A generalized Prony method for sparse approximation

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Reconstruction of sparse exponential sums (Prony's method)

Function $$f(x) = \sum_{j=1}^{M} c_j \, e^{T_j x}$$

We have $M,\ f(\ell),\ \ell = 0, \ldots, 2M-1$

We want $c_j, T_j \in \mathbb{C}$, where $-\pi \le \operatorname{Im} T_j < \pi$, $j = 1, \ldots, M$.

Consider the **Prony polynomial**

$$P(z) := \prod_{j=1}^{M} \left(z - e^{T_j}\right) = \sum_{\ell=0}^{M} p_\ell \, z^\ell$$

with unknown parameters T_j and $p_M = 1$.

$$\sum_{\ell=0}^{M} p_\ell f(\ell+m) = \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j e^{T_j(\ell+m)} = \sum_{j=1}^{M} c_j \, e^{T_j m} \sum_{\ell=0}^{M} p_\ell \, e^{T_j \ell}$$

$$= \sum_{j=1}^{M} c_j e^{T_j m} P(e^{T_j}) = 0, \qquad m = 0, \ldots, M-1.$$

Reconstruction algorithm

Input: $f(\ell)$, $\ell = 0, \ldots, 2M-1$

- Solve the Hankel system

$$\begin{pmatrix} f(0) & f(1) & \ldots & f(M-1) \\ f(1) & f(2) & \ldots & f(M) \\ \vdots & \vdots & & \vdots \\ f(M-1) & f(M) & \ldots & f(2M-2) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{M-1} \end{pmatrix} = - \begin{pmatrix} f(M) \\ f(M+1) \\ \vdots \\ f(2M-1) \end{pmatrix}.$$

- Compute the zeros of the Prony polynomial $P(z) = \sum_{\ell=0}^{M} p_\ell z^\ell$ and extract the parameters T_j from its zeros $z_j = e^{T_j}$, $j = 1, \ldots, M$.
- Compute c_j solving the linear system

$$f(\ell) = \sum_{j=1}^{M} c_j e^{T_j \ell}, \qquad \ell = 0, \ldots, 2M-1.$$

Output: Parameters T_j and c_j, $j = 1, \ldots, M$.

Literature

[Prony] (1795):	Reconstruction of a difference equation
[Schmidt] (1979):	**MUSIC** (Multiple Signal Classification)
[Roy, Kailath] (1989):	**ESPRIT** (Estimation of signal parameters via rotational invariance techniques)
[Hua, Sakar] (1990):	**Matrix-pencil method**
[Stoica, Moses] (2000):	**Annihilating filters**
[Potts, Tasche] (2010, 2011):	**Approximate Prony method**

Golub, Milanfar, Varah ('99); Vetterli, Marziliano, Blu ('02); Maravić, Vetterli ('04); Elad, Milanfar, Golub ('04); Beylkin, Monzon ('05,'10); Batenkov, Sarg, Yomdin ('12, '13); Filbir, Mhaskar, Prestin ('12); Peter, Potts, Tasche ('11,'12,'13); Plonka, Wischerhoff ('13); ...

Reconstruction of M-sparse polynomials
Ben-Or & Tiwari algorithm

Function $$f(x) = \sum_{j=1}^{M} c_j\, x^{d_j}$$

$$0 \le d_1 < d_2 < \ldots < d_M = N,\ d_j \in \mathbb{N}_0,\ c_j \in \mathbb{C}.$$

We have $M,\ f(x_0^\ell),\ \ell = 0, \ldots, 2M-1,\ (x_0^\ell$ pairwise different)

We want coefficients $c_j \in \mathbb{C} \setminus \{0\}$, indices d_j of "active" monomials

Consider the **Prony polynomial**

$$P(z) := \prod_{j=1}^{M} \left(z - x_0^{d_j}\right) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

with unknown zeros $x_0^{d_j}$ and $p_M = 1$.

Prony polynomial $$P(z) := \prod_{j=1}^{M} \left(z - x_0^{d_j}\right) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

Then

$$\sum_{\ell=0}^{M} p_\ell f(x_0^{\ell+m}) = \sum_{\ell=0}^{M} p_\ell \sum_{j=1}^{M} c_j x_0^{d_j(\ell+m)} = \sum_{j=1}^{M} c_j x_0^{d_j m} \sum_{\ell=0}^{M} p_\ell x_0^{d_j \ell}$$

$$= \sum_{j=1}^{M} c_j x_0^{d_j m} P(x_0^{d_j}) = 0, \qquad m = 0, \ldots, M-1.$$

Hence

$$\sum_{\ell=0}^{M-1} p_\ell f(x_0^{\ell+m}) = -f(x_0^{\ell+m}), \qquad m = 0, \ldots, M-1.$$

Compute p_ℓ $\Rightarrow$ $P(z)$ $\Rightarrow$ zeros $x_0^{d_j}$ $\Rightarrow$ d_j $\Rightarrow$ c_j.

