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A Note on Generalized Degenerate q-Bernoulli and q-Euler Matrices

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Abstract

In this paper, we explore the concepts of generalized degenerate q-Bernoulli and q-Euler polynomial matrices, elucidating their fundamental properties. Our primary focus is on investigating inversion-type formulas and matrix inversion formulas that are interconnected with these matrices.

Keywords and phrases: Generalized degenerate q-Bernoulli polynomials; generalized degenerate q-Euler polynomials; generalized degenerate q-Bernoulli matrix; generalized degenerate q-Euler matrix; generalized degenerate q-Pascal matrix.

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1 Introduction

The term `'quantum group'` was coined by the Fields Medalist V. G. Drinfel'd to describe a novel mathematical structure that first appeared in a technique for studying integrable systems in quantum field theory and statistical mechanics, known as the quantum inverse scattering method. Drinfel'd [7, 8] realized that the algebraic structure associated with the quantum inverse scattering method could be reproduced through a suitable algebraic quantization of Poisson Lie algebras. Similar relations were obtained by Jimbo [15] using a somewhat different approach. Today, quantum groups and their representations, closely related to well-known q-special functions, constitute a significant part of both mathematics and theoretical physics [31].

In this context, q-Pascal matrices [9, 10, 14, 27, 28, 34] can be used to construct specific representations of quantum groups. These representations are important as they provide a framework for understanding symmetries in quantum mechanics and quantum field theory in a non-commutative setting (see, for instance, [22]).

For example, the q-general linear group $GL_q(n,\mathbb{C})$ is a quantum group that arises by deforming the algebraic properties of the general linear group $GL(n,\mathbb{C})$ using a deformation parameter q. The q-special linear group $SL_q(n,\mathbb{C})$ is a subgroup of $GL_q(n,\mathbb{C})$ that plays a significant role in quantum algebra. It is possible to define certain quantum matrices related to Plücker coordinates in $SL_q(n,\mathbb{C})$. These quantum matrices have entries that are q-analogues of binomial coefficients and can be similar in nature to q-Pascal matrices. Furthermore, quantum matrices are used to construct representations of quantum groups like $SL_q(n,\mathbb{C})$, and often these matrices exhibit properties similar to those of q-Pascal matrices.

More recently, inversion formulas for various types of q-Pascal matrices, determinantal representations for polynomial sequences, identities involving q-Gaussian coefficients, and a novel general method of constructing q-analogues and other generalizations of Pascal-like matrices have been provided in [1, 29] (see also [11, 12] for some earlier approaches). Additionally, it has been proven that q-Pascal matrices allow the construction of q-analogues of certain Banach sequence spaces [32, 33].

Motivated by [23] and recent works such as [2, 3, 4, 5, 6, 13, 16, 19, 20, 30], we introduce a λ-degenerate deformation on q-Pascal matrices and provide corresponding factorizations for the generalized degenerate q-Bernoulli and q-Euler polynomial matrices, respectively. Furthermore, we present inversion-type formulas for the generalized degenerate q-Bernoulli and q-Euler polynomials. Furthermore, we show the inversion-type formulae for the generalized degenerate q-Bernoulli and q-Euler polynomials.

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The paper is organized as follows. Section 2 contains some notations, defnitions, and properties of the q-analogs and some other auxiliary results which we will use throughout the paper. In Section 3, we present the corresponding inversion-type formulas for the generalized degenerate q-Bernoulli and q-Euler polynomials, respectively, and establish novel properties for the generalized degenerate q-Bernoulli and q-Euler matrices (see Theorems 3.1, 3.3, 3.5, and 3.6). Finally, we provide concluding remarks in Section 4.

2 Background and previous results

Throughout this paper, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ denote, respectively, the sets of natural numbers, non-negative integers, integers, real numbers, and complex numbers. As usual, we will always use the principal branch for complex powers, in particular, $1^\alpha = 1$ for $\alpha \in \mathbb{C}$. Furthermore, the convention $0^0 = 1$ will be adopted.

