A Note on Generalized Degenerate $q$-Bernoulli and $q$-Euler Matrices

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Abstract

In this paper, we explore the concepts of generalized degenerate $q$-Bernoulli and $q$-Euler polynomial matrices, elucidating their fundamental properties. Our primary focus is on investigating inversion-type formulas and matrix inversion formulas that are interconnected with these matrices.

Keywords and phrases: Generalized degenerate $q$-Bernoulli polynomials; generalized degenerate $q$-Euler polynomials; generalized degenerate $q$-Bernoulli matrix; generalized degenerate $q$-Euler matrix; generalized degenerate $q$-Pascal matrix.

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1 Introduction

The term ‘quantum group’ was coined by the Fields Medalist V. G. Drinfel’d to describe a novel mathematical structure that first appeared in a technique for studying integrable systems in quantum field theory and statistical mechanics, known as the quantum inverse scattering method. Drinfel’d [7, 8] realized that the algebraic structure associated with the quantum inverse scattering method could be reproduced through a suitable algebraic quantization of Poisson Lie algebras. Similar relations were obtained by Jimbo [15] using a somewhat different approach. Today, quantum groups and their representations, closely related to well-known $q$-special functions, constitute a significant part of both mathematics and theoretical physics [31].

In this context, $q$-Pascal matrices [9, 10, 14, 27, 28, 34] can be used to construct specific representations of quantum groups. These representations are important as they provide a framework for understanding symmetries in quantum mechanics and quantum field theory in a non-commutative setting (see, for instance, [22]).

For example, the $q$-general linear group $GL_q(n, \mathbb{C})$ is a quantum group that arises by deforming the algebraic properties of the general linear group $GL(n, \mathbb{C})$ using a deformation parameter $q$. The $q$-special linear group $SL_q(n, \mathbb{C})$ is a subgroup of $GL_q(n, \mathbb{C})$ that plays a significant role in quantum algebra. It is possible to define certain quantum matrices related to Plücker coordinates in $SL_q(n, \mathbb{C})$. These quantum matrices have entries that are $q$-analogues of binomial coefficients and can be similar in nature to $q$-Pascal matrices. Furthermore, quantum matrices are used to construct representations of quantum groups like $SL_q(n, \mathbb{C})$, and often these matrices exhibit properties similar to those of $q$-Pascal matrices.

More recently, inversion formulas for various types of $q$-Pascal matrices, determinantal representations for polynomial sequences, identities involving $q$-Gaussian coefficients, and a novel general method of constructing $q$-analogues and other generalizations of Pascal-like matrices have been provided in [1, 29] (see also [11, 12] for some earlier approaches). Additionally, it has been proven that $q$-Pascal matrices allow the construction of $q$-analogues of certain Banach sequence spaces [32, 33].

Motivated by [23] and recent works such as [2, 3, 4, 5, 6, 13, 16, 19, 20, 30], we introduce a $\lambda$-degenerate deformation on $q$-Pascal matrices and provide corresponding factorizations for the generalized degenerate $q$-Bernoulli and $q$-Euler polynomial matrices, respectively. Furthermore, we present inversion-type formulas for the generalized degenerate $q$-Bernoulli and $q$-Euler polynomials. Furthermore, we show the inversion-type formulae for the generalized degenerate $q$-Bernoulli and $q$-Euler polynomials.

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The paper is organized as follows. Section 2 contains some notations, definitions, and properties of the q-analogs and some other auxiliary results which we will use throughout the paper. In Section 3, we present the corresponding inversion-type formulas for the generalized degenerate q-Bernoulli and q-Euler polynomials, respectively, and establish novel properties for the generalized degenerate q-Bernoulli and q-Euler matrices (see Theorems 3.1, 3.3, 3.5, and 3.6). Finally, we provide concluding remarks in Section 4.

2 Background and previous results

Throughout this paper, let \( \mathbb{N} \), \( \mathbb{N}_0 \), \( \mathbb{Z} \), \( \mathbb{R} \), and \( \mathbb{C} \) denote, respectively, the sets of natural numbers, non-negative integers, integers, real numbers, and complex numbers. As usual, we will always use the principal branch for complex powers, in particular, \( 1^\alpha = 1 \) for \( \alpha \in \mathbb{C} \). Furthermore, the convention \( 0^0 = 1 \) will be adopted.

