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On the Conjectures of the inequalities involving generalized trigonometric and hyperbolic functions with one-parameter

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Abstract

In this article, we present the best possible proof of the conjectures posed by re-searchers Li Yin et al. and Riku Klen et al., along with the appropriate bounds involv-ing generalized trigonometric and hyperbolic functions. These inequalities are further generalized with the best possible bounds. Alternative proofs of the Cusa-Huygens inequality were given using particular cases.

1 Introduction

For $0 \le t \le 1$, we know that,

$$\arcsin(t) = \int_0^t (1-x^p)^{\frac{-1}{2}}\,dx$$

And

$$\frac{\pi}{2} = \arcsin(1) = \int_0^1 (1-x^2)^{\frac{-1}{2}}\,dx.$$

From this result it is easy to define trigonometric *sine* function in the interval $[0, \frac{\pi}{2}]$ as the inverse of *arcsine* and it can be extended up to infinity. Also, In Dirichlet equation for the one-dimensional Laplacian, the eigenfunction is defines as the inverse of the function $\sigma_p : [0,1] \to \mathbb{R}$,

$$\sigma_p(t) = \int_0^t (1-x^p)^{\frac{-1}{p}}\,dx.$$

This eigenfunction is corresponding to the eigenvalue involving $\pi_p = \frac{2\pi}{p\sin(\frac{\pi}{p})}$. We can denote $\sigma_p(t)$ as $\arcsin_p(t)$.
In the year 1995, Lindqvist [7] defined a generalized version of the trigonometric and hyperbolic function. The generalized $\arcsin_p$ is defined as,

$$\arcsin_p(t) = \int_0^t (1-x^p)^{\frac{-1}{p}}\,dx,$$

and the inverse of $\arcsin_p(t)$ is said to be the generalized sine function $\sin_p(t)$ for all $t \in (0, \frac{\pi_p}{2}]$ and $1 < p < \infty$. The constant $\frac{\pi_p}{2}$ is defined as

$$\begin{aligned}\frac{\pi_p}{2} &= \arcsin_p(1) = \int_0^1 (1-x^p)^{\frac{-1}{p}}\,dx \\ &= \frac{\pi}{p}\left(\csc\frac{\pi}{p}\right) = \frac{1}{p}\Gamma\left(1-\frac{1}{p}\right)\Gamma\left(\frac{1}{p}\right),\end{aligned}$$

where, the Γ represents the gamma function leading to the result, $\pi_p = \frac{2\pi}{p\sin(\frac{\pi}{p})}$.
The generalized *sine* function is strictly increasing in $[0, \frac{\pi_p}{2}]$, since $\sin_p(0) = 0,\ \sin_p(\frac{\pi_p}{2}) = 1$. We can extend this function on $(-\infty, \infty)$ by the periodicity of $2\pi_p$. Now the generalized cosine function is defined as [12],

$$\cos_p(t) = \frac{d}{dt}[\sin_p(t)],\ 0 \le t \le \frac{\pi_p}{2}.$$

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And

$$\frac{d}{dt}[\cos_p(t)] = -[\cos_p^{2-p}(t)\sin_p^{p-1}(t)],\ 0 \le t \le \frac{\pi_p}{2}.$$

Similarly, the generalized tangent function is defined as,

$$\tan_p(t) = \frac{\sin_p(t)}{\cos_p(t)},\ t \in \mathbb{R} \setminus \{k\pi_p + \frac{\pi_p}{2} : k \in \mathbb{Z}\}.$$

And

$$\frac{d}{dt}(\tan_p(t)) = 1 + |\tan_p(t)|^p,\ t \in \left(-\frac{\pi_p}{2}, \frac{\pi_p}{2}\right).$$

The generalized secant and cosecant functions are also defined in the form of a generalized sine and cosine function.

$$\sec_p(t) = \frac{1}{\cos_p(t)},\ t \in [0, \frac{\pi_p}{2}).$$

And

$$\frac{d}{dt}(\sec_p(t)) = \sec_p(t)\tan_p^{p-1}(t),\ t \in [0, \frac{\pi_p}{2}).$$

Now, the generalized inverse hyperbolic sine function arcsinh_p is given as,

$$\text{arcsinh}_p(t) = \begin{cases} \int_0^t (1+x^p)^{\frac{-1}{p}} dx,\ t \in (0,\infty), \\ -\text{arcsinh}_p(-t),\ t \in (-\infty, 0). \end{cases}$$

The inverse of $\text{arcsinh}_p(t)$ is said to be the generalized hyperbolic p-sine function $\sinh_p(t)$.
Similarly, the generalized hyperbolic cosine function is given as,

$$\cosh_p(t) = \frac{d}{dt}[\sinh_p(t)].$$

And

$$\frac{d}{dt}[\cosh_p(t)] = \cosh_p^{2-p}(t)\sinh_p^{p-1}(t).$$

The generalized hyperbolic tangent, secant and cosecant functions are also defined using the generalized sine and cosine functions.