For $w \in \mathbb{C}$ and $k \in \mathbb{Z}$, we use the notations $w^{(k)}$ and $(w)_k$ for the rising and falling factorials, respectively, i.e.,

$$w^{(k)} = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^{k}(w+i-1), & \text{if } k \geq 1, \\ 0, & \text{if } k < 0, \end{cases}$$

and

$$(w)_k = \begin{cases} 1, & \text{if } k = 0, \\ \prod_{i=1}^{k}(w-i+1), & \text{if } k \geq 1, \\ 0, & \text{if } k < 0. \end{cases}$$

Next, we introduce some q-notations that will be needed frequently. The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n := \begin{cases} 1, & n = 0, \\ \prod_{k=0}^{n-1}\left(1-aq^k\right), & n \in \mathbb{N}, \end{cases} \tag{1}$$

where $a, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$, $m \in \mathbb{N}_0$. It is well know that there exists other notations for the q-shifted factorial (1), for instance, $(a)_{q,n}$, $[a]_n$, and even $(a)_n$, when the base q is understood. So, in order to avoid any ambiguity we only use the notation (1).

For any $z, q \in \mathbb{C}$ such that $q \neq 1$ and $q^z \neq 1$, the q-number $[z]_q$ is defined by (cf. [28])

$$[z]_q := \frac{q^z - 1}{q - 1}, \tag{2}$$

with the convention $[0]_q = 0$.

In particular, the q-analogue of $n \in \mathbb{N}$ is obtained from (2) taking $z = n$, i.e.,

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

The q-analogue of $n!$ is then defined by

$$[n]_q! := \begin{cases} 1, & \text{if } n = 0, \\ [n]_q[n-1]_q \cdots [2]_q[1]_q, & \text{if } n \in \mathbb{N}, \end{cases}$$

from which the q-binomial coefficient is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}, \quad n, k \in \mathbb{N}_0;\ 0 \leqq k \leqq n.$$

Any matrix is assumed an element of $M_{n+1}(\mathbb{R})$, the set of all $(n+1)$-square matrices over the real field $\mathbb{R}$. Moreover, for i, j, any nonnegative integers, and any matrix $A \in M_{n+1}(\mathbb{R})$ we adopt, respectively, the following conventions

$$\begin{bmatrix} i \\ j \end{bmatrix}_q = 0, \text{ whenever } j > i, \quad \text{and} \quad A^0 = I_{n+1} = \operatorname{diag}(1, 1, \ldots, 1),$$

where I_{n+1} denotes the identity matrix of order $n+1$.

From now on, the constraint $|q| < 1$ will be tacitly assumed. For $\lambda, x \in \mathbb{R}$, the degenerate q-exponentials are defined as follows (cf. [20]):

$$e^x_{q,\lambda}(z) = \begin{cases} \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{z^n}{[n]_q!}, & |z|<1, \quad \text{if } \lambda \in \mathbb{R}\setminus\{0\}, \\ \sum_{n=0}^{\infty} x^n \frac{z^n}{[n]_q!}, & |z|<1, \quad \text{if } \lambda = 0, \end{cases} \tag{3}$$

where the generalized falling factorials $(x)_{n,\lambda}$, are given by (cf. [16, 17, 18, 19, 20, 21]):

$$(x)_{n,\lambda} = \begin{cases} 1, & \text{if } n=0, \\ \prod_{i=1}^{n}(x-(i-1)\lambda), & \text{if } n \geq 1, \\ 0, & \text{if } n<0. \end{cases}$$

It is clear that $\lim_{\lambda\to 0} e^x_{q,\lambda}(z) = e^x_{q,0}(z)$. Furthermore, $e^1_{q,0}(z)$ coincides with a q-analogue of the classical exponential function [28, Equation (7)].

