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On the approximation of Lauricella–Saran's hypergeometric functions F_M and their ratios by branched continued fractions

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Abstract

The paper delves the problem of approximating Lauricella–Saran's hypergeometric functions by a special family of functions – branched continued fractions. Under certain conditions of the parameters of the Lauricella–Saran's hypergeometric functions F_M, new domains of the analytic continuation of these functions and their ratios are established, using their expansions into branched continued fractions, the elements of which are polynomials of three complex variables. At the end, several numerical experiments are presented that illustrate the efficient approximation of special function by branched continued fraction.

1 Introduction

Special functions, such as hypergeometric functions and their various generalizations, naturally arise when solving various problems in mathematics and physics, chemistry and biology, engineering and economics, etc. (see, [3, 12, 14, 26, 37]). As solutions to systems of equations describing complex processes, they are intendant to provide a better understanding of their properties and mechanisms of interaction [11, 13, 32, 35, 36]. However, the limitations of the series represented by these functions, in particular, the relatively small domains of their convergence, prompts the search for effective tools for their representation and research methods. One such tool is branched continued fractions which under certain conditions have wide domain of convergence and numerical stability (see, for example, [1, 15, 18, 21, 28]).

Recall that branched continued fractions are expressions of the form [6, 9]:

$$b_0+\sum_{i_1=1}^{N}\cfrac{a_{i_1}}{b_{i_1}+\sum\limits_{i_2=1}^{N}\cfrac{a_{i_1,i_2}}{b_{i_1,i_2}+\sum\limits_{i_3=1}^{N}\cfrac{a_{i_1,i_2,i_3}}{b_{i_1,i_2,i_3}+\ddots}}},$$

where N is a fixed natural number and the elements b_0, a_{i_1}, b_{i_1}, a_{i_1,i_2}, b_{i_1,i_2}, a_{i_1,i_2,i_3}, b_{i_1,i_2,i_3}, ... can be numbers, functions, matrices, operators, etc. Their properties can be found in [9]. It should be noted that when $N=1$, these are the continued fractions [27, 30].

This study delves hypergeometric functions of the Lauricella–Saran family [31, 39], in particular, the hypergeometric functions F_M, defined as follows:

$$F_M(\alpha_1,\alpha_2,\beta_1,\beta_2;\gamma_1,\gamma_2;\mathbf{z})=\sum_{p,q,r=0}^{+\infty}\frac{(\alpha_1)_p(\alpha_2)_{q+r}(\beta_1)_{p+r}(\beta_2)_q}{(\gamma_1)_p(\gamma_2)_{q+r}}\frac{z_1^pz_2^qz_3^r}{p!q!r!},\tag{1}$$

where $\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2\in\mathbb{C}$ herewith $\gamma_1,\gamma_2\notin\{0,-1,-2,\ldots\}$, $\mathbf{z}=(z_1,z_2,z_3)\in\mathbb{C}^3$, $(\cdot)_k$ is the Pochhammer symbol.

The problem of analytic continuation of Lauricella–Saran's hypergeometric functions F_M under certain conditions of the parameters through their integral representations and branched continued fraction expansions is considered in [22] and [19, 20], respectively. Our goal in this paper is to establish new domains of analytic continuation of these functions and their ratios through their branched continued fraction expansions. A study of other functions of the Lauricella–Saran family related to branched continued fractions can be found in [5, 10, 17, 23].

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Let $\mathfrak{I}=\{1,2\}$ and $\mathfrak{I}_k=\{i(k)=(i_0,i_1,i_2,\ldots,i_k):\ i_r\in\mathfrak{I},\ 0\le r\le k\}$, $k\ge 1$. In [34], it is shown that for each $i_0\in\mathfrak{I}$ the function

$$\frac{(1-\delta_{i_0}^2 z_1)F_M(\alpha_1,\alpha_2,\beta_1,\beta_2;\alpha_1,\gamma_2;\mathbf{z})}{F_M(\alpha_1,\alpha_2+1,\beta_1+\delta_{i_0}^1,\beta_2+\delta_{i_0}^2;\alpha_1,\gamma_2+1;\mathbf{z})}, \tag{2}$$

where $\alpha_1,\gamma_2\notin\{0,-1,-2,\ldots\}$ and δ_i^j is the Kronecker symbol, has a formal branched continued fraction

$$v_{i_0}(\mathbf{z})+\sum_{i_1=1}^{2}\frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z})+\sum_{i_2=1}^{2}\frac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z})+\sum_{i_3=1}^{2}\frac{u_{i(3)}(\mathbf{z})}{v_{i(3)}(\mathbf{z})+\ddots}}}, \tag{3}$$

where

$$v_{i_0}(\mathbf{z})=1-z_1-\frac{\alpha_2+\beta_{i_0}+1}{\gamma_2}(1-\delta_{i_0}^2 z_1)z_{4-i_0}-\frac{\beta_{3-i_0}}{\gamma_2}(1-\delta_{i_0}^1 z_1)z_{1+i_0} \tag{4}$$

and, for $i(k)\in\mathfrak{I}_k$, $k\ge 1$,

$$u_{i(k)}(\mathbf{z})=\frac{(\alpha_2+k)\left(\beta_{i_k}+\sum_{r=0}^{k-1}\delta_{i_r}^{i_k}\right)}{(\gamma_2+k-1)(\gamma_2+k)}(1-\delta_{i_k}^2 z_1)^2 z_{4-i_k}(1-\delta_{i_k}^1 z_1-z_{4-i_k}), \tag{5}$$

$$v_{i(k)}(\mathbf{z})=1-z_1-\frac{\alpha_2+\beta_{i_k}+k+1+\sum_{r=0}^{k-1}\delta_{i_r}^{i_k}}{\gamma_2+k}(1-\delta_{i_k}^2 z_1)z_{4-i_k}-\frac{\beta_{3-i_k}+\sum_{r=0}^{k-1}\delta_{i_r}^{3-i_k}}{\gamma_2+k}(1-\delta_{i_k}^1 z_1)z_{1+i_k}. \tag{6}$$

Note that (3) is a branched continued fraction with two branches of the branch, and its elements are polynomials in the variables z_1, z_2, and z_3. More about the branched continued fraction expansions of special functions and the problem of their convergence can be found in [6] and [7, 8], respectively.

