PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 18 · 2025 · Pages 72–90

A characterisation of the rate of approximation of Kantorovich sampling operators in weighted variable exponent Lebesgue spaces

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Communicated by Wolfgang Erb

Abstract

We establish a direct and a matching two-term converse estimate by a K-functional and moduli of smoothness for the rate of approximation by generalised Kantorovich sampling operators in weighted variable exponent Lebesgue spaces. They yield the saturation property and class of these operators. The weight is power-type with nonpositive exponents at infinity. We obtain an embedding inequality in weighted variable exponent Lebesgue spaces. We establish main properties of the moduli of smoothness. We demonstrate the general results on a sampling operator of that type whose kernel is supported on an arbitrarily fixed finite interval and which provides a rate of approximation of any power-type order given in advance.

1 Main results

Let f be a locally Lebesgue integrable function, defined on $\mathbb{R}$, and $\chi : \mathbb{R} \to \mathbb{R}$. Let $\{t_k\}_{k\in\mathbb{Z}}$ be a sequence of reals such that $t_k < t_{k+1}$ and $\lim_{k\to\pm\infty} t_k = \pm\infty$, as, moreover, $\theta \le \theta_k := t_{k+1} - t_k \le \Theta$ for all $k \in \mathbb{Z}$ with some constants $\theta, \Theta > 0$. Bardaro, Butzer, Stens and Vinti [11] introduced the Kantorovich-type sampling operators

$$(S_w^\chi f)(x) := \sum_{k\in\mathbb{Z}} \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du\,\chi(wx - t_k), \quad x \in \mathbb{R}, \quad w > 0. \tag{1.1}$$

A notable form of this operator is given by the choice $t_k = k$.

The function χ is called a kernel. We will need the following characteristics of the kernel χ:

- The discrete algebraic moment of χ of order $j \in \mathbb{N}_0$ w.r.t. $\{t_k\}_{k\in\mathbb{Z}}$, defined by
$$m_j(\chi,u) := \sum_{k\in\mathbb{Z}} (t_k - u)^j \chi(u - t_k), \quad u \in \mathbb{R},$$
provided the series is convergent for all $u \in \mathbb{R}$,
- The discrete absolute moment of χ of order $\sigma \ge 0$ w.r.t. $\{t_k\}_{k\in\mathbb{Z}}$, defined by
$$M_\sigma(\chi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |t_k - u|^\sigma |\chi(u - t_k)|.$$

In [11], where the operators S_w^χ were introduced, it was shown that if χ is Lebesgue measurable on $\mathbb{R}$, bounded in a neighbourhood of the origin, $m_0(\chi,u) \equiv 1$, and $M_\sigma(\chi) < \infty$ with some $\sigma > 0$, and f is a bounded Lebesgue measurable function on $\mathbb{R}$, then (see [11, Theorem 4.1])

$$\lim_{w\to\infty} S_w^\chi f(x) = f(x) \tag{1.2}$$

at every point x, at which f is continuous, and if, in addition, f is uniformly continuous and bounded on $\mathbb{R}$, then the convergence in (1.2) is uniform on $\mathbb{R}$ (see also [11, Remark 3.2]). The analogue of this result in $L_p(\mathbb{R})$, $1 \le p < \infty$ was established too (see

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[11, Corollary 5.2]). To be precise, in [11] the authors considered the more general modular convergence and convergence in the Luxemburg norm in Orlicz spaces.

We will consider approximation in variable exponent Lebesgue space with an exponent function $p:\mathbb{R}\to[1,+\infty]$, which is Lebesgue measurable. Following the usual notation, we will write $p(\cdot)$ to emphasise that p is generally nonconstant.

We set $p_* := \operatorname{ess\,inf}_{x\in\mathbb{R}} p(x)$, $p^* := \operatorname{ess\,sup}_{x\in\mathbb{R}} p(x)$ and

$$\mathbb{R}_\infty^{p(\cdot)} := \{x\in\mathbb{R} : p(x) = +\infty\}.$$

Next, for a Lebesgue measurable function f on $\mathbb{R}$, we set

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}\setminus\mathbb{R}_\infty^{p(\cdot)}} |f(x)|^{p(x)}dx + \operatorname*{ess\,sup}_{x\in\mathbb{R}_\infty^{p(\cdot)}} |f(x)|.$$

The variable exponent Lebesgue space $L_{p(\cdot)}(\mathbb{R})$ is defined as the set of all Lebesgue measurable functions f on $\mathbb{R}$, for which there exists $\lambda > 0$ (depending on f) such that

$$\rho_{p(\cdot)}(f/\lambda) < \infty.$$

It is a Banach space with the norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\}.$$

As is known, if $f\in L_{p(\cdot)}(\mathbb{R})$, then f is locally Lebesgue integrable (see, e.g., [18, Corollary 2.27]).

More generally, we will consider approximation by S_w^χ in variable exponent Lebesgue spaces with the weight

$$\rho_{\alpha,\beta}(x) := \begin{cases} |x|^{-\alpha}, & x < -1, \\ 1, & -1 \le x \le 1, \\ x^{-\beta}, & x > 1, \end{cases} \tag{1.3}$$

where $\alpha,\beta \ge 0$.

Equivalently, we can work with the weight given in the form

$$\rho_{\alpha,\beta}(x) := \begin{cases} \dfrac{1}{1+|x|^\alpha}, & x < 0, \\ \dfrac{1}{1+x^\beta}, & x \ge 0. \end{cases} \tag{1.4}$$

We define $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ to be the set of all Lebesgue measurable functions f on $\mathbb{R}$ such that $\rho_{\alpha,\beta} f \in L_{p(\cdot)}(\mathbb{R})$.

We will extensively use the Hardy-Littlewood maximal operator

$$\mathcal{M}f(x) := \sup_{\substack{t\in\mathbb{R}\\ t\ne x}} \frac{1}{t-x}\int_x^t |f(u)|\,du, \quad x\in\mathbb{R},$$

and the fact that it is a bounded operator that maps $L_{p(\cdot)}(\mathbb{R})$ into $L_{p(\cdot)}(\mathbb{R})$ under certain assumptions on the exponent function. One of them is that $1/p(\cdot)$ is log-Hölder continuous on $\mathbb{R}$.

To recall, we say that $r:\mathbb{R}\to[0,+\infty)$ is (globally) log-Hölder continuous and write $r(\cdot)\in LH(\mathbb{R})$ if there exists $c > 0$ such that

$$|r(x)-r(y)| \le \frac{c}{-\log|x-y|}, \quad x,y\in\mathbb{R},\ |x-y| < \frac{1}{2},$$

and

$$|r(x)-r(y)| \le \frac{c}{\log(e+|x|)}, \quad x\in\mathbb{R},\ |y| \ge |x|. \tag{1.5}$$

As is known, if $p(\cdot)$ is an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot)\in LH(\mathbb{R})$, then there exists $M > 0$ such that for all $f\in L_{p(\cdot)}(\mathbb{R})$ there holds (see [19, 20, 21, 35] or [18, Theorem 3.13])

$$\|\mathcal{M}f\|_{p(\cdot)} \le M\|f\|_{p(\cdot)}. \tag{1.6}$$

The value of the constant M depends only on $p(\cdot)$.

Condition (1.5) is equivalent to the condition that there exist constants r_∞ and c such that

$$|r(x)-r_\infty| \le \frac{c}{\log(e+|x|)}, \quad x\in\mathbb{R} \tag{1.7}$$

(cf. [18, Definition 2.2]). Therefore, if $1/p(\cdot) \in LH(\mathbb{R})$, then there exists $p_\infty := \lim_{x\to\pm\infty} p(x) \geq 1$, as it is possible that $p_\infty = +\infty$. Clearly, if $p_* > 1$, then $p_\infty > 1$ too.

In order to characterise the rate of approximation of S_w^χ in $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$, we will use a modulus of smoothness based on the standard forward difference operator. However, we cannot use the classical modulus of smoothness applied in problems in $L_p(\mathbb{R})$ because the shifted function does not necessarily remain in $L_{p(\cdot)}(\mathbb{R})$. Therefore, following Sharapudinov [37, 38], Israfilov, Kokilashvili and Samko [28], and Israfilov and Testici [29], we introduce the modulus of smoothness

$$\Omega_r(f,t)_{p(\cdot),\alpha,\beta} := \sup_{0<h\leq t} \left\| \frac{\rho_{\alpha,\beta}}{h} \int_0^h \Delta_u^r f\, du \right\|_{p(\cdot)},$$

where $\Delta_u f(x) := f(x+u) - f(x)$, $x, u \in \mathbb{R}$, is the classical forward difference of f with step u, and Δ_u^r is its rth iteration, that is, $\Delta_u^r := \Delta_u(\Delta_u^{r-1})$. Its expanded form is

$$\Delta_u^r f(x) = \sum_{\ell=0}^{r} (-1)^\ell \binom{r}{\ell} f(x+(r-\ell)u), \quad x \in \mathbb{R}.$$

By virtue of (1.6), $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ is defined and finite for every $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ provided that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. We will explicitly show that in Theorem 3.1(a).

We can use instead the modulus of smoothness, defined by (cf. [27])

$$\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta} := \sup_{0<h\leq t} \left\| \frac{\rho_{\alpha,\beta}}{h} \int_0^h |\Delta_u^r f|\, du \right\|_{p(\cdot)}.$$

In fact, as we will show in Corollary 3.3 below, for all $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $t \in (0,1]$ there holds

$$\Omega_r(f,t)_{p(\cdot),\alpha,\beta} \leq \overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta} \leq c\,\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$$

with some positive constant c, independent of f and t. Certainly, the left inequality above is trivial.

We will need, in addition, several other notations. Let $C(\mathbb{R})$ be the space of the real-valued functions that are continuous and *not* necessarily bounded on $\mathbb{R}$, and $C^r(\mathbb{R})$, $r \in \mathbb{N}_+$, be the space of the real-valued functions that are r-times continuously differentiable on $\mathbb{R}$, as we do *not* require that the derivatives be bounded. Let $AC_{loc}^r(\mathbb{R})$ be the space of the functions $f : \mathbb{R} \to \mathbb{R}$, which are r times differentiable on $\mathbb{R}$ and $f^{(j)}$, $j = 0, \ldots, r$ are absolutely continuous on any closed finite interval on the real line. Finally, let $W_{p(\cdot),\alpha,\beta}^r(\mathbb{R})$ denote the weighted variable exponent Sobolev-type space

$$W_{p(\cdot),\alpha,\beta}^r(\mathbb{R}) := \{f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R}) : f \in AC_{loc}^{r-1}(\mathbb{R}),\ f^{(r)} \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})\}.$$

One of our main results is the following direct estimate of the rate of approximation by S_w^χ. Below and throughout c denotes a positive constant, whose value is independent of the approximated function and the order of the operator w. It is not necessarily the same at each occurrence.

Theorem 1.1. *Let $r \in \mathbb{N}_+$ and $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha, \beta \geq 0$. Let $\chi \in C(\mathbb{R})$ be such that:*

(i) $\chi(u) = O(|u|^{-\gamma})$, *as* $u \to \pm\infty$, *where* $\gamma > r + 1 + \max\{\alpha, \beta\}$,

(ii) $m_0(\chi, u) \equiv 1$,

(iii) $\displaystyle\sum_{\ell=0}^{j} \binom{j+1}{\ell} \sum_{k\in\mathbb{Z}} \theta_k^{j-\ell} (t_k - u)^\ell \chi(u - t_k) \equiv 0, \quad j = 1, \ldots, r-1,$ *if* $r \geq 2$.

