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Numerical methods for hypersingular integrals on the real line

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Abstract

In the present paper the authors propose two numerical methods to approximate Hadamard transforms of the type

$$\mathbf{H}_p(f w_\beta, t) = \oint_{\mathbb{R}} \frac{f(x)}{(x-t)^{p+1}} w_\beta(x) dx,$$

where p is a nonnegative integer and $w_\beta(x) = e^{-|x|^\beta}$, $\beta > 1$, is a Freud weight. One of the procedures employed here is based on a simple tool like the "truncated" Gaussian rule conveniently modified to remove numerical cancellation and overflow phenomena. The second approach is a process of simultaneous approximation of the functions $\{\mathbf{H}_k(f w_\beta, t)\}_{k=0}^{p}$. This strategy can be useful in the numerical treatment of hypersingular integral equations. The methods are shown to be numerically stable and convergent and some error estimates in suitable Zygmund-type spaces are proved. Numerical tests confirming the theoretical estimates are given. Comparisons of our methods among them and with other ones available in literature are shown.

1 Introduction

In the present paper we propose some global strategies to approximate finite part integrals (shortly FP integrals)

$$\oint_{\mathbb{R}} \frac{f(x)}{(x-t)^{p+1}} w_\beta(x) dx, \qquad t \in \mathbb{R},$$

where p is a nonnegative integer, $w_\beta(x) = e^{-|x|^\beta}$, $\beta > 1$, is a Freud weight and the density function f is assumed in such a way that the integral exists in the Hadamard sense.

Integrals of this kind are of interest, for instance, in the solution of hypersingular BIE, which model many different Physical and Engineering problems (see [26] and the references therein, [8], [11], [18], [1], [31]).

The numerical treatment of FP integrals over finite ranges has been extensively treated [25], [16] and limiting ourselves to interpolatory rules based on the zeros of orthogonal polynomials, we cite among them [14], [29], [24] whenever the integrand contains Jacobi weights. For a comparison among different numerical approaches in $[-1, 1]$ see also [32]. On the other hand the use of the tools introduced for finite ranges to integrals over infinite ranges can be inadequate. For instance, non linear transformation can produce slower convergent results, since the density function can be significantly worsened. Moreover, the analysis of the stability of the rules and the estimates of the errors in different functional spaces can present some additional difficulties. For instance, the appealing Gaussian rule looses its high performance as well as product integration rules based on the zeros of orthogonal polynomials, and by a truncation strategy introduced in [19] (see also [7], [22]) the problem can be successfully overcome.

An additional hindrance depends on the weight w_β, since the coefficients of the three term recurrence relation of the corresponding orthogonal polynomials are unknown in the general case. However, it is possible to calculate them with high accuracy in extended arithmetic using *Wolfram Mathematica* routines [2] (details are given in Section 5).

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Following a very standard way, we start from the decomposition

$$\begin{aligned}\mathbf{H}_p(fw_\beta,t) &:= \int\!\!\!\!\!\!=_{\mathbb{R}} \frac{f(x)}{(x-t)^{p+1}} w_\beta(x)dx \\ &= \int_{\mathbb{R}} \frac{f(x)-\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} w_\beta(x)dx + \sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!} \int\!\!\!\!\!\!=_{\mathbb{R}} \frac{w_\beta(x)}{(x-t)^{p+1-k}}dx \\ &=: \mathbf{F}_p(f,t)+\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}\mathbf{H}_{p-k}(w_\beta,t) \\ &= \frac{1}{p!}\frac{d^p}{dt^p}\mathbf{F}_0(f,t)+\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}\mathbf{H}_{p-k}(w_\beta,t), \end{aligned} \tag{1}$$

where

$$\mathbf{F}_0(f,t)=\int_{\mathbb{R}} \frac{f(x)-f(t)}{x-t} w_\beta(x)dx, \tag{2}$$

focusing the attention on the first right-hand integral, since the remaining ones can be efficiently computed by standard routines for many choices of β (see Section 5). In the first method, for any fixed t, we select a suitable subsequence of Gaussian rules based on Freud zeros, which leads to a stable and fast convergent procedure. By this way, the severe numerical cancellation arising when t is "close" to a Gaussian node is avoided, without changes of variables which double the samples of f and depend on the singularity t (see for instance [15], [25]). In addition, since we use "truncated" Gaussian rules [20], the number of function evaluations is drastically reduced and possible overflow phenomena are overcome. To complete the framework, for t "large" (large in the sense we specify later) and f "sufficiently" smooth, we propose a shrewd use of the Gauss-Freud rule.

Since in some contexts the approximation of $\{\mathbf{H}_k(fw_\beta,t)\}_{k=0}^p$ for a "large" number of t and/or the uniform convergence of the approximation process are required, following an idea in [5] (see also [28], [21], [24], [4]), we approximate $\{\mathbf{F}_0^{(k)}(f)\}_{k=0}^p$ by means of the successive derivatives of a suitable Lagrange polynomial interpolating $\mathbf{F}_0(f)$ at Freud zeros. Since in the general case the samples of $\mathbf{F}_0(f)$ at the interpolation knots cannot be exactly computed, we approximate them by a "truncated" Gauss-Freud rule (see [20]). For a correct error estimate in weighted uniform spaces, we determine the class of $\mathbf{F}_0(f)$ depending on the Zygmund-type space f belongs to.

Moreover, whenever the derivatives $\{f^{(k)}(t)\}_{k=1}^p$ are not available, this method can be performed avoiding the differentiation of the density function f. Indeed, in this case, we propose to approximate $f^{(k)}(t)$ by the derivatives of a suitable Lagrange polynomial interpolating f, without requiring additional samples of the density function f.

The paper is organized as follows. In Section 2 some notations and basic results needed to introduce the main results are collected. In Sections 3 and 4 we describe the numerical methods proposed to approximate $\mathbf{H}_p(fw_\beta,t)$, getting some results about the stability and the rate of convergence. Section 5 contains some computational details useful in the implementation process and in Section 6 we present some numerical tests, by comparing both the procedures among them and with other methods available on the same topic. As we will show, the numerical results agree with the theoretical analysis and confirm the efficiency of the proposed procedures. Finally, in Section 7 we give the proofs of the main results.

2 Notations and preliminary results

In the sequel $\mathcal{C}$ will denote any positive constant which can be different in different formulas. Moreover $\mathcal{C} \neq \mathcal{C}(a,b,..)$ will be used to say that the constant $\mathcal{C}$ is independent of $a,b,\ldots$. The notation $A\sim B$, where A and B are positive quantities depending on some parameters, will be used if and only if $(A/B)^{\pm 1}\le \mathcal{C}$, with $\mathcal{C}$ positive constant independent of the above parameters.

Throughout the paper θ will denote a fixed real number belonging to the interval $(0,1)$ which can be different in different formulas. Moreover, $\mathbb{P}_m$ is the space of all algebraic polynomials of degree at most m.

Let $\{p_m(w_\beta)\}_m$ be the sequence of orthonormal polynomials w.r.t. the weight $w_\beta(x)=e^{-|x|^\beta}$, $\beta>1$, with positive leading coefficients, i.e.

$$p_m(w_\beta,x)=\gamma_m(w_\beta)x^m+\text{ terms of lower degree},\quad \gamma_m(w_\beta)>0.$$

Denote by $x_{m,k}$, $k=1,\ldots,\lfloor \frac{m}{2}\rfloor$, the positive zeros of $p_m(w_\beta)$ and by $x_{m,-k}=-x_{m,k}$ the negative ones, setting $x_{m,0}=0$ for m odd. The zeros of $p_m(w_\beta)$ satisfies

$$-a_m<x_{m,-\lfloor\frac{m}{2}\rfloor}<\cdots<x_{m,-1}<\cdots<x_{m,1}<x_{m,2}<\cdots<x_{m,\lfloor\frac{m}{2}\rfloor}<a_m,$$

where $a_m:=a_m(w_\beta)$ is the Mhaskar-Rachmanoff-Saff number w.r.t. w_β (in the sequel M-R-S number).

2.1 Function spaces and best approximation estimates

Setting $u(x)=e^{-\frac{|x|^\beta}{2}}$, $\beta>1$, we define the function space

$$C_u = \left\{f \;:\; fu\in C^0(\mathbb{R}),\ \lim_{x\to\pm\infty} f(x)u(x)=0\right\},$$

equipped with the norm

$$\|f\|_{C_u} := \|fu\| := \sup_{x\in\mathbb{R}} |f(x)|u(x).$$

In the sequel we will write $\|fu\|_E = \sup_{x\in E}|f(x)|u(x)$ for any $E\subset\mathbb{R}$. For smoother functions, we denote by $W_s(u)$, $s\in\mathbb{Z}^+$, the Sobolev space

$$W_s(u) = \left\{f\in C_u : f^{(s-1)}\in AC(\mathbb{R}) : \|f^{(s)}u\| < \infty\right\},$$

where $AC(\mathbb{R})$ denotes the set of the functions which are absolutely continuous on every closed subset of $\mathbb{R}$, equipped with the norm

$$\|f\|_{W_s(u)} = \|fu\| + \|f^{(s)}u\|.$$

Denoting by

$$E_m(f)_u = \inf_{P\in\mathbb{P}_m} \|(f-P)u\|$$

the error of the best approximation in C_u, the following weaker version of the Jackson theorem holds

$$E_m(f)_u \quad\le\quad \mathcal{C}\int_0^{\frac{a_m}{m}} \frac{\Omega^r(f,t)_u}{t}dt, \quad \mathcal{C}\ne\mathcal{C}(m,f), \tag{3}$$

where $a_m := a_m(u)$ denotes the M-R-S number w.r.t u and

$$\Omega^r(f,t)_u = \sup_{0<h\le t} \|(\Delta_h^r f)u\|_{I_{rh}},$$

is the main part of the r−th modulus of continuity (see [9]), with $I_{rh} = [-Arh^{-\frac{1}{\beta-1}}, Arh^{-\frac{1}{\beta-1}}]$, $A>0$ a constant and

$$\Delta_h f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right), \qquad \Delta_h^r = \Delta_h(\Delta_h^{r-1}).$$

Since $a_m(u)\sim a_m(w_\beta)\sim m^{\frac{1}{\beta}}$, throughout the paper we employ the same symbol a_m to denote both of them.

By means of the main part of the r−th modulus of continuity, the following Zygmund space $Z_s(u)$, with $s\in\mathbb{R}^+$, can be defined (see [27])

$$Z_s(u) \quad = \quad \left\{f\in C_u : \sup_{t>0}\frac{\Omega^r(f,t)_u}{t^s} < \infty, \quad r>s\right\},$$

equipped with the norm

$$\|f\|_{Z_s(u)} = \|fu\| + \sup_{t>0}\frac{\Omega^r(f,t)_u}{t^s}, \quad r>s.$$

By (3) it is easy to deduce

$$E_m(f)_u \le \mathcal{C}\left(\frac{a_m}{m}\right)^s \|f\|_{W_s(u)}, \quad \forall f\in W_s(u), \quad s\in\mathbb{Z}^+, \quad \mathcal{C}\ne\mathcal{C}(m,f), \tag{4}$$

$$E_m(f)_u \le \mathcal{C}\left(\frac{a_m}{m}\right)^s \|f\|_{Z_s(u)}, \quad \forall f\in Z_s(u), \quad s\in\mathbb{R}^+, \quad \mathcal{C}\ne\mathcal{C}(m,f). \tag{5}$$

2.2 Truncated Lagrange interpolation at Freud zeros

For any fixed $0<\theta<1$, let

$$x_{m,j} := x_{m,j(m)} = \min\left\{x_{m,k} : x_{m,k}\ge a_m\theta, \quad k=1,2,..,\left\lfloor\frac{m}{2}\right\rfloor\right\}. \tag{6}$$

Let $\mathcal{L}_{m+2}(w_\beta,g)$ be the Lagrange polynomial interpolating a given function g at the zeros of $p_m(w_\beta,x)(a_m^2-x^2)$ and denote by χ_j the characteristic function of the interval $(-x_{m,j},x_{m,j})$. The Lagrange polynomial $L_{m+2}(w_\beta,g) := \mathcal{L}_{m+2}(w_\beta,g\chi_j)$, introduced in [20], can take the following expression

$$L_{m+2}(w_\beta,g,x) = \sum_{k=-j}^{j} l_{m,k}(x)\frac{a_m^2-x^2}{a_m^2-x_{m,k}^2} f(x_{m,k}) =: \sum_{k=-j}^{j} \ell_{m,k}(x) f(x_{m,k}), \tag{7}$$

where $l_{m,k}(x) = \frac{p_m(w_\beta,x)}{p_m'(w_\beta,x_{m,k})(x-x_{m,k})}$. The polynomial $L_{m+2}(w_\beta,g)$ belongs to $\mathcal{P}_{m+1}^*\subset\mathbb{P}_{m+1}$, with

$$\mathcal{P}_{m+1}^* = \left\{p\in\mathbb{P}_{m+1} : p(x_{m,k}) = p(\pm a_m) = 0, \quad |k|>j\right\},$$

and the operator $L_{m+2}(w_\beta)$ projects C_u onto $\mathcal{P}_{m+1}^*$.

About the simultaneous approximation of a function and its successive derivatives, we state the following result.