For \( w \in \mathbb{C} \) and \( k \in \mathbb{Z} \), we use the notations \( w^{(k)} \) and \( (w)_k \) for the rising and falling factorials, respectively, i.e.,

\[
w^{(k)} = \begin{cases} 1, & \text{if } k = 0, \\
\prod_{i=1}^{k} (w + i - 1), & \text{if } k \geq 1, \\
0, & \text{if } k < 0,
\end{cases}
\]

and

\[
(w)_k = \begin{cases} 1, & \text{if } k = 0, \\
\prod_{i=1}^{k} (w - i + 1), & \text{if } k \geq 1, \\
0, & \text{if } k < 0.
\end{cases}
\]

Next, we introduce some q-notations that will be needed frequently. The q-shifted factorial \( (a; q)_n \) is defined by

\[
(a; q)_n := \begin{cases} 1, & n = 0, \\
\prod_{k=0}^{n-1} (1 - a q^k), & n \in \mathbb{N},
\end{cases}
\]

where \( a, q \in \mathbb{C} \) and it is assumed that \( a \neq q^{-m}, m \in \mathbb{N}_0 \). It is well known that there exists other notations for the q-shifted factorial (1), for instance, \( (a)_q, \lfloor a \rfloor_q \), and even \( (a)_i \), when the base \( q \) is understood. So, in order to avoid any ambiguity we only use the notation (1).

For any \( z, q \in \mathbb{C} \) such that \( q \neq 1 \) and \( q^z \neq 1 \), the q-number \( [z]_q \) is defined by (cf. [28])

\[
[z]_q := \frac{q^z - 1}{q - 1},
\]

with the convention \( [0]_q = 0 \).

In particular, the q-analogue of \( n \in \mathbb{N} \) is obtained from (2) taking \( z = n \), i.e.,

\[
[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.
\]

The q-analogue of \( n! \) is then defined by

\[
[n]_q! := \begin{cases} 1, & \text{if } n = 0, \\
[n]_q [n-1]_q \cdots [2]_q [1]_q, & \text{if } n \in \mathbb{N},
\end{cases}
\]

from which the q-binomial coefficient is given by

\[
\binom{n}{k}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!}, \quad n, k \in \mathbb{N}_0; \ 0 \leq k \leq n.
\]

Any matrix is assumed an element of \( M_{n+1} (\mathbb{R}) \), the set of all \( (n+1) \)-square matrices over the real field \( \mathbb{R} \). Moreover, for \( i, j \), any nonnegative integers, and any matrix \( A \in M_{n+1} (\mathbb{R}) \) we adopt, respectively, the following conventions

\[
\begin{pmatrix} i \\ j \end{pmatrix}_q = 0, \text{ whenever } j > i, \quad \text{and} \quad A^0 = I_{n+1} = \text{diag}(1, 1, \ldots, 1),
\]

where \( I_{n+1} \) denotes the identity matrix of order \( n + 1 \).

From now on, the constraint \( |q| < 1 \) will be tacitly assumed. For \( \lambda, x \in \mathbb{R} \), the degenerate q-exponentials are defined as follows (cf. [20]):

\[
\exp_q^\lambda (x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{[n]_q} x^n,
\]
where the generalized falling factorials \((x)_{n,\lambda}\), are given by (cf. [16, 17, 18, 19, 20, 21]):

\[
(x)_{n,\lambda} = \begin{cases} 
1, & \text{if } n = 0, \\
\prod_{i=1}^{n} (x - (i-1)\lambda), & \text{if } n \geq 1, \\
0, & \text{if } n < 0.
\end{cases}
\]

It is clear that \(\lim_{\lambda \to 0} e_{q,\lambda}^x(z) = e_{q,0}^x(z)\). Furthermore, \(e_{q,\lambda}^x(z)\) coincides with a \(q\)-analogue of the classical exponential function [28, Equation (7)].