$$\tanh_p(t) = \frac{\sinh_p(t)}{\cosh_p(t)},\ \text{sech}_p(t) = \frac{1}{\cosh_p(t)}.$$

And

$$\frac{d}{dt}[\tanh_p(t)] = 1 - \tanh_p^p(t), \quad \frac{d}{dt}[\text{sech}_p(t)] = -\text{sech}_p(t)\tanh_p^{p-1}(t).$$

When $p = 2$, all these results coincide with the classical trigonometric and hyperbolic functions.
These results were also used to establish some well-known classical inequalities. For more details (see [2, 3, 4]).
In the year 2012,Takeuchi [8] defined the two-parameter is said to be the generalized sine function with two-parameter $\sin_{p,q}(t)$.
This function is the inverse of:

$$\arcsin_{p,q}(t) = \int_0^t (1-x^q)^{\frac{-1}{p}} dx,$$

where $q, p \in (1, \infty)$ and $t \in [0, 1]$. If $p = q$ then it becomes $\sin_p(t)$ and expands to $(-\infty, \infty)$.. Similarly, using the generalized sine function of two-parameter, the generalized cosine and tangent functions are also defined.
The generalized hyperbolic sine function of two-parameter, $\sinh_{p,q}(t)$ is the inverse of,

$$\text{arcsinh}_{p,q}(t) = \int_0^t (1+x^q)^{\frac{-1}{p}} dx,\ t \in (0,\infty).$$

The generalized hyperbolic cosine and tangent functions with two-parameter are also defined in a similar way using the generalized hyperbolic sine function. In recent years, many researchers have studied these generalized trigonometric and hyperbolic functions. For more information (see [3, 9, 11, 12, 13, 14]).
In Section 2 we list preliminary results and lemma that are used in the proof of our main results. Section 3 we establish some classical inequalities with best possible bounds and particular cases. In Section 4 we have given the best possible proof of the three main conjectures posed by Huang et al and Riku Klen et al and Section 5 is the conclusion of the article.

2 Preliminaries and Lemma

In this section, we see some important results and lemmas, which play a crucial role in the proof of our main theorem.

Lemma 2.1. *[10] For $-\infty < a < b < \infty$ and the functions $f, g : [a, b] \to \mathbb{R}$ are continuous and differentiable on (a, b) with $g(a) = f(a) = 0$ or $g(b) = f(b) = 0$. Assume that, $g^{'}(t) \neq 0$ for all $t \in (a, b)$. If $\frac{f^{'}}{g^{'}}$ is decreasing (increasing) on (a, b) then $\frac{f}{g}$ is also decreasing (increasing) on (a, b) .*

Lemma 2.2. *For $p \in (1, 2]$ and for all $t \in (0, \frac{\pi_p}{2}]$, the function $\frac{\sin_p(t) - t\cos_p(t)}{t}$ increases strictly.*

Proof. Consider the function,

$$f(t) = \frac{\sin_p(t) - t\cos_p(t)}{t} = \frac{f_1(t)}{f_2(t)},$$

where $f_1(t) = \sin_p(t) - t\cos_p(t)$, $f_2(t) = t$ and clearly $f_1(0) = f_2(0) = 0$. Differentiation with respect to t we get,

$$\begin{aligned} \frac{f_1^{'}(t)}{f_2^{'}(t)} &= \cos_p(t) - \cos_p(t) + t\sin_p(t)\tan_p^{p-2}(t) \\ &= t\sin_p(t)\tan_p^{p-2}(t) > 0, \end{aligned}$$

which is increasing function, this implies that $\frac{f_1^{'}(t)}{f_2^{'}(t)}$ is strictly increasing. Hence, according to Lemma 2.1, the function $f(t)$ increases strictly. □

Lemma 2.3. *For all $t \in (0, \frac{\pi_p}{2})$ and $p \in [2, \infty)$, the function $f(t) = \frac{\tan_p(t) - t}{t - \tanh_p(t)}$ is strictly increasing.*

Proof. Consider the function,

$$g(t) = \frac{\tan_p(t) - t}{t - \tanh_p(t)} = \frac{g_1(t)}{g_2(t)},$$

where $g_1(t) = \tan_p(t) - t$ and $g_1(t) = t - \tanh_p(t)$ with $g_1(0) = g_2(0) = 0$. On Differentiation, we get

$$\begin{aligned} \frac{g_1^{'}(t)}{g_2^{'}(t)} &= \frac{\tan_p^p(t)}{\tanh_p^p(t)} \\ &= \tan_p^p(t)\coth_p^p(t), \end{aligned}$$

which is the increasing function, this implies that $\frac{g_1^{'}(t)}{g_2^{'}(t)}$ increases strictly. By Lemma 2.1, the function $g(t)$ also increases strictly. □