Then for all $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $w \geq 1$ the series defining $S_w^\chi f(x)$ is absolutely and uniformly convergent on the compact intervals of $\mathbb{R}$, $S_w^\chi f \in C(\mathbb{R})$, $S_w^\chi f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and there holds

$$\|\rho_{\alpha,\beta}(S_w^\chi f - f)\|_{p(\cdot)} \leq c\Omega_r(f, 1/w)_{p(\cdot),\alpha,\beta}.$$

The estimate remains valid with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ in place of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$.

The above direct estimate is best possible in the sense that the following converse estimate holds.

Theorem 1.2. *Let $r \in \mathbb{N}_+$ and $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha, \beta \geq 0$. Let $\chi \in C^{r+1}(\mathbb{R})$ be such that:*

(i) $M_0(\chi) < \infty$,

(ii) $\chi^{(r+1)}(u) = O(|u|^{-\gamma})$, *as* $u \to \pm\infty$, *where* $\gamma > 2r + 3 + \max\{\alpha, \beta\}$,

(iii) $m_0(\chi, u) \equiv 1$,

(iv) $\sum_{\ell=0}^{j}\binom{j+1}{\ell}\sum_{k\in\mathbb{Z}}\theta_k^{j-\ell}(t_k-u)^{\ell}\chi(u-t_k)\equiv 0,\quad j=1,\dots,r-1,$ *if* $r\geq 2$,

(v) $\sum_{\ell=0}^{r}\binom{r+1}{\ell}\sum_{k\in\mathbb{Z}}\theta_k^{r-\ell}(t_k-u)^{\ell}\chi(u-t_k)\equiv \mathrm{const}\neq 0.$

Then there exist constants $c,\varrho>0$ *such that for all* $f\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ *and all* $w,v\geq 1$ *with* $v\geq\varrho w$ *there holds*

$$\Omega_r(f,1/w)_{p(\cdot),\alpha,\beta}\leq c\left(\frac{v}{w}\right)^r\left(\|\rho_{\alpha,\beta}(S_w^{\chi}f-f)\|_{p(\cdot)}+\|\rho_{\alpha,\beta}(S_v^{\chi}f-f)\|_{p(\cdot)}\right).$$

In particular,

$$\Omega_r(f,1/w)_{p(\cdot),\alpha,\beta}\leq c\left(\|\rho_{\alpha,\beta}(S_w^{\chi}f-f)\|_{p(\cdot)}+\|\rho_{\alpha,\beta}(S_{\varrho w}^{\chi}f-f)\|_{p(\cdot)}\right).\tag{1.8}$$

The estimates remain valid with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ *in place of* $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$.

In the case $t_k=k$, assumptions (ii)-(iii) in Theorem 1.1 and (iii)-(v) in Theorem 1.2 can be checked by means of [3, Lemmas 5.3 and 5.4], respectively (see [26, Remark 1.1]). Analogues of Theorems 1.1 and 1.2 were established in [3, Theorems 3.6 and 3.7] for S_w^{χ} with $t_k=k$ in the unweighted case.

Theorems 1.1 and 1.2 directly imply the following O-equivalence characterisation of the error of the Kantorovich sampling operator S_w^{χ} in $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$.

Corollary 1.3. *Let the assumptions in Theorem 1.2 be satisfied and* $0<\lambda\leq r$. *Then*

$$\|\rho_{\alpha,\beta}(S_w^{\chi}f-f)\|_{p(\cdot)}=O(w^{-\lambda})\iff \Omega_r(f,t)_{p(\cdot),\alpha,\beta}=O(t^{\lambda}).$$

The assertion remains valid with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ *in place of* $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$.

We will use Theorems 1.1 and 1.2 along with properties of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ to show that the approximation process $\{S_w^{\chi}\}_{w\geq 1}$ is saturated in $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and further describe its saturation class and its trivial class. These results comprise the following theorem.

Theorem 1.4. *Let the assumptions in Theorem 1.2 be satisfied. Then the approximation process* $\{S_w^{\chi}\}_{w\geq 1}$ *is saturated in* $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ *with order* $O(w^{-r})$*, its saturation class is* $W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ *and its trivial class consists of the functions which are equal a.e. to:*

(a) 0 *if* $p_\infty\in\mathbb{R}$ *and either* $\alpha\leq 1/p_\infty$*, or* $\beta\leq 1/p_\infty$*;*

(b) *Algebraic polynomials of degree* s*, where* $s\leq r-1$ *and* $s<\alpha-1/p_\infty,\beta-1/p_\infty$*, if* $p_\infty\in\mathbb{R}$ *and* $\alpha,\beta>1/p_\infty$*;*

(c) *Algebraic polynomials of degree* s*, where* $s\leq r-1,\alpha,\beta$*, if* $p_\infty=+\infty$.

The definition of the approximation characteristics referred to in the last theorem can be found in, e.g., [14, Definition 12.0.2].

The results stated in the theorems above extend those established in [25] for the unweighted case $\alpha=\beta=0$, but now we impose stronger assumptions on the kernel to avoid technicalities. Nonetheless, they allow the results obtained to be applied for many important concrete operators of that type. In addition, we need to note that Theorem 1.4 corrects a mistake in the description of the trivial class in [25, Theorem 1.5]. The assertion of Theorem 1.1 was proved in [26] in the L_p-spaces with a constant exponent.

We refer the reader to the references in [25] for other related results in Lebesgue spaces with a constant or varying exponent and more general abstract spaces (we need to add [17, 30, 31] as well), and to the references in [26] for weighted approximation in $L_p(\mathbb{R})$. The Musielak-Orlicz spaces include weighted Lebesgue spaces with a variable exponent under certain conditions on the weight and the exponent.

Especially, a direct estimate for S_w^{χ} with $t_k=k$ in the uniform norm with the weight $\rho_{2,2}$ (as defined in (1.4)) by a modulus of continuity was established in [1, Theorem 4.1] and, similarly, for Durrmeyer-type sampling operators, which include S_w^{χ} with any equidistant t_k, in [9, Theorem 3]. The absolute constant in those estimates are given explicitly.

We add the following very recent results. Approximation by means of a general Durrmeyer-type sampling operators in weighted uniform norm was considered in [8] and a pointwise direct error estimate by a modulus of continuity was established. Similar results were obtained for compositions and linear combinations thereof of generalised sampling operators and Durrmeyer sampling operators in [39]. General weighted Korovkin, Jackson and Voronovskaya-type results were proved in [10]. Pointwise and (weighted) uniform norm convergence results and (weighted) Jackson and Voronovskaya-type estimates were proved for a generalised sampling series in [40]. A Steklov-type sampling operator was considered in [16], as pointwise and norm convergence results in $L_p(\mathbb{R})$ and in the space of the bounded uniformly continuous functions on $\mathbb{R}$ were obtained, including Jackson-type estimates. Exponential sampling series were considered in [4, 5, 33, 34, 36].

Finally, we refer the reader to the papers [12, 13], [15, Section 5.1] and [32] for brief accounts of the emergence and development of sampling series theory.

The contents of the paper are organised as follows. In the next section we will establish the basic properties of the Kantorovich sampling operator we need to prove Theorems 1.1 and 1.2. In Section 3 we establish several properties of the modulus of smoothness $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ as well as its equivalence to a K-functional. Then, in Section 4, we prove Theorems 1.1, 1.2 and 1.4. The last section contains a concrete example of the considered operator and the application of the general theorems to it.

2 Basic properties of the operator S_w^χ

We begin with an assertion that shows the operator S_w^χ is defined on the space $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ under certain assumptions on the kernel χ.

Proposition 2.1. *Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ and $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta \geq 0$. Let $\chi \in C(\mathbb{R})$ be such that $\chi(u) = O(|u|^{-\gamma})$, as $u \to \pm\infty$, where $\gamma > 1 + \max\{\alpha,\beta\}$. Then for each $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $w \geq 1$ the series, defining $S_w^\chi f(x)$ in (1.1), is absolutely and uniformly convergent on the compact intervals of $\mathbb{R}$ and $S_w^\chi f \in C(\mathbb{R})$.*

Proof. By virtue of Hölder's inequality [18, Theorem 2.32], we have

$$\left| \int_{t_k/w}^{t_{k+1}/w} f(u)\,du \right| \leq c\|\rho_{\alpha,\beta} f\|_{p(\cdot)[t_k/w,t_{k+1}/w]} \|1/\rho_{\alpha,\beta}\|_{q(\cdot)[t_k/w,t_{k+1}/w]}, \tag{2.1}$$

where $q(\cdot)$ is the conjugate exponent function of $p(\cdot)$, defined by the relation $1/p(x) + 1/q(x) = 1$, $x \in \mathbb{R}$, and $\|\circ\|_{p(\cdot)[a,b]}$ is the $L_{p(\cdot)}$-norm on the interval $[a,b]$.

Next, we apply [18, Theorem 2.26] to arrive at

$$\|1/\rho_{\alpha,\beta}\|_{q(\cdot)[t_k/w,t_{k+1}/w]} \leq (1+\Theta)\|1/\rho_{\alpha,\beta}\|_{\infty[t_k/w,t_{k+1}/w]}. \tag{2.2}$$

Clearly,

$$\rho_{\alpha,\beta}(x)^{-1} \leq \max\left\{\rho_{\alpha,\beta}(a)^{-1}, \rho_{\alpha,\beta}(b)^{-1}\right\}, \quad x \in [a,b]. \tag{2.3}$$

In addition, since $t_{k+1}/w \leq t_k/w + \Theta$, $k \in \mathbb{Z}$, $w \geq 1$, and $\rho_{\alpha,\beta}(x)^{-1} \geq 1$ for $x \in \mathbb{R}$, we have

$$\rho_{\alpha,\beta}\left(\frac{t_{k+1}}{w}\right)^{-1} \leq c\,\rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1}. \tag{2.4}$$

Therefore,

$$\|1/\rho_{\alpha,\beta}\|_{\infty[t_k/w,t_{k+1}/w]} \leq c\rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1}. \tag{2.5}$$

Now, (2.1), (2.2) and (2.5) imply

$$\left| \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du \right| \leq cw\rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} \|\rho_{\alpha,\beta} f\|_{p(\cdot)}. \tag{2.6}$$

We have with $\delta := \max\{\alpha,\beta\}$ that

$$\rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} \leq (1+|t_k|)^{\delta}. \tag{2.7}$$

Clearly, $|\chi(u)| \leq c(1+|u|)^{-\gamma}$, $u \in \mathbb{R}$. Let $[a,b]$ be an arbitrary closed finite interval on $\mathbb{R}$ and $c' := 1 + w\max\{|a|,|b|\}$. Then

$$|\chi(wx - t_k)| \leq c(c' + |t_k| - |wx|)^{-\gamma} \leq c''(1+|t_k|)^{-\gamma}, \quad x \in [a,b],\ k \in \mathbb{Z}, \tag{2.8}$$

where c'' is a positive constant, whose value is independent of x (but may depend on w).

By [26, Lemma 2.1] we have that there exist positive constants c_1 and c_2 such that

$$c_1(1+|k|) \leq 1 + |t_k| \leq c_2(1+|k|), \quad k \in \mathbb{Z}. \tag{2.9}$$

Estimates (2.6)-(2.8) and the left inequality in (2.9) yield

$$\left| \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du \right| |\chi(wx - t_k)| \leq c'''(1+|k|)^{\delta-\gamma}, \quad x \in [a,b],\ k \in \mathbb{Z}, \tag{2.10}$$

where c''' is a positive constant, whose value is independent of x and k (though, it generally depends on f and w).