The degenerate \(q\)-exponentials (3) do not satisfy the exponents product law like the exponentials functions, i.e.,

\[
e_{q,\lambda}^{xy}(z) \neq e_{q,\lambda}^x(z)e_{q,\lambda}^y(z).
\]

For \(x, \lambda \in \mathbb{R}\), we also consider, respectively, the \(\lambda\)-binomial and the degenerate \(q\)-binomial coefficients as follows [20, Equations (9), (27)]:

\[
\binom{x}{n}_{\lambda, q} = \begin{cases} 
\binom{n}{x}_{q,\lambda}, & \text{if } n \geq 1, \\
0, & \text{if } n < 0,
\end{cases}
\]

\[
\binom{x}{n}_{\lambda, q} = \begin{cases} 
\binom{n}{x}_{q,\lambda}, & \text{if } n \geq 1, \\
0, & \text{if } n < 0.
\end{cases}
\]

Since the following expression holds (cf. [13, Equation (7)])

\[
(x + y)_{n,\lambda} = \sum_{k=0}^{n} \binom{n}{k}_{\lambda, q} (x)_{k,\lambda} (y)_{n-k,\lambda}, \quad n \geq 0,
\]

it is straightforward that ([20, Equation (10)])

\[
\binom{x + y}{n}_{\lambda} = \sum_{k=0}^{n} \binom{x}{k}_{\lambda, q} \binom{y}{n-k}_{\lambda, q}, \quad n \geq 0.
\]

Finally, the connection between the \(\lambda\)-binomial and the degenerate \(q\)-binomial coefficients is given by

\[
\binom{x}{n}_{\lambda, q} = \frac{[n]_{\lambda}!}{n!} \binom{x}{n}_{q,\lambda}.
\]

For \(r \in \mathbb{N}, |q| < 1\) and \(|x| < |1 - q|^{-1}\), we consider the degenerate \(q\)-Bernoulli and \(q\)-Euler polynomials of order \(r\) as follows [20]:

\[
\left(\frac{z}{e_{q,\lambda}(z) - 1}\right)^r e_{q,\lambda}^x(z) = \sum_{a=0}^{\infty} B_{n,a,q}^{(r)}(x) \frac{z^n}{[n]_{\lambda}!}, \quad |z| < 1,
\]

\[
\left(\frac{2}{e_{q,\lambda}(z) + 1}\right)^r e_{q,\lambda}^x(z) = \sum_{a=0}^{\infty} C_{n,a,q}^{(r)}(x) \frac{z^n}{[n]_{\lambda}!}, \quad |z| < 1.
\]

These represent degenerate versions of the \(q\)-analogue of the classical Bernoulli and Euler polynomials, respectively. As usual, when \(x = 0\), \(B_{n,a,q}(0) = B_{n,a,q}^{(r)}\) and \(C_{n,a,q}(0) = C_{n,a,q}^{(r)}\) are called respectively, the degenerate \(q\)-Bernoulli and \(q\)-Euler numbers of order \(r\).

When \(r = 1\), \(B_{n,a,q}^{(1)}(x) = B_{n,a,q}(x)\) \((\epsilon_{n,a,q}^{(1)}(x) = \epsilon_{n,a,q}(x))\) which are the degenerate \(q\)-Bernoulli (q-Euler) polynomials and notice that \(\lim_{\lambda \to 0} B_{n,a,q}^{(r)}(x) = B_{n,a,q}^{(r)}(x)\) \(\lim_{\lambda \to 0} \epsilon_{n,a,q}^{(r)}(x) = \epsilon_{n,a,q}^{(r)}(x))\), which are \(q\)-Bernoulli (q-Euler) polynomials of order \(r\). Finally, \(B_{n,a,q}^{(0)}(x) = (x)_{n,\lambda} = \epsilon_{n,a,q}(x)\).
In [20] is proved the following addition formula:

\[
\binom{x + y}{n}_{q,\lambda} = \sum_{k=0}^{n} \binom{n}{k}_{q,\lambda} \binom{x}{k}_{q,\lambda} \binom{y}{n-k}_{q,\lambda}.
\]

### 3 Generalized degenerate \(q\)-Bernoulli and \(q\)-Euler matrices and their properties

Inversion formulae are typically used to compute the coefficients of a generating function or to count specific combinatorial structures. In contrast, inversion-type formulae are similar to the former but may involve more complex operations or dependencies on multiple parameters. In the context of generalized degenerate \(q\)-Pascal matrices, inversion-type formulae allow us to factorize these matrices in terms of degenerate \(q\)-Bernoulli (\(q\)-Euler) matrices. In this section, we present some novel properties for the generalized degenerate \(q\)-Bernoulli and \(q\)-Euler matrices. Before that, we demonstrate the corresponding inversion-type formulas for the generalized degenerate \(q\)-Bernoulli and \(q\)-Euler polynomials, respectively.