Lemma 2.4. *For $p \in (1, 2]$ and $t \in (0, \frac{\pi_p}{2}]$, the function $\frac{(p+\cos_p(t))(\tan_p(t)-t)}{t}$ is strictly increasing.*

Proof. Consider the function,

$$h(t) = \frac{(p + \cos_p(t))(\tan_p(t) - t)}{t} = \frac{h_1(t)}{h_2(t)},$$

where $h_1(t) = (\tan_p(t) - t)(p + \cos_p(t))$ and $h_2(t) = t$ with $h_1(0) = h_2(0) = 0$. On differentiation, we get

$$\begin{aligned} \frac{h_1^{'}(t)}{h_2^{'}(t)} &= p\tan_p^p(t) + \sin_p(t)\tan_p^{p-2}(t) \\ &= \sin_p(t)\tan_p^{p-2}(t)[p\tan_p(t)\sec_p(t) + 1], \end{aligned}$$

which is increasing for all $t \in (0, \frac{\pi_p}{2}]$, implies that $\frac{h_1^{'}(t)}{h_2^{'}(t)}$ strictly increasing. By Lemma 2.1, the function $h(t)$ is also strictly increasing. □

Lemma 2.5. *[1] Let $b_n > 0, n = 1, 2, \cdots, k$ then*

$$\frac{b_1 + b_2 + \cdots + b_k}{k} \geq \sqrt[k]{(1 + b_1)(1 + b_2)\cdots(1 + b_k)} - 1 \geq \sqrt[k]{b_1 b_2 \cdots b_k}.$$

Lemma 2.6. *[12] For $p > 0$ and $t \in (0, \frac{\pi_p}{2})$ the function $\frac{\tan_p(t)}{t}$ is strictly increasing.*

Lemma 2.7. *[12] For $p > 1$ and $t \in (0, \frac{\pi_p}{2}]$, the function $\frac{\sin_p(t)}{t}$ is strictly decreasing.*

Lemma 2.8. *For $p \in [2, \infty)$, the function $f(t) = \tan_p^{p-1}(t) - \tanh_p^{p-1}(t)$ is strictly increasing in $(0, \frac{\pi_p}{2})$.*

Proof. By differentiation, we get

$$f'(t) = (p-1)[\tan_p^{p-2}(t)(1+\tan_p^p(t)) - \tanh_p^{p-2}(t)(1-\tanh_p^p(t))].$$

Now, for $p \geq 2$,

$$f'(t) \geq (p-1)(\tan_p^{p-2}(t) - \tanh_p^{p-2}(t)) > 0,$$

since $\tan_p(t) > \tanh_p(t)$. Therefore, the function $f(t)$ is strictly increasing. □

3 Main Result:-1

Theorem 3.1. *For $p \in [2, \infty)$, $t \in (0, \infty)$ and $a_1 - pa_2 \leq 0$, $a_2 > 0$,*

$$\left(\frac{\sinh_p(t)}{t}\right)^{a_1} + \left(\frac{\tanh_p(t)}{t}\right)^{a_2} > 2. \tag{1}$$

Proof. From the A-G mean inequality as stated in [1] and [4],

$$\alpha^r + \beta^s \geq 2\alpha^{\frac{r}{2}}\beta^{\frac{s}{2}}.$$

We can write inequality (1) as,

$$\begin{aligned}
\left(\frac{\sinh_p(t)}{t}\right)^{a_1} + \left(\frac{\tanh_p(t)}{t}\right)^{a_2} &\geq 2\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1}{2}}\left(\frac{\tanh_p(t)}{t}\right)^{\frac{a_2}{2}} \\
&\geq 2\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1}{2}}\left(\frac{\sinh_p(t)}{t\cosh_p(t)}\right)^{\frac{a_2}{2}}
\end{aligned}$$

$$\implies \left(\frac{\sinh_p(t)}{t}\right)^{a_1} + \left(\frac{\tanh_p(t)}{t}\right)^{a_2} \geq 2\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1+a_2}{2}}\left(\frac{1}{\cosh_p(t)}\right)^{\frac{a_2}{2}} \tag{2}$$

using the inequality defined in [10] as,

$$\cosh_p^{\frac{1}{p+1}}(t) < \frac{\sinh_p(t)}{t}$$

i.e.