Literature

[Ben-Or, Tiwari] (1988):	Reconstruction of multivariate M-sparse polynomials
[Grigoriev, Karpinski, Singer] (1990):	Sparse polynomial interpolation over finite fields
[Dress, Grabmeir] (1991):	Interpolation of k-sparse character sums
[Lakshman, Saunders] (1995):	sparse polynomial interpolation in non-standard bases
[Kaltofen, Lee] (2003):	Early termination in sparse polynomial interpolation
[Giesbrecht, Labahn, Lee] (2008)	Sparse interpolation of multivariate polynomials

The generalized Prony method (Peter, Plonka (2013))

Let V be a normed vector space and let $\mathcal{A} : V \to V$ be a linear operator.

Let $\{e_n : n \in I\}$ be a set of eigenfunctions of $\mathcal{A}$ to **pairwise different** eigenvalues $\lambda_n \in \mathbb{C}$,

$$\mathcal{A}\, e_n = \lambda_n\, e_n.$$

Let

$$f = \sum_{j \in J} c_j\, e_j, \qquad J \subset I \text{ with } |J| = M,\ c_j \in \mathbb{C}.$$

Let $F : V \to \mathbb{C}$ be a linear functional with $F(e_n) \neq 0$ for all $n \in I$.

We have $M,\ F(\mathcal{A}^\ell f)$ for $\ell = 0, \ldots, 2M-1$

We want $J \subset I,\ c_j \in \mathbb{C}$ for $j \in J$

Prony polynomial

$$P(z) = \prod_{j\in J}(z-\lambda_j) = \sum_{\ell=0}^{M} p_\ell z^\ell$$

with unknown λ_j, i.e., with unknown J. Hence, for $m = 0, 1, \ldots$

$$\begin{aligned}\sum_{\ell=0}^{M} p_\ell F(\mathcal{A}^{\ell+m} f) &= \sum_{\ell=0}^{M} p_\ell F\left(\sum_{j\in J} c_j \mathcal{A}^{\ell+m} e_j\right) = \sum_{\ell=0}^{M} p_\ell F\left(\sum_{j\in J} c_j \lambda_j^{\ell+m} e_j\right) \\ &= \sum_{j\in J} c_j \lambda_j^m \left(\sum_{\ell=0}^{M} p_\ell \lambda_j^\ell\right) F(e_j) \\ &= \sum_{j\in J} c_j \lambda_j^m P(\lambda_j) F(e_j) = 0.\end{aligned}$$

Thus, if $F(\mathcal{A}^\ell f)$, $\ell = 0, \ldots, 2M-1$ is known, then we can compute the index set $J \subset I$ of the active eigenfunctions and the coefficients c_j.

Application to linear operators: shift operator

Choose the **shift operator** $\mathcal{S}_h : C(\mathbb{R}) \to C(\mathbb{R})$, $h > 0$

$$\mathcal{S}_h f(x) := f(x+h)$$

Eigenfunctions of $\mathcal{S}_h$

$$\mathcal{S}_h \, \mathrm{e}^{T_j x} = \mathrm{e}^{T_j (x+h)} = \mathrm{e}^{T_j h} \, \mathrm{e}^{T_j x}, \qquad T_j \in \mathbb{C}, \, \mathrm{Im}\, T_j \in [-\frac{\pi}{h}, \frac{\pi}{h}).$$

Prony method: For the reconstruction of

$$f(x) = \sum_{j=1}^{M} c_j \, \mathrm{e}^{T_j x} \quad \text{we need} \quad F(\mathcal{S}_h^\ell f) = F(f(\cdot + h\ell)), \; \ell = 0, \ldots, 2M-1.$$

Put $F(f) := f(x_0)$. $\qquad F(\mathcal{S}_h^\ell f) = f(x_0 + h\ell)$

Put $F(f) := \int_{x_0}^{x_0+h} f(x) dx$. $\qquad F(\mathcal{S}_h^\ell f) = \int\limits_{x_0+\ell h}^{x_0+(\ell+1)h} f(x) dx.$

Application to linear operators: dilation operator

Choose the **dilation operator** $\mathcal{D}_h : C(\mathbb{R}) \to C(\mathbb{R})$,

$$\mathcal{D}_h f(x) := f(hx)$$

Eigenfunctions of $\mathcal{D}_h$: $\mathcal{D}_h\, x^{p_j} = (hx)^{p_j} = h^{p_j} x^{p_j}$, $p_j \in \mathbb{C}$, $x \in \mathbb{R}$.
We need: h^{p_j} are pairwise different for all $j \in I$.