Theorem 2.1. *If $f\in Z_{p+\lambda}(u)$, $p\in\mathbb{N}$, $0<\lambda<1$, we have*

$$\|(f-L_{m+2}(w_\beta,f))^{(k)}u\| \quad\le\quad \mathcal{C}\,\|f\|_{Z_{p+\lambda}(u)}\log m\left(\frac{a_m}{m}\right)^{p+\lambda-k}, \quad k=1,\dots,p, \quad \mathcal{C}\ne\mathcal{C}(m,f).$$

In particular, if $f\in W_{p+r}(u)$, $p\in\mathbb{N}$, $r\in\mathbb{Z}^+$, we get

$$\|(f-L_{m+2}(w_\beta,f))^{(k)}u\| \quad\le\quad \mathcal{C}\,\|f\|_{W_{p+r}(u)}\log m\left(\frac{a_m}{m}\right)^{p+r-k}, \quad k=1,\dots,p, \quad \mathcal{C}\ne\mathcal{C}(m,f). \tag{8}$$

2.3 Truncated Gauss-Freud rule

In [20] (see also [7], [23]), it was introduced a truncated rule of the the following type

$$\int_{\mathbb{R}} f(x)w_\beta(x)dx = \sum_{|k|\le j} \lambda_{m,k} f(x_{m,k}) + \rho_m(f), \tag{9}$$

where $\lambda_{m,k}, k = -\lfloor \frac{m}{2} \rfloor, \ldots, \lfloor \frac{m}{2} \rfloor$, are the Christoffel numbers w.r.t. the weight w_β and $\rho_m(f)$ is the remainder term. We recall the following result useful in the next and proved in [20] in a more general context.

Theorem 2.2. *For any $f \in C_u$*

$$|\rho_m(f)| \le \mathcal{C}\left(E_M(f)_u + e^{-\mathcal{A}m}\|fu\|\right), \quad \mathcal{C} \ne \mathcal{C}(m,f), \quad \mathcal{A} \ne \mathcal{A}(m,f), \tag{10}$$

where $M = cm$, $0 < c < 1$ fixed.

3 The first method

Starting from the decomposition (1), let t be fixed in the interval $[-x_{m,j}, x_{m,j}]$, where the index j has been defined in (6). Since there exist efficient routines to compute $\mathbf{H}_{p-k}(w_\beta, t)$ for some choices of β (see Section 5), we focus our attention to the term $\mathbf{F}_p(f,t)$ assuming f s.t. the integral exists as an improper Riemann integral. By using the truncated Gauss-Freud rule in (9) we have

$$\mathbf{F}_p(f,t) = \sum_{|k|\le j} \lambda_{m,k} \frac{f(x_{m,k}) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x_{m,k}-t)^{p-k}}{(x_{m,k}-t)^{p+1}} + e_{p,m}(f,t) =: \mathbf{F}_{p,m}(f,t) + e_{p,m}(f,t).$$

However, as observed also in [25], formulae of this kind are very appealing since its coefficients have a very simple form, but they can present severe numerical cancellation when t is "close" to one of the Gaussian nodes.

Thus, to overcome this problem we propose to select, for any fixed t, a proper subsequence of the truncated Gauss-Freud sequence $\{\mathbf{F}_{p,m}(f,t)\}_m$, in such a way that the distances $|x_{m,k} - t|$, $k = -j, \ldots, j$, are always large enough.

For m sufficiently large (say $m \ge m_0 \in \mathbb{N}$), there exists an index d s.t. $x_{m,d} \le t < x_{m,d+1}$. Since the zeros of $p_m(w_\beta)$ interlace those of $p_{m+1}(w_\beta)$, two cases are possible:

$$(1) \quad x_{m,d} \le t < x_{m+1,d+1}, \qquad (2) \quad x_{m+1,d+1} \le t < x_{m,d+1}.$$

In the case (1), if

$$t \in \left[x_{m,d}, \ \frac{x_{m+1,d+1} + x_{m,d}}{2}\right[$$

then we approximate $\mathbf{F}_p(f,t)$ with $\mathbf{F}_{p,m+1}(f,t)$ and we set $m^* = m+1$, while if

$$t \in \left[\frac{x_{m+1,d+1} + x_{m,d}}{2}, \ x_{m+1,d+1}\right[$$

then we approximate $\mathbf{F}_p(f,t)$ with $\mathbf{F}_{p,m}(f,t)$ and we set $m^* = m$. Similarly we proceed in the case (2).

By this way we have

$$\mathbf{F}_p(f,t) \quad := \quad \mathbf{F}_{p,m^*}(f,t) + e_{p,m^*}(f,t),$$

where $e_{p,m^*}(f,t)$ is the remainder term and therefore

$$\mathbf{H}_p(fw_\beta, t) \quad := \quad \mathbf{F}_{p,m^*}(f,t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w_\beta, t) + e_{p,m^*}(f,t). \tag{11}$$

In order to highlight the possible instability of the ordinary truncated Gaussian rule, we want to propose just an example. Consider the integral

$$\mathbf{H}_1(fw_\beta), t) = \oint_{\mathbb{R}} \frac{\sin(x+5)}{(x-t)^2} e^{-x^2} dx.$$

In Figure 1 we compare the absolute errors obtained by implementing the truncated Gauss-Freud rule $\{\mathbf{F}_{p,m}(f,t)\}_m$ (red curve) and the modified Gauss-Freud rule $\{\mathbf{F}_{p,m^*}(f,t)\}_m$ (blue curve), for increasing values of m.

As the graph shows, the highest errors are attained by using the ordinary truncated Gaussian rule and, for the same t, also for different values of m. Since the function f is smooth, this bad behavior depends on the closeness of t to some of the Gaussian abscissae. The shortcoming is successfully overcome by using the method in (11).

Next theorem deals with the stability and the convergence of the proposed quadrature rule.

Theorem 3.1. *Let $p \in \mathbb{N}_0$, $0 < \lambda < 1$ and $0 < \theta < 1$ fixed. For any fixed $t \in (-\theta a_m, \theta a_m)$, if $f \in Z_{p+\lambda}(u)$ then*

$$u(t)|\mathbf{F}_{p,m^*}(f,t)| \quad \le \quad \mathcal{C}\,\|f\|_{Z_{p+\lambda}(u)} \log m \tag{12}$$

and

$$u(t)|\mathbf{F}_p(f,t) - \mathbf{F}_{p,m^*}(f,t)| \quad \le \quad \mathcal{C}\,\|f\|_{Z_{p+\lambda}(u)} \log m \left(\frac{a_m}{m}\right)^{\lambda}, \tag{13}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m,f,t)$.

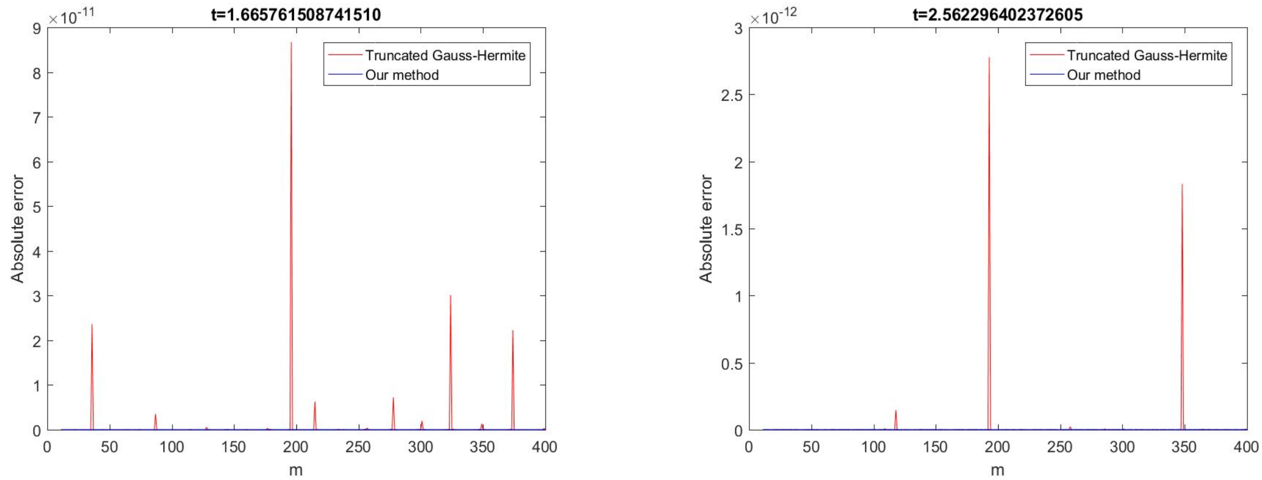

Figure 1: Comparison between the absolute errors of the rules $\mathbf{F}_{p,m}(f,t)$ and $\mathbf{F}_{p,m^*}(f,t)$

For smoother functions the following result holds.

Corollary 3.2. *Let $p \in \mathbb{N}_0$, $r \in \mathbb{N}$, $0 < \lambda < 1$ and $0 < \theta < 1$ fixed. For any fixed $t \in (-\theta a_m, \theta a_m)$, if $f \in Z_{p+r+\lambda}(u)$ then*

$$u(t)|\mathbf{F}_p(f,t) - \mathbf{F}_{p,m^*}(f,t)| \quad \le \quad \mathcal{C}\, \|f\|_{Z_{p+r+\lambda}(u)} \log m \left(\frac{a_m}{m}\right)^{r+\lambda}, \quad 0 < \mathcal{C} \ne \mathcal{C}(m,f,t), \tag{14}$$

and if $f \in W_{p+r}(u)$ then

$$u(t)|\mathbf{F}_p(f,t) - \mathbf{F}_{p,m^*}(f,t)| \quad \le \quad \mathcal{C}\, \|f\|_{W_{p+r}(u)} \log m \left(\frac{a_m}{m}\right)^{r}, \quad 0 < \mathcal{C} \ne \mathcal{C}(m,f,t).$$

3.1 The case t "large"

The method in (11) can be used when t stays between two Gaussian nodes, i.e. for $|t| < \theta a_m$, or equivalently, $m > \left(\frac{|t|}{\theta}\right)^{\beta}$. Thus, the "larger" t is, the "larger" m will be. For instance, for $t = 100$, $\beta = 2$ and $\theta = \frac{1}{2}$, it should be chosen $m > 2500$. In such cases the computation of weights and nodes in the Gaussian rule is too much expensive, and sometimes unfeasible. For all these reasons, we propose here an alternative procedure, which is essentially a shrewd application of the Gaussian rule again, in the sense we go to precise. For any fixed t, let m be such that

$$|t| > x_{m,j} + 1,$$

where j is defined in (6). Setting $G_t(x) = \frac{f(x)}{(x-t)^{p+1}}$, we approximate the integral by the m−th truncated Gauss-Freud rule, i.e.

$$\mathbf{H}_p(f w_\beta, t) = \sum_{|i| \le j} G_t(x_{m,i})\lambda_{m,i} + R_m(G_t). \tag{15}$$

Since f and $G_t(f)$ for $|t| > x_{m,j} + 1$ have the same smoothness, by (10) and (4), if $f \in W_r(u), r \in \mathbb{N}$, we get

$$|R_m(G_t)| \le \mathcal{C}\left(\|G_t^{(r)} u\| \left(\frac{a_m}{m}\right)^r + e^{-\mathcal{A}m}\|G_t u\|\right), \tag{16}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m, G_t)$ and $0 < \mathcal{A} \ne \mathcal{A}(m, G_t)$.

We observe that this procedure can be successfully applied when f is "smooth" and t is "large", say $|t| > x_{m,j} + 1$, where j is defined in (6). Moreover, smoother is the function f and smaller is the value of m to achieve the desired accuracy. Of course, the error bound in (16) holds for a fixed m and therefore the limit on m of $R_m(F_t)$ has no meaning.

In conclusion, according to the position of t we use (11) or (15), i.e.

$$\mathbf{H}_p(f,t) = \Psi_{p,m}(f,t) + \tau_{p,m}(f,t), \tag{17}$$

where

$$\Psi_{p,m}(t) = \begin{cases} \mathbf{F}_{p,m^*}(f,t) + \displaystyle\sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w_\beta, t), & |t| \le \theta a_m, \\ \displaystyle\sum_{|i| \le j} G_t(x_{m,i})\lambda_{m,i} + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w_\beta, t), & |t| > x_{m,j} + 1, \end{cases}$$

approximating $\mathbf{H}_p(f w_\beta, t)$ for a "wide" range of t.