Since $\delta - \gamma < -1$, the series

$$\sum_{k\in\mathbb{Z}} (1+|k|)^{\delta-\gamma}$$

is convergent. We apply the Weierstrass M-test to get that the series, defining $S_w^\chi f(x)$, is absolutely and uniformly convergent on $[a,b]$; hence $S_w^\chi f \in C(\mathbb{R})$. □

Henceforward, we will often apply the following auxiliary result, established in [26, Lemma 2.2].

Lemma A. *Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta \ge 0$. Let $\eta \in C(\mathbb{R})$ be such that $\eta(u) = O(|u|^{-\lambda})$, as $u \to \pm\infty$, where $\lambda > 1 + \max\{\alpha,\beta\}$. Then for all $x \in \mathbb{R}$ and $w \ge 1$ there holds*

$$\sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} |\eta(wx - t_k)| \le c\rho_{\alpha,\beta}(x)^{-1}.$$

Above c is a positive constant whose value is independent of x and w.

Next, we will establish that, under certain assumptions, $\{S_w^\chi\}_{w\ge1}$ is a family of uniformly bounded operators that map $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ into itself.

Proposition 2.2. *Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta \ge 0$. Let $\chi \in C(\mathbb{R})$ be such that $\chi(u) = O(|u|^{-\gamma})$, as $u \to \pm\infty$, where $\gamma > 2 + \max\{\alpha,\beta\}$. Then for each $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $w \ge 1$ we have $S_w^\chi f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ as, moreover,*

$$\|\rho_{\alpha,\beta} S_w^\chi f\|_{p(\cdot)} \le c\|\rho_{\alpha,\beta} f\|_{p(\cdot)}. \tag{2.11}$$

Proof. As we have already shown in the previous proposition, $S_w^\chi(x)$ is defined on $\mathbb{R}$ and is continuous under the present assumptions. To complete the proof of the proposition, it remains to establish (2.11). To this end, we will estimate the coefficients of the operator by means of the maximal function of $\rho_{\alpha,\beta} f$ and apply (1.6).

We have

$$\left|\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\right| \le \left|\frac{w}{\theta_k}\int_x^{t_k/w} |f(u)|\,du\right| + \left|\frac{w}{\theta_k}\int_x^{t_{k+1}/w} |f(u)|\,du\right|. \tag{2.12}$$

We estimate the first integral above as follows

$$\begin{aligned}\left|\frac{w}{\theta_k}\int_x^{t_k/w} |f(u)|\,du\right| &\le cw \max_{\substack{u \text{ is between}\\ x \text{ and } t_k/w}} \rho_{\alpha,\beta}(u)^{-1} \left|\int_x^{t_k/w} |\rho_{\alpha,\beta}(u) f(u)|\,du\right| \\ &\le cw\left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1}\right)\left|\frac{t_k}{w} - x\right| \mathcal{M}(\rho_{\alpha,\beta} f)(x) \\ &\le c\left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1}\right)|t_k - wx|\mathcal{M}(\rho_{\alpha,\beta} f)(x), \quad x \in \mathbb{R},\end{aligned} \tag{2.13}$$

where we have used inequality (2.3).

By the last relation with t_{k+1} in place of t_k, the trivial estimate

$$|t_{k+1} - wx| \le |t_k - wx| + \Theta$$

and (2.4), we arrive at

$$\left|\frac{w}{\theta_k}\int_x^{t_{k+1}/w} |f(u)|\,du\right| \le c\left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1}\right)\big(|t_k - wx| + 1\big)\mathcal{M}(\rho_{\alpha,\beta} f)(x), \quad x \in \mathbb{R}. \tag{2.14}$$

Estimates (2.12)-(2.14) yield

$$\left|\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\right| \le c\left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1}\right)\big(|t_k - wx| + 1\big)\mathcal{M}(\rho_{\alpha,\beta} f)(x),\ x \in \mathbb{R},\ k \in \mathbb{Z}; \tag{2.15}$$

hence

$$\begin{aligned}|\rho_{\alpha,\beta}(x) S_w^\chi f(x)| &\le c\mathcal{M}(\rho_{\alpha,\beta} f)(x) \sum_{k\in\mathbb{Z}} \big(|t_k - wx| + 1\big)|\chi(wx - t_k)| \\ &\quad + c\mathcal{M}(\rho_{\alpha,\beta} f)(x)\,\rho_{\alpha,\beta}(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1}\big(|t_k - wx| + 1\big)|\chi(wx - t_k)|, \quad x \in \mathbb{R}.\end{aligned}$$

We deduce that each of the two sums on the right hand-side of the inequality above is bounded on $x \in \mathbb{R}$ (the second one with the weight $\rho_{\alpha,\beta}$) by means of Lemma A with $\eta(u) = (|u| + 1)\chi(u)$ and $\lambda = \gamma - 1$, as for the first sum we set $\alpha = \beta = 0$ in the lemma.

Consequently,

$$|\rho_{\alpha,\beta}(x) S_w^\chi f(x)| \le c\mathcal{M}(\rho_{\alpha,\beta} f)(x), \quad x \in \mathbb{R};$$

hence, by virtue of (1.6) with $\rho_{\alpha,\beta} f$ in place of f, we establish the proposition. □

Next, we will establish a Jackson-type estimate.

Proposition 2.3. *Let $r \in \mathbb{N}_+$ and $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha, \beta \geq 0$. Let $\chi \in C(\mathbb{R})$ be such that:*

(i) $\chi(u) = O(|u|^{-\gamma})$, *as* $u \to \pm\infty$, *where* $\gamma > r + 1 + \max\{\alpha, \beta\}$,

(ii) $m_0(\chi, u) \equiv 1$,

(iii) $\displaystyle\sum_{\ell=0}^{j} \binom{j+1}{\ell} \sum_{k\in\mathbb{Z}} \theta_k^{j-\ell} (t_k - u)^\ell \chi(u - t_k) \equiv 0, \quad j = 1, \ldots, r-1,$ *if* $r \geq 2$.

Then for all $f \in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ *and* $w \geq 1$ *there holds*

$$\|\rho_{\alpha,\beta}(S^\chi_w f - f)\|_{p(\cdot)} \leq \frac{c}{w^r} \|\rho_{\alpha,\beta} f^{(r)}\|_{p(\cdot)}.$$

Proof. In the proof of an analogous result in the unweighted case considered in [25, Proposition 2.3], we arrived at the formula (see [25, (2.6)])

$$S^\chi_w f(x) - f(x) = \sum_{k\in\mathbb{Z}} R_{r,k,w} f(x) \chi(wx - t_k), \quad x \in \mathbb{R}, \tag{2.16}$$

where

$$R_{r,k,w} f(x) := \frac{1}{(r-1)!} \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} \left(\int_x^u (u - v)^{r-1} f^{(r)}(v)\, dv \right) du.$$

Further, we derive the following representation of $R_{r,k,w} f(x)$ (see [25, p. 8])

$$R_{r,k,w} f(x) = \frac{1}{r!} \left(\frac{\theta_k}{w}\right)^{r-1} \int_x^{t_{k+1}/w} f^{(r)}(v) \sum_{\ell=0}^{r-1} \binom{r}{\ell} \left(\frac{t_k - wv}{\theta_k}\right)^\ell dv + \frac{1}{r!} \left(\frac{\theta_k}{w}\right)^{r-1} \int_{t_k/w}^{t_{k+1}/w} f^{(r)}(v) \left(\frac{t_k - wv}{\theta_k}\right)^r dv.$$

Next, by (2.15) with $f^{(r)}$ in place of f, we arrive at the estimates

$$\left| \int_{t_k/w}^{t_{k+1}/w} f^{(r)}(v) \left(\frac{t_k - wv}{\theta_k}\right)^r dv \right| \leq \int_{t_k/w}^{t_{k+1}/w} |f^{(r)}(v)|\, dv \leq \frac{c}{w} \left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} \right) (|t_k - wx| + 1) \mathcal{M}(\rho_{\alpha,\beta} f^{(r)})(x)$$

and

$$\begin{aligned}
\left| \int_x^{t_{k+1}/w} f^{(r)}(v) \sum_{\ell=0}^{r-1} \binom{r}{\ell} \left(\frac{t_k - wv}{\theta_k}\right)^\ell dv \right| &\leq c \sum_{\ell=0}^{r-1} \left(|t_k - wx|^\ell + 1\right) \left| \int_x^{t_{k+1}/w} |f^{(r)}(v)|\, dv \right| \\
&\leq c \left(|t_k - wx|^{r-1} + 1\right) \left| \int_x^{t_{k+1}/w} |f^{(r)}(v)|\, dv \right| \\
&\leq \frac{c}{w} \left(|t_k - wx|^{r-1} + 1\right) \left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} \right) (|t_k - wx| + 1) \mathcal{M}(\rho_{\alpha,\beta} f^{(r)})(x) \\
&\leq \frac{c}{w} (|t_k - wx|^r + 1) \left(\rho_{\alpha,\beta}(x)^{-1} + \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} \right) \mathcal{M}(\rho_{\alpha,\beta} f^{(r)})(x)
\end{aligned}$$

for all $x \in \mathbb{R}$, $k \in \mathbb{Z}$.

Consequently,

$$|\rho_{\alpha,\beta}(x) R_{r,k,w} f(x)| \leq \frac{c}{w^r} (|t_k - wx|^r + 1) \left(1 + \rho_{\alpha,\beta}(x) \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} \right) \mathcal{M}(\rho_{\alpha,\beta} f^{(r)})(x), \quad x \in \mathbb{R}. \tag{2.17}$$

Then, in view of (2.16), we arrive at

$$\begin{aligned}
|\rho_{\alpha,\beta}(x)(S^\chi_w f(x) - f(x))| &\leq \sum_{k\in\mathbb{Z}} |\rho_{\alpha,\beta}(x) R_{r,k,w} f(x)|\, |\chi(wx - t_k)| \\
&\leq \frac{c}{w^r} \mathcal{M}(\rho_{\alpha,\beta} f^{(r)})(x) \sum_{k\in\mathbb{Z}} (|t_k - wx|^r + 1) |\chi(wx - t_k)| \\
&\quad + \frac{c}{w^r} \mathcal{M}(\rho_{\alpha,\beta} f^{(r)})(x) \rho_{\alpha,\beta}(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} (|t_k - wx|^r + 1) |\chi(wx - t_k)|,
\end{aligned}$$

where $x \in \mathbb{R}$.

We see that each of the two sums on the right hand-side of the inequality above is bounded on $x \in \mathbb{R}$ by means of Lemma A with $\eta(u) = (|u|^r + 1)\chi(u)$ and $\lambda = \gamma - r$, as for the first sum we set $\alpha = \beta = 0$ in the lemma.

Thus we establish that

$$|\rho_{\alpha,\beta}(x)(S^\chi_w f(x) - f(x))| \leq \frac{c}{w^r} \mathcal{M}(\rho_{\alpha,\beta} f^{(r)})(x), \quad x \in \mathbb{R},$$

which, by virtue of (1.6) with $\rho_{\alpha,\beta} f^{(r)}$ in place of f, yields the estimate in the proposition. □

Next, we prove a Voronovskaya-type inequality.