The degenerate q-exponentials (3) do not satisfy the exponents product law like the exponentials functions, i.e.,

$$e^{x+y}_{q,\lambda}(z) \neq e^x_{q,\lambda}(z)\, e^y_{q,\lambda}(z).$$

For $x, \lambda \in \mathbb{R}$, we also consider, respectively, the λ-binomial and the degenerate q-binomial coefficients as follows [20, Equations (9), (27)]:

$$\begin{aligned} \binom{x}{n}_{\lambda} &= \begin{cases} 1, & \text{if } n=0, \\ \frac{(x)_{n,\lambda}}{n!}, & \text{if } n\geq 1, \\ 0, & \text{if } n<0, \end{cases} \\ \left(\begin{array}{c} x \\ n \end{array}\right)_{q,\lambda} &= \begin{cases} 1, & \text{if } n=0, \\ \frac{(x)_{n,\lambda}}{[n]_q!}, & \text{if } n\geq 1, \\ 0, & \text{if } n<0. \end{cases} \end{aligned} \tag{4}$$

Since the following expression holds (cf. [13, Equation (7)])

$$(x+y)_{n,\lambda} = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) (x)_{k,\lambda}(y)_{n-k,\lambda}, \quad n \geq 0,$$

it is straightforward that ([20, Equation (10)])

$$\binom{x+y}{n}_{\lambda} = \sum_{k=0}^{n} \binom{x}{k}_{\lambda} \binom{y}{n-k}_{\lambda}, \quad n \geq 0.$$

Finally, the connection between the λ-binomial and the degenerate q-binomial coefficients is given by

$$\binom{x}{n}_{\lambda} = \frac{[n]_q!}{n!} \left(\begin{array}{c} x \\ n \end{array}\right)_{q,\lambda}.$$

For $r \in \mathbb{N}$, $|q|<1$ and $|x|<|1-q|^{-1}$, we consider the degenerate q-Bernoulli and q-Euler polynomials of order r as follows [20]:

$$\left(\frac{z}{e_{q,\lambda}(z)-1}\right)^r e^x_{q,\lambda}(z) = \sum_{n=0}^{\infty} \mathcal{B}^{(r)}_{n,q,\lambda}(x) \frac{z^n}{[n]_q!}, \quad |z|<1, \tag{5}$$

$$\left(\frac{2}{e_{q,\lambda}(z)+1}\right)^r e^x_{q,\lambda}(z) = \sum_{n=0}^{\infty} \mathcal{E}^{(r)}_{n,q,\lambda}(x) \frac{z^n}{[n]_q!}, \quad |z|<1. \tag{6}$$

These represent degenerate versions of the q-analogue of the classical Bernoulli and Euler polynomials, respectively. As usual, when $x=0$, $\mathcal{B}^{(r)}_{n,q,\lambda}(0) = \mathcal{B}^{(r)}_{n,q,\lambda}$ and $\mathcal{E}^{(r)}_{n,q,\lambda}(0) = \mathcal{E}^{(r)}_{n,q,\lambda}$ are called respectively, the degenerate q-Bernoulli and q-Euler numbers of order r.

When $r=1$, $\mathcal{B}^{(1)}_{n,q,\lambda}(x) = \mathcal{B}_{n,q,\lambda}(x)$ $\;(\mathcal{E}^{(1)}_{n,q,\lambda}(x) = \mathcal{E}_{n,q,\lambda}(x))$ which are the degenerate q-Bernoulli (q-Euler) polynomials and notice that $\lim_{\lambda\to 0} \mathcal{B}^{(r)}_{n,q,\lambda}(x) = \mathcal{B}^{(r)}_{n,q}(x)$ $\;(\lim_{\lambda\to 0} \mathcal{E}^{(r)}_{n,q,\lambda}(x) = \mathcal{E}^{(r)}_{n,q}(x))$, which are q-Bernoulli (q-Euler) polynomials of order r. Finally, $\mathcal{B}^{(0)}_{n,q,\lambda}(x) = (x)_{n,\lambda} = \mathcal{E}^{(0)}_{n,q,\lambda}(x)$.

In [20] is proved the following addition formula:

$$\binom{x+y}{n}_{q,\lambda}=\sum_{k=0}^{n}\frac{\binom{n}{k}}{\left[\begin{array}{c}n\\k\end{array}\right]_q}\binom{x}{k}_{q,\lambda}\binom{y}{n-k}_{q,\lambda}.$$

3 Generalized degenerate q-Bernoulli and q-Euler matrices and their properties

Inversion formulae are typically used to compute the coefficients of a generating function or to count specific combinatorial structures. In contrast, inversion-type formulae are similar to the former but may involve more complex operations or dependencies on multiple parameters. In the context of generalized degenerate q-Pascal matrices, inversion-type formulae allow us to factorize these matrices in terms of degenerate q-Bernoulli (q-Euler) matrices. In this section, we present some novel properties for the generalized degenerate q-Bernoulli and q-Euler matrices. Before that, we demonstrate the corresponding inversion-type formulas for the generalized degenerate q-Bernoulli and q-Euler polynomials, respectively.