The main results of this study are the following:

Theorem 1.1. *Let α_1 be complex number herewith $\alpha_1\notin\{0,-1,-2,\ldots\}$ and α_2, β_1, β_2, and γ_2 be real numbers such that*

$$0<\alpha_2+1\le\gamma_2,\ 0\le\beta_{i_0},\ \beta_1^2+\beta_2^2\ne 0,\ \beta_{i_0}\le\gamma_2,\ i_0\in\mathfrak{I}. \tag{7}$$

Then, for each $i_0\in\mathfrak{I}$, we have the following:

(i) *The branched continued fraction (3) converges to a finite value $f^{(i_0)}(\mathbf{z})$ for each $\mathbf{z}\in\mathfrak{G}_\lambda$, where*

$$\mathfrak{G}_\lambda=\Big\{\mathbf{z}\in\mathbb{R}^3:\ z_1<1,\ z_3(1-z_1-z_3)\ge 0,\ 0\le z_2\le 1,$$
$$z_1+2(1-\delta_k^2 z_1)z_{4-k}+(1-\delta_k^1 z_1)z_{1+k}\le\lambda,\ k=1,2\Big\},\ 0<\lambda<1. \tag{8}$$

(ii) *The convergence is uniform on every compact subset of the domain $\mathrm{Int}(\mathfrak{G}_\lambda)$, and $f^{(i_0)}(\mathbf{z})$ is holomorphic on $\mathrm{Int}(\mathfrak{G}_\lambda)$.*

(iii) *If $f_n^{(i_0)}(\mathbf{z})$ denotes the nth approximant of (3), then for each $\mathbf{z}\in\mathfrak{G}_\lambda$*

$$|f^{(i_0)}(\mathbf{z})-f_n^{(i_0)}(\mathbf{z})|\le\frac{\mu^{n+1}(\mathbf{z})}{(1-\nu(\mathbf{z}))((1-\nu(\mathbf{z}))^2+\mu(\mathbf{z}))^n},\ n\ge 1,$$

where

$$\nu(\mathbf{z})=\max\{z_1+2(1-z_1)z_2+z_3,\ z_1+(1-z_1)z_2+2z_3\} \tag{9}$$

and

$$\mu(\mathbf{z})=z_3(1-z_1-z_3)+(1-z_1)^2 z_2(1-z_2). \tag{10}$$

(iv) *The function $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of (2) in $\mathrm{Int}(\mathfrak{G}_\lambda)$.*

Remark 1. If, in particular,

$$z_1<\frac{\lambda}{2},\ 0\le z_2<\frac{\lambda}{4(2-\lambda)},\ 0\le z_3<\frac{\lambda}{16},$$

then the inequalities in (8) are satisfied.

Theorem 1.2. *Let the conditions of Theorem 1.1 be satisfied. Then, for each $i_0\in\mathfrak{I}$, we have the following:*

(i) *The branched continued fraction (3) converges uniformly on every compact subset of the domain*

$$\mathfrak{Q}_\eta = \bigcup_{-\pi/2<\alpha<\pi/2} \mathfrak{Q}_{\eta,\alpha}, \ 0<\eta<1, \tag{11}$$

where

$$\mathfrak{Q}_{\eta,\alpha} = \Big\{ \mathbf{z} \in \mathbb{C}^3 : \ \mathrm{Re}((z_1 + 2(1-\delta_k^2 z_1) z_{4-k} + (1-\delta_k^1 z_1) z_{1+k}) e^{-i\alpha}) < \eta \cos\alpha, \ k=1,2,$$
$$2\sum_{k=1}^{2} (|(1-\delta_k^2 z_1)^2 z_{4-k}(1-\delta_k^1 z_1 - z_{4-k})| - \mathrm{Re}((1-\delta_k^2 z_1)^2 z_{4-k}(1-\delta_k^1 z_1 - z_{4-k}) e^{-2i\alpha})) < (1-\eta)^2 \cos^2\alpha \Big\}, \tag{12}$$

to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in $\mathfrak{Q}_\eta$.

(ii) *The function $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of the function (2) in the domain (11).*

2 Proofs of the Main Results

Let us present the necessary notation and formulas related to branched continued fraction (3) (see, [9]).

We set

$$F_{i(n)}^{(n)}(\mathbf{z}) = v_{i(n)}(\mathbf{z}), \ i(n) \in \mathfrak{I}_n, \ n \ge 1, \tag{13}$$

and

$$F_{i(k)}^{(n)}(\mathbf{z}) = v_{i(k)}(\mathbf{z}) + \sum_{i_{k+1}=1}^{2} \cfrac{u_{i(k+1)}(\mathbf{z})}{v_{i(k+1)}(\mathbf{z}) + \sum\limits_{i_{k+2}=1}^{2} \cfrac{u_{i(k+2)}(\mathbf{z})}{v_{i(k+2)}(\mathbf{z}) + \ddots + \sum\limits_{i_n=1}^{2} \cfrac{u_{i(n)}(\mathbf{z})}{v_{i(n)}(\mathbf{z})}}},$$

where $i(k) \in \mathfrak{I}_k$, $1 \le k \le n-1$, $n \ge 2$. Then

$$F_{i(k)}^{(n)}(\mathbf{z}) = v_{i(k)}(\mathbf{z}) + \sum_{i_{k+1}=1}^{2} \frac{u_{i(k+1)}(\mathbf{z})}{F_{i(k+1)}^{(n)}(\mathbf{z})}, \ i(k) \in \mathfrak{I}_k, \ 1 \le k \le n-1, \ n \ge 2. \tag{14}$$

If $f_n^{(i_0)}(\mathbf{z})$ denotes nth approximant of (3), then

$$f_n^{(i_0)}(\mathbf{z}) = v_{i_0}(\mathbf{z}) + \sum_{i_1=1}^{2} \frac{u_{i(1)}(\mathbf{z})}{F_{i(1)}^{(n)}(\mathbf{z})}, \tag{15}$$

where $n \ge 1$. In addition, if

$$F_{i(k)}^{(n)}(\mathbf{z}) \ne 0, \ i(k) \in \mathfrak{I}_k, \ 1 \le k \le n, \ n \ge 1, \tag{16}$$

then (see, [9]), for $n \ge 1$ and $k \ge 1$,

$$f_{n+k}^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z}) = (-1)^n \sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\dots\sum_{i_{n+1}=1}^{2} \frac{\prod\limits_{r=1}^{n+1} u_{i(r)}(\mathbf{z})}{\prod\limits_{r=1}^{n+1} F_{i(r)}^{(n+k)}(\mathbf{z}) \prod\limits_{r=1}^{n} F_{i(r)}^{(n)}(\mathbf{z})}. \tag{17}$$

Proof of Theorem 1.1. Let i_0 be an arbitrary index in $\mathfrak{I}$.

(i) Let $\mathbf{z}$ be an arbitrary fixed point in (8). Then under conditions (7), the elements (5) are nonnegative.

In that follows, we will estimate the elements (6). Since, for any $i(k) \in \mathfrak{I}_k$, $k \ge 1$,

$$v_{i(k)}(\mathbf{z}) \ge 1 - z_1 - \frac{\alpha_2 + \beta_{i_k} + 2k + 1}{\gamma_2 + k}(1-\delta_{i_k}^2 z_1) z_{4-i_k} - \frac{\beta_{3-i_k} + k}{\gamma_2 + k}(1-\delta_{i_k}^1 z_1) z_{1+i_k},$$

then by (7) and (8)

$$v_{i(k)}(\mathbf{z}) \ge 1 - z_1 - 2(1-\delta_{i_k}^2 z_1) z_{4-i_k} - (1-\delta_{i_k}^1 z_1) z_{1+i_k} \ge 1 - \nu(\mathbf{z}) \ge 1 - \lambda > 0.$$

Thus, for any $n \ge 1$ and $i(k) \in \mathfrak{I}_k$, $1 \le k \le n$, by (13) and (14) we have

$$F_{i(k)}^{(n)}(\mathbf{z}) \ge v_{i(k)}(\mathbf{z}) \ge 1 - \nu(\mathbf{z}) \ge 1 - \lambda > 0. \tag{18}$$

In that follows, for $n \geq 1$ and $k \geq 1$ we will estimate

$$|f_{n+k}^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})|.$$

For convenient, we will write (17) in the form

$$f_{n+k}^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z}) = (-1)^n \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_{n+1}=1}^{2} \frac{u_{i(1)}(\mathbf{z})}{F_{i(1)}^{(q)}(\mathbf{z})} \prod_{r=1}^{[(n+1)/2]} \frac{u_{i(2r)}(\mathbf{z})}{F_{i(2r-1)}^{(p)}(\mathbf{z}) F_{i(2r)}^{(p)}(\mathbf{z})} \prod_{r=1}^{[n/2]} \frac{u_{i(2r+1)}(\mathbf{z})}{F_{i(2r)}^{(q)}(\mathbf{z}) F_{i(2r+1)}^{(q)}(\mathbf{z})},$$

where $q = n + k$, $p = n$, if $n = 2s$, and $q = n$, $p = n + k$, if $n = 2s - 1$, $s \geq 1$.