Proposition 2.4. *Let $r \in \mathbb{N}_+$ and $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha, \beta \ge 0$. Let $\chi \in C(\mathbb{R})$ be such that:*

(i) $\chi(u) = O(|u|^{-\gamma})$, *as* $u \to \pm\infty$, *where* $\gamma > r + 2 + \max\{\alpha, \beta\}$,

(ii) $m_0(\chi, u) \equiv 1$,

(iii) $\displaystyle\sum_{\ell=0}^{j} \binom{j+1}{\ell} \sum_{k\in\mathbb{Z}} \theta_k^{j-\ell} (t_k - u)^\ell \chi(u - t_k) \equiv 0, \quad j = 1, \ldots, r-1,$ *if* $r \ge 2$,

(iv) $\displaystyle\varphi_r := \sum_{\ell=0}^{r} \binom{r+1}{\ell} \sum_{k\in\mathbb{Z}} \theta_k^{r-\ell} (t_k - u)^\ell \chi(u - t_k) \equiv \text{const} \neq 0.$

Then for all $f \in W^{r+1}_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $w \ge 1$ there holds

$$\left\| \rho_{\alpha,\beta} \left(S_w^\chi f - f - \frac{\varphi_r}{(r+1)!\, w^r} f^{(r)} \right) \right\|_{p(\cdot)} \le \frac{c}{w^{r+1}} \|\rho_{\alpha,\beta} f^{(r+1)}\|_{p(\cdot)}.$$

A Voronovskaya-type pointwise estimate for S_w^χ with $t_k = k$ was established in [1, Theorem 5.2] with the weight $\rho_{2,2}$ in the form (1.4). Such an estimate for Durrmeyer-type sampling operators, which include S_w^χ with any equidistant t_k, was obtained in [9, Theorem 4] in the same setting.

Proof of Proposition 2.4. By [25, (2.9)]

$$S_w^\chi f(x) - f(x) - \frac{\varphi_r}{(r+1)!\, w^r} f^{(r)}(x) = \sum_{k\in\mathbb{Z}} R_{r+1,k,w} f(x) \chi(wx - t_k), \quad x \in \mathbb{R},$$

which along with (2.17) with $r+1$ in place of r, implies

$$\begin{aligned} \left| \rho_{\alpha,\beta}(x) \left(S_w^\chi f(x) - f(x) - \frac{\varphi_r}{(r+1)!\, w^r} f^{(r)}(x) \right) \right| &\le \frac{c}{w^{r+1}} \mathcal{M}\big(\rho_{\alpha,\beta} f^{(r+1)}\big)(x) \sum_{k\in\mathbb{Z}} (|t_k - wx|^{r+1} + 1)|\chi(wx - t_k)| \\ &+ \frac{c\rho_{\alpha,\beta}(x)}{w^{r+1}} \mathcal{M}\big(\rho_{\alpha,\beta} f^{(r+1)}\big)(x) \sum_{k\in\mathbb{Z}} \rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1} (|t_k - wx|^{r+1} + 1)|\chi(wx - t_k)|, \end{aligned}$$

where $x \in \mathbb{R}$.

Next, as in the previous proofs, we apply Lemma A to show that the sums of the series (for the second one with the weight $\rho_{\alpha,\beta}$) are bounded on $\mathbb{R}$. We use the lemma with $\eta(u) = (|u|^{r+1} + 1)\chi(u)$ and $\lambda = \gamma - r - 1$.

Thus we arrive at

$$\left| \rho_{\alpha,\beta}(x) \left(S_w^\chi f(x) - f(x) - \frac{\varphi_r}{(r+1)!\, w^r} f^{(r)}(x) \right) \right| \le \frac{c}{w^{r+1}} \mathcal{M}\big(\rho_{\alpha,\beta} f^{(r+1)}\big)(x), \quad x \in \mathbb{R}.$$

Hence, the estimate in the proposition follows from (1.6). □

The last two basic estimates for S_w^χ are Bernstein-type inequalities. We will make use of the following auxiliary result established in [3, Lemma 4.1].

Lemma B. *Let $r \in \mathbb{N}_+$. Let $\chi \in C^r(\mathbb{R})$ be such that $M_0(\chi) < \infty$ and*

$$\chi^{(r)}(u) = O(|u|^{-\gamma}), \quad u \to \pm\infty,$$

with some real $\gamma > r$. Then

$$\chi^{(j)}(u) = O(|u|^{-\gamma+r-j}), \quad u \to \pm\infty, \quad j = 0, \ldots, r-1.$$

Proposition 2.5. *Let $r \in \mathbb{N}_+$. Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha, \beta \ge 0$. Let $\chi \in C^{(r)}(\mathbb{R})$ be such that $M_0(\chi) < \infty$ and $\chi^{(r)}(u) = O(|u|^{-\gamma})$, as $u \to \pm\infty$, where $\gamma > r + 1 + \max\{\alpha, \beta\}$.*

Then $S_w^\chi f \in C^r(\mathbb{R})$ and $(S_w^\chi f)^{(r)} \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ for all $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $w \ge 1$, as, moreover,

$$\|\rho_{\alpha,\beta}(S_w^\chi f)^{(r)}\|_{p(\cdot)} \le c w^r \|\rho_{\alpha,\beta} f\|_{p(\cdot)}.$$

Proof. First, we note that each of the series

$$\sum_{k\in\mathbb{Z}} \frac{w}{\theta_k} \int_{t_k/w}^{t_{k+1}/w} f(u)\,du\,\chi^{(j)}(wx-k), \quad j=0,\ldots,r, \tag{2.18}$$

is uniformly convergent on the compact intervals of $\mathbb{R}$.

Indeed, by virtue of Lemma B, we have $\chi^{(j)}(u)=O(|u|^{-\gamma+r-j})$ for $j=0,\ldots,r-1$. Similarly to (2.10), we have

$$\left|\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\right| |\chi^{(j)}(wx-t_k)| \le c'''(1+|k|)^{\delta-\gamma+r-j}, \quad x\in[a,b],\ k\in\mathbb{Z},$$

where $\delta := \max\{\alpha,\beta\}$ and c''' is a positive constant, whose value is independent of x and k (but may depend on f and w). Since $\delta-\gamma+r-j<-1$ for $j=0,\ldots,r$, we conclude, as at the end of the proof of Proposition 2.1, that each of the series (2.18) is uniformly convergent on $[a,b]$.

Consequently, $S_w^\chi f \in C^r(\mathbb{R})$ and

$$(S_w^\chi f)^{(r)}(x) = w^r \sum_{k\in\mathbb{Z}} \frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\,\chi^{(r)}(wx-t_k), \quad x\in\mathbb{R}. \tag{2.19}$$

Next, we apply estimate (2.15) to derive from (2.19)

$$\begin{aligned}|\rho_{\alpha,\beta}(x)(S_w^\chi f)^{(r)}(x)| &\le cw^r\mathcal{M}(\rho_{\alpha,\beta}f)(x)\sum_{k\in\mathbb{Z}}\big(|t_k-wx|+1\big)|\chi^{(r)}(wx-t_k)|\\ &\quad + cw^r\mathcal{M}(\rho_{\alpha,\beta}f)(x)\,\rho_{\alpha,\beta}(x)\sum_{k\in\mathbb{Z}}\rho_{\alpha,\beta}\left(\frac{t_k}{w}\right)^{-1}\big(|t_k-wx|+1\big)|\chi^{(r)}(wx-t_k)|\end{aligned}$$

for all $x\in\mathbb{R}$.

As earlier, we apply Lemma A with $\eta(u)=(|u|+1)\chi^{(r)}(u)$ and $\lambda=\gamma-1$ to deduce

$$|\rho_{\alpha,\beta}(x)(S_w^\chi f)^{(r)}(x)| \le cw^r\mathcal{M}(\rho_{\alpha,\beta}f)(x), \quad x\in\mathbb{R}.$$

By virtue of (1.6) with $\rho_{\alpha,\beta}f$ in place of f, we complete the proof of the proposition. □

Proposition 2.6. *Let $r\in\mathbb{N}_+$. Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_*>1$ and $1/p(\cdot)\in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta\ge 0$. Let $\chi\in C^{r+1}(\mathbb{R})$ be such that:*

(i) $M_0(\chi)<\infty$,

(ii) $\chi^{(r+1)}(u)=O(|u|^{-\gamma})$, *as* $u\to\pm\infty$, *where* $\gamma>r+2+\max\{\alpha,\beta\}$,

(iii) $\displaystyle\sum_{\ell=0}^{j}\binom{j+1}{\ell}\sum_{k\in\mathbb{Z}}\theta_k^{j-\ell}(t_k-u)^\ell\chi^{(r+1)}(u-t_k)\equiv 0, \quad j=0,\ldots,r-1.$

Then $S_w^\chi f\in C^{r+1}(\mathbb{R})$ and $(S_w^\chi f)^{(r+1)}\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ for all $f\in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $w\ge 1$, as, moreover,

$$\|\rho_{\alpha,\beta}(S_w^\chi f)^{(r+1)}\|_{p(\cdot)} \le cw\,\|\rho_{\alpha,\beta}f^{(r)}\|.$$

Proof. Just as in the proof of the previous proposition we show that $S_w^\chi f\in C^{r+1}(\mathbb{R})$ and

$$(S_w^\chi f)^{(r+1)}(x) = w^{r+1}\sum_{k\in\mathbb{Z}}\frac{w}{\theta_k}\int_{t_k/w}^{t_{k+1}/w} f(u)\,du\,\chi^{(r+1)}(wx-t_k), \quad x\in\mathbb{R}.$$

In the proof of [25, Proposition 2.11], we established the formula

$$(S_w^\chi f)^{(r+1)}(x) = w^{r+1}\sum_{k\in\mathbb{Z}} R_{r,k,w}f(x)\chi^{(r+1)}(wx-t_k), \quad x\in\mathbb{R}.$$

Then we use (2.17) to complete the proof just in the same way as for Proposition 2.3 but with $\chi^{(r+1)}$ in the place of χ. □

3 Moduli of smoothness and K-functionals

We will extend the properties of $\Omega_r(f,t)_{p(\cdot),0,0}$ and $\overline{\Omega}_r(f,t)_{p(\cdot),0,0}$, established in [25, Section 3], to their weighted forms. Several properties of an analogue of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ for Muckenhoupt weights were given in [6, p. 206]. The properties of a function characteristic, based on Steklov means and quite similar to a modulus of smoothness, were established in [7].

Theorem 3.1. *Let $r\in\mathbb{N}_+$. Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_*>1$ and $1/p(\cdot)\in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta\ge 0$. Then there hold:*

(a) $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}\le 2^r M(1+rt)^\beta\|\rho_{\alpha,\beta}f\|_{p(\cdot)}$, $f\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$, $t>0$;

(b) $\Omega_r(f+g,t)_{p(\cdot),\alpha,\beta}\le \Omega_r(f,t)_{p(\cdot),\alpha,\beta}+\Omega_r(g,t)_{p(\cdot),\alpha,\beta}$, $f,g\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$, $t>0$;

(c) $\Omega_r(cf,t)_{p(\cdot),\alpha,\beta}=|c|\,\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$, $c\in\mathbb{R}$ *and* $f\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$, $t>0$;

(d) $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}\le M(1+rt)^\beta\, t^r\,\|\rho_{\alpha,\beta}f^{(r)}\|_{p(\cdot)}$, $f\in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$, $t>0$.