4 The second method

Let $\mathbf{F}_0(f)$ be defined in (2). For any fixed $\theta \in (0,1)$, denoting by $\{x_{m+1,k}\}_{k=-\lfloor \frac{m+1}{2} \rfloor}^{\lfloor \frac{m+1}{2} \rfloor}$ the zeros of $p_{m+1}(w_\beta)$, let us define the index $q := q(m+1)$ by

$$x_{m+1,q} = \min_{k=1,2,..,\lfloor \frac{m+1}{2} \rfloor} \left\{ x_{m+1,k} : x_{m+1,k} \geq \theta a_{m+1} \right\},$$

and set

$$\mathbf{F}_{0,m+1}(f,t) = \sum_{i=-q}^{q} \lambda_{m+1,i} \frac{f(x_{m+1,i}) - f(t)}{x_{m+1,i} - t}. \tag{18}$$

By approximating $\mathbf{F}_0(f)$ by the Lagrange polynomial $L_{m+2}(w_\beta, \mathbf{F}_0(f))$ defined in (7), in view of (1)

$$\mathbf{F}_p(f,t) = \frac{1}{p!} L_{m+2}^{(p)}(w_\beta, \mathbf{F}_0(f), t) + \eta_{p,m}(f,t).$$

However, since the samples $\mathbf{F}_0(f, x_{m,k})$ are integrals containing the Freud weight w_β, we use formula (18) to evaluate them, i.e.

$$\mathbf{F}_p(f,t) = \frac{1}{p!} L_{m+2}^{(p)}(w_\beta, \mathbf{F}_{0,m+1}(f), t) + \rho_{p,m}(f,t).$$

Therefore we set

$$\begin{aligned} \mathbf{H}_p(f w_\beta, t) &= \frac{1}{p!} L_{m+2}^{(p)}(w_\beta, \mathbf{F}_{0,m+1}(f), t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w_\beta, t) + \rho_{p,m}(f,t) \\ &=: \Phi_{p,m}(f,t) + \rho_{p,m}(f,t), \end{aligned} \tag{19}$$

where $\rho_{p,m}(f,t)$ is the remainder term.

We remark that in view of [27, Proposition 1], Gaussian knots and interpolation nodes are sufficiently far among them. This good "distance" prevents numerical cancellation when computing $x_{m+1,i} - x_{m,k}$. About the stability and the convergence of the procedure, we are able to prove the following results

Theorem 4.1. *Let $p \in \mathbb{N}_0$ and $0 < \lambda < 1$. Then, for any function $f \in Z_{p+\lambda}(u)$*

$$\|\Phi_{p,m}(f)u\| \leq \mathcal{C}\, \|f\|_{Z_{p+\lambda}(u)} \log^2 m, \quad 0 < \mathcal{C} \neq \mathcal{C}(m,f). \tag{20}$$

Moreover, if $f \in Z_{p+r+\lambda}(u), r \in \mathbb{N}$, then

$$\|\rho_{p,m}(f)u\| \leq \mathcal{C}\, \|f\|_{Z_{p+r+\lambda}(u)} \log^2 m \left(\frac{a_m}{m}\right)^r, \quad 0 < \mathcal{C} \neq \mathcal{C}(m,f). \tag{21}$$

In particular, for f belonging to Sobolev spaces, we get

Corollary 4.2. *Let $p \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Then, for any function $f \in W_{p+1}(u)$*

$$\|\Phi_{p,m}(f)u\| \leq \mathcal{C}\, \|f\|_{W_{p+1}(u)} \log^2 m, \quad 0 < \mathcal{C} \neq \mathcal{C}(m,f),$$

and if $f \in W_{p+r+1}(u)$ then

$$\|\rho_{p,m}(f)u\| \leq \mathcal{C}\, \|f\|_{W_{p+r+1}(u)} \log^2 m \left(\frac{a_m}{m}\right)^r, \quad 0 < \mathcal{C} \neq \mathcal{C}(m,f).$$

As we have announced, it is possible to implement the method without computing the derivatives $\{f^{(k)}(t)\}_{k=1}^{p}$. Indeed, we can approximate $f^{(k)}(t)$ with $L_{m+2}^{(k)}(w_\beta, f, t)$ for $k = 0,1,2,\dots,p$, where $L_{m+2}(w_\beta, f)$ is the truncated Lagrange polynomial defined in (7). By this way, reusing the same samples involved in the evaluation of $\Phi_{p,m}(f,t)$ and taking into account that $\mathbf{H}_{p-k}(w,t)$ can be computed with the desired accuracy, we get

$$\mathbf{H}_p(f w_\beta, t) = \Phi_{p,m}(f,t) + \sum_{k=0}^{p} \binom{p}{k} L_{m+2}^{(k)}(w_\beta, f, t) \mathbf{H}_{p-k}(w_\beta, t) + \mathbf{T}_{p,m}(f,t), \tag{22}$$

where

$$\mathbf{T}_{p,m}(f,t) = \rho_{p,m}(f,t) + \sum_{k=0}^{p} \binom{p}{k} \tau_{k,m}(f,t) \mathbf{H}_{p-k}(w_\beta, t),$$

with

$$\tau_{k,m}(f,t) = (f(t) - L_{m+2}(w_\beta, f, t))^{(k)}.$$

The next theorem deals with the pointwise estimate of the error $\mathbf{T}_{p,m}(f,t)$:

Theorem 4.3. *Let $p \geq 0$. Then, for any function $f \in Z_{p+r+\lambda}(u)$ with $0 < \lambda < 1$, $r \geq 1$, we have*

$$|\mathbf{T}_{p,m}(f,t)u(t)| \leq \frac{\mathcal{C}}{|t|^p} \|f\|_{Z_{p+r+\lambda}(u)} \left(\frac{a_m}{m}\right)^r \log^2 m, \quad 0 < \mathcal{C} \neq \mathcal{C}(m,f).$$

Remark 1. For $|t| \geq \mathcal{C}$, $\mathcal{C}$ being an arbitrary positive constant, we get

$$|\mathbf{T}_{p,m}(f,t)u(t)| \leq \mathcal{C}\, \|f\|_{Z_{p+r+\lambda}(u)} \left(\frac{a_m}{m}\right)^r \log^2 m, \quad 0 < \mathcal{C} \neq \mathcal{C}(m,f).$$

5 Computational aspects

First of all, we point out that since $w_\beta(x)=e^{-|x|^\beta}$, $\beta>1$, is not a classical weight, the coefficients of the three-term recurrence relation of the corresponding orthonormal polynomials are unknown, in the general case. However, it is possible to calculate them, for instance, by the Chebyshev algorithm, which requires the moments

$$\mu_k=\int_{\mathbb{R}} x^k w_\beta(x)dx, \quad k=0,1,\dots.$$

Even if the algorithm suffers of ill conditioning, we have successfully implemented it in *Wolfram Mathematica Language*, in extended arithmetic with high accuracy by using the functions `aChebyshevAlgorithm` and `aGaussianNodesWeights` of the software package `OrthogonalPolynomials` by Cvetkovic and G.V. Milovanovic (see [2]).

Now we give some details about the computation of the Hadamard transforms of the weight $w_\beta(x)=e^{-|x|^\beta}$. In order to use the relation

$$\mathbf{H}_{p-k}(w_\beta,t):=\frac{1}{(p-k)!}\frac{d^{p-k}}{dt^{p-k}}\int\!\!\!\!\!\!-_{\mathbb{R}}\frac{w_\beta(x)}{x-t}dx, \tag{23}$$

we first deduce the expression of $\mathbf{H}_0(w_\beta,t), k=0,1,\dots,p$, for some choices of the parameter $\beta>1$. Since $\mathbf{H}_0(w_\beta,t)=-\mathbf{H}_0(w_\beta,-t)$, we assume $t>0$. Then, we have

$$\mathbf{H}_0(w_\beta,t)=\int\!\!\!\!\!\!-_0^{+\infty}\frac{e^{-x^\beta}}{x-t}dx-\int_0^{+\infty}\frac{e^{-x^\beta}}{x+t}dx=\frac{1}{\beta}\left\{\int\!\!\!\!\!\!-_0^{+\infty}\frac{e^{-x}x^{\frac{1}{\beta}-1}}{x^{\frac{1}{\beta}}-t}dx-\int_0^{+\infty}\frac{e^{-x}x^{\frac{1}{\beta}-1}}{x^{\frac{1}{\beta}}+t}dx\right\}.$$

In the case $\beta=p/q,\ p,q\in\mathbb{N}, p>q$, we have for p even

$$\mathbf{H}_0(w_\beta,t)=2\frac{q}{p}\sum_{j=1}^{\frac{p}{2}}t^{2j-1}\int\!\!\!\!\!\!-_0^{+\infty}\frac{x^{\frac{q}{p}(p-2j+1)-1}}{x^q-t^p}e^{-x}dx \tag{24}$$

and for p odd

$$\mathbf{H}_0(w_\beta,t)=\frac{q}{p}\sum_{j=1}^{p}t^{j-1}\left\{\int\!\!\!\!\!\!-_0^{+\infty}\frac{x^{\frac{q}{p}(p-j+1)-1}}{x^q-t^p}e^{-x}dx+(-1)^j\int_0^{+\infty}\frac{x^{\frac{q}{p}(p-j+1)-1}}{x^q+t^p}e^{-x}dx\right\}. \tag{25}$$

Integrals in (24), (25) can be found in a closed form for some values of q. In particular for $q=1$, taking into account [30, p. 325 n. 16]

$$\begin{aligned}\int\!\!\!\!\!\!-_0^{+\infty}\frac{e^{-x}x^\alpha}{x-t}dx &= -\pi t^\alpha e^{-t}\cot((1+\alpha)\pi)+\Gamma(\alpha)\,{}_1F_1(1,1-\alpha,-t), \quad \alpha\neq 0\\ \int\!\!\!\!\!\!-_0^{+\infty}\frac{e^{-x}}{x-t}dx &= -e^{-t}Ei(t),\end{aligned}$$

and [10, p.217, n.(17)]

$$\int_0^{+\infty}\frac{e^{-x}x^\alpha}{x+t}dx=\Gamma(1+\alpha)\,t^\alpha e^t\Gamma(-\alpha,t),$$

where ${}_1F_1(a,b,x)$ is the Confluent Hypergeometric function, $Ei(t)$ is the exponential integral function, and $\Gamma(a,b)$ is the incomplete Gamma function, we get for p even

$$\begin{aligned}\mathbf{H}_0(w_\beta,t) &= \frac{2}{p}\sum_{j=1}^{\frac{p}{2}}t^{2j-1}\int\!\!\!\!\!\!-_0^{+\infty}\frac{e^{-x}x^{\frac{1-2j}{p}}}{x-t^p}dx\\ &= -\frac{2\pi}{p}\sum_{j=1}^{\frac{p}{2}-1}e^{-t^p}\cot\left(\left(1+\frac{1-2j}{p}\right)\pi\right)+\Gamma\left(\frac{1-2j}{p}\right)\,{}_1F_1\left(1,1-\frac{1-2j}{p},-t^p\right)-\frac{1}{p}e^{-t^p}Ei(t^p)\end{aligned}$$

and for p odd

$$\begin{aligned}\mathbf{H}_0(w_\beta,t) &= \frac{1}{p}\sum_{j=1}^{p}t^{j-1}\left\{\int\!\!\!\!\!\!-_0^{+\infty}\frac{e^{-x}x^{\frac{1-j}{p}}}{x-t^p}dx+(-1)^j\int_0^{+\infty}\frac{e^{-x}x^{\frac{1-j}{p}}}{x+t^p}dx\right\}\\ &= -\frac{1}{p}e^{-t^p}Ei(t^p)+\frac{1}{p}\sum_{j=2}^{p}\left[-\pi e^{-t^p}\cot\left(\left(1+\frac{1-j}{p}\right)\pi\right)+t^{j-1}\Gamma\left(\frac{1-j}{p}\right)\,{}_1F_1\left(1,1-\frac{1-j}{p},-t^p\right)\right]\\ &+ \frac{1}{p}\sum_{j=1}^{p}(-1)^j\Gamma\left(1+\frac{1-j}{p}\right)e^{t^p}\Gamma\left(-\frac{1-j}{p},t^p\right).\end{aligned}$$

Thus, taking into account [13, p. 1086, 9.213]

$$\frac{d}{dt}Ei(t) = -\frac{d}{dt}\int_{-t}^{+\infty}\frac{e^{-x}}{x}dx = \frac{e^t}{t}, \qquad \frac{d}{dt}{}_1F_1(a,b;t) = \frac{a}{b}{}_1F_1(a+1,b+1,t),$$

by (23), the functions $\{\mathbf{H}_{p-k}(w_\beta,t)\}_{k=0}^{p}$ are completely determined. We point out that the computation of the function Ei and ${}_1F_1$ have been performed by the *Wolfram Mathematica* routines `ExpIntegralEi` and `Hypergeometric1F1`, respectively.

In the case $q=2$ and $q=4$, the function $\mathbf{H}_0(w_\beta)$ can be obtained by combining integrals in [13, 3.354.1 p. 359 and 3.358 p. 361].

Finally, we give some details about the computation of the derivatives of the fundamental Lagrange polynomials $\{\ell_{m,k}^{(i)}(t)\}_{k=-j}^{j}$, $i=1,2,\dots,p$. Setting

$$S_{r,k}(t) = \sum_{\substack{|h|\le\lfloor\frac{m}{2}\rfloor\\ h\neq k}} \frac{r!}{[(t-x_{m,h})(t^2-a_m^2)]^{r+1}}, \quad t\neq\pm a_m, \quad t\neq\{x_{m,h}\}_{h=-\lfloor\frac{m}{2}\rfloor}^{\lfloor\frac{m}{2}\rfloor},$$

it is no hard to prove that

$$\ell_{m,k}^{(i)}(t) = \sum_{r=0}^{i-1}\binom{i-1}{r}(-1)^r\ell_{m,k}^{(i-1-r)}(t)S_{r,k}(t),$$

for $-j\le k\le j$ and $i=1,2,\dots,p$.