By (13), (14) and (18) it follows that for any $l \geq 1$

$$\begin{aligned} \sum_{i_1=1}^{2} \frac{u_{i(1)}(\mathbf{z})}{F_{i(1)}^{(l)}(\mathbf{z})} &\leq \sum_{i_1=1}^{2} \frac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z})} \\ &\leq \sum_{i_1=1}^{2} \frac{u_{i(1)}(\mathbf{z})}{v(\mathbf{z})} \end{aligned}$$

and for any $i(k) \in \mathfrak{I}_k$, $1 \leq k \leq l$,

$$\begin{aligned} \sum_{i_{k+1}=1}^{2} \frac{u_{i(k+1)}(\mathbf{z})}{F_{i(k)}^{(l+1)}(\mathbf{z}) F_{i(k+1)}^{(l+1)}(\mathbf{z})} &= \frac{\displaystyle\sum_{i_{k+1}=1}^{2} \frac{u_{i(k+1)}(\mathbf{z})}{F_{i(k+1)}^{(l+1)}(\mathbf{z})}}{v_{i(k)}(\mathbf{z}) + \displaystyle\sum_{i_{k+1}=1}^{2} \frac{u_{i(k+1)}(\mathbf{z})}{F_{i(k+1)}^{(l+1)}(\mathbf{z})}} \\ &\leq \frac{\displaystyle\sum_{i_{k+1}=1}^{2} \frac{u_{i(k+1)}(\mathbf{z})}{v_{i(k)}(\mathbf{z}) v_{i(k+1)}(\mathbf{z})}}{1 + \displaystyle\sum_{i_{k+1}=1}^{2} \frac{u_{i(k+1)}(\mathbf{z})}{v_{i(k)}(\mathbf{z}) v_{i(k+1)}(\mathbf{z})}} \\ &\leq \frac{\displaystyle\sum_{i_{k+1}=1}^{2} u_{i(k+1)}(\mathbf{z})}{(1 - v(\mathbf{z}))^2 + \displaystyle\sum_{i_{k+1}=1}^{2} u_{i(k+1)}(\mathbf{z})}. \end{aligned}$$

Then, for any $i(k) \in \mathfrak{I}_k$, $k \geq 1$, by (5), (7) and (8) we have

$$\begin{aligned} \sum_{i_k=1}^{2} u_{i(k)}(\mathbf{z}) &= \sum_{i_k=1}^{2} \frac{(\alpha_2 + k)\left(\beta_{i_k} + \displaystyle\sum_{r=0}^{k-1} \delta_{i_r}^{i_k}\right)}{(\gamma_2 + k - 1)(\gamma_2 + k)} (1 - \delta_{i_k}^2 z_1)^2 z_{4-i_k} (1 - \delta_{i_k}^1 z_1 - z_{4-i_k}) \\ &\leq \sum_{i_k=1}^{2} (1 - \delta_{i_k}^2 z_1)^2 z_{4-i_k} (1 - \delta_{i_k}^1 z_1 - z_{4-i_k}) \\ &= z_3(1 - z_1 - z_3) + (1 - z_1)^2 z_2 (1 - z_2). \end{aligned}$$

Thus, for arbitrary $n \geq 1$ and $k \geq 1$

$$|f_{n+k}^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})| \leq \frac{\mu^{n+1}(\mathbf{z})}{(1 - v(\mathbf{z}))((1 - v(\mathbf{z}))^2 + \mu(\mathbf{z}))^n}, \tag{19}$$

where $v(\mathbf{z})$ and $\mu(\mathbf{z})$ are defined by (9) and (10), respectively. Hence, due to the arbitrariness of k and taking into account that for arbitrary fixed $\mathbf{z} \in \mathfrak{G}_\lambda$,

$$\frac{\mu^{n+1}(\mathbf{z})}{(1 - v(\mathbf{z}))((1 - v(\mathbf{z}))^2 + \mu(\mathbf{z}))^n} \to 0 \text{ as } n \to +\infty,$$

it follows that (3) converges to a finite value $f^{(i_0)}(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{G}_\lambda$.

(ii) Let $\mathfrak{K}$ be an arbitrary compact subset of $\mathrm{Int}(\mathfrak{G}_\lambda)$. Then there exists $\mu > 0$ such that $\mu \geq \mu(\mathbf{z})$, where $\mu(\mathbf{z})$ is defined by (10), and, therefore, from (19) for $n \geq 1$ and $k \geq 1$

$$|f_{n+k}^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})| < \frac{\mu^{n+1}}{(1 - \lambda)((1 - \lambda)^2 + \mu)^n}$$

for all $\mathbf{z} \in \mathfrak{K}$. In addition, if p and q are arbitrary natural numbers such that $p \geq n \geq 1$ and $q \geq 1$, then, for all $\mathbf{z} \in \mathfrak{K}$,

$$|f_{p+q}^{(i_0)}(\mathbf{z}) - f_p^{(i_0)}(\mathbf{z})| \leq |f_{p+q}^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})| + |f_p^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})|.$$

Hence, taking into account that

$$\frac{\mu^{n+1}}{(1-\lambda)((1-\lambda)^2+\mu)^n} \to 0 \text{ as } n \to +\infty,$$

it follows that (3) converges uniformly on every compact subset of $\mathbf{z} \in \mathrm{Int}(\mathfrak{G}_\lambda)$.

(iii) follows from (19), passing to the limit as $k \to +\infty$.