Here M is the constant in inequality (1.6).

The assertions remain valid with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ in place of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$.

Proof. We will verify the properties regarding $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$. Their proof for $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ is either the same, or quite similar.

To prove (a) we use (2.3) to get

$$\begin{aligned}\left|\frac{\rho_{\alpha,\beta}(x)}{h}\int_0^h \Delta_u^r f(x)\,du\right| &\le \sum_{\ell=0}^r\binom{r}{\ell}\frac{\rho_{\alpha,\beta}(x)}{h}\int_0^h|f(x+(r-\ell)u)|\,du\\ &=|\rho_{\alpha,\beta}(x)f(x)|+\sum_{\ell=1}^r\binom{r}{\ell}\frac{\rho_{\alpha,\beta}(x)}{\ell h}\int_x^{x+\ell h}|f(u)|\,du\\ &=|\rho_{\alpha,\beta}(x)f(x)|+\sum_{\ell=1}^r\binom{r}{\ell}\max\left\{1,\frac{\rho_{\alpha,\beta}(x)}{\rho_{\alpha,\beta}(x+\ell h)}\right\}\frac{1}{\ell h}\int_x^{x+\ell h}|\rho_{\alpha,\beta}(u)f(u)|\,du\\ &\le|\rho_{\alpha,\beta}(x)f(x)|+\mathcal{M}(\rho_{\alpha,\beta}f)(x)\sum_{\ell=1}^r\binom{r}{\ell}\max\left\{1,\frac{\rho_{\alpha,\beta}(x)}{\rho_{\alpha,\beta}(x+\ell h)}\right\}.\end{aligned}$$

We again take into consideration (2.3) to arrive at

$$\rho_{\alpha,\beta}(x+\ell h)^{-1}\le\max\{\rho_{\alpha,\beta}(x)^{-1},\rho_{\alpha,\beta}(x+rt)^{-1}\}$$

for $\ell=1,\dots,r$ and $0<h\le t$.

Therefore,

$$\sum_{\ell=1}^r\binom{r}{\ell}\max\left\{1,\frac{\rho_{\alpha,\beta}(x)}{\rho_{\alpha,\beta}(x+\ell h)}\right\}\le(2^r-1)\max\left\{1,\frac{\rho_{\alpha,\beta}(x)}{\rho_{\alpha,\beta}(x+rt)}\right\}.$$

Elementary arguments show that

$$\frac{\rho_{\alpha,\beta}(x)}{\rho_{\alpha,\beta}(x+rt)}\le(1+rt)^\beta,\quad x\in\mathbb{R}. \tag{3.1}$$

Now, assertion (a) follows from (1.6) and $M\ge 1$.

Properties (b) and (c) follow directly from the definition of the modulus of smoothness.

To verify (d), we apply the formula (see, e.g., [22, p. 45])

$$\Delta_u^r f(x)=u^r\int_0^r M_r(v)f^{(r)}(x+uv)\,dv,\quad x\in\mathbb{R},$$

where $M_r(v)$ is the r-fold convolution of the characteristic function of $[0,1]$ with itself.

We have $0\le M_r(v)\le 1$, $v\in[0,r]$ (see [22, p. 45]). Then, similarly to the proof of (a), we get for $0<h\le t$

$$\begin{aligned}\left|\frac{\rho_{\alpha,\beta}(x)}{h}\int_0^h \Delta_u^r f(x)\,du\right| &\le\frac{t^{r-1}}{r}\rho_{\alpha,\beta}(x)\int_x^{x+rt}|f^{(r)}(v)|\,dv\\ &\le t^r\mathcal{M}\big(\rho_{\alpha,\beta}f^{(r)}\big)(x)\max\left\{1,\frac{\rho_{\alpha,\beta}(x)}{\rho_{\alpha,\beta}(x+rt)}\right\}\\ &\le(1+rt)^\beta\,t^r\mathcal{M}\big(\rho_{\alpha,\beta}f^{(r)}\big)(x),\quad x\in\mathbb{R}.\end{aligned}$$

Assertion (d) follows from (1.6). □

We will use the relation of the moduli of smoothness to the K-functional

$$K_r(f,t)_{p(\cdot),\alpha,\beta} := \inf_{g\in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})} \left\{\|\rho_{\alpha,\beta}(f-g)\|_{p(\cdot)} + t\|\rho_{\alpha,\beta}g^{(r)}\|_{p(\cdot)}\right\}, \quad t>0.$$

That relation is quite similar to the one between the classical moduli of smoothness and the K-functional above in the case of the constant exponent Lebesgue spaces (e.g., [22, Chapter 6, Theorem 2.4]).

Theorem 3.2. *Let $r\in\mathbb{N}_+$. Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_*>1$ and $1/p(\cdot)\in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta\ge 0$. Then for all $f\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $t\in(0,1]$ there holds*

$$\frac{1}{2^r M(r+1)^\beta}\Omega_r(f,t)_{p(\cdot),\alpha,\beta} \le K_r(f,t^r)_{p(\cdot),\alpha,\beta} \le (2r)^{r+2}\Omega_r(f,t)_{p(\cdot),\alpha,\beta},$$

where M is the constant in inequality (1.6).

The assertion remains valid with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ in place of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$.

Proof. The left inequality for both types of moduli follows easily from properties (a), (b) and (d) of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ in Theorem 3.1—see the proof of the analogous result for the classical moduli in, e.g., [22, p. 177].

To prove the other inequality we follow verbatim the argument we used in the unweighted case in [25, pp. 20–21]—we only need to multiply by $\rho_{\alpha,\beta}(x)$ the identities there and then derive from them the corresponding estimates in the $L_{p(\cdot)}$-norm. Thus, we show with

$$f_{r,t}(x) := \frac{r+1}{t^{r+1}}\int_0^t \sum_{\ell=0}^{r-1}(-1)^{r-\ell-1}\binom{r}{\ell}\left(\int_0^u\cdots\int_0^u f\left(x+\frac{r-\ell}{r}(u_1+\cdots+u_r)\right)du_1\cdots du_r\right)du, \quad x\in\mathbb{R},$$

the estimates:

$$\|\rho_{\alpha,\beta}(f-f_{r,t})\|_{p(\cdot)} \le 4r\,\Omega_r(f,t)_{p(\cdot),\alpha,\beta}, \quad t>0,$$

and

$$t^r\|\rho_{\alpha,\beta}f^{(r)}_{r,t}\|_{p(\cdot)} \le (r+1)(2r)^r\Omega_r(f,t)_{p(\cdot),\alpha,\beta}, \quad t>0.$$

Consequently,

$$\begin{aligned} K_r(f,t^r)_{p(\cdot),\alpha,\beta} &\le \|\rho_{\alpha,\beta}(f-f_{r,t})\|_{p(\cdot)} + t^r\|\rho_{\alpha,\beta}f^{(r)}_{r,t}\|_{p(\cdot)} \\ &\le (2r)^{r+2}\Omega_r(f,t)_{p(\cdot),\alpha,\beta}, \quad t>0. \end{aligned}$$

□

The previous theorem and the definition of the moduli of smoothness imply the following assertion.

Corollary 3.3. *Let $r\in\mathbb{N}_+$. Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_*>1$ and $1/p(\cdot)\in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta\ge 0$. Then for all $f\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $t\in(0,1]$ there holds*

$$\Omega_r(f,t)_{p(\cdot),\alpha,\beta} \le \overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta} \le (4r)^{r+2}M(r+1)^\beta\Omega_r(f,t)_{p(\cdot),\alpha,\beta}.$$

At the end of the section, we will extend [25, Theorem 3.4] to the weighted case, we consider. We need to note a mistake in the statement of [25, Theorem 3.4(i)]—it is valid under the additional assumption that $p^*<\infty$. For the statement of the theorem below, we recall that we have set $p_\infty := \lim_{x\to\pm\infty}p(x)$ (see p. 74). Since $p(x)\ge 1$ for all x, then $p_\infty\ge 1$ too.

Theorem 3.4. *Let $r\in\mathbb{N}_+$. Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_*>1$ and $1/p(\cdot)\in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta\ge 0$. Let $f\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}=o(t^r)$. Then:*

(a) If $p_\infty\in\mathbb{R}$ and either $\alpha\le 1/p_\infty$, or $\beta\le 1/p_\infty$, then $f=0$ a.e.;

(b) If $p_\infty\in\mathbb{R}$ and $\alpha,\beta>1/p_\infty$, then f coincides a.e. with an algebraic polynomial of degree s, where $s\le r-1$ and $s<\alpha-1/p_\infty,\beta-1/p_\infty$;

(c) If $p_\infty=+\infty$, then f coincides a.e. with an algebraic polynomial of degree s, where $s\le r-1,\alpha,\beta$.

The assertions remain valid with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ in place of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$.

Proof. By virtue of the right inequality in Theorem 3.2, we have $K_r(f,t)_{p(\cdot),\alpha,\beta} = o(t)$; hence f coincides a.e. with an algebraic polynomial of degree at most $r-1$.

Clearly, if $f = 0$ a.e., then $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$. Since $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ is a linear space, it is sufficient to determine which monomials x^n, $n = 0,\dots,r-1$, are in it, that is, there exists $\lambda > 0$ such that

$$\int_{\mathbb{R}\setminus\mathbb{R}_\infty^{p(\cdot)}} \left|\frac{\rho_{\alpha,\beta}(x)x^n}{\lambda}\right|^{p(x)} dx < \infty \tag{3.2}$$

and

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}_\infty^{p(\cdot)}} |\rho_{\alpha,\beta}(x)x^n| < \infty. \tag{3.3}$$

(a)-(b) Let $p_\infty \in \mathbb{R}$. Then there exists a positive real p_0 such that $p(x)$ is finite if $|x| \ge p_0$. Hence the set $\mathbb{R}_\infty^{p(\cdot)}$ is bounded and (3.3) is satisfied for any $n \in \mathbb{N}_0$.

Further, if the set T is bounded and $\lambda > \max_{x\in\overline{T}} |\rho_{\alpha,\beta}(x)x^n|$, where $\overline{T}$ is the closure of T, then the function $|\rho_{\alpha,\beta}(x)x^n/\lambda|^{p(x)}$ is bounded on T. Thus, to determine for which n relation (3.2) holds with some $\lambda > 0$, it is sufficient to see when there exists $\lambda > 0$ such that

$$\int_{p_0}^{+\infty} \left(\frac{x}{\lambda}\right)^{(n-\tilde{\delta})p(x)} dx < \infty, \tag{3.4}$$

where $\tilde{\delta} := \min\{\alpha,\beta\}$. We will show that the latter holds iff $(n-\tilde{\delta})p_\infty < -1$ regardless of λ.

Straightforward considerations demonstrate that if $(n-\tilde{\delta})p_\infty < -1$, then (3.4) is valid, and if $(n-\tilde{\delta})p_\infty > -1$, then it is not.

Let $(n-\tilde{\delta})p_\infty = -1$. By virtue of (1.7) with $r(x) = 1/p(x)$, we have

$$|p(x) - p_\infty| \le \frac{c}{\log x}, \quad x \ge e, p_0,$$

with some positive constant c; hence

$$(n-\tilde{\delta})p(x) \ge -1 - \frac{c'}{\log x}, \quad x \ge e, p_0,$$

where $c' > 0$ is a constant.