Theorem 3.1. *For every $n \geq 0$, $\lambda \in \mathbb{R}$ and $r = 1$, the degenerate q-Bernoulli polynomials satisfy the following inversion-type formula:*

$$\binom{x}{n}_{q,\lambda} = \frac{1}{[n+1]_q!}\sum_{k=0}^{n}\left[\begin{array}{c}n+1\\k+1\end{array}\right]_q (1)_{k+1,\lambda}\mathcal{B}_{n-k,q,\lambda}(x) \tag{7}$$

$$= \frac{1}{[n+1]_q!}\sum_{k=0}^{n}\left[\begin{array}{c}n+1\\k+1\end{array}\right]_q (1-\lambda)_{k,\lambda}\mathcal{B}_{n-k,q,\lambda}(x). \tag{8}$$

Proof. Let $\lambda \in \mathbb{R}$. In view of (3) and (5), and the identity

$$z\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{z^n}{[n]_q!}=\sum_{n=0}^{\infty}[n+1]_q(x)_{n,\lambda}\frac{z^{n+1}}{[n+1]_q!},$$

we have

$$\sum_{n=0}^{\infty}[n+1]_q(x)_{n,\lambda}\frac{z^{n+1}}{[n+1]_q!} = \left[\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{z^n}{[n]_q!}-1\right]\left[\sum_{n=0}^{\infty}\mathcal{B}_{n,q,\lambda}(x)\frac{z^n}{[n]_q!}\right]$$

$$= \left[\sum_{n=0}^{\infty}(1)_{n+1,\lambda}\frac{z^{n+1}}{[n+1]_q!}\right]\left[\sum_{n=0}^{\infty}\mathcal{B}_{n,q,\lambda}(x)\frac{z^n}{[n]_q!}\right]. \tag{9}$$

From the use of the Cauchy product rule on the right-hand side of (9), it follows that

$$\sum_{n=0}^{\infty}[n+1]_q(x)_{n,\lambda}\frac{z^{n+1}}{[n+1]_q!}=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\left[\begin{array}{c}n+1\\k+1\end{array}\right]_q(1)_{k+1,\lambda}\mathcal{B}_{n-k,q,\lambda}(x)\right]\frac{z^{n+1}}{[n+1]_q!}. \tag{10}$$

Hence, comparing the coefficients of z^{n+1} on both sides of (10) and using the identity (4), we obtain (7).

Finally, (8) is a simple consequence of the identity $(1)_{k+1,\lambda}=(1-\lambda)_{k,\lambda}$, for all $k \in \mathbb{N}_0$. □

Example 3.1. The first three degenerate q-Bernoulli polynomials are

$$\mathcal{B}_{0,q,\lambda}(x)=1,\quad \mathcal{B}_{1,q,\lambda}(x)=x+\frac{\lambda-1}{[2]_q},\quad \mathcal{B}_{2,q,\lambda}(x)=x^2-x+\frac{\lambda^2-2\lambda+1}{[2]_q}-\frac{2\lambda^2-3\lambda+1}{[3]_q}.$$

Remark 1. Notice that the substitution of $\lambda=0$ into (7) recovers the classical q-Bernoulli polynomials (cf. [14, 24, 25]).

From a matrix framework, Theorem 3.1 has the following consequence.