6 Numerical Tests

In this section we propose some numerical experiments obtained by implementing the above introduced methods. To be more precise, we show the numerical results obtained approximating $\mathbf{H}_p(w_\beta,f)$ by the sequence $\{\Psi_{p,m}(f,t)\}_m$ of the first method (17) and by $\{\Phi_{p,m}(f,t)\}_m$ of the second method (19). Since the exact values of the integrals are unknown, we will retain exact the values computed with $m=1000$, setting

$$\bar{\tau}_{p,m}(f,t) = u(t)|\Psi_{p,m}(f,t)-\Psi_{p,1000}(f,t)|,$$
$$\bar{\rho}_{p,m}(f,t) = u(t)|\Phi_{p,m}(f,t)-\Phi_{p,1000}(f,t)|,$$

as weighted absolute errors.

The "truncation intervals", depending on a fixed θ, have been empirically detected. In particular, denoting by eps the machine precision, in the first method we set

$$j := j_1 - j_2$$

with

$$j_1 := \begin{cases} \max\limits_{k=0,\dots,\lfloor\frac{m^*}{2}\rfloor}\left\{k \;:\; \lambda_{m^*,k}\left|\dfrac{f(x_{m^*,k})-\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}(x_{m^*,k}-t)^{p-k}}{(x_{m^*,k}-t)^{p+1}}\right|\ge eps\right\} & \text{in (11)}\\[2ex] \max\limits_{k=0,\dots,\lfloor\frac{m^*}{2}\rfloor}\left\{k \;:\; |G_t(x_{m,k})|\lambda_{m,k}\ge eps\right\} & \text{in (15)}\end{cases}$$

$$j_2 := \begin{cases} \min\limits_{k=-\lfloor\frac{m^*}{2}\rfloor,\dots,-1}\left\{k \;:\; \lambda_{m^*,k}\left|\dfrac{f(x_{m^*,k})-\sum_{k=0}^{p}\frac{f^{(k)}(t)}{k!}(x_{m^*,k}-t)^{p-k}}{(x_{m^*,k}-t)^{p+1}}\right|\ge eps\right\} & \text{in (11)}\\[2ex] \min\limits_{k=-\lfloor\frac{m^*}{2}\rfloor,\dots,-1}\left\{k \;:\; |G_t(x_{m,k})|\lambda_{m,k}\ge eps\right\} & \text{in (15)}\end{cases}$$

and for the second method in (19), setting

$$\bar{q}_k := \max_{i=0,\dots,\lfloor\frac{m+1}{2}\rfloor}\left\{i \;:\; \left|\lambda_{m+1,i}\frac{f(x_{m+1,i})-f(x_{m,k})}{x_{m+1,i}-x_{m,k}}\right|\ge eps\right\},$$

$$\hat{q}_k := \min_{i=-\lfloor\frac{m+1}{2}\rfloor,\dots,-1}\left\{i \;:\; \left|\lambda_{m+1,i}\frac{f(x_{m+1,i})-f(x_{m,k})}{x_{m+1,i}-x_{m,k}}\right|\ge eps\right\},$$

$$\bar{q} := \max_{k=j_2,\dots,j_1}\bar{q}_k, \quad \hat{q} := \min_{k=j_2,\dots,j_1}\hat{q}_k,$$

$$\hat{\imath}_1 := \max_{k=0,\dots,\lfloor\frac{m}{2}\rfloor}\left\{k \;:\; \left|\ell_{m,k}^{(p)}(t)\sum_{i=\hat{q}_k}^{\bar{q}_k}\lambda_{m+1,i}\frac{f(x_{m+1,i})-f(x_{m,k})}{x_{m+1,i}-x_{m,k}}\right|\ge eps\right\},$$

$$\hat{\imath}_2 := \min_{k=-\lfloor\frac{m}{2}\rfloor,\dots,-1}\left\{k \;:\; \left|\ell_{m,k}^{(p)}(t)\sum_{i=\hat{q}_k}^{\bar{q}_k}\lambda_{m+1,i}\frac{f(x_{m+1,i})-f(x_{m,k})}{x_{m+1,i}-x_{m,k}}\right|\ge eps\right\},$$

we let

$$\hat{\imath} := \hat{\imath}_1 - \hat{\imath}_2, \quad q := \bar{q} - \hat{q}.$$

We point out that for the simultaneous approximation of $\mathbf{H}_0(f w_\beta, t), \ldots, \mathbf{H}_p(f w_\beta, t)$ at s values of t, the first method requires $j + (p+1)s$ samples of the density function f, while the second needs $q + \hat{\imath} + (p+1)s$ evaluations of f. In both the cases, the effort doesn't depend on s.

In the numerical tests proposed below, we compare our results among them and with those achieved by a method discussed in [25, p. 14-15] for finite part integrals on bounded intervals. Thus we have properly adapted this procedure, in the sense we go to specify. Starting from the decomposition

$$\mathbf{H}_p(f w_\beta, t) \quad = \quad \mathbf{F}_p(f, t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w_\beta, t),$$

by an idea in [15] (see also [25]), to approximate $\mathbf{F}_p(f, t)$, the interval $(-\infty, +\infty)$ is broken up into $(-\infty, t)$ and $(t, +\infty)$

$$\mathbf{F}_p(f, t) \quad = \quad \int_0^{+\infty} \frac{f(t-y) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(-y)^k}{(-y)^{p+1}} e^{-|t-y|^\beta + y} e^{-y} dy + \int_0^{+\infty} \frac{f(y+t) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(y)^k}{(y)^{p+1}} e^{-|y+t|^\beta + y} e^{-y} dy,$$

and using the m-th truncated Gauss-Laguerre rule w.r.t. the weight e^{-y} [19], we obtain

$$\begin{aligned}
\mathbf{H}_p(f w_\beta, t) \quad &= \quad \sum_{i=1}^{k_1} \frac{f(t - y_{m,i}) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(-y_{m,i})^k}{(-y_{m,i})^{p+1}} e^{-|t-y_{m,i}|^\beta + y_{m,i}} \bar{\lambda}_{m,i} + \sum_{i=1}^{k_2} \frac{f(y_{m,i} + t) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(y_{m,i})^k}{(y_{m,i})^{p+1}} e^{-|y_{m,i} - t|^\beta + y_{m,i}} \bar{\lambda}_{m,i} \\
&+ \quad \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathbf{H}_{p-k}(w_\beta, t) + \sigma_{p,m}(f, t) \\
&:= \quad J_{p,m}(f, t) + \sigma_{p,m}(f, t), \qquad (26)
\end{aligned}$$

where $y_{m,i}$ and $\bar{\lambda}_{m,i}$, $i = 1, \ldots, m$, are the knots and the weights of the m−th Gauss-Laguerre rule and the parameters k_1 and k_2 are empirically detected with criteria similar to those used for the rule (11). We will set

$$\hat{\sigma}_{p,m}(f, t) = |J_{p,m}(f, t) - J_{p,1000}(f, t)|.$$

We point out that by implementing formula (26), for any value of t two changes of variables depending on t are required and none of the samples of the function f can be saved. This means that the computation $\{\mathbf{H}_i(f, t)\}_{i=0}^{p}$ at s values of t requires almost $s(k_1 + k_2 + p + 1)$ samples of f, which is far greater than the number of samples required in our methods.

All the computations have been performed in double-machine precision ($eps \approx 2.22044e - 16$) and in all the tables the symbol "-" means that the machine accuracy has been achieved.

Example 6.1.

$$\mathbf{H}_p(f w_2, t) = \int_{\mathbb{R}} \frac{\sin\left(\frac{x}{2}\right)\cos(x - e)}{(x-t)^{p+1}} e^{-x^2} dx, \quad p = 0, 1, 2.$$

Since the function $f(x) = \sin\left(\frac{x}{2}\right)\cos(x - e)$ is very smooth we expect for a fast convergence. As the test shows, the sequence $\{J_{p,m}(f, t)\}_m$ appears slower convergent w.r.t. both the sequences $\{\Psi_{p,m}(f, t)\}_m$ and $\{\Phi_{p,m}(f, t)\}_m$. Indeed, looking at Tables 1-2, $\mathbf{H}_0(f; t), \mathbf{H}_1(f; t), \mathbf{H}_2(f; t)$ are computed for different values of t, attaining the machine precision with $m = 30$ in the first method and with $m = 40$ in the second one. On the contrary, as Table 3 evidences, by the method (26) the machine precision is not always achieved for $m = 160$.

In Figure 2 we show the graphs of the approximating functions $\Phi_{p,40}(f, t)$, $p = 0, 1, 2$.

m	j	$\bar{\tau}_{0,m}(f, -3)$	$\bar{\tau}_{0,m}(f, -0.5)$	$\bar{\tau}_{0,m}(f, 4)$	$\bar{\tau}_{0,m}(f, 10)$
10	8	$5.11e-7$	$3.67e-6$	$1.07e-12$	−
20	18	$8.50e-16$	$1.55e-14$	−	−
30	27	−	−	−	−
m	j	$\bar{\tau}_{1,m}(f, -3)$	$\bar{\tau}_{1,m}(f, -0.5)$	$\bar{\tau}_{1,m}(f, 4)$	$\bar{\tau}_{1,m}(f, 10)$
10	8	$9.71e-6$	$1.01e-6$	$5.87e-9$	−
20	18	$8.47e-16$	$7.90e-15$	−	−
30	27	−	−	−	−
m	j	$\bar{\tau}_{2,m}(f, -3)$	$\bar{\tau}_{2,m}(f, -0.5)$	$\bar{\tau}_{2,m}(f, 4)$	$\bar{\tau}_{2,m}(f, 10)$
10	8	$5.85e-5$	$8.35e-7$	$1.51e-6$	−
20	18	−	$4.77e-15$	−	−
30	27	−	−	−	−

Table 1: Example 6.1: $\bar{\tau}_{p,m}(f, t)$, $p = 0, 1, 2$, with $t = -3, -0.5, 4, 10$

m	$\hat{\imath}$	q	$\bar{\rho}_{0,m}(f,-3)$	$\bar{\rho}_{0,m}(f,-0.5)$	$\bar{\rho}_{0,m}(f,4)$	$\bar{\rho}_{0,m}(f,10)$
10	10	10	$3.66e-5$	$5.86e-5$	$4.14e-5$	–
20	20	20	$9.50e-11$	$2.73e-10$	$2.57e-10$	–
30	30	29	$1.18e-13$	$4.58e-14$	$5.83e-14$	–
40	40	35	–	–	–	–
m	$\hat{\imath}$	q	$\bar{\rho}_{1,m}(f,-3)$	$\bar{\rho}_{1,m}(f,-0.5)$	$\bar{\rho}_{1,m}(f,4)$	$e\bar{r}r_{1,m}(f,10)$
10	10	10	$3.73e-5$	$5.19e-5$	$1.50e-5$	–
20	20	20	$1.62e-10$	$8.02e-12$	$7.44e-11$	–
30	30	29	$1.26e-14$	$8.10e-15$	$1.27e-14$	–
40	40	35	–	–	–	–
m	$\hat{\imath}$	q	$\bar{\rho}_{2,m}(f,-3)$	$\bar{\rho}_{2,m}(f,-0.5)$	$\bar{\rho}_{2,m}(f,4)$	$\bar{\rho}_{2,m}(f,10)$
10	10	10	$1.48e-5$	$8.89e-6$	$2.09e-6$	–
20	20	20	$2.98e-11$	$1.37e-11$	$3.36e-12$	–
30	30	29	$2.52e-15$	$1.24e-15$	–	–
40	40	35	–	–	–	–

Table 2: Example 6.1: $\bar{\rho}_{p,m}(f,t)$, $p=0,1,2$, with $t=-3,-0.5,4,10$

m	k_1	k_2	$\hat{\sigma}_{0,m}(f,-3)$	$\hat{\sigma}_{0,m}(f,-0.5)$	$\hat{\sigma}_{0,m}(f,4)$	$\hat{\sigma}_{0,m}(f,10)$
20	9	9	$1.02e-4$	$3.01e-4$	$2.40e-6$	–
40	12	13	$2.40e-7$	$5.28e-9$	$1.21e-7$	–
80	17	18	$1.16e-9$	$1.57e-10$	$1.24e-9$	–
160	25	26	$4.22e-14$	–	$1.58e-14$	–
m	k_1	k_2	$\hat{\sigma}_{1,m}(f,-3)$	$\hat{\sigma}_{1,m}(f,-0.5)$	$\hat{\sigma}_{1,m}(f,4)$	$e\hat{r}r_{1,m}(f,10)$
20	9	9	$4.19e-5$	$1.21e-5$	$2.60e-6$	–
40	12	13	$4.62e-7$	$3.78e-7$	$2.33e-8$	–
80	17	18	$7.18e-10$	$4.54e-11$	$1.70e-10$	–
160	25	26	$5.74e-15$	–	$4.16e-15$	–
m	k_2	k_2	$\hat{\sigma}_{2,m}(f,-3)$	$\hat{\sigma}_{2,m}(f,-0.5)$	$\hat{\sigma}_{2,m}(f,4)$	$e\hat{r}r_{2,m}(f,10)$
20	9	9	$2.16e-5$	3.86 e - 5	$5.46e-7$	–
40	12	13	$1.94e-7$	1.48 e - 7	$9.68e-10$	–
80	17	18	$1.59e-10$	3.60 e - 12	$1.90e-11$	–
160	25	26	–	3.8601 e - 15	$6.37e-16$	–

Table 3: Example 6.1: $\hat{\sigma}_{p,m}(f,t)$, $p=0,1,2$, with $t=-3,-0.5,4,10$

Example 6.2.

$$\mathbf{H}_p(fw_4,t)=\int_{\mathbb{R}}\frac{\sinh\left(\frac{x}{5}\right)\left|x+\frac{1}{4}\right|^{\frac{11}{2}}}{(x-t)^{p+1}}e^{-x^4}dx,\quad p=0,1.$$

In this case the density function $f(x)=|x-1|^{\frac{9}{2}}\cosh(x)$ belongs to $Z_{5.5}(u)$ and, according to estimates in (14) and (21), the rate of convergence is $\frac{\log m}{m^{(1-1/4)(5.5-p)}}$ for the first method and $\frac{\log^2 m}{m^{(1-1/4)(5-p)}}$ for the second method. The results in Tables 4 and 5 agree with the theoretical expectations. Also in this case, as Table 6 shows, the sequence $\{J_{p,m}(f,t)\}_m$ converges slower than the sequences $\{\Psi_{p,m}(f,t)\}_m$ and $\{\Phi_{p,m}(f,t)\}_m$, since for the same value $m=350$ until 5 exact digits are lost w.r.t. both our methods (compare the values for $t=1.5$).