$$\frac{1}{\cosh_p(t)} > \left(\frac{\sinh_p(t)}{t}\right)^{-(p+1)},$$

we can write inequality (2) as,

$$\begin{aligned}
\left(\frac{\sinh_p(t)}{t}\right)^{a_1} + \left(\frac{\tanh_p(t)}{t}\right)^{a_2} &> 2\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1+a_2}{2}}\left(\frac{\sinh_p(t)}{t}\right)^{\frac{-(p+1)a_2}{2}} \\
&> 2\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1+a_2-pa_2-a_2}{2}} \\
&> 2\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1-pa_2}{2}}.
\end{aligned}$$

Since $a_1 - pa_2 \leq 0$, $a_2 > 0$, the above inequality becomes,

$$\left(\frac{\sinh_p(t)}{t}\right)^{a_1} + \left(\frac{\tanh_p(t)}{t}\right)^{a_2} > 2.$$

□

Inequality (1) holds for all $a_1, a_2 > 0$ and $p \in [2, \infty)$.

Example 3.1.1 If $p = 2$ and $a_1 = 0.1, a_2 = 0.1 > 0$ are such that $a_1 - pa_2 = -0.1 < 0$ with $t = 1$, then inequality (1) gives,

$$[\sinh_2(1)]^{0.1} + [\tanh_2(1)]^{0.1} = 1.1231 + 0.9731 = 2.0962 > 2.$$

Corollary 3.2. *If $a_1 = 2, a_2 = 1$ the inequality defined in Theorem 3.1 becomes*

$$\left(\frac{\sinh_p(t)}{t}\right)^2 + \left(\frac{\tanh_p(t)}{t}\right) > 2$$

which is the first Wilker's inequality [5].

Corollary 3.3. *If $a_1 = 2p, a_2 = p$ and $p \geq 2$ then the above inequality reduces to,*

$$\left(\frac{\sinh_p(t)}{t}\right)^{2p} + \left(\frac{\tanh_p(t)}{t}\right)^p > 2.$$

Theorem 3.4. *For $p \in [2, \infty)$, $t \in (0, \infty)$ and $a_1 - pa_2 \leq 0$, $a_2 > 0$,*

$$\left[1 + \left(\frac{\sinh_p(t)}{t}\right)^{a_1}\right]\left[1 + \left(\frac{\tanh_p(t)}{t}\right)^{a_2}\right] > 4. \tag{3}$$

Proof. For $k = 2$ the inequality defined in Lemma 2.5 becomes,

$$\sqrt{(1 + b_1)(1 + b_2)} \geq \sqrt{b_1 b_2} + 1$$
$$\Longrightarrow (1 + b_1)(1 + b_2) \geq \left(\sqrt{b_1 b_2} + 1\right)^2.$$

For $b_1 = \left(\frac{\sinh_p(t)}{t}\right)^{a_1}$ and $b_2 = \left(\frac{\tanh_p(t)}{t}\right)^{a_2}$ we get,

$$\begin{aligned}\left[1 + \left(\frac{\sinh_p(t)}{t}\right)^{a_1}\right]\left[1 + \left(\frac{\tanh_p(t)}{t}\right)^{a_2}\right] &\geq \left[\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1}{2}} \left(\frac{\tanh_p(t)}{t}\right)^{\frac{a_2}{2}} + 1\right]^2 \\ &\geq \left[\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1}{2}} \left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_2}{2}} \left(\frac{1}{\cosh_p(t)}\right)^{\frac{a_2}{2}} + 1\right]^2 \\ &\geq \left[\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1 + a_2}{2}} \left(\frac{1}{\cosh_p(t)}\right)^{\frac{a_2}{2}} + 1\right]^2.\end{aligned}$$

Using the inequality $\frac{1}{\cosh_p(t)} > \left(\frac{\sinh_p(t)}{t}\right)^{-(p+1)}$ we get,

$$\left[1 + \left(\frac{\sinh_p(t)}{t}\right)^{a_1}\right]\left[1 + \left(\frac{\tanh_p(t)}{t}\right)^{a_2}\right] > \left[\left(\frac{\sinh_p(t)}{t}\right)^{\frac{a_1 - pa_2}{2}} + 1\right]^2$$

Since $a_1 - pa_2 < 0$, $a_2 > 0$ and $\frac{\sinh_p(t)}{t} < 1$, the inequality reduces to

$$\left[1 + \left(\frac{\sinh_p(t)}{t}\right)^{a_1}\right]\left[1 + \left(\frac{\tanh_p(t)}{t}\right)^{a_2}\right] > 4.$$

□

Inequality (3) holds for all $a_1, a_2 > 0$ and $p \in [2, \infty)$.