Corollary 3.2. *For $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$, the matrix $\mathbf{T}_\lambda(x)=\begin{pmatrix}1 & (x)_{1,\lambda} & \cdots & (x)_{n,\lambda}\end{pmatrix}^T$ can be expressed as follows:*

$$\mathbf{T}_\lambda(x)=\mathbf{M}_\lambda\mathbf{B}_{q,\lambda}(x)$$

$$=\begin{pmatrix} \begin{bmatrix}1\\1\end{bmatrix}_q(1)_{1,\lambda} & 0 & \cdots & 0\\ \frac{1}{[2]_q}\begin{bmatrix}2\\2\end{bmatrix}_q(1)_{2,\lambda} & \frac{1}{[2]_q}\begin{bmatrix}2\\1\end{bmatrix}_q(1)_{1,\lambda} & \cdots & 0\\ \frac{1}{[3]_q}\begin{bmatrix}3\\3\end{bmatrix}_q(1)_{3,\lambda} & \frac{1}{[3]_q}\begin{bmatrix}3\\2\end{bmatrix}_q(1)_{2,\lambda} & \ddots & 0\\ \vdots & \vdots & \cdots & \vdots\\ \frac{1}{[n+1]_q}\begin{bmatrix}n+1\\n+1\end{bmatrix}_q(1)_{n+1,\lambda} & \frac{1}{[n+1]_q}\begin{bmatrix}n+1\\n\end{bmatrix}_q(1)_{n,\lambda} & \cdots & \frac{1}{[n+1]_q}\begin{bmatrix}n+1\\1\end{bmatrix}_q(1)_{1,\lambda}\end{pmatrix}\mathbf{B}_{q,\lambda}(x)$$

$$=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ \frac{1}{[2]_q}(1)_{2,\lambda} & 1 & 0 & \cdots & 0\\ \frac{1}{[3]_q}(1)_{3,\lambda} & (1)_{2,\lambda} & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{1}{[n+1]_q}(1)_{n+1,\lambda} & (1)_{n,\lambda} & \frac{[n]_q}{[2]_q}(1)_{n-1,\lambda} & \cdots & 1\end{pmatrix}\mathbf{B}_{q,\lambda}(x),$$

where $\mathbf{B}_{q,\lambda}(x)=\left(\mathcal{B}_{0,q,\lambda}(x)\quad \mathcal{B}_{1,q,\lambda}(x)\quad \cdots \quad \mathcal{B}_{n,q,\lambda}(x)\right)^T$.

Theorem 3.3. *For every $n\geq 0$ and $\lambda\in\mathbb{R}$ and $r=1$, the degenerate q-Euler polynomials satisfy the following inversion-type formula:*

$$\binom{x}{n}_{q,\lambda}=\frac{1}{2[n]_q!}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q(1+a_k)(1)_{k,\lambda}\mathcal{E}_{n-k,q,\lambda}(x), \tag{11}$$

where

$$a_k=\begin{cases}1, & \text{if } k=0,\\ 0, & \text{if } 1\leq k\leq n.\end{cases}$$

Proof. From (3) and (6) we have

$$\begin{aligned} 2\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{z^n}{[n]_q!} &= \left[\sum_{n=0}^{\infty}(1)_{n,\lambda}\frac{z^n}{[n]_q!}+1\right]\left[\sum_{n=0}^{\infty}\mathcal{E}_{n,q,\lambda}(x)\frac{z^n}{[n]_q!}\right]\\ &= \left[\sum_{n=0}^{\infty}(1+a_n)(1)_{n,\lambda}\frac{z^n}{[n]_q!}\right]\left[\sum_{n=0}^{\infty}\mathcal{E}_{n,q,\lambda}(x)\frac{z^n}{[n]_q!}\right]\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(1+a_k)\begin{bmatrix}n\\k\end{bmatrix}_q(1)_{k,\lambda}\mathcal{E}_{n-k,q,\lambda}(x)\right]\frac{z^n}{[n]_q!}, \end{aligned}$$

where

$$a_k=\begin{cases}1, & \text{if } k=0,\\ 0, & \text{if } 1\leq k\leq n.\end{cases}$$

Therefore, by comparing the coefficients of z^n on both sides and using the identity (4), we obtain the identity (11) . □

Theorem 3.3 has the following consequence.