The graphs of the approximating functions $\Phi_{p,351}(f,t)$, $p=0,1$ are shown in Figure 3.

Example 6.3.

$$\mathbf{H}_p(fw_3,t)=\int_{\mathbb{R}}\frac{|x-1|^{\frac{9}{2}}\cosh(x)}{(x-t)^{p+1}}e^{-x^3}dx,\quad p=0,1,2.$$

Here the function $f(x)=|x-1|^{\frac{9}{2}}\cosh(x)$ belongs to $Z_{4.5}(u)$ and according to estimates (14) and (21) the convergence order is $\frac{\log m}{m^{(1-1/3)(4.5-p)}}$ for the first method and $\frac{\log^2 m}{m^{(1-1/3)(4-p)}}$ for the second method. By inspecting Tables 7-8, also in this case the theoretical estimates are confirmed by the numerical tests. Moreover, as Table 9 shows, the sequence $\{J_{p,m}(f,t)\}_m$ converges slower than the sequences $\{\Psi_{p,m}(f,t)\}_m$ and $\{\Phi_{p,m}(f,t)\}_m$. In particular we note that, differently from the numerical results in our method, the behavior of the errors depends on the singularity t.

In Figure 3 you can see the graphs of the approximating functions $\Phi_{p,451}(f,t)$, $p=0,1,2$.

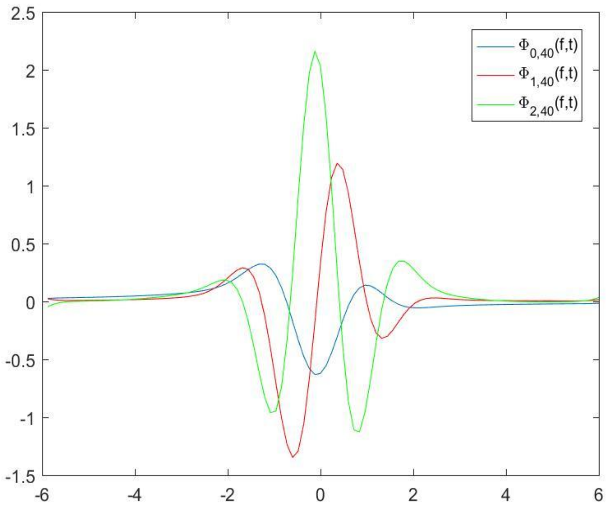

Figure 2: Example 1

m	j	$\bar{\tau}_{0,m}(f,-2)$	$\bar{\tau}_{0,m}(f,-0.249999)$	$\bar{\tau}_{0,m}(f,1.5)$	$\bar{\tau}_{0,m}(f,2.2)$
50	47	$1.09e-13$	$1.70e-9$	$2.39e-11$	$1.84e-15$
101	78	$1.22e-15$	$2.44e-10$	$2.27e-13$	–
150	104	–	$4.57e-11$	$3.98e-14$	–
201	127	–	$1.31e-11$	$1.24e-14$	–
250	149	–	$6.82e-12$	$1.91e-15$	–
301	168	–	$3.02e-12$	$3.02e-15$	–
351	189	–	$9.01e-13$	$1.11e-15$	–
m	j	$\bar{\tau}_{1,m}(f,-2)$	$\bar{\tau}_{1,m}(f,-0.249999)$	$\bar{\tau}_{1,m}(f,1.5)$	$\bar{\tau}_{1,m}(f,2.2)$
50	47	$6.30e-14$	$4.49e-8$	$1.33e-11$	$1.84e-15$
101	78	$6.03e-16$	$2.14e-9$	$1.51e-13$	$7.34e-16$
150	104	–	$3.66e-10$	$2.05e-14$	–
201	127	–	$1.53e-10$	$6.70e-15$	–
250	149	–	$1.46e-11$	–	–
301	168	–	$3.80e-11$	–	–
351	189	–	$3.67e-11$	–	–

Table 4: Example 6.2: $\bar{\tau}_{p,m}(f,t)$, $p=0,1$ with $t=-2,-0.249999,1.5,2.2$

7 Proofs

Proof of Theorem 2.1 . Let $Q_{m+1}\in\mathcal{P}^*_{m+1}$ s. t. $\|(f-Q_{m+1})u\|=\inf_{P_{m+1}\in\mathcal{P}^*_{m+1}}\|(f-P_{m+1})u\|=:\widetilde{E}_{m+1}(f)_u$. We have

$$\|(f-L_{m+2}(w_\beta,f))^{(k)}u\| \quad\leq\quad \mathcal{C}\left(\|(f-Q_{m+1})^{(k)}u\|+\|L_{m+2}(w_\beta,f-Q_{m+1})^{(k)}u\|\right)=:I_1+I_2. \tag{27}$$

By the Bernstein inequality

$$\left(\frac{a_m}{m}\right)^k\|P_m^{(k)}u\|\leq\mathcal{C}\,\|P_mu\|,\quad \forall P\in\mathbb{P}_m,\quad \mathcal{C}\neq\mathcal{C}(m), \tag{28}$$

and by [20, Theorem 3.2] and (5), we get

$$\begin{aligned} I_2 &\leq \mathcal{C}\left(\frac{m}{a_m}\right)^k\log m\; E_M(f)_u\\ &\leq \mathcal{C}\left(\frac{a_m}{m}\right)^{p+\lambda-k}\log m\;\|f\|_{Z_s(u)}. \end{aligned} \tag{29}$$

To estimate I_1, since [20, (3.2)]

$$\widetilde{E}_{m+1}(f)_u\leq\mathcal{C}\left[E_M(f)_u+e^{-\mathcal{A}m}\|fu\|\right], \tag{30}$$

the sequence $\{Q_m\}_m$ uniformly converges to $f\in C_u$ and therefore

$$f-Q_{m+1}=\sum_{k=0}^{\infty}(Q_{2^{k+1}(m+1)}-Q_{2^k(m+1)}).$$

m	$\hat{\imath}$	q	$\bar{\rho}_{0,m}(f,-2)$	$\bar{\rho}_{0,m}(f,-0.249999)$	$\bar{\rho}_{0,m}(f,1.5)$	$\bar{\rho}_{0,m}(f,2.2)$
50	50	49	$8.94e-9$	$1.04e-7$	$4.46e-9$	$9.08e-9$
100	100	81	$1.05e-10$	$9.82e-9$	$4.14e-11$	$1.81e-10$
150	136	107	$7.37e-12$	$1.52e-9$	$1.28e-11$	$6.40e-12$
200	167	131	$1.11e-12$	$4.36e-10$	$1.91e-12$	$2.99e-12$
250	197	153	$9.77e-13$	$1.89e-10$	$3.63e-13$	$2.29e-12$
300	222	174	$1.92e-12$	$8.95e-11$	$2.69e-12$	$3.89e-13$
350	247	194	$5.20e-13$	$3.03e-11$	$7.21e-13$	$2.38e-13$
m	$\hat{\imath}$	q	$\bar{\rho}_{1,m}(f,-2)$	$\bar{\rho}_{1,m}(f,-0.249999)$	$\bar{\rho}_{1,m}(f,1.5)$	$\bar{\rho}_{1,m}(f,2.2)$
50	50	49	$5.06e-7$	$5.68e-8$	$7.40e-7$	$3.66e-7$
100	100	81	$1.06e-8$	$2.37e-7$	$1.86e-8$	$1.48e-9$
150	139	107	$2.96e-11$	$9.07e-8$	$1.32e-9$	$1.00e-9$
200	171	131	$2.19e-10$	$3.59e-8$	$4.73e-10$	$1.02e-10$
250	199	153	$2.60e-10$	$1.47e-8$	$3.44e-10$	$1.57e-11$
300	229	174	$1.01e-10$	$5.32e-9$	$1.71e-11$	$1.83e-10$
350	255	194	$1.82e-10$	$1.44e-9$	$1.86e-10$	$1.39e-10$

Table 5: Example 6.2: $\bar{\rho}_{p,m}(f,t)$, $p=0,1$, with $t=-2,-0.249999,1.5,2.2$

m	k_1	k_2	$\hat{\sigma}_{0,m}(f,-2)$	$\hat{\sigma}_{0,m}(f,-0.249999)$	$\hat{\sigma}_{0,m}(f,1.5)$	$\hat{\sigma}_{0,m}(f,2.2)$
50	9	10	$4.57e-6$	$5.37e-3$	$2.43e-3$	$4.36e-6$
100	13	14	$1.05e-6$	$4.51e-4$	$1.08e-4$	$4.97e-7$
150	17	17	$4.88e-8$	$3.33e-5$	$2.87e-5$	$7.45e-8$
200	19	20	$3.58e-8$	$7.42e-6$	$3.58e-6$	$1.60e-8$
250	21	22	$7.12e-9$	$1.33e-6$	$1.14e-6$	$4.65e-9$
300	24	24	$3.05e-9$	$1.03e-7$	$1.57e-7$	$1.65e-10$
350	25	26	$5.31e-10$	$5.68e-8$	$3.03e-8$	$4.69e-10$
m	k_1	k_2	$\hat{\sigma}_{1,m}(f,-2)$	$\hat{\sigma}_{1,m}(f,-0.249999)$	$\hat{\sigma}_{1,m}(f,1.5)$	$\hat{\sigma}_{1,m}(f,2.2)$
50	9	10	$5.82e-6$	$3.48e-3$	$8.64e-3$	$1.07e-5$
100	13	14	$7.82e-6$	$2.17e-4$	$3.69e-4$	$1.24e-6$
150	16	17	$5.81e-7$	$1.56e-5$	$1.16e-4$	$1.81e-7$
200	19	20	$2.90e-7$	$3.80e-6$	$1.38e-5$	$4.15e-8$
250	21	22	$6.34e-8$	$5.54e-7$	$6.19e-6$	$1.20e-8$
300	23	24	$1.36e-8$	$4.83e-9$	$6.44e-7$	$2.73e-10$
350	25	26	$3.62e-9$	$2.79e-8$	$6.81e-8$	$1.22e-9$

Table 6: Example 6.2: $\hat{\sigma}_{p,m}(f,t)$, $p=0,1$, with $t=-2,-0.249999,1.5,2.2$

Then, by the Bernstein inequality (28)

$$\begin{aligned} I_1 &\leq \sum_{i=0}^{\infty} \|(Q_{2^{i+1}(m+1)}-Q_{2^i(m+1)})^{(k)}u\| \\ &\leq \mathcal{C}\sum_{i=0}^{\infty}\left(\frac{2^{i+1}(m+1)}{a_{2^{i+1}(m+1)}}\right)^k \|(Q_{2^{i+1}(m+1)}-Q_{2^i(m+1)})u\| \leq \mathcal{C}\sum_{i=0}^{\infty}\left(\frac{2^{i+1}(m+1)}{a_{2^{i+1}(m+1)}}\right)^k E_{2^{i+1}(m+1)}(f)_u \end{aligned}$$

and by (5) we get

$$I_1 \leq \mathcal{C}\,\|f\|_{Z_{p+\lambda}(u)}\sum_{i=0}^{\infty}\left(\frac{a_{2^{i+1}(m+1)}}{2^{i+1}(m+1)}\right)^{p+\lambda-k}.$$

Taking into account that $a_{2^{i+1}(m+1)}\sim(m+1)^{\frac{1}{\beta}}2^{\frac{i+1}{\beta}}$, we have, for any polynomial in $\mathcal{P}^*_{m+1}$,

$$\begin{aligned} I_1=\|(f-Q_{m+1})^{(k)}u\| &\leq \mathcal{C}\,\|f\|_{Z_{p+\lambda}(u)}m^{\left(\frac{1}{\beta}-1\right)^{p+\lambda-k}}\sum_{i=0}^{\infty}\left(2^{(i+1)\left(\frac{1}{\beta}-1\right)}\right)^{p+\lambda-k} \\ &\leq \mathcal{C}\,\|f\|_{Z_{p+\lambda}(u)}\left(\frac{a_m}{m}\right)^{p+\lambda-k}, \end{aligned} \tag{31}$$

since the assumption $\beta>1$ assures $2^{(i+1)\left(\frac{1}{\beta}-1\right)}<1$. The theorem follows by combining (29), (31) with (27). □