Consequently, we have with $a := \max\{p_0, e, \lambda\}$

$$\begin{aligned}\int_a^{+\infty} \left(\frac{x}{\lambda}\right)^{(n-\tilde{\delta})p(x)} dx &\ge \int_a^{+\infty} \left(\frac{x}{\lambda}\right)^{-1-\frac{c'}{\log x}} dx \\ &\ge \min\left\{\lambda, \lambda^{1+\frac{c'}{\log a}}\right\} \int_a^{+\infty} x^{-1-\frac{c'}{\log x}} dx \\ &= \min\left\{\lambda, \lambda^{1+\frac{c'}{\log a}}\right\} e^{-c'} \int_{\log a}^{+\infty} dy = +\infty,\end{aligned}$$

where we have made the change of the variable $x = e^y$. Therefore, (3.4) is not satisfied for any $\lambda > 0$ if $(n-\tilde{\delta})p_\infty = -1$. That completes the proof of (a) and (b).

(c) Let $p_\infty = +\infty$. Since $p(x) \ge 1$ and at least one of the sets $\mathbb{R}_\infty^{p(\cdot)}$ or $\mathbb{R}\setminus\mathbb{R}_\infty^{p(\cdot)}$ is unbounded in Lebesgue measure, then in order to have (3.2)-(3.3), it is necessary that $n - \tilde{\delta} \le 0$.

Clearly, if $n - \tilde{\delta} \le 0$, then (3.3) holds. We will show that, if $n - \tilde{\delta} \le 0$, then (3.2) is also valid with some $\lambda > 0$. Again, as in the proof of (a)-(b), we have that if the set T is bounded and $\lambda > \max_{x\in\overline{T}} |\rho_{\alpha,\beta}(x)x^n|$, then the function $|\rho_{\alpha,\beta}(x)x^n/\lambda|^{p(x)}$ is bounded on T and to verify (3.2) with some $\lambda > 0$, it remains to show that there exists $\lambda > 0$ such that

$$\int_{T_1} \left(\frac{x^{n-\tilde{\delta}}}{\lambda}\right)^{p(x)} dx < \infty,$$

where $T_1 := [1,+\infty)\setminus\mathbb{R}_\infty^{p(\cdot)}$.

Since $p_\infty = +\infty$, relation (1.7) with $r(x) = 1/p(x)$ implies $r_\infty = 0$ and then

$$p(x) \ge \frac{\log x}{c_\infty}, \quad x \ge 1,$$

with some constant $c_\infty > 0$. Here we also take into account that $p(x) \ge 1$ for $x \in \mathbb{R}$. Therefore, we have with $\lambda = e^{2c_\infty}$

$$\int_{T_1} \left(\frac{x^{n-\tilde{\delta}}}{\lambda}\right)^{p(x)} dx \le \int_{T_1} e^{-2c_\infty p(x)} dx \le \int_{T_1} \frac{dx}{x^2} < \infty,$$

which completes the proof of (c).

The assertions with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ in place of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ are reduced to the ones for $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ because $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta} = o(t^r)$ implies $\Omega_r(f,t)_{p(\cdot),\alpha,\beta} = o(t^r)$ in view of the trivial inequality $\Omega_r(f,t)_{p(\cdot),\alpha,\beta} \le \overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$. □

Next, we will extend [25, Theorem 3.4(ii)] to the weighted case.

Theorem 3.5. *Let $r\in\mathbb{N}_+$. Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_*>1$ and $1/p(\cdot)\in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta\ge 0$. Let $f\in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$. If $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}=O(t^r)$, then $f\in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$.*

The assertions remain valid with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ in place of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$.

The assertion can be established just as the analogous result for the non-variable exponent Lebesgue spaces (see, e.g., [22, Chapter 2, Theorem 9.3]). We followed that approach in the unweighted case stated in [25, Theorem 3.4(ii)]. However, it can also be derived from [25, Theorem 3.4(ii)]. We will present this argument below. It includes an embedding inequality, which is of independent interest.

Proposition 3.6. *Let $r\in\mathbb{N}_+$, $r\ge 2$. Let $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_*>1$ and $1/p(\cdot)\in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha,\beta\ge 0$. Then for any $g\in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ there hold*

$$\|\rho_{\alpha,\beta}g^{(j)}\|_{p(\cdot)}\le c\big(\|\rho_{\alpha,\beta}g\|_{p(\cdot)}+\|\rho_{\alpha,\beta}g^{(r)}\|_{p(\cdot)}\big),\quad j=1,\dots,r-1.$$

Proof. We mostly follow the argument that verifies the assertion for the constant exponent L_p-space (see, e.g., [22, p. 38]), as, in addition, we avail ourselves of the boundedness of the maximal operator in variable Lebesgue spaces whose exponent function satisfies the assumptions above.

Let $u>0$ and $x\in\mathbb{R}$. We expand $g(x+u)$ by Taylor's formula at x up to the derivative of order r:

$$g(x+u)=\sum_{j=0}^{r-1}\frac{u^j}{j!}g^{(j)}(x)+R_rg(x,u),\tag{3.5}$$

where

$$R_rg(x,u):=\frac{1}{(r-1)!}\int_x^{x+u}(x+u-v)^{r-1}g^{(r)}(v)\,dv.$$

We integrate (3.5) on u from 0 to $t>0$, multiply it by $\rho_{\alpha,\beta}(x)/t$ and rearrange the terms to arrive at

$$\sum_{j=1}^{r-1}\frac{t^j}{(j+1)!}\rho_{\alpha,\beta}(x)g^{(j)}(x)=\frac{\rho_{\alpha,\beta}(x)}{t}\int_x^{x+t}g(u)\,du-\rho_{\alpha,\beta}(x)g(x)-\frac{\rho_{\alpha,\beta}(x)}{t}\int_0^t R_rg(x,u)\,du.\tag{3.6}$$

We set $t=1,\dots,r-1$ in (3.6) to get a linear system with unknowns $\rho_{\alpha,\beta}(x)g^{(j)}(x)$, $j=1,\dots,r-1$, whose determinant is non-zero because it is a positive multiple of the Vandermonde one. In view of Cramer's rule, to complete the proof of the proposition, it remains to show that the first and the last term on the right side of (3.6) as functions of x are in $L_{p(\cdot)}(\mathbb{R})$ and their $L_{p(\cdot)}(\mathbb{R})$-norm is bounded by $\|\rho_{\alpha,\beta}g\|_{p(\cdot)}$ and $\|\rho_{\alpha,\beta}g^{(r)}\|_{p(\cdot)}$, respectively, for each positive t. To this end, by virtue of (1.6), it is sufficient to estimate those terms in (3.6) by the maximal functions of $\rho_{\alpha,\beta}g$ and $\rho_{\alpha,\beta}g^{(r)}$, respectively.

As we have done numerous times already, we get for the first one by means of (2.3) and (3.1) that

$$\begin{aligned}\left|\frac{\rho_{\alpha,\beta}(x)}{t}\int_x^{x+t}g(u)\,du\right|&\le\frac{\rho_{\alpha,\beta}(x)}{t}\big(\rho_{\alpha,\beta}(x)^{-1}+\rho_{\alpha,\beta}(x+t)^{-1}\big)\int_x^{x+t}|\rho_{\alpha,\beta}(u)g(u)|\,du\\&\le\frac{c_0}{t}\int_x^{x+t}|\rho_{\alpha,\beta}(u)g(u)|\,du\le c_0\mathcal{M}(\rho_{\alpha,\beta}g)(x),\quad x\in\mathbb{R}.\end{aligned}$$

Above and below until the end of the proof, c_0 denotes a positive constant whose value is independent of x and g, but depends on t. Its growth w.r.t. t can be estimated by (3.1), but we do not need that.

We estimate the last term on the right of (3.6) in a similar way. First, we get for $u\in(0,t]$ and any x

$$\begin{aligned}|\rho_{\alpha,\beta}(x)R_rg(x,u)|&\le\frac{u^{r-1}\rho_{\alpha,\beta}(x)}{(r-1)!}\big(\rho_{\alpha,\beta}(x)^{-1}+\rho_{\alpha,\beta}(x+t)^{-1}\big)\int_x^{x+u}|\rho_{\alpha,\beta}(v)g^{(r)}(v)|\,dv\\&\le c_0\mathcal{M}(\rho_{\alpha,\beta}g^{(r)})(x);\end{aligned}$$

hence,

$$\left|\frac{\rho_{\alpha,\beta}(x)}{t}\int_0^t R_rg(x,u)\,du\right|\le c_0\mathcal{M}(\rho_{\alpha,\beta}g^{(r)})(x),\quad x\in\mathbb{R}.$$

This completes the proof of the proposition. □

We proceed to the proof of the theorem. We set $\Omega_r(f,t)_{p(\cdot)}:=\Omega_r(f,t)_{p(\cdot),0,0}$, where $f\in L_{p(\cdot)}(\mathbb{R})$, and $W^r_{p(\cdot)}(\mathbb{R}):=W^r_{p(\cdot),0,0}(\mathbb{R})$.

Proof of Theorem 3.5. We will establish the assertion for $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$. Then we can get it for $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ just as in the proof of the previous theorem.

Let $\tilde{\rho}_{\alpha,\beta} \in C^r(\mathbb{R})$ be such that $\tilde{\rho}_{\alpha,\beta}(x) = \rho_{\alpha,\beta}(x)$ if $|x| \ge 1$ and $\tilde{\rho}_{\alpha,\beta}(x) > 0$ on $[-1,1]$.

We will show that if $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $0 < t \le 1$, then

$$\Omega_r(\tilde{\rho}_{\alpha,\beta}f,t)_{p(\cdot)} \le c\big(\Omega_r(f,t)_{p(\cdot),\alpha,\beta} + t^r\|\rho_{\alpha,\beta}f\|_{p(\cdot)}\big). \tag{3.7}$$

We have for any $g \in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $0 < t \le 1$ that

$$\begin{aligned}\Omega_r(\tilde{\rho}_{\alpha,\beta}f,t)_{p(\cdot)} &\le \Omega_r(\tilde{\rho}_{\alpha,\beta}(f-g),t)_{p(\cdot)} + \Omega_r(\tilde{\rho}_{\alpha,\beta}g,t)_{p(\cdot)}\\ &\le c\big(\|\tilde{\rho}_{\alpha,\beta}(f-g)\|_{p(\cdot)} + t^r\|(\tilde{\rho}_{\alpha,\beta}g)^{(r)}\|_{p(\cdot)}\big).\end{aligned}$$

Clearly,

$$\tilde{\rho}^{(j)}_{\alpha,\beta}(x) \le c\rho_{\alpha,\beta}(x), \quad x \in \mathbb{R},\ j = 0,\dots,r. \tag{3.8}$$

Thus, we get

$$\|\tilde{\rho}_{\alpha,\beta}(f-g)\|_{p(\cdot)} \le c\|\rho_{\alpha,\beta}(f-g)\|_{p(\cdot)}$$

and, in addition, by virtue of Proposition 3.6,

$$\begin{aligned}\|(\tilde{\rho}_{\alpha,\beta}g)^{(r)}\|_{p(\cdot)} &\le c\sum_{j=0}^{r}\|\tilde{\rho}^{(r-j)}_{\alpha,\beta}g^{(j)}\|_{p(\cdot)} \le c\sum_{j=0}^{r}\|\rho_{\alpha,\beta}g^{(j)}\|_{p(\cdot)}\\ &\le c\big(\|\rho_{\alpha,\beta}g\|_{p(\cdot)} + \|\rho_{\alpha,\beta}g^{(r)}\|_{p(\cdot)}\big)\\ &\le c\big(\|\rho_{\alpha,\beta}(f-g)\|_{p(\cdot)} + \|\rho_{\alpha,\beta}g^{(r)}\|_{p(\cdot)} + \|\rho_{\alpha,\beta}f\|_{p(\cdot)}\big).\end{aligned}$$

We combine the above estimates to arrive at

$$\Omega_r(\tilde{\rho}_{\alpha,\beta}f,t)_{p(\cdot)} \le c\big(\|\rho_{\alpha,\beta}(f-g)\|_{p(\cdot)} + t^r\|\rho_{\alpha,\beta}g^{(r)}\|_{p(\cdot)}\big) + ct^r\|\rho_{\alpha,\beta}f\|_{p(\cdot)}$$

for any $g \in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $t \in (0,1]$. We take the infimum on g, to arrive at

$$\Omega_r(\tilde{\rho}_{\alpha,\beta}f,t)_{p(\cdot)} \le c\big(K_r(f,t^r)_{p(\cdot),\alpha,\beta} + t^r\|\rho_{\alpha,\beta}f\|_{p(\cdot)}\big), \quad 0 < t \le 1.$$

Now, (3.7) follows from the right inequality in Theorem 3.2.