(iv) Recall that the hypergeometric series (1) converges in the domain (see, [39])

$$\mathfrak{D}_{\kappa,\tau} = \{\mathbf{z} \in \mathbb{C}^3 : \ |z_1| < \kappa, \ |z_2| < 1, \ |z_3| < \tau\},$$

where κ and τ are positive numbers such that $\kappa + \tau = 1$. In addition, it is obvious that

$$F_M(\alpha_1, \alpha_2, \alpha_1, \beta_2; \gamma_1, \gamma_2; \mathbf{0}) = 1$$

and

$$F_M(\alpha_1, \alpha_2+1, \beta_1+\delta_{i_0}^1, \beta_2+\delta_{i_0}^2; \alpha_1, \gamma_2+1; \mathbf{0}) = 1.$$

Hence, there exists $0 < \varepsilon < 1$ such that function (2) is holomorphic in domain

$$\mathfrak{D}_{\kappa,\tau,\lambda,\varepsilon} = \left\{\mathbf{z} \in \mathbb{R}^3 : \ 0 < z_1 < \varepsilon \min\left\{\kappa, \frac{\lambda}{2}\right\}, \ 0 < z_2 < \frac{\varepsilon\lambda}{4(2-\lambda)}, \ 0 < z_3 < \varepsilon \min\left\{\tau, \frac{\lambda}{16}\right\}\right\},$$

and, in addition, (see, Remark 1)

$$\mathfrak{D}_{\kappa,\tau,\lambda,\varepsilon} \subset (\mathfrak{D}_{\kappa,\tau} \cap \mathrm{Int}(\mathfrak{G}_\lambda)),$$

where $\mathfrak{G}_\lambda$ is defined by (8), in particular,

$$\mathfrak{D}_{\kappa,\tau,\lambda,1/2} \subset (\mathfrak{D}_{\kappa,\tau} \cap \mathrm{Int}(\mathfrak{G}_\lambda)).$$

Let $\mathbf{z}$ be an arbitrary fixed point in $\mathfrak{D}_{\kappa,\tau,\lambda,\varepsilon}$. From proof (i) it follows that under conditions (7) the elements (4)–(6) have positive values. This means that the property of fork holds for the approximants of the branched continued fraction (3) (see, [2, 9]), e.i.,

$$f_{2n}^{(i_0)}(\mathbf{z}) < f_{2n+2}^{(i_0)}(\mathbf{z}) < f_{2n+1}^{(i_0)}(\mathbf{z}) < f_{2n-1}^{(i_0)}(\mathbf{z}), \ n \geq 1,$$

so that the even and odd approximants of (3) converge to a finite value $f^{(i_0)}(\mathbf{z})$.

In that follows, for $n \geq 1$ we will consider

$$\frac{(1-\delta_{i_0}^2 z_1) F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})}{F_M(\alpha_1, \alpha_2+1, \beta_1+\delta_{i_0}^1, \beta_2+\delta_{i_0}^2; \alpha_1, \gamma_2+1; \mathbf{z})} - f_n^{(i_0)}(\mathbf{z}),$$

where (see [34])

$$\frac{(1-\delta_{i_0}^2 z_1) F_M(\alpha_1, \alpha_2, \beta_1, \beta_2; \alpha_1, \gamma_2; \mathbf{z})}{F_M(\alpha_1, \alpha_2+1, \beta_1+\delta_{i_0}^1, \beta_2+\delta_{i_0}^2; \alpha_1, \gamma_2+1; \mathbf{z})} = v_{i_0}(\mathbf{z}) + \sum_{i_1=1}^{2} \cfrac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z}) + \sum_{i_2=1}^{2} \cfrac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z}) + \ddots + \sum_{i_{n+1}=1}^{2} \cfrac{u_{i(n+1)}(\mathbf{z})}{R_{i(n+1)}^{(n+1)}(\mathbf{z})}}},$$

where, for $i(n+1) \in \mathfrak{I}_{n+1}$, $n \geq 1$,

$$R_{i(n+1)}^{(n+1)}(\mathbf{z}) = \frac{(1-\delta_{i_{n+1}}^2 z_1) F_M\left(\alpha_1, \alpha_2+n+1, \beta_1+\sum_{r=0}^{n}\delta_{i_r}^1, \beta_2+\sum_{r=0}^{n}\delta_{i_r}^2; \alpha_1, \gamma_2+n+1; \mathbf{z}\right)}{F_M\left(\alpha_1, \alpha_2+n+2, \beta_1+\sum_{r=0}^{n+1}\delta_{i_r}^1, \beta_2+\sum_{r=0}^{n+1}\delta_{i_r}^2; \alpha_1, \gamma_2+n+2; \mathbf{z}\right)}.$$

We set

$$R_{i(k)}^{(n+1)}(\mathbf{z}) = v_{i(k)}(\mathbf{z}) + \sum_{i_{k+1}=1}^{2} \cfrac{u_{i(k+1)}(\mathbf{z})}{v_{i(k+1)}(\mathbf{z}) + \sum_{i_{k+2}=1}^{2} \cfrac{u_{i(k+2)}(\mathbf{z})}{v_{i(k+2)}(\mathbf{z}) + \ddots + \sum_{i_{n+1}=1}^{2} \cfrac{u_{i(n+1)}(\mathbf{z})}{R_{i(n+1)}^{(n+1)}(\mathbf{z})}}},$$

where $i(k)\in\mathfrak{I}_k$, $1\le k\le n$, $n\ge 1$. Then

$$R_{i(k)}^{(n+1)}(\mathbf{z})=v_{i(k)}(\mathbf{z})+\sum_{i_{k+1}=1}^{2}\frac{u_{i(k+1)}(\mathbf{z})}{R_{i(k+1)}^{(n+1)}(\mathbf{z})},\ i(k)\in\mathfrak{I}_k,\ 1\le k\le n,\ n\ge 1. \tag{20}$$

It is clear that $F_{i(k)}^{(n)}(\mathbf{z})\ne 0$ and $R_{i(k)}^{(n)}(\mathbf{z})\ne 0$ for all indices and for all $\mathbf{z}\in\mathfrak{D}_{\kappa,\tau,\lambda,\varepsilon}$. Using (13), (14), and (20), from (17) for $n\ge 1$ we have

$$\frac{(1-\delta_{i_0}^2 z_1)F_M(\alpha_1,\alpha_2,\beta_1,\beta_2;\alpha_1,\gamma_2;\mathbf{z})}{F_M(\alpha_1,\alpha_2+1,\beta_1+\delta_{i_0}^1,\beta_2+\delta_{i_0}^2;\alpha_1,\gamma_2+1;\mathbf{z})}-f_n^{(i_0)}(\mathbf{z})=(-1)^n\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\ldots\sum_{i_{n+1}=1}^{2}\frac{\prod\limits_{r=1}^{n+1}u_{i(r)}(\mathbf{z})}{\prod\limits_{r=1}^{n+1}R_{i(r)}^{(n+1)}(\mathbf{z})\prod\limits_{r=1}^{n}F_{i(r)}^{(n)}(\mathbf{z})}.$$

Thus, for any $n\ge 1$ and for all $\mathbf{z}\in\mathfrak{D}_{\kappa,\tau,\lambda,\varepsilon}$,

$$f_{2n}^{(i_0)}(\mathbf{z})<\frac{(1-\delta_{i_0}^2 z_1)F_M(\alpha_1,\alpha_2,\beta_1,\beta_2;\alpha_1,\gamma_2;\mathbf{z})}{F_M(\alpha_1,\alpha_2+1,\beta_1+\delta_{i_0}^1,\beta_2+\delta_{i_0}^2;\alpha_1,\gamma_2+1;\mathbf{z})}<f_{2n-1}^{(i_0)}(\mathbf{z}).$$