Example 3.2.1 If $p = 2$ and $a_1 = 0.1, a_2 = 0.1 > 0$ are such that $a_1 - pa_2 = -0.1 < 0$ with $t = 1$, then inequality (3) gives:

$$[1 + (\sinh_2(1))^{0.1}][1 + (\tanh_2(1))^{0.1}] = (2.1231)(1.9731) = 4.1890 > 4.$$

We now present an alternate proof of the Cusa-type inequality [6] with the best possible bounds involving trigonometric functions which was previously proved by Huang et al. [1].

Theorem 3.5. *For $p \in (1, 2]$ and $t \in (0, \frac{\pi_p}{2}]$ the function $\frac{\ln\left(\frac{\sin_p(t)}{t}\right)}{\ln\left(\frac{p + \cos_p(t)}{p+1}\right)}$ is strictly increasing. In particular for all $p \in (1, 2]$ and $t \in (0, \frac{\pi_p}{2}]$,*

$$\left(\frac{p + \cos_p(t)}{p + 1}\right)^a < \frac{\sin_p(t)}{t} < \left(\frac{p + \cos_p(t)}{p + 1}\right)^b \tag{4}$$

with the best possible values of $a = 1$ and $b = \frac{\ln\left(\frac{2\sin_p(\frac{\pi_p}{2})}{\pi_p}\right)}{\ln\left(\frac{p + \cos_p(\frac{\pi_p}{2})}{p+1}\right)}$.

Proof. To prove this result, consider the function,

$$g(t) = \frac{\ln\left(\frac{\sin_p(t)}{t}\right)}{\ln\left(\frac{p+\cos_p(t)}{p+1}\right)} = \frac{g_1(t)}{g_2(t)},$$

where $g_1(t) = \ln\left(\frac{\sin_p(t)}{t}\right)$ and $g_2(t) = \ln\left(\frac{p+\cos_p(t)}{p+1}\right)$. Then by differentiation, we have

$$\begin{aligned}\frac{g_1^{'}(t)}{g_2^{'}(t)} &= \frac{[t\cos_p(t) - \sin_p(t)][p+\cos_p(t)]}{(-t)(\tan_p^{p-2}(t))\sin_p^2(t)}\\ &= \frac{[\tan_p(t) - t][p+\cos_p(t)]}{(t)\tan_p^{p-1}(t)\sin_p(t)}\\ &= \frac{1}{\tan_p^{p-1}(t)\sin_p(t)}\left[\frac{(\tan_p(t)-t)(p+\cos_p(t))}{t}\right].\end{aligned}$$

Now, by Lemma 2.4 and monotonicity of $\tan_p(t)$, $\sin_p(t)$, the ratio $\frac{g_1^{'}(t)}{g_2^{'}(t)}$ is strictly increasing on $t \in (0, \frac{\pi_p}{2}]$, $g^{'}(t)$ is strictly increases since, $g^{'}(t) > g^{'}(0) > 0$. Hence, using Lemma 2.1, the function $g(t)$ also strictly increasing.
The values of bounds follows from the l'Hospital rule,

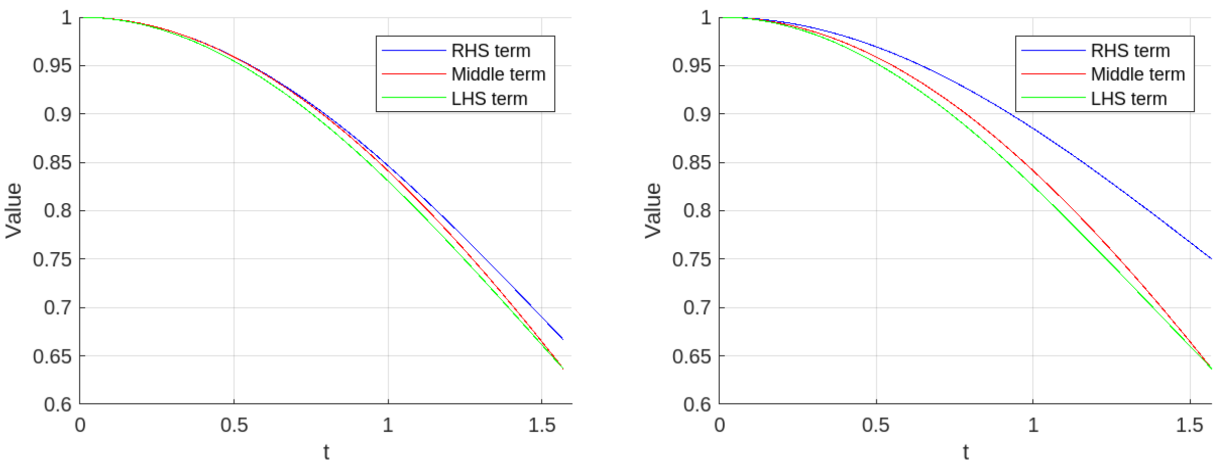
$$g(0^+) = 1$$

$$g(\frac{\pi_p}{2}) = \frac{\ln\left(\frac{2\sin_p(\frac{\pi_p}{2})}{\pi_p}\right)}{\ln\left(\frac{p+\cos_p(\frac{\pi_p}{2})}{p+1}\right)}.$$