Next, $\Omega_r(f,t)_{p(\cdot),\alpha,\beta} = O(t^r)$ yields, by virtue of (3.7), that $\Omega_r(\tilde{\rho}_{\alpha,\beta}f,t)_{p(\cdot)} = O(t^r)$ and then [25, Theorem 3.4(ii)] implies $\tilde{\rho}_{\alpha,\beta}f \in W^r_{p(\cdot)}(\mathbb{R})$ (in the proof of [25, Theorem 3.4(ii)] the assumption that $p(x)$ is finite was not used). That, in particular, yields $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $f \in AC^{r-1}_{loc}(\mathbb{R})$. It remains to show that $\rho_{\alpha,\beta}f^{(r)} \in L_{p(\cdot)}(\mathbb{R})$.

By virtue of Proposition 3.6 with $\alpha = \beta = 0$ and $g = \tilde{\rho}_{\alpha,\beta}f$, we deduce that $(\tilde{\rho}_{\alpha,\beta}f)^{(j)} \in L_{p(\cdot)}(\mathbb{R})$ for all $j = 1,\dots,r$. That along with the Leibniz rule, (3.8) and $\rho_{\alpha,\beta}f \in L_{p(\cdot)}(\mathbb{R})$ enables us to get consecutively for $j = 1,\dots,r$ that $\rho_{\alpha,\beta}f^{(j)} \in L_{p(\cdot)}(\mathbb{R})$. □

4 Proof of the main results

Proof of Theorem 1.1. The assertions that the series, defining $S^\chi_w f(x)$, is uniformly convergent on the compact intervals of $\mathbb{R}$ and, hence, $S^\chi_w f \in C(\mathbb{R})$ follow from Proposition 2.1. In addition, Proposition 2.2 implies $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$.

The direct estimate follows from Propositions 2.2 and 2.3 by a short standard argument. We have for any $g \in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and any $w \ge 1$ that

$$\begin{aligned}\|\rho_{\alpha,\beta}(S^\chi_w f - f)\|_{p(\cdot)} &\le \|\rho_{\alpha,\beta}S^\chi_w(f-g)\|_{p(\cdot)} + \|\rho_{\alpha,\beta}(S^\chi_w g - g)\|_{p(\cdot)} + \|\rho_{\alpha,\beta}(g-f)\|_{p(\cdot)}\\ &\le c\left(\|\rho_{\alpha,\beta}(f-g)\|_{p(\cdot)} + \frac{1}{w^r}\|\rho_{\alpha,\beta}g^{(r)}\|_{p(\cdot)}\right).\end{aligned}$$

Taking the infimum on g, we arrive at

$$\|\rho_{\alpha,\beta}(S^\chi_w f - f)\|_{p(\cdot)} \le cK_r(f,w^{-r})_{p(\cdot),\alpha,\beta}, \quad w \ge 1.$$

Now, the assertion of the theorem for each of the two moduli of smoothness follows from the right inequality in Theorem 3.2. □

Proof of Theorem 1.2. We apply the method developed by Ditzian and Ivanov [23] for establishing strong converse inequalities. More precisely, we apply [23, Theorem 3.2] to the operators S^χ_w in the setting: $n = w$, $k = \nu$, $X = L_{p(\cdot),\alpha,\beta}(\mathbb{R})$, $Y = W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $Z = W^{r+1}_{p(\cdot),\alpha,\beta}(\mathbb{R})$.

By virtue of Lemma B, we have that $\chi(u) = O(|u|^{r+1-\gamma})$ and $\chi^{(r)}(u) = O(|u|^{1-\gamma})$ as $u \to \pm\infty$.

Proposition 2.2 shows that [23, (3.3)] is satisfied. Note that the proposition is applicable since $\chi(u) = O(|u|^{r+1-\gamma})$ and $\gamma - r - 1 > 2 + \max\{\alpha,\beta\}$.

By virtue of Proposition 2.4, we have that [23, (3.4)] holds with $Df = -[\operatorname{sgn}\varphi_r]f^{(r)}$, $\Phi(f) = \|\rho_{\alpha,\beta}f^{(r+1)}\|_{p(\cdot)}$,

$$\lambda(w) = \frac{|\varphi_r|}{(r+1)!\,w^r} \quad \text{and} \quad \lambda_1(w) = \frac{c}{w^{r+1}},$$

where c is the constant on the right hand-side of the estimate in Proposition 2.4. The assumption on the decay of the kernel at infinity is satisfied—we have $\gamma - r - 1 > r + 2 + \max\{\alpha,\beta\}$.

Next, Proposition 2.5 implies that $S_w^\chi f \in W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$. The assumptions in this proposition about the kernel are satisfied—we have $\chi^{(r)}(u) = O(|u|^{1-\gamma})$ as $u \to \pm\infty$ and $\gamma - 1 > r + 1 + \max\{\alpha,\beta\}$. In addition, [25, Lemma 2.12] implies that assumption (iii) in Proposition 2.6 is satisfied. Here to show that the series

$$\sum_{k\in\mathbb{Z}} |t_k - u|^{r-1}|\chi(u - t_k)| \quad \text{and} \quad \sum_{k\in\mathbb{Z}} |t_k - u|^{r-1}|\chi^{(r+1)}(u - t_k)|$$

are uniformly convergent on the compact intervals, we use (2.8) to get

$$|\chi(wx - t_k)| \le c''(1 + |t_k|)^{r+1-\gamma}, \quad x \in [a,b],\ k \in \mathbb{Z},$$

and

$$|\chi^{(r+1)}(wx - t_k)| \le c''(1 + |t_k|)^{-\gamma}, \quad x \in [a,b],\ k \in \mathbb{Z},$$

where c'' is a positive constant, whose value is independent of x (but may depend on w), and then take into account the left hand-side inequality in (2.9) to arrive at

$$|\chi(wx - t_k)| \le \tilde{c}''(1 + |k|)^{r+1-\gamma}, \quad x \in [a,b],\ k \in \mathbb{Z},$$

and

$$|\chi^{(r+1)}(wx - t_k)| \le \tilde{c}''(1 + |k|)^{-\gamma}, \quad x \in [a,b],\ k \in \mathbb{Z},$$

with some positive constant $\tilde{c}''$, whose value is independent of x. We have $r + 1 - \gamma < -1$ and then the uniform convergence of the series on the compact intervals follows from the Weierstrass M-test.

Thus Proposition 2.6 is applicable with $S_w^\chi f$ in place of f and we get [23, (3.5)] with $m = 2$ and $\ell = 1$.

Finally, Proposition 2.5 implies [23, (3.6)].

Now, [23, Theorem 3.2] yields

$$K_r(f, w^{-r})_{p(\cdot),\alpha,\beta} \le c\left(\frac{v}{w}\right)^r \left(\|\rho_{\alpha,\beta}(S_w^\chi f - f)\|_{p(\cdot)} + \|\rho_{\alpha,\beta}(S_v^\chi f - f)\|_{p(\cdot)}\right)$$

with some positive constant c, whose value is independent of f, w and $v \ge \varrho w$, where ϱ is also a positive constant.

To get the first converse estimate Theorem 1.2 it remains to apply the left inequality in Theorem 3.2 with $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ or $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$, respectively, and $t = w^{-1}$. □

Proof of Theorem 1.4. If $\|\rho_{\alpha,\beta}(S_w^\chi f - f)\| = o(w^{-r})$, then, by virtue of (1.8), we have $\Omega_r(f,t)_{p(\cdot),\alpha,\beta} = o(t^r)$. Then, by Theorem 3.4, we get that f coincides a.e. with an algebraic polynomial as described in (a)-(c) of Theorem 3.4. Then Proposition 2.3 implies $S_w^\chi f = f$ for all $w \ge 1$; hence f is an invariant element of $\{S_w^\chi\}_{w\ge1}$. In this way, we get the description of the trivial class of the approximation family, given in Theorem 1.4(a)-(c).

Next, again Proposition 2.3 implies that for any function in $W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ there holds $\|\rho_{\alpha,\beta}(S_w^\chi f - f)\|_{p(\cdot)} = O(w^{-r})$. Since $W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ contains non-invariant elements of $\{S_w^\chi\}_{w\ge1}$, we get that $\{S_w^\chi\}_{w\ge1}$ possesses the saturation property, its optimal approximation order is $O(w^{-r})$ and $W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ is a subset of its saturation class. The assertion that the saturation class is exactly $W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$ follows from (1.8) and Theorem 3.5. □

5 An example

In [26, Section 5], following Butzer and Stens [15, Section 5.2.2], we defined the Kantorovich-type sampling operator

$$(S_w^\psi f)(x) := \sum_{k\in\mathbb{Z}} w \int_{k/w}^{(k+1)/w} f(u)\,du\,\psi(wx - k),$$

where the kernel $\psi(t)$ is a linear combination of translates of central B-splines:

$$\psi(t) := \sum_{\mu=0}^{r-1} a_\mu B_{r+s}(t - b_\mu). \tag{5.1}$$

Above, $r, s \in \mathbb{N}_0$ $r \geq 2$, $B_\ell(x)$ is the central B-spline of order $\ell \geq 2$

$$B_\ell(x) := \frac{1}{(\ell-1)!}\sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}\left(\frac{\ell}{2}+x-j\right)_+^{\ell-1}, \quad x \in \mathbb{R},$$

$b_\mu \in \mathbb{R}$ are such that $b_0 < \cdots < b_{r-1}$, and $a_\mu \in \mathbb{R}$ are the unique solution of the linear system

$$\sum_{\mu=0}^{r-1}\left(\frac{1}{2}-b_\mu\right)^j a_\mu = (-i)^j\left(\frac{1}{\widehat{B_{r+s+1}}}\right)^{(j)}(0), \quad j = 0, \ldots, r-1. \tag{5.2}$$

Here, $\hat{f}$ denotes the Fourier transform of $f \in L_1(\mathbb{R})$ given by

$$\hat{f}(v) := \int_{\mathbb{R}} f(u)e^{-ivu}du, \quad v \in \mathbb{R}.$$

To recall,

$$\widehat{B_\ell}(v) = \left(\frac{\sin(v/2)}{v/2}\right)^\ell, \quad v \in \mathbb{R},\ \ell \in \mathbb{N}_+. \tag{5.3}$$

The central B-spline of order 1 is defined to be the characteristic function of the interval $[-1/2, 1/2]$.

