} \\ 0 & \frac{t(t^2-12+6\lambda)}{288\sqrt{6}} \end{bmatrix}.$$

Proof. We substitute $n+1$ into the given expression for $\tilde{B}_n(t)$ and $\tilde{C}_n(t)$. We expand each term using asymptotic expansions around infinity, we obtain:

$$\tilde{B}_{n+1}(t) = {}_1b_t^a n + {}_1b_t^a + {}_{\frac{1}{2}}b_t^a \sqrt{n} + \frac{{}_{\frac{1}{2}}b_t^a}{2\sqrt{n}} + {}_0b_t^a + \frac{{}_{-\frac{1}{2}}b_t^a}{\sqrt{n}} + \frac{{}_{-1}b_t^a}{n} + \mathcal{O}(n^{-3/2}),$$

$$\tilde{C}_{n+1}(t) = {}_{\frac{3}{2}}c_t^a n^{\frac{3}{2}} + {}_1c_t^a n + \left({}_{\frac{1}{2}}c_t^a + \frac{3}{2}\,{}_{\frac{3}{2}}c_t^a\right)\sqrt{n} + ({}_1c_t^a + {}_0c_t^a) + \left(\frac{3}{8}\,{}_{\frac{3}{2}}c_t^a + \frac{1}{2}\,{}_{\frac{1}{2}}c_t^a + {}_{-\frac{1}{2}}c_t^a\right)\frac{1}{\sqrt{n}} + \mathcal{O}\left(n^{-1}\right).$$

Thus:

$$\tilde{B}_n(t) - \tilde{B}_{n+1}(t) = -{}_1b_t^a - {}_{\frac{1}{2}}b_t^a \frac{1}{2\sqrt{n}} + \mathcal{O}(n^{-3/2}),$$

$$\tilde{C}_n(t) - \tilde{C}_{n+1}(t) = -\frac{3}{2}\,{}_{\frac{3}{2}}c_t^a \sqrt{n} - {}_1c_t^a - \left(\frac{3}{8}\,{}_{\frac{3}{2}}c_t^a - \frac{1}{2}\,{}_{\frac{1}{2}}c_t^a\right)\frac{1}{\sqrt{n}} + \mathcal{O}\left(n^{-1}\right).$$

To finish the proof, we use the relation (10).

□

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