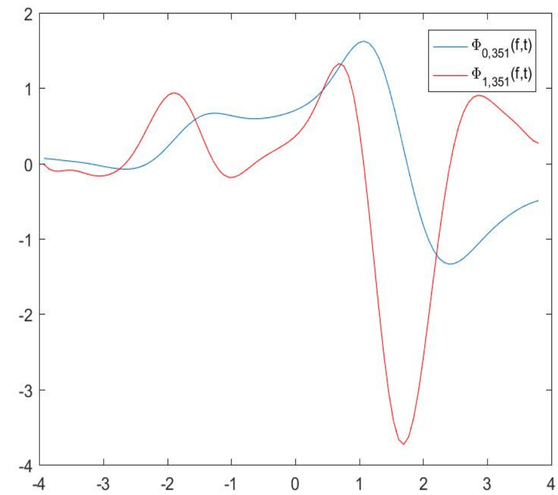

Figure 3: Example 2

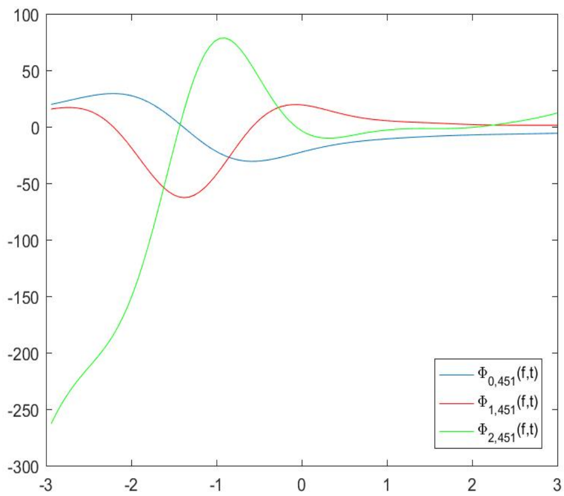

Figure 4: Example 3

Now we state some lemmas useful in the successive proofs.

Lemma 7.1. *For $0 < \lambda < 1$ and $p \in \mathbb{N}$, $f^{(p)} \in Z_\lambda(u)$ implies $f \in Z_{\lambda+p}(u)$ and viceversa.*

Proof. We omit the proof since it follows by arguments similar to those used in the proof of Lemma 2.1 in [5]. □

With $R_p(f,x,t) = f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k$, we recall the Peano form of the Taylor's remainder term

$$R_p(f,x,t) = \frac{1}{(p-1)!} \int_t^x [f^{(p)}(\tau) - f^{(p)}(t)](x-\tau)^{p-1} d\tau. \tag{32}$$

Lemma 7.2. *Let $p \in \mathbb{N}$, $0 < \lambda < 1$ and $t \in \mathbb{R}$. If $f^{(p)} \in Z_\lambda(u)$ we have*

$$u(t) \left| \int_{|x-t|<1} \frac{R_p(f,x,t)}{(x-t)^{p+1}} w_\beta(x) dx \right| \leq \mathcal{C} \int_0^1 \frac{\Omega(f^{(p)},\sigma)_u}{\sigma} d\sigma,$$

where $0 < \mathcal{C} \neq \mathcal{C}(t,f)$.

m	j	$\bar{\tau}_{0,m}(f,-3.5)$	$\bar{\tau}_{0,m}(f,-1)$	$\bar{\tau}_{0,m}(f,0.99999)$	$\bar{\tau}_{0,m}(f,4.5)$
50	46	–	$2.34e-9$	$8.47e-7$	$1.84e-15$
100	74	–	$4.23e-9$	$4.76e-8$	–
150	96	–	$4.93e-10$	$3.15e-8$	–
200	116	–	$1.94e-10$	$1.34e-8$	–
250	133	–	$1.36e-10$	$3.88e-9$	–
300	150	–	$1.73e-11$	$3.83e-9$	–
351	165	–	$7.50e-12$	$2.09e-9$	–
401	180	–	$1.72e-11$	$4.06e-10$	–
451	195	–	$1.09e-11$	$1.45e-9$	–
m	j	$\bar{\tau}_{1,m}(f,-3.5)$	$\bar{\tau}_{1,m}(f,-1)$	$\bar{\tau}_{1,m}(f,0.99999)$	$\bar{\tau}_{1,m}(f,4.5)$
50	46	–	$2.97e-9$	$1.95e-6$	–
100	74	–	$2.14e-9$	$1.68e-6$	–
150	96	–	$2.28e-10$	$4.22e-7$	–
200	116	–	$9.20e-11$	$2.21e-7$	–
250	133	–	$6.91e-11$	$1.65e-7$	–
300	150	–	$9.58e-12$	$1.62e-8$	–
351	165	–	$4.17e-12$	$5.73e-9$	–
401	180	–	$8.73e-12$	$3.55e-8$	–
451	195	–	$5.29e-12$	$2.95e-8$	–
m	j	$\bar{\tau}_{2,m}(f,-3.5)$	$\bar{\tau}_{2,m}(f,-1)$	$\bar{\tau}_{2,m}(f,0.99999)$	$\bar{\tau}_{2,m}(f,4.5)$
50	46	–	$2.35e-9$	$1.25e-3$	–
100	74	–	$1.08e-9$	$1.63e-5$	–
150	96	–	$1.05e-10$	$1.96e-5$	–
200	114	–	$4.33e-11$	$1.27e-5$	–
250	133	–	$3.50e-11$	$3.01e-6$	–
300	150	–	$5.23e-12$	$4.73e-6$	–
351	165	–	$2.29e-12$	$3.03e-6$	–
401	180	–	$4.40e-12$	$1.90e-7$	–
451	193	–	$2.55e-12$	$3.76e-6$	–

Table 7: Example 6.3: $\bar{\tau}_{p,m}(f,t)$, $p=0,1,2$, with $t=-3.5,-1,0.99999,4.5$

Proof. We have

$$\begin{aligned} I(t) &:= \int_{|x-t|<1} \frac{R_p(f,x,t)}{(x-t)^{p+1}} w_\beta(x)dx = \int_{t-1}^{t+1} \frac{R_p(f,x,t)}{(x-t)^{p+1}} w_\beta(x)dx \\ &= \frac{1}{(p-1)!}\int_{t-1}^{t}\left[\int_x^t [f^{(p)}(t)-f^{(p)}(\tau)](\tau-x)^{p-1}d\tau\right]\frac{w_\beta(x)}{(t-x)^{p+1}}dx \\ &+ \frac{1}{(p-1)!}\int_{t}^{t+1}\left[\int_t^x [f^{(p)}(\tau)-f^{(p)}(t)](x-\tau)^{p-1}d\tau\right]\frac{w_\beta(x)}{(x-t)^{p+1}}dx \end{aligned}$$

and, by the changes of variables $x=t-\sigma$, $\tau=t-z$ in the first integral and $x=t+\sigma, \tau=t+z$ in the second integral, we get

$$\begin{aligned} I(t) &= \frac{1}{(p-1)!}\int_0^1\left[\int_0^\sigma [f^{(p)}(t)-f^{(p)}(t-z)](\sigma-z)^{p-1}dz\right]\frac{w_\beta(t-\sigma)}{\sigma^{p+1}}d\sigma \\ &+ \frac{1}{(p-1)!}\int_0^1\left[\int_0^\sigma [f^{(p)}(t+z)-f^{(p)}(t)](\sigma-z)^{p-1}dz\right]\frac{w_\beta(t+\sigma)}{\sigma^{p+1}}d\sigma. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} |I(t)| &\le \frac{\mathcal{C}}{u(t)}\int_0^1 \frac{\Omega(f^{(p)},\sigma)_u}{\sigma}\left[w_\beta(t-\sigma)+w_\beta(t+\sigma)\right]d\sigma \\ &\le \frac{\mathcal{C}}{u(t)}\int_0^1 \frac{\Omega(f^{(p)},\sigma)_u}{\sigma}d\sigma, \qquad (33) \end{aligned}$$

being $w_\beta(t-\sigma)<\mathcal{C}$ and $w_\beta(t+\sigma)<\mathcal{C}$, i.e. the thesis follows. □

m	$\hat{\imath}$	q	$\bar{\rho}_{0,m}(f,-3.5)$	$\bar{\rho}_{0,m}(f,-1)$	$\bar{\rho}_{0,m}(f,0.99999)$	$\bar{\rho}_{0,m}(f,4.5)$
50	50	47	$8.33e-7$	$3.34e-6$	$5.17e-6$	$4.91e-8$
100	100	76	$4.18e-8$	$1.83e-7$	$4.53e-6$	$1.21e-8$
150	148	100	$2.89e-8$	$8.01e-8$	$7.13e-7$	$2.05e-8$
200	182	120	$3.59e-9$	$5.19e-9$	$3.62e-7$	$4.83e-9$
250	213	139	$1.97e-9$	$3.65e-9$	$2.81e-8$	$6.01e-9$
300	240	156	$3.22e-9$	$4.78e-9$	$5.55e-9$	$3.57e-9$
350	266	173	$1.31e-9$	$2.73e-9$	$3.20e-9$	$6.08e-10$
400	290	189	$1.08e-9$	$8.14e-10$	$7.22e-9$	$4.46e-10$
450	314	204	$2.70e-10$	$3.42e-10$	$1.31e-8$	$5.03e-10$
m	$\hat{\imath}$	q	$\bar{\rho}_{1,m}(f,-3.5)$	$\bar{\rho}_{1,m}(f,-1)$	$\bar{\rho}_{1,m}(f,0.99999)$	$\bar{\rho}_{1,m}(f,4.5)$
50	50	47	$1.53e-5$	$4.64e-5$	$9.13e-5$	$1.10e-6$
100	100	76	$1.91e-6$	$1.67e-6$	$5.21e-5$	$1.28e-6$
150	148	100	$1.68e-7$	$1.55e-6$	$2.54e-7$	$1.41e-6$
200	184	120	$2.46e-8$	$3.80e-7$	$2.43e-5$	$1.94e-7$
250	216	139	$2.80e-7$	$7.66e-7$	$8.56e-6$	$2.72e-7$
300	243	156	$2.60e-8$	$4.30e-7$	$2.07e-6$	$5.09e-8$
350	270	173	$8.22e-8$	$2.78e-7$	$1.43e-6$	$1.81e-7$
400	296	189	$1.38e-8$	$2.04e-7$	$2.56e-6$	$1.21e-7$
450	318	204	$4.49e-8$	$9.75e-8$	$3.72e-6$	$2.12e-8$
m	$\hat{\imath}$	q	$\bar{\rho}_{2,m}(f,-3.5)$	$\bar{\rho}_{2,m}(f,-1)$	$\bar{\rho}_{2,m}(f,0.99999)$	$\bar{\rho}_{2,m}(f,4.5)$
50	50	47	$8.92e-6$	$5.69e-4$	$5.16e-3$	$1.15e-5$
100	100	76	$4.92e-6$	$8.34e-5$	$2.43e-3$	$2.77e-5$
150	149	100	$2.92e-5$	$5.81e-5$	$9.20e-4$	$1.56e-5$
200	186	120	$6.04e-6$	$6.17e-6$	$6.77e-4$	$2.84e-6$
250	218	139	$9.05e-6$	$4.22e-6$	$2.63e-4$	$5.75e-6$
300	247	156	$7.60e-6$	$8.15e-6$	$2.15e-4$	$8.65e-6$
350	274	173	$5.45e-6$	$5.72e-6$	$1.68e-4$	$3.35e-6$
400	299	189	$4.06e-6$	$1.87e-6$	$1.24e-4$	$1.85e-6$
450	324	204	$2.83e-7$	$1.17e-6$	$1.43e-4$	$2.92e-6$

Table 8: Example 6.3: $\bar{\rho}_{p,m}(f,t),\ p=0,1,2$, with $t=-3.5,-1,0.99999,4.5$

Lemma 7.3. *Let $0<\lambda<1$ and $s\in\mathbb{N}_0$. If $f\in Z_{s+\lambda}(u)$ then $\mathbf{F}_0(f)\in W_s(u)$ and*

$$\|\mathbf{F}_0^{(s)}(f)u\|\le\mathcal{C}\left\{\|fu\|+\|f^{(s)}u\|+\int_0^1\frac{\Omega(f^{(s)},\sigma)_u}{\sigma}d\sigma\right\}\le\mathcal{C}\,\|f\|_{Z_{s+\lambda}(u)},\quad 0<\mathcal{C}\neq\mathcal{C}(m,f). \tag{34}$$

Proof. Start from

$$\mathbf{F}_0(f,t)\quad=\quad\left(\int_{|x-t|>1}+\int_{|x-t|<1}\right)\frac{f\ (x)-f\ (t)}{x-t}w_\beta(x)dx=:A(t)+B(t). \tag{35}$$