It gives the inequality defined in (4) as,

$$\left(\frac{p+\cos_p(t)}{p+1}\right)^a < \frac{\sin_p(t)}{t} < \left(\frac{p+\cos_p(t)}{p+1}\right)^b$$

with best possible value of a and b. We can easily see in Figure 1 that the inequality holds for different values of parameter p.



(a) $p = 2$ **(b)** $p. = 3$

Figure 1: Inequalities for the different values of parameter p.

□

Corollary 3.6. *For $p = 2$ the inequality defined in (4) is reduced into the inequality,*

$$\frac{2+\cos_p(t)}{3} > \frac{\sin_p(t)}{t}$$

which is the Cusa type inequality.

4 Main Result:-2

In this section we proved the conjecture and particular cases posed by Huang et al. [1] and two conjectures posed by Riku Klen et al.[12]

Theorem 4.1 (Conjecture 4.1[1]). *For all* $t \in (0, \frac{\pi_p}{2}]$ *and* $p \in (1,2]$, *the function* $\frac{\ln(\frac{t}{\sin_p(t)})}{\ln(\cosh_p(t))}$ *is strictly increasing.*

Proof. To prove this result, consider the function,

$$h(t) = \frac{\ln(\frac{t}{\sin_p(t)})}{\ln(\cosh_p(t))} = \frac{h_1(t)}{h_2(t)},$$

where $h_1(t) = \ln(\frac{t}{\sin_p(t)})$ and $h_2(t) = \ln(\cosh_p(t))$ with $h_1(0) = 0 = h_2(0)$. On Differentiation we get,

$$\begin{aligned}
\frac{h_1^{'}(t)}{h_2^{'}(t)} &= \frac{(\sin_p(t) - (t)\cos_p(t))}{(t)\sin_p(t)\tanh_p^{p-1}(t)} \\
&= \frac{1}{\sin_p(t)\tanh_p^{p-1}(t)}\left[\frac{\sin_p(t) - (t)\cos_p(t)}{t}\right] \\
(\sin_p(t)\tanh_p^{p-1}(t))h_1^{'}(t) &= \frac{\sin_p(t) - t\cos_p(t)}{t},
\end{aligned}$$

which is positive since the monotonicity of $\sin_p(t), \tanh_p(t)$ and Lemma 2.2. This implies that $\frac{h_1^{'}(t)}{h_2^{'}(t)}$ is increasing. Using Lemma 2.1, the ratio $\frac{h_1(t)}{h_2(t)}$ also increases strictly for all $t \in (0, \frac{\pi_p}{2}]$. Therefore, the function $h(t)$ is strictly increasing. □

Theorem 4.2 (Conjecture (3.12) [12]). *For* $p \in [2, \infty)$ *and* $t \in (0, \frac{\pi_p}{2})$, *the function* $\frac{\ln(\frac{t}{\sin_p(t)})}{\ln(\frac{\sinh_p(t)}{t})}$ *is strictly increasing and*

$$\left(\frac{\sinh_p(t)}{t}\right)^{a_1} < \frac{t}{\sin_p(t)} < \left(\frac{\sinh_p(t)}{t}\right)^{a_2}$$

holds for the best possible constant $a_1 = 1$ *and* $a_2 = \frac{2\sinh_p(\frac{\pi_p}{2})}{\pi_p}$.

Proof. For the proof of result, consider the function,

$$f(t) = \frac{\ln(\frac{t}{\sin_p(t)})}{\ln(\frac{\sinh_p(t)}{t})} = \frac{f_1(t)}{f_2(t)},$$

where $f_1(t) = \ln(\frac{t}{\sin_p(t)})$ and $f_2(t) = \ln(\frac{\sinh_p(t)}{t})$ with $f_1(0) = 0 = f_2(0)$. Differentiation gives

$$\begin{aligned}
\frac{f_1^{'}(t)}{f_2^{'}(t)} &= \frac{\sinh_p(t)[\sin_p(t) - (t)\cos_p(t)]}{\sin_p(t)[(t)\cosh_p(t) - \sinh_p(t)]} \\
&= \left[\frac{\sinh_p(t)}{\sin_p(t)}\right].\left[\frac{\sin_p(t) - (t)\cos_p(t)}{(t)\cosh_p(t) - \sinh_p(t)}\right] \\
&= \left[\frac{\tanh_p(t)}{\tan_p(t)}\right].\left[\frac{\tan_p(t) - t}{t - \tanh_p(t)}\right] \\
(\tan_p(t)\coth_p(t))(\frac{f_1^{'}(t)}{f_2^{'}(t)}) &= \frac{\tan_p(t) - t}{t - \tanh_p(t)},
\end{aligned}$$