Ben-Or & Tiwari method: For reconstruction of

$$f(x) = \sum_{j=1}^{M} c_j\, x^{p_j},$$

we need $F(\mathcal{D}_h^\ell f) = F(f(h^\ell \cdot))$, $\ell = 0, \ldots, 2M-1$.

Put $F(f) := f(x_0)$. $F(\mathcal{D}_h^\ell f) = f(h^\ell x_0)$

Put $F(f) := \int_0^1 f(x)dx$. $F(\mathcal{D}_h^\ell f) = \frac{1}{h^\ell} \int\limits_0^{h^\ell} f(x)dx$.

Application to linear operators: Sturm-Liouville operator

Choose the **Sturm-Liouville operator** $\mathcal{L}_{p,q} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$,

$$\mathcal{L}_{p,q} f(x) := p(x) f''(x) + q(x) f'(x),$$

where $p(x), q(x)$ are polynomials of degree 2 and 1, respectively.

Eigenfunctions are orthogonal polynomials, where $\mathcal{L}_{p,q} Q_n = \lambda_n Q_n$.

$p(x)$	$q(x)$	λ_n	name	symbol
$(1-x^2)$	$(\beta-\alpha-(\alpha+\beta+2)x)$	$-n(n+\alpha+\beta+1)$	Jacobi	$P_n^{(\alpha,\beta)}$
$(1-x^2)$	$-(2\alpha+1)x$	$-n(n+2\alpha)$	Gegenbauer	$C_n^{(\alpha)}$
$(1-x^2)$	$-2x$	$-n(n+1)$	Legendre	P_n
$(1-x^2)$	$-x$	$-n^2$	Chebyshev 1. kind	T_n
$(1-x^2)$	$-3x$	$-n(n+2)$	Chebyshev 2. kind	U_n
1	$-2x$	$-2n$	Hermite	H_n
x	$(\alpha+1-x)$	$-n$	Laguerre	$L_n^{(\alpha)}$

Sparse sums of orthogonal polynomials

Function
$$f(x) = \sum_{j=1}^{M} c_{n_j} Q_{n_j}(x).$$

We want: $c_{n_j} \in \mathbb{C} \setminus \{0\}$, indices n_j of "active" basis polynomials Q_{n_j}

Now, f can be uniquely recovered from

$$F(\mathcal{L}_{p,q}^k f) = \mathcal{L}_{p,q}^k f(x_0) = \sum_{j=1}^{M} c_{n_j} \lambda_{n_j}^k Q_{n_j}(x_0), \quad k = 0, \ldots, 2M-1.$$

Theorem.

For each polynomial f and each $x \in \mathbb{R}$, the values $\mathcal{L}_{p,q}^k f(x)$, $k = 0, \ldots, 2M-1$, are uniquely determined by $f^{(m)}(x)$, $m = 0, \ldots, 4M-2$.

If $p(x_0) = 0$, then $\mathcal{L}_{p,q} f(x_0)$ reduces to $\mathcal{L}_{p,q} f(x_0) = q(x_0) f'(x_0)$, and the values $\mathcal{L}_{p,q}^k f(x_0)$, $k = 0, \ldots, 2M-1$, can be determined uniquely by $f^{(m)}(x_0)$, $m = 0, \ldots, 2M-1$.

Example: Sparse Laguerre expansion

Operator equation

$$x(L_n^{(\alpha)})''(x) + (\alpha + 1 - x)(L_n^{(\alpha)})'(x) = -n\, L_n^{(\alpha)}(x)$$

Sparse Laguerre expansion

$$f(x) = \sum_{j=1}^{6} c_{n_j}\, L_{n_j}^{(0)}(x) \qquad \text{(with } \alpha = 0\text{)}$$

Given values: $f(0),\ f'(0),\ \ldots,\ f^{(11)}(0)$.

j	n_j	c_{n_j}	$\tilde{n}_j$	$\tilde{c}_{n_j}$
1	142	-3	142.0000000018223	-2.99999999999987
2	125	-1	125.0000000494359	-1.00000000000034
3	91	2	90.9999998114290	2.00000000000063
4	69	-3	69.0000003316075	-3.00000000000058
5	53	-1	53.0000003445395	-0.99999999999988
6	11	2	10.9999999973030	2.00000000000004

Application to linear operators: Recovery of sparse vectors

Choose the operator $\mathbf{D} : \mathbb{C}^N \to \mathbb{C}^N$

$$\mathbf{D}\mathbf{x} := \mathrm{diag}(d_0, \ldots, d_{N-1})\,\mathbf{x}$$

with pairwise different d_j.