We note that we have interchanged the roles of r and s above compared to the definition of ψ in [26, Section 5].

By [26, Theorem 5.1] with $n = 0$, and s replaced with r, we have

$$\|\rho_{\alpha,\beta}(S_w^\psi f - f)\|_p \leq c\,\omega_r(f, 1/w)_{p,\alpha,\beta}$$

for all $f \in L_{p,\alpha,\beta}(\mathbb{R})$, $1 \leq p \leq \infty$, and $w \geq 1$. We still assume that $\alpha, \beta \geq 0$. Here, $\omega_r(f,t)_{p,\alpha,\beta}$ is the classical weighted modulus of smoothness of order r, defined by

$$\omega_r(f,t)_{p,\alpha,\beta} := \sup_{0<h\leq t}\|\rho_{\alpha,\beta}\Delta_h^r f\|_p.$$

The important feature of S_w^ψ is that the support of the kernel is bounded; hence $S_w^\psi f(x)$ reduces to a finite sum for any fixed x and w. In addition, the parameters b_μ can be fixed in such a way that the samplings of the functions are taken only from the past w.r.t. the moment x at which we calculate its approximate value $S_w^\psi f(x)$.

We will demonstrate that the general theorems stated in Section 1 are applicable to that operator. Thus we have the following characterisation of its rate of approximation in $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$.

Theorem 5.1. *Let $r, s \in \mathbb{N}_0$, $r \geq 2$ and $p(\cdot)$ be an exponent function on $\mathbb{R}$ such that $p_* > 1$ and $1/p(\cdot) \in LH(\mathbb{R})$. Let $\rho_{\alpha,\beta}$ be defined by (1.3) with $\alpha, \beta \geq 0$. Let ψ be defined by (5.1)-(5.2). Then for all $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $w \geq 1$ there holds*

$$\|\rho_{\alpha,\beta}(S_w^\psi f - f)\|_{p(\cdot)} \leq c\,\Omega_r(f,t)_{p(\cdot),\alpha,\beta}.$$

Conversely, if, in addition, $s \geq 3$ and

$$\sum_{\mu=0}^{r-1}\left(\frac{1}{2}-b_\mu\right)^j a_\mu \neq (-i)^r\left(\frac{1}{\widehat{B_{r+s+1}}}\right)^{(r)}(0), \tag{5.4}$$

then there exists $\varrho > 0$ such that for all $f \in L_{p(\cdot),\alpha,\beta}(\mathbb{R})$ and $w \geq 1$ there holds

$$\Omega_r(f,1/w)_{p(\cdot),\alpha,\beta} \leq c\left(\|\rho_{\alpha,\beta}(S_w^\psi f - f)\|_{p(\cdot)} + \|\rho_{\alpha,\beta}(S_{\varrho w}^\psi f - f)\|_{p(\cdot)}\right).$$

The assertions remain valid with $\overline{\Omega}_r(f,t)_{p(\cdot),\alpha,\beta}$ in place of $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$.

Proof. Clearly, the kernel $\psi(u)$ satisfies assumption (i) in Theorem 1.1 and $\psi \in C(\mathbb{R})$. It satisfies assumptions (ii) and (iii) with $t_k = k$ as well, as it was verified in the proof of [26, Theorem 5.1] (we apply the argument there with $r = 0$ and s replaced with r), but we will establish it now for the sake of completeness and to be able to more clearly present the argument that proves the second assertion in the theorem.

By [3, Lemma 5.3] with $r = 0$ and s replace with r, the system (ii)-(iii) in Theorem 1.1 with $t_k = k$ and $\chi = \psi$ is equivalent to

$$\begin{cases} \widehat{\psi}(0) = 1, \\ \displaystyle\sum_{\ell=0}^{j}(-i)^\ell\binom{j+1}{\ell}\widehat{\psi}^{(\ell)}(0) = 0, \quad j = 1, \ldots, r-1, \\ \widehat{\psi}^{(j)}(2\pi k) = 0, \quad k \in \mathbb{Z},\ k \neq 0,\ j = 0, \ldots, r-1. \end{cases} \tag{5.5}$$

We will verify that (5.5) holds. Then the direct estimate for S_w^ψ will follow from Theorem 1.1.

We have

$$\widehat{\psi}(v) = \widehat{B_{r+s}}(v) \sum_{\mu=0}^{r-1} a_\mu e^{-ib_\mu v}, \quad v \in \mathbb{R}. \tag{5.6}$$

Then, clearly, in view of (5.3) with $\ell = r + s > r - 1$, we get the third set of relations in (5.5).

It is known that $B_{\ell+1} = B_1 * B_\ell$, where $f_1 * f_2$ is the convolution of $f_1, f_2 \in L_1(\mathbb{R})$, defined by

$$f_1 * f_2(x) := \int_{\mathbb{R}} f_1(x-y) f_2(y)\, dy, \quad x \in \mathbb{R}.$$

We set

$$P(v) := \sum_{\mu=0}^{r-1} a_\mu e^{i(\frac{1}{2}-b_\mu)v}, \quad v \in \mathbb{R},$$

and

$$\Psi(v) := g(v)\widehat{\psi}(v), \quad g(v) := \frac{e^{iv}-1}{iv}, \quad v \in \mathbb{R}.$$

We will make use of the following relation, which can be established by direct computations, based on the convolution theorem and (5.3),

$$\Psi(v) = P(v)\widehat{B_{r+s+1}}(v), \quad v \in \mathbb{R}. \tag{5.7}$$

By virtue of (5.2), we have

$$P^{(j)}(0) = \left(\frac{1}{\widehat{B_{r+s+1}}}\right)^{(j)}(0), \quad j = 0, \dots, r-1. \tag{5.8}$$

We use formulas (5.7) and (5.8) to deduce

$$\begin{aligned} \Psi^{(j)}(0) &= \sum_{n=0}^{j} \binom{j}{n} P^{(n)}(0) \widehat{B_{r+s+1}}^{(j-n)}(0) \\ &= \left(\frac{1}{\widehat{B_{r+s+1}}}\widehat{B_{r+s+1}}\right)^{(j)}(0) = \begin{cases} 1, & j = 0, \\ 0, & j = 1, \dots, r-1. \end{cases} \end{aligned} \tag{5.9}$$

On the other hand, we have

$$\Psi^{(j)}(0) = \sum_{\ell=0}^{j} \binom{j}{\ell} g^{(j-\ell)}(v) \widehat{\psi}^{(\ell)}(v), \quad j \in \mathbb{N}_0. \tag{5.10}$$

Since

$$g(v) = \sum_{n=0}^{\infty} \frac{i^n}{(n+1)!} v^n,$$

then

$$g^{(n)}(0) = \frac{i^n}{n+1}, \quad n \in \mathbb{N}_0.$$

Therefore, (5.10) yields

$$\begin{aligned} \Psi^{(j)}(0) &= \sum_{\ell=0}^{j} \binom{j}{\ell} \frac{i^{j-\ell}}{j-\ell+1} \widehat{\psi}^{(\ell)}(0) \\ &= \frac{i^j}{j+1} \sum_{\ell=0}^{j} (-i)^\ell \binom{j+1}{\ell} \widehat{\psi}^{(\ell)}(0), \quad j \in \mathbb{N}_0. \end{aligned} \tag{5.11}$$

Now, the first condition and the second group of conditions in (5.5) readily follow from (5.9). That completes the proof of the first assertion in the theorem.

To prove the second one, we apply Theorem 1.2 with $\chi = \psi$ and $t_k = k$. We have that $B_{r+s} \in C^{r+s-2}(\mathbb{R})$; hence, in view of $s \geq 3$, we have $\psi \in C^{r+1}(\mathbb{R})$.

It is straightforward to see that ψ satisfies assumptions (i) and (ii) in Theorem 1.2, and we have already shown that it satisfies (iii) and (iv).

It remains to check assumption (v). To this end, we apply [3, Lemma 5.4]) with $r = 0$ and s replace with r. In view of the considerations concerning the verification of Theorem 1.1(ii)-(iii) above, we only need to check that

$$\begin{cases} \displaystyle\sum_{\ell=0}^{r} (-i)^\ell \binom{r+1}{\ell} \widehat{\psi}^{(\ell)}(0) \neq 0, \\ \widehat{\psi}^{(r)}(2\pi k) = 0, \quad k \in \mathbb{Z},\ k \neq 0. \end{cases}$$

Just as in the first part of the proof we see by (5.6) and (5.3) with $\ell = r + s > r$ that the second relation above holds. To verify the first one, we again use (5.11) with $j = r$, (5.7) and (5.8) to get

$$\begin{aligned}\frac{i^r}{r+1}\sum_{\ell=0}^{r}(-i)^\ell\binom{r+1}{\ell}\widehat{\psi}^{(\ell)}(0) &= \Psi^{(r)}(0)\\ &= \sum_{n=0}^{r}\binom{r}{n}P^{(n)}(0)\widehat{B_{r+s+1}}^{(r-n)}(0)\\ &= \sum_{n=0}^{r}\binom{r}{n}\left(\frac{1}{\widehat{B_{r+s+1}}}\right)^{(n)}(0)\widehat{B_{r+s+1}}^{(r-n)}(0) - \left(\frac{1}{\widehat{B_{r+s+1}}}\right)^{(r)}(0) + P^{(r)}(0)\\ &= P^{(r)}(0) - \left(\frac{1}{\widehat{B_{r+s+1}}}\right)^{(r)}(0)\\ &= i^r\sum_{\mu=0}^{r-1}\left(\frac{1}{2}-b_\mu\right)^j a_\mu - \left(\frac{1}{\widehat{B_{r+s+1}}}\right)^{(r)}(0) \neq 0,\end{aligned}$$

as, at the last step, we applied (5.4).

This completes the proof of the second assertion in the theorem. □

It can happen that parameters r, s, a_μ and b_μ that satisfy (5.2) for $j = 0, \ldots, r-1$ also satisfy it for $j = r$, that is, they do not satisfy (5.4)—see [2, Remark 6.3] in view of the relation between the classical sampling operator and its Kantorovich modification (e.g., [3, p. 17]).

By the last theorem and properties of the modulus $\Omega_r(f,t)_{p(\cdot),\alpha,\beta}$ (or by Theorem 1.4), we get that the family of approximation operators $\{S_w^\psi\}_{w\geq 1}$ is saturated with the rate $O(w^{-r})$, its saturation class is $W^r_{p(\cdot),\alpha,\beta}(\mathbb{R})$, and its trivial class consists of the algebraic polynomials of degree at most $r-1$, which are in $L_{p(\cdot),\alpha,\beta}(\mathbb{R})$. We refer to Theorems 1.4 or 3.4 for a detailed description of the latter.

Similarly, Theorem 1.1 implies a direct estimate for the rate of approximation of the operators introduced in [24].

In conclusion, let us note that the converse inequality in Theorem 1.2 is established under more restrictive assumptions on the smoothness and decay of the kernel than the direct one in Theorem 1.1. It seems worth investigating to what extend they can be relaxed.

Acknowledgements

I am thankful to the referees, whose comments improved the exposition and corrected mistakes.

Funding

This study is financed by the European Union-NextGenerationEU, through the National Recovery and Resilience Plan of the Republic of Bulgaria, project No BG-RRP-2.004-0008.

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