which is clearly increasing since $\tan_p(t) > \tanh_p(t)$ and by Lemma 2.3. This implies that $\frac{f_1^{'}(t)}{f_2^{'}(t)}$ is increasing. Hence, the function $f(t)$ is strictly increasing by Lemma 2.1. Using the l'Hospital rule, we easily find the best possible values of a_1 and a_2.

$$f(0) = a_1 = 1, f(\frac{\pi_p}{2}) = a_2 = \frac{2\sinh(\frac{\pi_p}{2})}{\pi_p}.$$

The inequality holds for all values of $p \in [2, \infty)$, for the particular value of $p = 2$, the inequality is shown in Figure 2. □

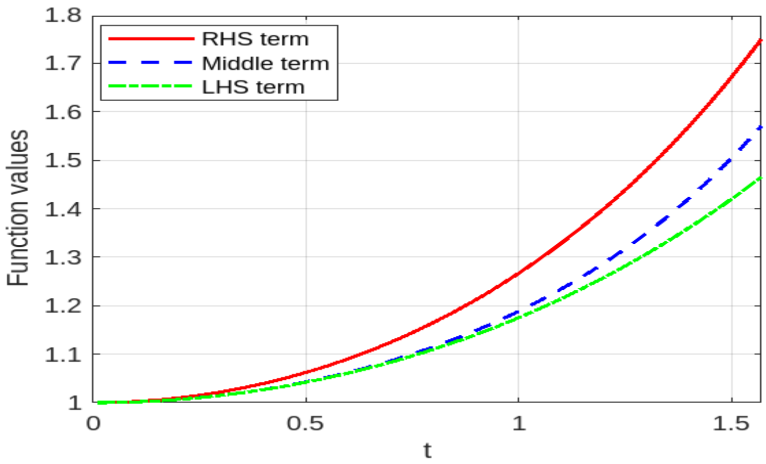


Figure 2: Inequality for the particular case $p = 2$

Theorem 4.3 (Conjecture 3.29 [12]). *For* $p \in [2, \infty)$ *and* $t \in (0, \frac{\pi_p}{2})$,

$$\frac{\sinh_p(t)}{t} < \frac{p+1}{p + \cos_p(t)}. \tag{5}$$

Proof. For the proof of inequality, let us take the function,

$$f(t) = (p+1)(t) - (\sin_p(t))(p + \cos_p(t)).$$

By Differentiation, we have

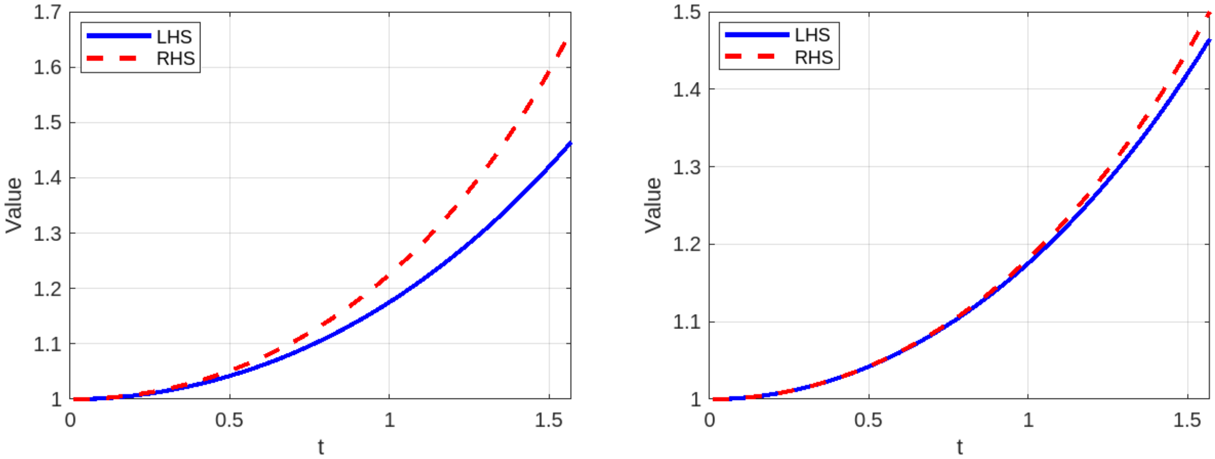
$$\begin{aligned} f'(t) &= (p+1) - \cosh_p(t)(p + \cos_p(t)) - \sinh_p(t)(\cos_p^{2-p}(t)\sin_p^{p-1}(t)) \\ &= (p+1) - \cosh_p(t)(p + \cos_p(t)) - \sinh_p(t)\sin_p(t)\tan_p^{p-2}(t), \end{aligned}$$