Hence, taking into account that for all $\mathbf{z}\in\mathfrak{D}_{\kappa,\tau,\lambda,\varepsilon}$

$$\lim_{n\to+\infty}f_{2n}^{(i_0)}(\mathbf{z})=\lim_{n\to+\infty}f_{2n-1}^{(i_0)}(\mathbf{z})=f^{(i_0)}(\mathbf{z}),$$

it follows that for all $\mathbf{z}\in\mathfrak{D}_{\kappa,\tau,\lambda,\varepsilon}$,

$$f^{(i_0)}(\mathbf{z})=\frac{(1-\delta_{i_0}^2 z_1)F_M(\alpha_1,\alpha_2,\beta_1,\beta_2;\alpha_1,\gamma_2;\mathbf{z})}{F_M(\alpha_1,\alpha_2+1,\beta_1+\delta_{i_0}^1,\beta_2+\delta_{i_0}^2;\alpha_1,\gamma_2+1;\mathbf{z})}.$$

and hence, by Theorem 2 [2], $f^{(i_0)}(\mathbf{z})$ provides the analytic continuation of the function (2) into the domain $\mathrm{Int}(\mathfrak{G}_\lambda)$. □

Setting $\alpha_2=\beta_1=0$ and replacing γ_2 by γ_2-1, we have the following:

Corollary 2.1. *Let $i_0=1$, α_1 be complex number herewith $\alpha_1\notin\{0,-1,-2,\ldots\}$, β_2 and γ_2 be real numbers such that $0<\beta_2\le\gamma_2-1$, $\gamma_2\ge 2$. Then we have the following:*

(i) *The branched continued fraction*

$$\cfrac{1}{v_{i_0}(\mathbf{z})+\sum\limits_{i_1=1}^{2}\cfrac{u_{i(1)}(\mathbf{z})}{v_{i(1)}(\mathbf{z})+\sum\limits_{i_2=1}^{2}\cfrac{u_{i(2)}(\mathbf{z})}{v_{i(2)}(\mathbf{z})+\ddots}}} \tag{21}$$

converges to a finite value $f^{(i_0)}(\mathbf{z})$ for each $\mathbf{z}\in\mathfrak{G}_\lambda$, where $\mathfrak{G}_\lambda$ is defined by (8),

$$v_{i_0}(\mathbf{z})=1-z_1-\frac{\beta_{i_0}+1}{\gamma_2-1}(1-\delta_{i_0}^2 z_1)z_{4-i_0}-\frac{\beta_{3-i_0}}{\gamma_2-1}(1-\delta_{i_0}^1 z_1)z_{1+i_0}$$

and, for $i(k)\in\mathfrak{I}_k$, $k\ge 1$,

$$u_{i(k)}(\mathbf{z})=\frac{k\left(\beta_{i_k}+\sum\limits_{r=0}^{k-1}\delta_{i_r}^{i_k}\right)}{(\gamma_2+k-2)(\gamma_2+k-1)}(1-\delta_{i_k}^2 z_1)^2 z_{4-i_k}(1-\delta_{i_k}^1 z_1-z_{4-i_k}),$$

$$v_{i(k)}(\mathbf{z})=1-z_1-\frac{\beta_{i_k}+k+1+\sum\limits_{r=0}^{k-1}\delta_{i_r}^{i_k}}{\gamma_2+k-1}(1-\delta_{i_k}^2 z_1)z_{4-i_k}-\frac{\beta_{3-i_k}+\sum\limits_{r=0}^{k-1}\delta_{i_r}^{3-i_k}}{\gamma_2+k-1}(1-\delta_{i_k}^1 z_1)z_{1+i_k}$$

herewith $\beta_1=0$.

(ii) *The convergence is uniform on every compact subset of the domain $\mathrm{Int}(\mathfrak{G}_\lambda)$, and $f^{(i_0)}(\mathbf{z})$ is holomorphic on $\mathrm{Int}(\mathfrak{G}_\lambda)$.*

(iii) *If $f_n^{(i_0)}(\mathbf{z})$ denotes the nth approximant of (21), then for each $\mathbf{z}\in\mathfrak{G}_\lambda$*

$$|f^{(i_0)}(\mathbf{z})-f_n^{(i_0)}(\mathbf{z})|\le\frac{\mu^n(\mathbf{z})}{(1-z_1-(1-z_1)z_2-z_3)^3((1-\nu(\mathbf{z}))^2+\mu(\mathbf{z}))^{n-1}},\ n\ge 1,$$

where $\nu(\mathbf{z})$ and $\mu(\mathbf{z})$ are defined by (9) and (10), respectively.

(iv) *The function $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of the function*

$$F_M(\alpha_1, 1, 1, \beta_2; \alpha_1, \gamma_2; \mathbf{z}) \tag{22}$$

in the domain $\mathrm{Int}(\mathfrak{G}_\lambda)$.

Remark 2. Theorem 1.1(i),(iii) is also true without the condition $z_1 < 1$ in (8). Corollary 2.1(i),(iii) is also true if this condition is replaced by $z_1 + (1 - z_1)z_2 + z_3 \le \lambda$.

In that follows, we will use the method of extending the convergence domain, which is based on the convergence continuation theorem (see [17]).

Proof of Theorem 1.2. (i) Let i_0 be an arbitrary index in $\mathfrak{I}$, α be an arbitrary real number in $(-\pi/2, \pi/2)$, and $\mathbf{z}$ be an arbitrary fixed point in (12).

In that follows, we will prove the following inequalities

$$\mathrm{Re}(F^{(n)}_{i(k)}(\mathbf{z})e^{-i\alpha}) > \frac{(1-\eta)\cos\alpha}{2} = c > 0,\ i(k) \in \mathfrak{I}_k,\ 1 \le k \le n,\ n \ge 1, \tag{23}$$

where $F^{(n)}_{i(k)}(\mathbf{z})$, $i(k) \in \mathfrak{I}_k$, $1 \le k \le n$, $n \ge 1$, are defined by (13) and (14).

Let n be an arbitrary natural number. Using (7) and (12), from (13) for arbitrary $i(n) \in \mathfrak{I}_n$ we have

$$\begin{aligned}
\mathrm{Re}(F^{(n)}_{i(n)}(\mathbf{z})e^{-i\alpha}) &= \mathrm{Re}(v_{i(n)}(\mathbf{z})e^{-i\alpha}) \\
&= \mathrm{Re}(e^{-i\alpha}) - \mathrm{Re}(z_1 e^{-i\alpha}) - \frac{\alpha_2 + \beta_{i_n} + 2n + 1}{\gamma_2 + n}\mathrm{Re}((1 - \delta^2_{i_n} z_1)z_{4-i_n}e^{-i\alpha}) - \frac{\beta_{3-i_n} + n}{\gamma_2 + n}\mathrm{Re}((1 - \delta^1_{i_n} z_1)z_{1+i_n}e^{-i\alpha}) \\
&\ge \cos\alpha - \mathrm{Re}(z_1 + 2(1 - \delta^2_{i_n} z_1)z_{4-i_n} + (1 - \delta^1_{i_n} z_1)z_{1+i_n})e^{-i\alpha}) \\
&> (1-\eta)\cos\alpha > c.
\end{aligned}$$