We have

$$u(t)|A(t)|\quad\le\quad\mathcal{C}u(t)\|fu\|\le\mathcal{C}\,\|fu\| \tag{36}$$

and, by Lemma 7.2 with $p=0$,

$$u(t)|B(t)|\quad\le\quad\mathcal{C}\int_0^1\frac{\Omega(f,\sigma)_u}{\sigma}d\sigma\le\mathcal{C}\sup_{\sigma\ge0}\frac{\Omega(f,\sigma)_u}{\sigma^\lambda}\le\mathcal{C}\,\|f\|_{Z_\lambda(u)}. \tag{37}$$

Consequently, combining (36) and (37) with (35), we deduce

$$|\mathbf{F}_0(f,t)u(t)|\le\mathcal{C}\,\|f\|_{Z_\lambda(u)},\qquad\lim_{t\to\pm\infty}\mathbf{F}_0(f,t)u(t)=0.$$

Now we prove (34) with $s\ge1$. With $R_s(f,x,t)$ in (32),

$$\mathbf{F}_0^{(s)}(f,t)=\left\{\int_{|x-t|<1}+\int_{|x-t|\ge1}\right\}\frac{R_s(f,x,t)}{(x-t)^{s+1}}w_\beta(x)dx.$$

By Lemmas 7.1 and 7.2 and the inequality

$$u(t)\int_{|x-t|\ge1}\frac{R_s(f,x,t)}{(x-t)^{s+1}}w_\beta(x)dx\le\mathcal{C}\left\{\|fu\|+\sum_{k=0}^s\|f^{(k)}u\|\right\}\le\mathcal{C}\left\{\|fu\|+\|f^{(s)}u\|\right\},$$

(34) follows. □

m	k_1	k_2	$\hat{\sigma}_{0,m}(f,-3.5)$	$\hat{\sigma}_{0,m}(f,-1)$	$\hat{\sigma}_{0,m}(f,0.99999)$	$\hat{\sigma}_{0,m}(f,4.5)$
50	12	10	$1.71e-8$	$8.60e-3$	$2.50e-2$	–
100	17	14	$2.69e-9$	$1.90e-3$	$6.26e-4$	–
150	21	17	$8.87e-10$	$1.33e-3$	$3.73e-4$	–
200	24	19	$6.95e-10$	$1.33e-4$	$1.35e-5$	–
250	27	22	$4.53e-10$	$2.44e-4$	$1.17e-5$	–
300	29	24	$2.55e-10$	$9.49e-5$	$6.48e-6$	–
350	32	26	$3.36e-10$	$1.13e-4$	$3.70e-7$	–
400	34	28	$1.55e-10$	$2.15e-4$	$6.21e-6$	–
450	36	29	$2.29e-10$	$1.30e-5$	$2.50e-6$	–
m	k_1	k_2	$\hat{\sigma}_{1,m}(f,-3.5)$	$\hat{\sigma}_{1,m}(f,-1)$	$\hat{\sigma}_{1,m}(f,0.99999)$	$\hat{\sigma}_{1,m}(f,4.5)$
50	12	10	$3.18e-8$	$2.45e-2$	$6.88e-3$	–
100	17	14	$4.62e-9$	$3.61e-3$	$1.85e-4$	–
150	21	17	$1.55e-9$	$2.74e-3$	$2.72e-5$	–
200	24	19	$1.17e-9$	$3.36e-4$	$1.26e-5$	–
250	27	22	$7.70e-10$	$5.45e-4$	$1.10e-5$	–
300	29	24	$4.42e-10$	$2.19e-4$	$5.79e-6$	–
350	32	26	$5.74e-10$	$2.26e-4$	$1.07e-6$	–
400	34	27	$2.64e-10$	$4.55e-4$	$6.35e-6$	–
450	36	29	$3.92e-10$	$3.77e-5$	$2.05e-6$	–
m	k_1	k_2	$\hat{\sigma}_{2,m}(f,-3.5)$	$\hat{\sigma}_{2,m}(f,-1)$	$\hat{\sigma}_{2,m}(f,0.99999)$	$\hat{\sigma}_{2,m}(f,4.5)$
50	12	10	$2.77e-8$	$2.99e-2$	$1.11e-3$	–
100	17	14	$3.77e-9$	$3.60e-3$	$5.85e-5$	–
150	21	17	$1.28e-9$	$2.84e-3$	$2.93e-5$	–
200	24	19	$9.55e-10$	$3.88e-4$	$1.05e-5$	–
250	27	22	$6.24e-10$	$5.88e-4$	$1.01e-5$	–
300	29	24	$3.64e-10$	$2.41e-4$	$4.98e-6$	–
350	32	26	$4.68e-10$	$2.30e-4$	$1.77e-6$	–
400	34	27	$2.14e-10$	$4.78e-4$	$6.42e-6$	–
450	36	29	$3.20e-10$	$4.53e-5$	$1.57e-6$	–

Table 9: Example 6.3: $\hat{\sigma}_{p,m}(f,t)$, $p=0,1,2$, with $t=-3.5,-1,0.99999,4.5$

In order to prove next lemma, we recall the Posse-Markov-Stieltjes inequalities [12, p.33]. For any function g s.t. $g^{(k)}(x)\geq 0$, $k=0,1,\dots 2m-1, m>1$, for $x\in(0,x_d)$, $d=2,3,\dots,m$, we have

$$\sum_{k=-\lfloor m/2\rfloor}^{d-1}\lambda_{m,k}g(x_k)\leq\int_{-\infty}^{x_d}g(x)w_\beta(x)dx\leq\sum_{k=-\lfloor m/2\rfloor}^{d}\lambda_{m,k}g(x_k). \tag{38}$$

For any function g s.t. $(-1)^k g^{(k)}(x)\geq 0$, $k=0,1,\dots 2m-1, m>1$, for $x\in(x_d,+\infty)$, $d=1,2,\dots,m-1$, we have

$$\sum_{k=d+1}^{\lfloor m/2\rfloor}\lambda_{m,k}g(x_k)\leq\int_{x_d}^{+\infty}g(x)w_\beta(x)dx\leq\sum_{k=d}^{\lfloor m/2\rfloor}\lambda_{m,k}g(x_k). \tag{39}$$

Lemma 7.4. *Let $0<\lambda<1$. If $f\in Z_{s+\lambda}(u)$ with $s\in\mathbb{N}_0$, then for any fixed m, $\mathbf{F}_{0,m}(f)\in W_s(u)$ and*

$$\|\mathbf{F}_{0,m}^{(s)}(f)u\|\leq\mathcal{C}\left\{\|f^{(s)}u\|\log m+\int_0^{\frac{a_m}{m}}\frac{\Omega(f^{(s)},\sigma)_u}{\sigma}d\sigma\right\}\leq\mathcal{C}\log m\|f\|_{Z_{s+\lambda}(u)}, \tag{40}$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$.

Proof. To prove (40) with $s=0$ we start from

$$\mathbf{F}_{0,m}(f,t)\quad=\quad\left(\sum_{|x_{m,k}-t|\geq 1}+\sum_{|x_{m,k}-t|<1}\right)\frac{f\,(x_{m,k})-f\,(t)}{x_{m,k}-t}\lambda_{m,k}=:\Sigma_1(t)+\Sigma_2(t). \tag{41}$$

We have

$$\begin{aligned} u(t)|\Sigma_1(t)| &\leq \mathcal{C}u(t)\|fu\|\sum_{|k|\leq j}\Delta x_{m,k}\frac{w_\beta(x_{m,k})}{u(x_{m,k})}+|f(t)|u(t)\sum_{|k|\leq j}\lambda_{m,k}\\ &\leq \mathcal{C}\;\|fu\|, \end{aligned} \tag{42}$$

being, with $\Delta x_{m,k}=x_{m,k+1}-x_{m,k}$, [17]

$$\lambda_{m,k}\sim\Delta x_{m,k}w_\beta(x_{m,k}),\quad k=-\left\lfloor\frac{m}{2}\right\rfloor,-\left\lfloor\frac{m}{2}\right\rfloor+1,\dots\left\lfloor\frac{m}{2}\right\rfloor-1.$$

Denoted by $x_{m,d}$ the closest knot to t, we have

$$\begin{aligned}u(t)|\Sigma_2(t)| &\le \mathcal{C}\,\|fu\|\sum_{\substack{|x_{m,k}-t|<1\\k\ne d}}\frac{\lambda_{m,k}}{|x_{m,k}-t|}\frac{u(t)}{u(x_{m,k})}+|f(t)|u(t)\sum_{\substack{|x_{m,k}-t|<1\\k\ne d}}\frac{\lambda_{m,k}}{|x_{m,k}-t|}\\&+u(t)\frac{|f(x_{m,d})|}{|x_{m,d}-t|}\lambda_{m,d}+|f(t)|u(t)\frac{\lambda_{m,d}}{|x_{m,d}-t|}=:\sum_{i=1}^4 S_i(t).\end{aligned}\tag{43}$$

To estimate $S_1(t)$ and $S_2(t)$ we use [27, (56)] (see, also, [3, Proof of Th. 3.1, p. 1184]) to obtain

$$S_1(t)\le\mathcal{C}\,\|fu\|\sum_{\substack{|k|\le j\\k\ne d}}\frac{\Delta x_{m,k}}{|x_{m,k}-t|}\le\mathcal{C}\,\|fu\|\log m\tag{44}$$

and

$$S_2(t)\le\mathcal{C}\,\|fu\|\sum_{\substack{|k|\le j\\k\ne d}}\frac{\Delta x_{m,k}}{|x_{m,k}-t|}\le\mathcal{C}\,\|fu\|\log m.\tag{45}$$

Taking into account [20] (see also [3, Lemma 4.3])

$$\frac{\lambda_{m,d}}{|x_{m,d}-t|}\le\mathcal{C}w_\beta(x_{m,d})\le\mathcal{C}w_\beta(t),\tag{46}$$

we get

$$S_3(t)+S_4(t)\le\mathcal{C}\,\|fu\|w_\beta(t)\le\mathcal{C}\,\|fu\|.\tag{47}$$

Combining (44), (45), (47) with (43),

$$u(t)|\Sigma_2(t)|\le\mathcal{C}\,\|fu\|\log m$$

and thus (40) with $s=0$ follows by combining last estimate and (42) with (41).

Now we prove (40) with $s\ge1$. Let $x_{m,d}<t<x_{m,d+1}$, being $x_{m,d}$ the zero of $p_m(w_\beta)$ closest to t. Then, we will use the following decomposition

$$\begin{aligned}\mathbf{F}^{(s)}_{0,m}(fw_\beta,t) &= \left\{\sum_{k=-j}^{d-1}+\sum_{k=d}^{d+1}+\sum_{k=d+2}^{j}\right\}\frac{f(x_{m,k})-\sum_{j=0}^s\frac{f^{(j)}(t)}{j!}(x_{m,k}-t)^j}{(x_{m,k}-t)^{s+1}}\lambda_{m,k}\\&=: A(t)+B(t)+C(t).\end{aligned}\tag{48}$$

To estimate $A(t)$, by (32), since $x_{m,k}\le x_{m,d}<t$, we have

$$\begin{aligned}\left|f(x_{m,k})-\sum_{j=0}^s\frac{f^{(j)}(t)}{j!}(x_{m,k}-t)^j\right| &\le \frac{1}{(s-1)!}\int_{x_{m,k}}^t|f^{(s)}(\tau)-f^{(s)}(t)|(\tau-x_{m,k})^{s-1}d\tau\\&\le \mathcal{C}\,\|f^{(s)}u\|\int_{x_{m,k}}^t\frac{(\tau-x_{m,k})^{s-1}}{u(\tau)}d\tau\\&+\mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\int_{x_{m,k}}^t(\tau-x_{m,k})^{s-1}d\tau\le\mathcal{C}\,\|f^{(s)}u\|\frac{(t-x_{m,k})^s}{u(t)}\end{aligned}$$

and therefore

$$\begin{aligned}|A(t)| &\le \mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\sum_{k=-j}^{d-1}\frac{\lambda_{m,k}}{(t-x_{m,k})}\le\mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\int_{-x_{m,j+1}}^{x_d}\frac{e^{-|x|^\beta}}{(t-x)}dx\\&\le \mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\log\frac{t-x_{m,d}}{t-x_{m,-j-1}}\le\mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\log m,\end{aligned}\tag{49}$$

since by (38),

$$\sum_{k=-j}^{d-1}\frac{\lambda_{m,k}}{(t-x_{m,k})}\le\int_{-x_{m,j+1}}^{x_{m,d}}\frac{e^{-|x|^\beta}}{(t-x)}dx.$$

To estimate $B(t)$, start from

$$f(x_d)-\sum_{j=0}^{s}\frac{f^{(j)}(t)}{j!}(x_{m,d}-t)^j = \frac{1}{(s-1)!}\int_{x_{m,d}}^{t}(f^{(s)}(\tau)-f^{(s)}(t))(x_{m,d}-\tau)^{s-1}d\tau.$$

and by the change of variable $\tau=t-z$, we get

$$\begin{aligned}\left|f(x_{m,d})-\sum_{j=0}^{s}\frac{f^{(j)}(t)}{j!}(x_{m,d}-t)^j\right| &\le \frac{1}{(s-1)!}\int_0^{t-x_{m,d}}\left|f^{(s)}(t-z)-f^{(s)}(t)\right|(t-x_{m,d}-z)^{s-1}dz\\ &= \frac{1}{(s-1)!u(t)}\int_0^{t-x_{m,d}}|\Delta_z f^{(s)}(t)u(t)|(t-x_{m,d}-z)^{s-1}dz.\end{aligned}$$