and

$$\begin{aligned} f''(t) = \; & \cosh_p(\sin_p(t)\tan_p^{p-2}(t)) - \sinh_p(t)\tanh_p^{p-2}(t)(p + \cos_p(t)) + \cos_p(t)\sinh_p(t)\tan_p^{p-2}(t) \\ & + \sin_p(t)\cosh_p(t)\tan_p^{p-2}(t) - (p-2)\sin_p(t)\sinh_p(t)\tan_p^{p-3}(t)\sec_p^p(t) \\ \geq \; & \cos_p(t)\sinh_p(t)[\tan_p^{p-2}(t) - \tanh_p^{p-2}(t)] + \sin_p(t)\cosh_p(t)\tan_p^{p-3}(t)[2\tan_p(t) + \\ & (p-2)\tanh_p(t)\sec_p^p(t)] \\ \geq \; & 0, \end{aligned}$$

where this inequality follows from the monotonicity of functions and Lemma 2.6, 2.8, now we can see that, $f'(t) > f'(0) > 0$. This implies that the function $f(t)$ is strictly increasing, which implies the inequality (5).
Figure 3 illustrate the inequalities define in (5) for particular value of p.

The inequality (5) holds for all values of $p \geq 2$ and it is more closer at $p = 2$ and the inequality will be more strict when $p \geq 3$. □



(a) $p = 3$ **(b)** $p = 2$

Figure 3: Inequalities for the particular values of p

5 Conclusion

In this paper, we provided proofs of the inequalities involving generalized hyperbolic and trigonometric functions with one parameter establishing the best possible bounds. Additionally, we made efforts to address the conjectures posed by researchers.

References

[1] Huang, L. G., Yin, L., Wang, Y. L. and Lin, X. L. (2018). Some Wilker and Cusa type inequalities for generalized trigonometric and hyperbolic functions. Journal of Inequalities and Applications, 2018, 1-8.

[2] Nantomah, K., Okpoti, C. A., and Nasiru, S. (2020). On a generalized sigmoid function and its properties. Asian J. Math. Appl, 2020.

[3] Nantomah, K., and Prempeh, E. (2020). Some inequalities for generalized hyperbolic functions. Moroccan Journal of Pure and Applied Analysis, 6(1), 76-92.

[4] Anderson, G. D., Vamanamurthy, M. K., and Vuorinen, M. (1997). Conformal invariants, inequalities, and quasiconformal maps.

[5] Alzer, H., and Qiu, S. L. (2004). Monotonicity theorems and inequalities for the complete elliptic integrals. Journal of Computational and Applied Mathematics, 172(2), 289-312.

[6] Gradshteyn, I. S., and Ryzhik, I. M. (2014). Table of integrals, series, and products. Academic press.

[7] Lindqvist, P. (1995). Some remarkable sine and cosine functions. *Ricerche Mat.*, 44, 269.

[8] Takeuchi, S. (2012). Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian. Journal of Mathematical Analysis and Applications, 385(1), 24-35.

[9] Xiaoyan Ma, Xiangbin Si, Zhong Genhong and Jianhui H.(2020) "Inequalities for the generalized trigonometric and hyperbolic functions" Open Mathematics, 18(1):1580-1589. https://doi.org/10.1515/math-2020-0096.

[10] Li Yin, Li-Guo Huang and and Feng Qi(2014). Some Inequalities for the Generalized Trigonometric and Hyperbolic Functions. Turkish Journal of Analysis and Number Theory. 2(3):96-101. doi: 10.12691/tjant-2-3-8.

[11] Wang, M. K., Hong, M. Y., Xu, Y. F., Shen, Z. H. and Chu, Y. M. (2020). Inequalities for generalized trigonometric and hyperbolic functions with one parameter. J. Math. Inequal, 14(1), 1-21.

[12] KlÃ©n, R., Vuorinen, M. and Zhang, X. (2014). Inequalities for the generalized trigonometric and hyperbolic functions. Journal of Mathematical Analysis and Applications, 409(1), 521-529.

[13] Yin, L., Huang, L. and Lin, X. (2021). Inequalities and bounds for the p-generalized trigonometric functions. AIMS Mathematics, 6(10), 11097-11108.

[14] Mohammed, F., William, R., Clemente, C., and Stiven, D. (2025). The 2-variable truncated Tricomi functions. Dolomites Research Notes on Approximation, 18(1), 49-55.