Let inequalities (23) hold for $k = s+1$ and for all $i(s+1) \in \mathfrak{I}_{s+1}$ such that $s + 1 \le n$. Using (7), (12) and Corollary 2 [2], from (14) for $k = s$ and for arbitrary $i(s) \in \mathfrak{I}_s$ we have

$$\begin{aligned}
\mathrm{Re}(F^{(n)}_{i(s)}(\mathbf{z})e^{-i\alpha}) &= \mathrm{Re}(v_{i(s)}(\mathbf{z})e^{-i\alpha}) + \sum_{i_{s+1}=1}^{2} \frac{(\alpha_2 + s + 1)\left(\beta_{i_{s+1}} + \sum_{r=0}^{s}\delta^{i_{s+1}}_{i_r}\right)}{(\gamma_2 + s)(\gamma_2 + s + 1)}\mathrm{Re}\left(\frac{(1 - \delta^2_{i_{s+1}} z_1)^2 z_{4-i_{s+1}}(1 - \delta^1_{i_{s+1}} z_1 - z_{4-i_{s+1}})e^{-2i\alpha}}{F^{(n)}_{i(s+1)}(\mathbf{z})e^{-i\alpha}}\right) \\
&> (1-\eta)\cos\alpha - \sum_{i_{s+1}=1}^{2} \frac{(\alpha_2 + s)\left(\beta_{i_{s+1}} + \sum_{r=0}^{s}\delta^{i_{s+1}}_{i_r}\right)}{(\gamma_2 + s)(\gamma_2 + s + 1)2\,\mathrm{Re}(F^{(n)}_{i(s+1)}(\mathbf{z})e^{-i\alpha})}(|(1 - \delta^2_{i_{s+1}} z_1)^2 z_{4-i_{s+1}}(1 - \delta^1_{i_{s+1}} z_1 - z_{4-i_{s+}})| \\
&\quad - \mathrm{Re}((1 - \delta^2_{i_{s+1}} z_1)^2 z_{4-i_{s+1}}(1 - \delta^1_{i_{s+1}} z_1 - z_{4-i_{s+1}})e^{-2i\alpha})) \\
&> (1-\eta)\cos\alpha - \sum_{i_{s+1}=1}^{2} \frac{1}{(1-\eta)\cos\alpha}(|(1 - \delta^2_{i_{s+1}} z_1)^2 z_{4-i_{s+1}}(1 - \delta^1_{i_{s+1}} z_1 - z_{4-i_{s+1}})| \\
&\quad - \mathrm{Re}((1 - \delta^2_{i_{s+1}} z_1)^2 z_{4-i_{s+1}}(1 - \delta^1_{i_{s+1}} z_1 - z_{4-i_{s+1}})e^{-2i\alpha})) \\
&> (1-\eta)\cos\alpha - \frac{(1-\eta)\cos\alpha}{2} \\
&= c.
\end{aligned}$$

It follows that for all $\mathbf{z} \in \mathfrak{Q}_{\eta,\alpha}$ the inequality (16) holds. This gives that approximants of branched continued fraction (3) form a sequence of functions holomorphic in the domain (12), and, consequently, in (11) by virtue of arbitrariness α.

Let $\mathfrak{K}$ be an arbitrary compact subset of (11). Then there exists an open ball $\mathfrak{L}$ with center at the origin and radius l such that $\mathfrak{K} \subset \mathfrak{L}$. We cover $\mathfrak{K}$ with domains of the form $\mathfrak{Q}_{\eta,\alpha,l} = \mathfrak{Q}_{\eta,\alpha} \cap \mathfrak{L}$ and from this cover we choose the finite subcover $\mathfrak{Q}_{\eta,\alpha_1,l}, \mathfrak{Q}_{\eta,\alpha_2,l}, \ldots, \mathfrak{Q}_{\eta,\alpha_k,l}$. Then again, using (4), (5), (7), and (23), from (15) for any $s \in \{1, 2, \ldots, k\}$ and $\mathbf{z} \in \mathfrak{Q}_{\eta,\alpha_s,l}$ we have, for $n \ge 1$,

$$\begin{aligned}
|f^{(i_0)}_n(\mathbf{z})| &\le 1 + |z_1| + \frac{\alpha_2 + \beta_{i_0} + 1}{\gamma_2}(1 + \delta^2_{i_0}|z_1|)|z_{4-i_0}| + \frac{\beta_{3-i_0}}{\gamma_2}(1 + \delta^1_{i_0}|z_1|)|z_{1+i_0}| \\
&\quad + \sum_{i_1=1}^{2} \frac{(\alpha_2 + 1)(\beta_{i_1} + \delta^{i_1}_{i_0})}{\gamma_2(\gamma_2 + 1)} \frac{(1 + \delta^2_{i_1}|z_1|)^2|z_{4-i_1}|(1 + \delta^1_{i_1}|z_1| + |z_{4-i_1}|)}{\mathrm{Re}(F^{(n)}_{i(1)}(\mathbf{z})e^{-i\alpha_s})} \\
&< 1 + l + 2(1 + \delta^2_{i_0} l)l + (1 + \delta^1_{i_0} l)l + \frac{2(l(1+2l) + l(1+l)^3)}{(1-\eta)\cos\alpha_s} \\
&= C(\mathfrak{Q}_{\eta,\alpha_s,l}).
\end{aligned}$$

We set

$$C(\mathfrak{K}) = \max_{1\le s\le k} C(\mathfrak{Q}_{\eta,\alpha_s,l}).$$

Then for any $z \in \mathfrak{K}$ we obtain $|f_n^{(i_0)}(\mathbf{z})| \le C(\mathfrak{K})$ for $n \ge 1$, i.e. the $\{f_n^{(i_0)}(\mathbf{z})\}$ is uniformly bounded on every compact subset of (11). It is clear that there exists a nonempty set $\mathfrak{E}$ such that $\mathfrak{E} \subset \mathrm{Int}(\mathfrak{G}_\lambda) \cap \mathfrak{Q}_\eta$, where $\mathfrak{G}_\lambda$ is defined by (8), and which is a real neighborhood of the point $\mathbf{z}^0$ in $\mathrm{Int}(\mathfrak{G}_\lambda) \cap \mathfrak{Q}_\eta$. From Theorem 1.1 it follows that (3) converges in the domain $\mathfrak{E}$, and, therefore, according to Theorem 5 [2], the convergence of (3) is uniform on compact subsets of (11). This proves (i).

(ii) is proved similarly to the proof of Theorem 1.1(iv); hence it is omitted. □

Finally, we have the following:

Corollary 2.2. *Let the conditions of Corollary 2.1 be satisfied. Then the branched continued fraction (21) converges uniformly on every compact subset of the domain*

$$\mathfrak{H}_\eta = \bigcup_{-\pi/2<\alpha<\pi/2} \left(\mathfrak{Q}_{\eta,\alpha} \bigcap \mathfrak{H}_{\eta,\alpha}\right), \; 0 < \eta < 1, \tag{24}$$

where $\mathfrak{Q}_{\eta,\alpha}$ is defined by (12) and

$$\mathfrak{H}_{\eta,\alpha} = \left\{\mathbf{z} \in \mathbb{C}^3 : \; \mathrm{Re}((z_1 + z_3 + (1-z_1)z_2)e^{-i\alpha}) < \eta\cos\alpha\right\},$$

to the function $f^{(i_0)}(\mathbf{z})$ holomorphic in $\mathfrak{H}_\eta$, and, in addition, the $f^{(i_0)}(\mathbf{z})$ is an analytic continuation of the function (22) in the domain (24).