Corollary 3.4. *For $n\in\mathbb{N}_0$ and $\lambda\in\mathbb{R}$, the matrix $\mathbf{T}_\lambda(x)=\left(1\quad (x)_{1,\lambda}\quad \cdots \quad (x)_{n,\lambda}\right)^T$ can be expressed as follows:*

$$\mathbf{T}_\lambda(x)=\mathbf{M}_\lambda \mathbf{E}_{q,\lambda}(x)$$
$$=\frac{1}{2}\begin{pmatrix} \begin{bmatrix} 0\\0\end{bmatrix}_q(1+a_0)(1)_{0,\lambda} & 0 & \cdots & 0\\ \begin{bmatrix} 1\\1\end{bmatrix}_q(1+a_1)(1)_{1,\lambda} & \begin{bmatrix} 1\\0\end{bmatrix}_q(1+a_0)(1)_{0,\lambda} & \cdots & 0\\ \begin{bmatrix} 2\\2\end{bmatrix}_q(1+a_2)(1)_{2,\lambda} & \begin{bmatrix} 2\\1\end{bmatrix}_q(1+a_2)(1)_{1,\lambda} & \ddots & 0\\ \vdots & \vdots & \cdots & \vdots\\ \begin{bmatrix} n\\n\end{bmatrix}_q(1+a_n)(1)_{n,\lambda} & \begin{bmatrix} n\\n-1\end{bmatrix}_q(1+a_n)(1)_{n-1,\lambda} & \cdots & \begin{bmatrix} n\\0\end{bmatrix}_q(1+a_0)(1)_{0,\lambda}\end{pmatrix}\mathbf{E}_{q,\lambda}(x)$$

$$=\frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & \cdots & 0\\ (1)_{1,\lambda} & 2 & 0 & \cdots & 0\\ (1)_{2,\lambda} & (1)_{1,\lambda} & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ (1)_{n,\lambda} & (1)_{n-1,\lambda} & (1)_{n-2,\lambda} & \cdots & 2\end{pmatrix}\mathbf{E}_{q,\lambda}(x),$$

where $\mathbf{E}_{q,\lambda}(x)=\left(\mathcal{E}_{0,q,\lambda}(x) \quad \mathcal{E}_{1,q,\lambda}(x) \quad \cdots \quad \mathcal{E}_{n,q,\lambda}(x)\right)^T$.

The degenerate q-Pascal matrices corresponding to the generalized falling factorials can be defined as follows:

Definition 3.1. Let x be any nonzero real number. For $\lambda\in\mathbb{R}$ and $|q|<1$, the generalized degenerate q-Pascal matrix of first kind $P_{q,\lambda}[x]$, is an $(n+1)\times(n+1)$ matrix whose entries are given by

$$p_{i,j,q,\lambda}(x):=\begin{cases}\begin{bmatrix} i\\j\end{bmatrix}_q (x)_{i-j,\lambda}, & i\geq j,\\[2ex] 0, & \text{otherwise.}\end{cases} \tag{12}$$

Remark 2.

(i) It is clear that the matrix $P_{q,\lambda}[x]$ tends to the q-Pascal matrix of first kind $P_q[x]$ as $\lambda\to 0$ (cf. [10] (Equation (8))).

(ii) It is worth mentioning that $P_{q,\lambda}[x]$ is a lower triangular matrix with nonnull determinant and hence, it is a nonsingular matrix.

(iii) The identity (4) says us that the entries of the generalized degenerate q-Pascal matrix of first kind in (12) can be written as

$$p_{i,j,q,\lambda}(x)=\begin{cases}\dfrac{[i]_q!}{[j]_q!}\dbinom{x}{i-j}_{q,\lambda}, & i\geq j,\\[2ex] 0, & \text{otherwise.}\end{cases}$$

(iv) For $x,y\in\mathbb{R}$, the generalized degenerate q-Pascal matrix do not satisfy the addition law like the generalized degenerate Pascal matrix, i.e.,

$$P_{q,\lambda}[x+y]\neq P_{q,\lambda}[x]P_{q,\lambda}[y].$$

(v) If the convention $(0)_{0,\lambda}=1$ is adopted, then it is possible to define

$$P_{q,\lambda}[0]:=I_{n+1}.$$

Definition 3.2. The generalized degenerate $(n+1)\times(n+1)$ q-Bernoulli matrix $\mathscr{B}^{(r)}_{q,\lambda}(x)$ of real order r is defined by the entries

$$\mathscr{B}^{(r)}_{i,j,q,\lambda}(x)=\begin{cases}\begin{bmatrix} i\\j\end{bmatrix}_q \mathcal{B}^{(r)}_{i-j,q,\lambda}(x), & i\geq j,\\[2ex] 0, & \text{otherwise.}\end{cases}$$

Remark 3.