Eigenvectors of $\mathbf{D}$: $\quad \mathbf{D}\,\mathbf{e}_j = d_j \mathbf{e}_j \qquad j = 0, \ldots, N-1.$

We want to reconstruct

$$\mathbf{x} = \sum_{j=1}^{M} c_{n_j}\,\mathbf{e}_{n_j} \quad c_{n_j} \in \mathbb{C},\ 0 \le n_1 < \ldots < n_M \le N-1.$$

We need $F(\mathbf{D}^\ell \mathbf{x})$, $\ell = 0, \ldots 2M-1$.

Let $F(\mathbf{x}) := \mathbf{1}^{\mathrm{T}}\mathbf{x} := \sum_{j=0}^{N-1} x_j$. Then $F(\mathbf{D}^\ell \mathbf{x}) = \mathbf{1}^{\mathrm{T}}\mathbf{D}^\ell \mathbf{x} = \mathbf{a}_\ell^{\mathrm{T}} \mathbf{x}$,

where $\mathbf{a}_\ell^{\mathrm{T}} = (d_0^\ell, \ldots, d_{N-1}^\ell)$, $\ell = 0, \ldots 2M-1$.

Example: Sparse vectors

Choose

$$\mathbf{D} = \operatorname{diag}(\omega_N^0, \omega_N^1, \ldots, \omega_N^{N-1}),$$

where $\omega_N := \mathrm{e}^{-2\pi \mathrm{i}/N}$ denotes the N-th root of unity.

Then an M-sparse vector $\mathbf{x}$ can be recovered from

$$\mathbf{y} = \mathbf{F}_{2M,N}\,\mathbf{x},$$

where $\mathbf{F}_{2M,N} = (\omega_N^{k\ell})_{k,\ell=0}^{2M-1,N-1} \in \mathbb{C}^{2M\times N}$.

More generally, choose $\sigma \in \mathbb{N}$ with $(\sigma, N) = 1$ and

$$\mathbf{D} = \operatorname{diag}(\omega_N^0, \omega_N^{\sigma}, \ldots, \omega_N^{\sigma(N-1)}),$$

Then an M-sparse vector $\mathbf{x}$ can be recovered from

$$\mathbf{y} = \widetilde{\mathbf{F}}_{2M,N}\,\mathbf{x},$$

where $\widetilde{\mathbf{F}}_{2M,N} = (\omega_N^{\sigma k\ell})_{k,\ell=0}^{2M-1,N-1} \in \mathbb{C}^{2M\times N}$.

Numerical issues

Let $f = \sum_{j\in J} c_j e_j$ with $J \subset I$ and $|J| = M$, $\mathcal{A} e_n = \lambda_n v_n$ for $n \in I$.

Algorithm (Recovery of f)

Input: M, $F(\mathcal{A}^k f)$, $k = 0, \dots, 2M-1$.

- Solve the linear system

$$\sum_{k=0}^{M-1} p_k F(\mathcal{A}^{k+m} f) = -F(\mathcal{A}^{M+m} f), \qquad m = 0, \dots, M-1. \quad (1)$$

- Form the Prony polynomial $P(z) = \sum_{k=0}^{M} p_k z^k$. Compute the zeros λ_j, $j \in J$, of $P(z)$ and determine v_j, $j \in J$.
- Compute the coefficients c_j by solving the overdetermined system

$$F(\mathcal{A}^k f) = \sum_{j\in J} c_j \lambda_j^k v_j \qquad k = 0, \dots, 2M-1.$$

Output: c_j, v_j, $j \in J$, determining f.

Numerical issues II (Generalization of [Potts,Tasche '13])

Numerically stable procedure for steps 1 and 2:

Let $\mathbf{g}_k := (F(\mathcal{A}^{k+m} f))_{m=0}^{M-1}$, $k = 0, \ldots, M-1$ and

$\mathbf{G} := (F(\mathcal{A}^{k+m} f))_{k,m=0}^{M-1} = (\mathbf{g}_0 \ldots \mathbf{g}_{M-1})$ and $\widetilde{\mathbf{G}} = (\mathbf{g}_1 \ldots \mathbf{g}_M)$.

Then (1) yields

$$\mathbf{G}\mathbf{p} = -\mathbf{g}_M,$$

Let $\mathbf{C}(P)$ be the companion matrix of the Prony polynomial $P(z)$ with eigenvalues λ_j, $j \in J$. Then

$$\mathbf{G}\,\mathbf{C}(P) = \widetilde{\mathbf{G}},$$

Hence λ_j are the eigenvalues of the matrix pencil

$$z\mathbf{G} - \widetilde{\mathbf{G}}, \qquad z \in \mathbb{C}.$$

Now apply a QR-decomposition or SVD-decomposition to $(\mathbf{g}_0 \ldots \mathbf{g}_M)$.

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