Remark 3. The corollaries similar Corollaries 2.1 and 2.2 are valid if $i_0 = 2$, $\alpha_2 = \beta_2 = 0$, and γ_2 is replaced by $\gamma_2 - 1$.

3 Applications

This section provides the use of branched continued fractions to approximate special functions, examining the consistency with the results obtained in Section 2.

In [34] it is shown that the function

$$\ln\frac{1-z_1-z_3}{(1-z_1)(1-z_2)} \tag{25}$$

has the following two representations in the formal triple power series

$$((1-z_1)z_2 - z_3)\sum_{p,q,r=0}^{+\infty} \frac{(1)_{q+r}(1)_{p+r}(1)_q}{(2)_{q+r}}\frac{z_1^p z_2^q z_3^r}{p!q!r!}. \tag{26}$$

and in the formal branched continued fraction

$$\cfrac{(1-z_1)z_2 - z_3}{(1-z_1)(1-z_2) - z_3 + \cfrac{\frac{z_3(1-z_1-z_3)}{2}}{(1-z_1)\left(1-\frac{z_2}{2}\right) - \frac{3z_3}{2} + \ddots} + \cfrac{\frac{(1-z_1)^2 z_2(1-z_2)}{2}}{(1-z_1)\left(1-\frac{3z_2}{2}\right) - \frac{z_3}{2} + \ddots}}. \tag{27}$$

From the Corollary 2.2 it follows that the branched continued fraction (27) converges and represents a single-valued branch of the function (25) in the domain $\mathfrak{T}_\eta = \mathfrak{H}_\eta \cap \mathfrak{R}$, where $\mathfrak{H}_\eta$ is defined by (24) and

$$\mathfrak{R} = \left\{\mathbf{z} \in \mathbb{C}^3 : \; \frac{1-z_1-z_3}{(1-z_1)(1-z_2)} \notin (-\infty, 0]\right\}.$$

The results in Table 1 show that the relative errors of approximation of the function (25) by 5th approximants of the branched continued fraction (27) is better than by 5th partial sums of the triple power series (26) at points close to the origin. At points distant from the origin, the branched continued fraction (27) still converges, albeit worse, while the triple power series diverges (26).

The plots in Figure 1 shows the curves of the nth approximants $f_n(\mathbf{z})$ $(3 \le n \le 8)$ of the branched continued fraction (27) for fixed variables $z_2 = 0.4$ and $z_3 = 0.5$ (Fig. 1(a)), $z_1 = z_3 = 0.1$ (Fig. 1(b)), and $z_1 = z_2 = 0.4$ (Fig. 1(c)). On all selected segments where the elements of (27) have positive values we observe the property of fork (see, [2, 9]), that is

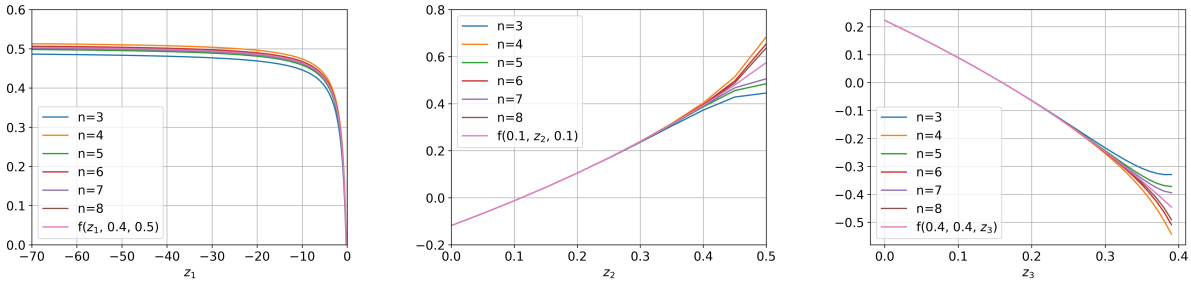
$$f_3(\mathbf{z}) < f_5(\mathbf{z}) < f_7(\mathbf{z}) < f_8(\mathbf{z}) < f_6(\mathbf{z}) < f_4(\mathbf{z}).$$

In plots of Figures 2–4 depict domains in twelve different planes with fixed $z_3 = 0.1$ (Fig. 2(a)–(d)), $z_2 = 0.1$ (Fig. 3(a)–(d)), and $z_1 = 0.1$ (Fig. 4(a)–(d)), where the fifth approximation of the branched continued fraction (27) guarantees certain truncation error bounds for the function (25).

Calculations and plots were performed using Python 3.11.9, mpmath 1.3.0.

Table 1: Relative error of 5th partial sum and 5th approximant.

z	(25)	(27)	(26)
$(-0.01, 0.01, 0.01)$	0.0001	2.1832×10^{-13}	4.8787×10^{-13}
$(-0.1, 0.1, 0.1)$	0.0101	1.5485×10^{-6}	1.6423×10^{-6}
$(0.65, 0.24, 0.12)$	-0.1454	9.3719×10^{-3}	1.1755×10^{-1}
$(-0.3, -0.3, -0.3)$	-0.05472	7.0649×10^{-5}	4.424×10^{-4}
$(-0.1+0.1i, 0.1-0.1i, -0.1-0.1i)$	$0.1821 - 0.02i$	1.6833×10^{-6}	6.5464×10^{-6}
$(5, 0.2, 0.1)$	0.2478	2.8046×10^{-5}	$2.1897 \times 10^{+4}$
$(3+5i, 0.2+0.5i, 0.3)$	$0.0801 + 0.508i$	5.1474×10^{-3}	$1.2686 \times 10^{+5}$
$(-10+10i, -10-10i, 10+10i)$	$-2.4011 - 1.5208i$	2.5556×10^{-1}	$3.0581 \times 10^{+20}$
$(72, 0.1, 0.1)$	0.2245	3.1828×10^{-5}	$2.1234 \times 10^{+11}$
$(75, -5, 0.1)$	-1.7904	9.0111×10^{-2}	$3.5935 \times 10^{+14}$
$(-100, -100, -100)$	-3.9269	4.1728×10^{-1}	$5.7102 \times 10^{+34}$
$(1000, 0.2, 0.1)$	0.2232	3.206×10^{-5}	$1.4032 \times 10^{+18}$



(a) At $z_2 = 0.4$ and $z_3 = 0.5$. **(b)** At $z_1 = z_3 = 0.1$. **(c)** At $z_1 = z_2 = 0.4$.

Figure 1: The plots of values of the nth approximants of (27) for (25).

4 Conclusions

For each $i_0 \in \mathfrak{I}$ and $\mathbf{z} \in \mathrm{Int}(\mathfrak{G}_\lambda)$, where $\mathfrak{G}_\lambda$ is defined by (8), we have established that under conditions (7) the branched continued fraction (3) converges to the function $f^{(i_0)}(\mathbf{z})$ which is holomorphic in $\mathrm{Int}(\mathfrak{G}_\lambda)$, at least as fast as geometric series with ratio

$$\rho(\mathbf{z}) = \frac{\mu(\mathbf{z})}{(1-\nu(\mathbf{z}))^2 + \mu(\mathbf{z})},$$

where $\nu(\mathbf{z})$ and $\mu(\mathbf{z})$ are defined by (9) and (10), respectively. Thus,

$$\limsup_{n\to+\infty} |f^{(i_0)}(\mathbf{z}) - f_n^{(i_0)}(\mathbf{z})|^{1/n} \le \rho(\mathbf{z}),$$

where $f_n^{(i_0)}(\mathbf{z})$ is nth approximant of (3). It is also established that for each $i_0 \in \mathfrak{I}$ under conditions (7), the function $f^{(i_0)}(\mathbf{z})$, to which the branched continued fraction (3) converges, is an analytic extension of the function (2) in the domain (11). Numerical experiments provide the effectiveness and feasibility of using branched continued fraction (27) as a tool for approximating special function (25) compared to the triple power series (26).