(i) We denote by $\mathscr{B}_{q,\lambda}(x)$ the degenerate q-Bernoulli matrix $\mathscr{B}^{(1)}_{q,\lambda}(x)$.

Definition 3.2 and the inversion-type formula (7) lead to the following result:

Theorem 3.5. *The generalized degenerate q-Pascal matrix of the first kind $P_{q,\lambda}[x]$ can be factorized in terms of $\mathcal{B}_{q,\lambda}(x)$ as follows:*

$$P_{q,\lambda}[x] = \mathcal{B}_{q,\lambda}(x)\mathscr{H}_{q,\lambda}, \tag{13}$$

where $\mathscr{H}_{q,\lambda}$ is an $(n+1)\times(n+1)$ invertible matrix with entries

$$\mathscr{H}_{i,j,q,\lambda} = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix}_q \dfrac{(1)_{i-j+1,\lambda}}{[i-j+1]_q}, & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Let us consider $n \in \mathbb{N}_0$ and $0 \leq i, j \leq n$ such that $i \geq j$. From Definition 3.2 and the inversion-type formula (7), we have

$$\begin{aligned} p_{i,j,q,\lambda}(x) &= \begin{bmatrix} i \\ j \end{bmatrix}_q (x)_{i-j,\lambda} = \begin{bmatrix} i \\ j \end{bmatrix}_q \frac{1}{[i-j+1]_q!} \sum_{k=0}^{i-j} \begin{bmatrix} i-j+1 \\ k+1 \end{bmatrix}_q (1)_{k+1,\lambda} \mathcal{B}_{i-j-k,q,\lambda}(x) \\ &= \sum_{k=0}^{i-j} \left[\begin{bmatrix} i-j \\ k \end{bmatrix}_q \mathcal{B}_{i-j-k,q,\lambda}(x) \right] \left[\begin{bmatrix} i \\ j \end{bmatrix}_q \frac{(1)_{k+1,\lambda}}{[k+1]_q} \right]. \end{aligned} \tag{14}$$

Since the right hand member of (14) is the (i,j)-th entry of matrix product $\mathcal{B}_{q,\lambda}(x)\mathscr{H}_{q,\lambda}$, we conclude that (13) holds. Notice that the matrix $\mathscr{H}_{q,\lambda}$ is a lower triangular matrix with nonnull determinant and hence, it is a nonsingular matrix. □

The following example shows the validity of Theorem 3.5.

Example 3.2. Let us consider $n = 2$. It follows from Definition 3.1, (13), and a simple computation that

$$\begin{aligned} P_{q,\lambda}[x] &= \begin{pmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q (x)_{0,\lambda} & 0 & 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q (x)_{1,\lambda} & \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q (x)_{0,\lambda} & 0 \\ \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q (x)_{2,\lambda} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q (x)_{1,\lambda} & \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q (x)_{0,\lambda} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \mathcal{B}_{0,q,\lambda}(x) & 0 & 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \mathcal{B}_{1,q,\lambda}(x) & \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q \mathcal{B}_{0,q,\lambda}(x) & 0 \\ \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q \mathcal{B}_{2,q,\lambda}(x) & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \mathcal{B}_{1,q,\lambda}(x) & \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \mathcal{B}_{0,q,\lambda}(x) \end{pmatrix}}_{\mathcal{B}_{q,\lambda}(x)} \underbrace{\begin{pmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q (1)_{1,\lambda} & 0 & 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \frac{(1)_{2,\lambda}}{[2]_q} & \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q (1)_{1,\lambda} & 0 \\ \begin{bmatrix} 2 \\ 0 \end{bmatrix}_q \frac{(1)_{3,\lambda}}{[3]_q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \frac{(1)_{2,\lambda}}{[2]_q} & \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q (1)_{1,\lambda} \end{pmatrix}}_{\mathscr{H}_{q,\lambda}}. \end{aligned}$$

Definition 3.3. The generalized degenerate $(n+1)\times(n+1)$ q-Euler matrix $\mathscr{E}^{(r)}_{q,\lambda}(x)$ is defined by the entries

$$\mathscr{E}^{(r)}_{i,j,q,\lambda}(x) = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix}_q \mathcal{E}^{(r)}_{i-j,q,\lambda}(x), & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

We denote by $\mathscr{E}_{q,\lambda}(x)$ the degenerate q-Euler matrix $\mathscr{E}^{(1)}_{q,\lambda}(x)$.