Since $\Delta x_{m,k}\sim\frac{a_m}{m}$, for $|k|\le j$, it results $t-x_{m,d}\sim\Delta x_{m,d}\sim\frac{a_m}{m}$ (see, for instance, [27, (6)]), then

$$\begin{aligned}\left|f(x_{m,d})-\sum_{j=0}^{s}\frac{f^{(j)}(t)}{j!}(x_{m,d}-t)^j\right| &\le \frac{\mathcal{C}}{u(t)}\sup_{0<z\le\frac{a_m}{m}}\|\Delta_z f^{(s)}u\|\int_0^{t-x_{m,d}}(t-x_{m,d}-z)^{s-1}dz\\ &\le \frac{\mathcal{C}}{u(t)}\Omega\left(f^{(s)},\frac{a_m}{m}\right)_u(t-x_{m,d})^s.\end{aligned}$$

Therefore,

$$\frac{\left|f(x_{m,d})-\sum_{j=0}^{s}\frac{f^{(j)}(t)}{j!}(x_{m,d}-t)^j\right|\lambda_{m,d}}{(t-x_{m,d})^{s+1}}\le\frac{\mathcal{C}}{u(t)}\Omega\left(f^{(s)},\frac{a_m}{m}\right)_u\frac{\lambda_{m,d}}{(t-x_{m,d})}$$

and, using (46) and

$$\Omega\left(f^{(s)},\frac{a_m}{m}\right)_u\le\mathcal{C}\int_0^{\frac{a_m}{m}}\frac{\Omega\left(f^{(s)},t\right)_u}{t}dt,$$

we obtain

$$\frac{\left|f(x_{m,d})-\sum_{j=0}^{s}\frac{f^{(j)}(t)}{j!}(x_{m,d}-t)^j\right|\lambda_{m,d}}{(t-x_{m,d})^{s+1}}\le\mathcal{C}u(t)\int_0^{\frac{a_m}{m}}\frac{\Omega\left(f^{(s)},t\right)_u}{t}dt.$$

Since last estimate holds replacing $t-x_{m,d}\sim x_{m,d+1}-t$, we conclude

$$|B(t)|\le\mathcal{C}\int_0^{\frac{a_m}{m}}\frac{\Omega\left(f^{(s)},t\right)_u}{t}dt. \tag{50}$$

It remains to estimate $C(t)$. For $t<x_{m,d+2}\le x_k$, we have

$$\begin{aligned}\left|f(x_{m,k})-\sum_{j=0}^{s}\frac{f^{(j)}(t)}{j!}(x_{m,k}-t)^j\right| &\le \frac{1}{(s-1)!}\int_t^{x_{m,k}}|f^{(s)}(\tau)-f^{(s)}(t)|(x_{m,k}-\tau)^{s-1}d\tau\\ &\le \mathcal{C}\,\|f^{(s)}u\|\int_t^{x_{m,k}}\frac{(x_{m,k}-\tau)^{s-1}}{u(\tau)}d\tau+\mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\int_t^{x_{m,k}}(x_{m,k}-\tau)^{s-1}d\tau\\ &\le \mathcal{C}\,\|f^{(s)}u\|(x_{m,k}-t)^s e^{\frac{|x_{m,k}|^\beta}{2}}+\mathcal{C}\,\|f^{(s)}u\|\frac{(x_{m,k}-t)^s}{u(t)}.\end{aligned}$$

Then, by (39),

$$\begin{aligned}|C(t)| &\le \mathcal{C}\,\|f^{(s)}u\|\sum_{k=d+2}^{j}\frac{\Delta x_{m,k}e^{-\frac{|x_{m,k}|^\beta}{2}}}{(x_{m,k}-t)}+\mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\sum_{k=d+2}^{j}\frac{\lambda_{m,k}}{(x_{m,k}-t)}\\ &\le \mathcal{C}\,\|f^{(s)}u\|\int_{x_{m,d+1}}^{x_{m,j+1}}\frac{e^{-\frac{|x|^\beta}{2}}}{(x-t)}dx+\mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\int_{x_{m,d+1}}^{x_{m,j+1}}\frac{e^{-|x|^\beta}}{(x-t)}dx\\ &\le \mathcal{C}\,\|f^{(s)}u\|\log\frac{x_{m,j+1}-t}{x_{m,d+1}-t}+\mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\log\frac{x_{m,j+1}-t}{x_{m,d+1}-t}\\ &\le \mathcal{C}\,\|f^{(s)}u\|\log m+\mathcal{C}\frac{\|f^{(s)}u\|}{u(t)}\log m.\end{aligned} \tag{51}$$

The thesis follows by combining (49), (50) and (51) with (48) and taking into account Lemma 7.1. □

Proof of Theorem 3.1. Recalling that

$$\mathbf{F}_{p,m^*}(f,t) = \frac{1}{p!}\frac{d^p}{dt^p}\mathbf{F}_{0,m^*}(f,t),$$

(12) is a consequence of Lemma 7.4 with $s = p$.

Concerning (13), denoting by $Q \in \mathcal{P}^*_{m+1}$ s. t. $\|(f-Q)u\| = \inf_{P_{m+1}\in\mathcal{P}^*_{m+1}} \|(f-P_{m+1})u\| =: \widetilde{E}_{m+1}(f)_u$. We have

$$\begin{aligned}\mathbf{F}_p(f,t) - \mathbf{F}_{p,m^*}(f,t) &= \mathbf{F}_p(f-Q,t) - \mathbf{F}_{p,m^*}(f-Q,t) \\ &=: E_1(t) - E_2(t),\end{aligned}$$

being the ordinary Gaussian rule exact for polynomials of degree at most $2m-1$.

By (34)

$$|E_1(t)u(t)| \le \mathcal{C}\left\{\widetilde{E}_{m+1}(f)_u + \|(f-Q)^{(p)}u\| + \int_0^1 \frac{\Omega((f-Q)^{(p)},\sigma)}{\sigma}d\sigma\right\}.$$

Then, using (30), (5) and (31) and recalling that, by [27, Lemma 2] and Lemma 7.1,

$$\int_0^1 \frac{\Omega((f-Q)^{(p)},\sigma)_u}{\sigma}d\sigma \le \mathcal{C}\log m \int_0^{\frac{a_m}{m}} \frac{\Omega^r(f^{(p)},\sigma)_u}{\sigma}d\sigma \le \mathcal{C}\|f\|_{Z_{p+\lambda}(u)}\log m\left(\frac{a_m}{m}\right)^\lambda,$$

we deduce

$$|E_1(t)u(t)| \le \mathcal{C}\ \|f\|_{Z_{p+\lambda}(u)}\log m\left(\frac{a_m}{m}\right)^\lambda.$$

Since, using (40), analogously it is possible to prove that

$$|E_2(t)u(t)| \le \mathcal{C}\ \|f\|_{Z_{p+\lambda}(u)}\log m\left(\frac{a_m}{m}\right)^\lambda,$$

(13) easily follows. □

Proof of Theorem 4.1. We consider only the case $p \ge 1$, being the case $p = 0$ simpler. We first prove (21). Start from

$$\begin{aligned}|\rho_{p,m}(f,t)|u(t) &\le \|[\mathbf{F}_0(f) - L_{m+2}(w_\beta,\mathbf{F}_0(f))]^{(p)}u\| \\ &+ \|L_{m+2}(w_\beta,\mathbf{F}_0(f) - \mathbf{F}_m(f))^{(p)}u\| =: S_1 + S_2.\end{aligned} \tag{52}$$

Since $f \in Z_{p+r+\lambda}(u)$, by Lemma 7.3, we have $\mathbf{F}_0(f) \in W_{p+r}(u)$. Then, by (8) and (34)

$$S_1 \le \mathcal{C}\ \|\mathbf{F}_0(f)\|_{W_{p+r}(u)}\log m\left(\frac{a_m}{m}\right)^r \le \mathcal{C}\ \|f\|_{Z_{p+r+\lambda}(u)}\log m\left(\frac{a_m}{m}\right)^r. \tag{53}$$

By the Bernstein inequality (28) and by [20, Th. 3.2], we get

$$S_2 \le \mathcal{C}\sqrt{m^p}\|L_{m+2}(w_\beta,\mathbf{F}_0(f) - \mathbf{F}_{0,m}(f))u\| \le \mathcal{C}\sqrt{m^p}\log m[E_M(\mathbf{F}_0(f))_u + E_M(\mathbf{F}_{0,m}(f))_u].$$

Using Lemmas 7.3 and 7.4 and (5) we deduce

$$S_2 \le \mathcal{C}\ \|f\|_{Z_{p+r+\lambda}(u)}\log^2 m\left(\frac{a_m}{m}\right)^r.$$

Then (21) follows combining last estimate and (53) with (52).

Now we prove (20). Since

$$\|L_{m+2}(w_\beta,\mathbf{F}_{0,m}(f))^{(p)}u\| \le \|(\mathbf{F}_{0,m}(f) - L_{m+2}(w_\beta,\mathbf{F}_{0,m}(f)))^{(p)}u\| + \|(\mathbf{F}^{(p)}_{0,m}(f))u\|,$$

by (8) and (40), (20) follows. □

Lemma 7.5. *For $p \in \mathbb{N}_0$, $t \in \mathbb{R}\backslash\{0\}$,*

$$|\mathbf{H}_p(w_\beta,t)| \le \mathcal{C}w_\beta(t)\begin{cases}|t|^{-p} & p \ge 1 \\ |t|^{-1} & p = 0\end{cases}.$$

Proof. Since $\mathbf{H}_p(w_\beta,t)$ is an odd (even) function for p even (odd), assume $t > 0$, and use

$$\mathbf{H}_p(w_\beta,t) = \int_0^\infty \frac{e^{-x^\beta}}{(x-t)^{p+1}}dx + (-1)^{p+1}\int_0^\infty \frac{e^{-x^\beta}}{(x+t)^{p+1}}dx =: J_1(t) + J_2(t)$$

To estimate $J_1(t)$ we recall the following result in [6]

$$|J_1(t)| \le \mathcal{C}e^{-t^\beta}\begin{cases}t^{-p} & p \ge 1 \\ \log t^{-1} & p = 0\end{cases}. \tag{54}$$

For $t > 1$,

$$|J_2(t)| \le \int_0^\infty \frac{e^{-x^\beta}}{(x+1)^{p+1}} dx \le \mathcal{C}$$

and for $0 < t < 1$

$$\begin{aligned} |J_2(t)| &\le \int_0^t \frac{e^{-x^\beta}}{(x+t)^{p+1}} dx + \int_t^\infty \frac{e^{-x^\beta}}{(x+t)^{p+1}} dx \le \mathcal{C} \int_0^t \frac{dx}{(x+t)^{p+1}} + \int_t^\infty \frac{e^{-x^\beta} dx}{(x+t)^{p+1}} dx \\ &\le \frac{\mathcal{C}}{t^p} + \int_t^\infty \frac{e^{-x^\beta} dx}{(x+t)^{p+1}} dx. \end{aligned} \tag{55}$$

Since

$$\int_t^\infty \frac{e^{-x^\beta} dx}{(x+t)^{p+1}} dx \le \mathcal{C} \frac{e^{-t^\beta}}{t^p}, \quad p \ge 1$$

and

$$\int_t^\infty \frac{e^{-x^\beta} dx}{(x+t)} dx \le \frac{1}{2t} \int_0^\infty e^{-x^\beta} dx \le \frac{\mathcal{C}}{t},$$

the Lemma follows combining last estimates with (55) and (54).

□

Proof of Theorem 4.3. By (21)

$$\begin{aligned} |\mathbf{T}_{p,m}(f,t)u(t)| &\le \|\rho_{p,m}(f)u\| + \sum_{k=1}^p \binom{p}{k} \|(f - L_{m+2}(w_\beta, f))^{(k)} u\| \sup_{t \in \mathbb{R}\setminus\{0\}} \left|\mathbf{H}_{p-k}(w_\beta, t)\right| \\ &\le \mathcal{C} \, \|f\|_{Z_{p+r+\lambda}(u)} \log^2 m \left(\frac{a_m}{m}\right)^r + \sum_{k=1}^p \binom{p}{k} \|(f - L_{m+2}(w_\beta, f))^{(k)} u\| \sup_{\mathbb{R}\setminus\{0\}} \left|\mathbf{H}_{p-k}(w_\beta, t)\right|. \end{aligned}$$

Taking into account Lemma 7.5 and Theorem 2.1,

$$|\mathbf{T}_{p,m}(f,t)u(t)| \le \mathcal{C} \, \|f\|_{Z_{p+r+\lambda}(u)} \log^2 m \left(\frac{a_m}{m}\right)^r + \frac{\mathcal{C}}{|t|^p} \|f\|_{Z_{p+r+\lambda}(u)} \log m \left(\frac{a_m}{m}\right)^{p+\lambda+r}$$

the thesis follows. □

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