Further direction is the study of numerical stability (see, [16, 24, 25, 29]) of branched continued fractions (3) and (21). Another direction is the application of quantum calculus to branched continued fractions (some results related to polynomial can be found in [33, 38]).

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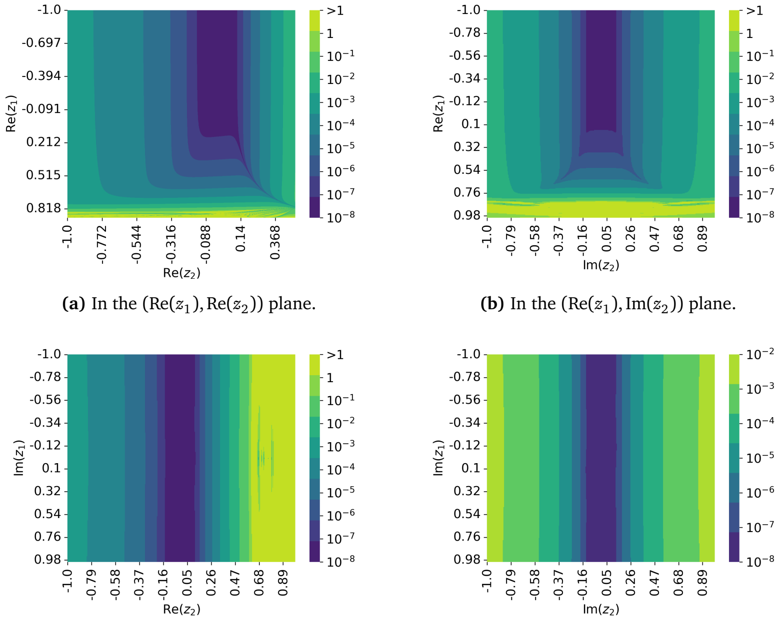


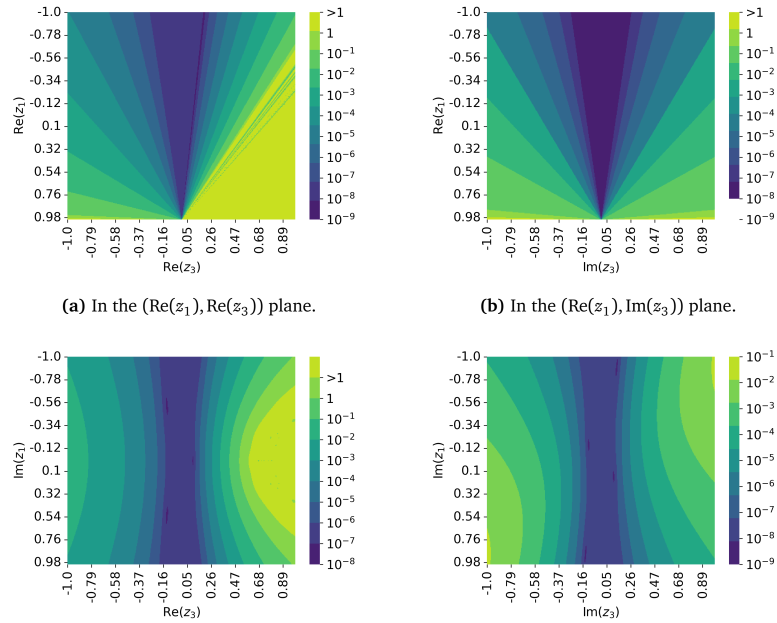
(a) In the $(\mathrm{Re}(z_1), \mathrm{Re}(z_2))$ plane.

(b) In the $(\mathrm{Re}(z_1), \mathrm{Im}(z_2))$ plane.

(c) In the $(\mathrm{Im}(z_1), \mathrm{Re}(z_2))$ plane.

(d) In the $(\mathrm{Im}(z_1), \mathrm{Im}(z_2))$ plane.

Figure 2: The plots where the approximant $f_5(z_1, z_2, 0.1)$ of (27) guarantees certain truncation error bounds for (25).



(a) In the $(\mathrm{Re}(z_1), \mathrm{Re}(z_3))$ plane.

(b) In the $(\mathrm{Re}(z_1), \mathrm{Im}(z_3))$ plane.

(c) In the $(\mathrm{Im}(z_1), \mathrm{Re}(z_3))$ plane.

(d) In the $(\mathrm{Im}(z_1), \mathrm{Im}(z_3))$ plane.

Figure 3: The plots where the approximant $f_5(z_1, 0.1, z_3)$ of (27) guarantees certain truncation error bounds for (25).

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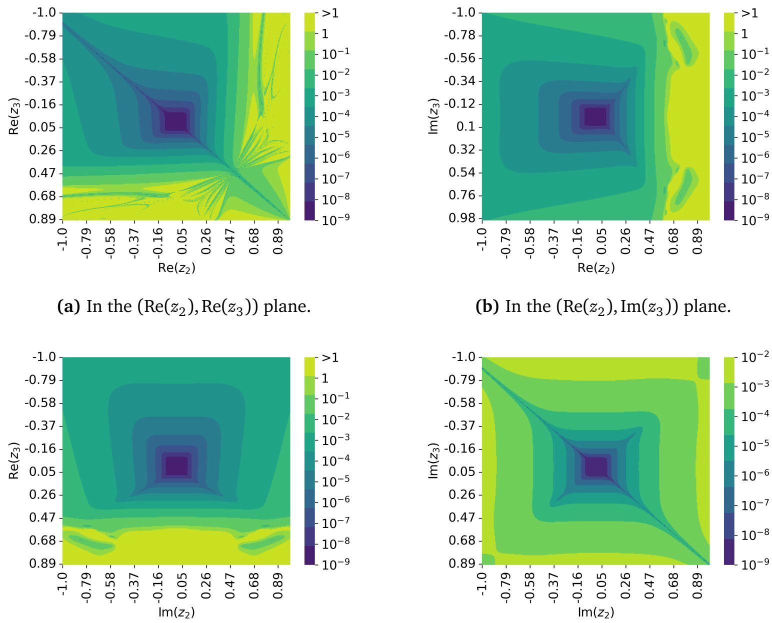

(a) In the $(\mathrm{Re}(z_2), \mathrm{Re}(z_3))$ plane.

(b) In the $(\mathrm{Re}(z_2), \mathrm{Im}(z_3))$ plane.

(c) In the $(\mathrm{Im}(z_2), \mathrm{Re}(z_3))$ plane.

(d) In the $(\mathrm{Im}(z_2), \mathrm{Im}(z_3))$ plane.

Figure 4: The plots where the approximant $f_5(0.1, z_2, z_3)$ of (27) guarantees certain truncation error bounds for (25).

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