Definition 3.3 and the inversion-type formula (11) lead to the following result:

Theorem 3.6. *The generalized degenerate q-Pascal matrix of the first kind $P_{q,\lambda}[x]$ can be factorized in terms of $\mathcal{E}_{q,\lambda}(x)$ as follows:*

$$P_{q,\lambda}[x] = \mathcal{E}_{q,\lambda}(x)\mathscr{T}_{q,\lambda}, \tag{15}$$

where $\mathscr{T}_{q,\lambda}$ is an $(n+1)\times(n+1)$ invertible matrix with entries

$$\mathscr{T}_{i,j,q,\lambda} = \begin{cases} \begin{bmatrix} i \\ j \end{bmatrix}_q \dfrac{\left(1+a_{i-j}\right)(1)_{i-j,\lambda}}{2}, & i \geq j, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Let us consider $n \in \mathbb{N}_0$ and $0 \le i, j \le n$ such that $i \ge j$. From Definition 3.3 and the inversion-type formula (11), we have

$$\begin{aligned} p_{,i,jq,\lambda}(x) &= \begin{bmatrix} i \\ j \end{bmatrix}_q (x)_{i-j,\lambda} = \begin{bmatrix} i \\ j \end{bmatrix}_q \frac{1}{2}\sum_{k=0}^{i-j} \begin{bmatrix} i-j \\ k \end{bmatrix}_q (1+a_k)(1)_{k,\lambda}\mathcal{E}_{i-j-k,q,\lambda}(x) \\ &= \sum_{k=0}^{i-j}\left[\begin{bmatrix} i-j \\ k \end{bmatrix}_q \mathcal{E}_{i-j-k,q,\lambda}(x)\right]\left[\begin{bmatrix} i \\ j \end{bmatrix}_q \frac{(1+a_k)(1)_{k,\lambda}}{2}\right]. \end{aligned} \tag{16}$$

Since the right-hand member of (16) is the (i, j)-th entry of matrix product $\mathcal{E}_{q,\lambda}(x)\mathcal{T}_{q,\lambda}$, we conclude that (15) holds. Notice that the matrix $\mathcal{T}_{q,\lambda}$ is a lower triangular matrix with nonnull determinant and hence, it is a nonsingular matrix. □

Combining Theorems 3.5 and 3.6 gives the following connection formula.

Corollary 3.7. *For any $\lambda, x \in \mathbb{R}$, we have*

$$\mathcal{E}_{q,\lambda}(x) = \mathcal{B}_{q,\lambda}(x)\mathcal{H}_{q,\lambda}\mathcal{T}_{q,\lambda}^{-1}.$$

4 Conclusion

Diverse kinds of q-Pascal matrices can be used to construct certain representations of quantum groups. These representations are essential for understanding symmetries in quantum mechanics and quantum field theory in a non-commutative setting.

The aim of our research was to determine some novel properties of generalized degenerate q-Bernoulli and q-Euler polynomials and their matrices. Firstly, we focused our attention on some inversion-type formulae for the generalized degenerate q-Bernoulli and q-Euler polynomials and their matrices. Secondly, we introduced the generalized degenerate q-Pascal matrix of the first kind and provided factorizations for the generalized degenerate q-Bernoulli and q-Euler polynomial matrices in terms of the generalized degenerate q-Pascal matrix of the first kind.

Finally, it is noteworthy that under the suitable constraints of parameters associated with the generalized Apostol-type polynomial matrices given in [26], it is possible to provide a λ-degenerate deformation for some q-analogues of these matrices. The proof of this statement is not provided here; the interested reader is strongly encouraged to follow the above arguments to prove it.

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