**Theorem 3.1.** For every \(n \geq 0\), \(\lambda \in \mathbb{R}\) and \(r = 1\), the degenerate \(q\)-Bernoulli polynomials satisfy the following inversion-type formula:

\[
\binom{x}{n}_{q,\lambda} = \frac{1}{[n+1]_q!} \sum_{k=0}^{n} \binom{n+1}{k+1}_{q,\lambda} (1)_{k+1,\lambda} B_{n-k,\lambda}(x) \tag{7}
\]

\[
\binom{x}{n}_{q,\lambda} = \frac{1}{[n+1]_q!} \sum_{k=0}^{n} \binom{n+1}{k+1}_{q,\lambda} (1-\lambda)_{k,\lambda} B_{n-k,\lambda}(x). \tag{8}
\]

**Proof.** Let \(\lambda \in \mathbb{R}\). In view of (3) and (5), and the identity

\[
x \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{z^n}{[n]_q!} = \sum_{n=0}^{\infty} [n+1]_{q,\lambda} (x)_{n,\lambda} \frac{z^{n+1}}{[n+1]_q!},
\]

we have

\[
\sum_{n=0}^{\infty} [n+1]_{q,\lambda} (x)_{n,\lambda} \frac{z^{n+1}}{[n+1]_q!} = \left[ \sum_{n=0}^{\infty} (1)_{n,\lambda} \frac{z^n}{[n]_q!} - 1 \right] \left[ \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{z^n}{[n]_q!} \right].
\]

\[
= \left[ \sum_{n=0}^{\infty} (1)_{n,\lambda+1} \frac{z^{n+1}}{[n+1]_q!} \right] \left[ \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{z^n}{[n]_q!} \right]. \tag{9}
\]

From the use of the Cauchy product rule on the right-hand side of (9), it follows that

\[
\sum_{n=0}^{\infty} [n+1]_{q,\lambda} (x)_{n,\lambda} \frac{z^{n+1}}{[n+1]_q!} = \sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{n+1}{k+1}_{q,\lambda} (1)_{k+1,\lambda} B_{n-k,\lambda}(x) \frac{z^{n+1}}{[n+1]_q!}. \tag{10}
\]

Hence, comparing the coefficients of \(z^{n+1}\) on both sides of (10) and using the identity (4), we obtain (7).

Finally, (8) is a simple consequence of the identity \( (1)_{k+1,\lambda} = (1-\lambda)_{k,\lambda} \), for all \(k \in \mathbb{N}_0\). \(\square\)

**Example 3.1.** The first three degenerate \(q\)-Bernoulli polynomials are

\[B_{0,\lambda}(x) = 1, \quad B_{1,\lambda}(x) = x, \quad B_{2,\lambda}(x) = x^2 - x + \frac{\lambda - 1}{[2]_q},\]

\[B_{3,\lambda}(x) = x^3 - 3x^2 + \frac{\lambda^2 - 2\lambda + 1}{[3]_q} - 2\lambda^2 - 3\lambda + 1.\]

**Remark 1.** Notice that the substitution of \(\lambda = 0\) into (7) recovers the classical \(q\)-Bernoulli polynomials (cf. [14, 24, 25]).

**Corollary 3.2.** For \(n \in \mathbb{N}_0\) and \(\lambda \in \mathbb{R}\), the matrix \(T(\lambda)(x) = [1 \quad (x)_{1,\lambda} \cdots (x)_{n,\lambda}]^T\) can be expressed as follows:
Theorem 3.3. For every \( n \geq 0 \) and \( \lambda \in \mathbb{R} \) and \( r = 1 \), the degenerate q-Euler polynomials satisfy the following inversion-type formula:

\[
\binom{x}{n}_{\lambda,q} = \frac{1}{2[n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q (1 + a_k)(1)_k \lambda \mathcal{E}^{n-k}_q,(\lambda,x),
\]

where

\[
a_k = \begin{cases} 
1, & \text{if } k = 0, \\
0, & \text{if } 1 \leq k \leq n.
\end{cases}
\]

Proof. From (3) and (6) we have

\[
2 \sum_{n=0}^{\infty} (x)_n \lambda \frac{z^n}{[n]_q!} = \sum_{k=0}^{\infty} \left( \sum_{n=0}^{\infty} (1)_n \lambda \frac{z^n}{[n]_q!} \right)^k + 1 \left( \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x) \frac{z^n}{[n]_q!} \right)
\]

\[
= \sum_{k=0}^{\infty} (1 + a_k)(1)_k \lambda \left( \sum_{n=0}^{\infty} \mathcal{E}_{n,\lambda}(x) \frac{z^n}{[n]_q!} \right)^k.
\]

Therefore, by comparing the coefficients of \( z^n \) on both sides and using the identity (4), we obtain the identity (11).

Theorem 3.3 has the following consequence.

Corollary 3.4. For \( n \in \mathbb{N}_0 \) and \( \lambda \in \mathbb{R} \), the matrix \( T_\lambda(x) = \begin{pmatrix} (x)_1 \lambda & \cdots & (x)_n \lambda \end{pmatrix} \) can be expressed as follows:
Remark

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Definition 3.1. Let \( E \in \mathbb{R}^{(q)} \) be any nonzero real number. For \( \lambda \in \mathbb{R} \) and \( |q| < 1 \), the generalized degenerate \( q \)-Pascal matrix of first kind \( P_{q,\lambda}[x] \), is an \((n + 1) \times (n + 1)\) matrix whose entries are given by

\[
P_{i,j,q,\lambda}(x) := \begin{cases} \frac{1}{q} \binom{x}{i-j,q,\lambda}, & i \geq j, \\ 0, & \text{otherwise} \end{cases} \tag{12}
\]

Remark 2.

(i) It is clear that the matrix \( P_{q,\lambda}[x] \) tends to the \( q \)-Pascal matrix of first kind \( P_{\lambda}[x] \) as \( \lambda \to 0 \) (cf. [10] (Equation (8))).

(ii) It is worth mentioning that \( P_{q,\lambda}[x] \) is a lower triangular matrix with nonnull determinant and hence, it is a nonsingular matrix.

(iii) The identity (4) says us that the entries of the generalized degenerate \( q \)-Pascal matrix of first kind in (12) can be written as

\[
P_{i,j,q,\lambda}(x) = \frac{\lfloor x \rfloor_{q,\lambda}!}{\lfloor i-j \rfloor_{q,\lambda}!} \binom{x}{i-j,q,\lambda}, \quad i \geq j,
\]

\[
0, \quad \text{otherwise}.
\]

(iv) For \( x, y \in \mathbb{R} \), the generalized degenerate \( q \)-Pascal matrix do not satisfy the addition law like the generalized degenerate Pascal matrix, i.e.,

\[
P_{q,\lambda}[x + y] \neq P_{q,\lambda}[x]P_{q,\lambda}[y].
\]

(v) If the convention \((0)_{0,\lambda} = 1\) is adopted, then it is possible to define

\[
P_{q,\lambda}[0] := I_{n+1}.
\]

Definition 3.2. The generalized degenerate \((n + 1) \times (n + 1)\) \( q \)-Bernoulli matrix \( B_{q,\lambda}(x) \) of real order \( r \) is defined by the entries

\[
B_{i,j,q,\lambda}(x) = \begin{cases} \frac{i!}{j!} B_{i-j,q,\lambda}(x), & i \geq j, \\ 0, & \text{otherwise} \end{cases}
\]

Remark 3.

(i) We denote by \( B_{q,\lambda}(x) \) the degenerate \( q \)-Bernoulli matrix \( B_{q,\lambda}(x) \).
Definition 3.2 and the inversion-type formula (7) lead to the following result:

**Theorem 3.5.** The generalized degenerate q-Pascal matrix of the first kind $P_{q,\lambda}[x]$ can be factorized in terms of $B_{q,\lambda}(x)$ as follows:

$$P_{q,\lambda}[x] = B_{q,\lambda}(x)\mathcal{H}_{q,\lambda},$$

(13)

where $\mathcal{H}_{q,\lambda}$ is an $(n+1) \times (n+1)$ invertible matrix with entries

$$\mathcal{H}_{i,j,q,\lambda} = \begin{cases} \frac{(1)}{q} \begin{pmatrix} i \end{pmatrix}_{q} \frac{(1)}{q} \begin{pmatrix} i-j+1 \end{pmatrix}_{q}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let us consider $n \in \mathbb{N}_{0}$ and $0 \leq i, j \leq n$ such that $i \geq j$. From Definition 3.2 and the inversion-type formula (7), we have

$$P_{i,j,q,\lambda}(x) = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] B_{0,q,\lambda}(x) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right) B_{0,q,\lambda}(x).$$

Since the right-hand member of (14) is the $(i,j)$-th entry of the matrix product $B_{q,\lambda}(x)\mathcal{H}_{q,\lambda}$, we conclude that (13) holds. Notice that the matrix $\mathcal{H}_{q,\lambda}$ is a lower triangular matrix with nonnull determinant and hence, it is a nonsingular matrix.

The following example shows the validity of Theorem 3.5.

**Example 3.2.** Let us consider $n = 2$. It follows from Definition 3.1, (13), and a simple computation that

$$P_{q,\lambda}[x] = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right] B_{0,q,\lambda}(x) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \right) B_{0,q,\lambda}(x).$$

Definition 3.3. The generalized degenerate $(n+1) \times (n+1)$ q-Euler matrix $\mathcal{E}^{(1)}_{q,\lambda}(x)$ is defined by the entries

$$\mathcal{E}^{(1)}_{i,j,q,\lambda}(x) = \begin{cases} \frac{(1)}{q} \begin{pmatrix} i \end{pmatrix}_{q} \frac{(1)}{q} \begin{pmatrix} i-j+1 \end{pmatrix}_{q} \lambda, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{E}_{q,\lambda}(x)$ the degenerate q-Euler matrix $\mathcal{E}^{(1)}_{q,\lambda}(x)$.

Definition 3.3 and the inversion-type formula (11) lead to the following result:

**Theorem 3.6.** The generalized degenerate q-Pascal matrix of the first kind $P_{q,\lambda}[x]$ can be factorized in terms of $\mathcal{E}_{q,\lambda}(x)$ as follows:

$$P_{q,\lambda}[x] = \mathcal{E}_{q,\lambda}(x)\mathcal{G}_{q,\lambda},$$

(15)

where $\mathcal{G}_{q,\lambda}$ is an $(n+1) \times (n+1)$ invertible matrix with entries

$$\mathcal{G}_{i,j,q,\lambda} = \begin{cases} \frac{(1)}{q} \begin{pmatrix} i \end{pmatrix}_{q} \frac{(1)}{q} \begin{pmatrix} i-j+1 \end{pmatrix}_{q}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$
Corollary 3.7. For any \( \lambda, x \in \mathbb{R} \), we have
\[
\mathcal{E}_{q,\lambda}(x) = B_{q,\lambda}(x)\mathcal{W}_{q,\lambda}\mathcal{W}_{q,\lambda}^{-1}.
\]

4 Conclusion

Diverse kinds of \( q \)-Pascal matrices can be used to construct certain representations of quantum groups. These representations are essential for understanding symmetries in quantum mechanics and quantum field theory in a non-commutative setting.

The aim of our research was to determine some novel properties of generalized degenerate \( q \)-Bernoulli and \( q \)-Euler polynomials and their matrices. Firstly, we focused our attention on some inversion-type formulae for the generalized degenerate \( q \)-Bernoulli and \( q \)-Euler polynomials and their matrices. Secondly, we introduced the generalized degenerate \( q \)-Pascal matrix of the first kind and provided factorizations for the generalized degenerate \( q \)-Bernoulli and \( q \)-Euler polynomial matrices in terms of the generalized degenerate \( q \)-Pascal matrix of the first kind.

Finally, it is noteworthy that under the suitable constraints of parameters associated with the generalized Apostol-type polynomial matrices given in [26], it is possible to provide a \( \lambda \)-degenerate deformation for some \( q \)-analogues of these matrices. The proof of this statement is not provided here; the interested reader is strongly encouraged to follow the above arguments to prove it.

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