Approximation methods in MPI
A tutorial and introduction

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The material of this tutorial talk is published in [1].

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History of the scientific network MathMPI

Goal of the scientific network MathMPI

This project addresses for the first time the systematic study of mathematical questions related to MPI. The central goal of this network is the development, analysis and application of mathematical methods to improve the reconstruction quality in MPI. In particular, taylored to the specific needs of MPI we develop elaborate reconstruction algorithms, analyze and refine the underlying MPI models and test the new methods numerically on real biomedical data. (from the application to the DFG)

- 07/2012 First contacts at a summer school in Munich
- 02/2014 Submission of the application to the DFG
- 08/2014 MathMPI starts
- 11/2014 First meeting in Lübeck
- 02/2015 Study group in Munich
- 06/2015 Study group in Ettlingen
- 09/2015 Forth workshop in Hamburg
- 07/2016 Last conference in Osnabrück

Participants in the project

Mathematicians

- Christina Brandt (Osnabrück, Hamburg)
- Wolfgang Erb (Lübeck, Hawaii)
- Jürgen Frikel (Munich, Regensburg)
- Thomas März (Oxford)
- Martin Storath (Lausanne, Heidelberg)
- Andreas Weinmann (Munich, Darmstadt)

Medical engineers

- Mandy Ahlborg (Lübeck)
- Thorsten M. Buzug (Lübeck)
- Gael Bringout (Lübeck, Karlsruhe)
- Martin Hofmann (Hamburg)
- Christian Kaethner (Lübeck)
- Tobias Knopp (Hamburg)

Partners from industry

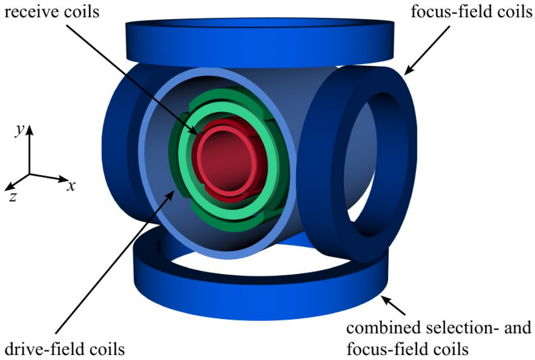
- Jürgen Rahmer (Philips)
- Alexander Weber (Bruker)

Main objectives of the project

- **Mathematical Analysis of the MPI System Function.** Mathematical analysis of the reconstruction process. The obtained information is used to get algorithms specifically tailored to the needs of MPI.
- **Modeling and Determination of High Quality System Functions.** Accurate description of the involved physical processes by removing simplifications of the current MPI models.
- **Development of Elaborate Reconstruction Algorithms.** Design of stable and efficient reconstruction algorithms which are tailored to clinical needs.
- **Application to Biomedical Data.** Evaluation of the performance of reconstruction algorithms on real biomedical data.

Introduction to 1D modelling in Magnetic Particle Imaging

What is Magnetic Particle Imaging (MPI)?



Knopp, Buzug: *Magnetic Particle Imaging*, Springer, 2012 [2]

Basic Principle of Magnetic Particle Imaging

Principle: Recover the density of paramagnetic particles from their non-linear magnetization response in an applied magnetic field.

What is measured? In the receive coils the time-dependent change of the particle magnetization is measured via induced voltage .

General model equation:

$$u(t) = -\mu_0 \sigma_0 \frac{d}{dt} \int_{\mathrm{Object}} c(x) \bar{\mathbf{m}}(H(x,t)) dx$$

- $u(t)$ induced voltage in the receive coils
- $c(x)$ particle density
- $\bar{\mathbf{m}}$ magnetization response for magnetic field $H(x,t)$.

General 1D model for Magnetic Particle Imaging

Simplifications in the 1D case:

- neglect constants μ_0, σ_0.
- $\text{Object} = \mathbb{R}$, $x \in \mathbb{R}$.

Then the 1D-model equation is given by [3]:

$$u(t) = -\frac{d}{dt}\int_{\mathbb{R}} c(x)\bar{\mathbf{m}}(H(x,t))dx = \int_{\mathbb{R}} c(x)s(x,t)dx$$

with the system kernel

$$s(x,t) = -\frac{d}{dt}\bar{\mathbf{m}}(H(x,t)) = -\frac{d\bar{\mathbf{m}}}{dH}\frac{d}{dt}H(x,t).$$

1. mathematical formulation for 1D model

Further simplification: $H(x,t)$ is periodic and even in time t with period T and frequency $\omega_0 = \frac{2\pi}{T}$. Then, $u(t)$ is an odd periodic signal with period T.

Now, introduce the Hilbert spaces

$$X^{\mathrm{space}} = L^2(\mathbb{R}), \quad X^{\mathrm{time}} = L^2(\left[0, \tfrac{T}{2}\right])$$

and the operator

$$S^{\mathrm{t}} : X^{\mathrm{space}} \to X^{\mathrm{time}}, \quad S^{\mathrm{t}} c(t) = \int_{\mathbb{R}} c(x) s(x,t) dx.$$

1D-MPI reconstruction problem

Given $u \in R(S^{\mathrm{t}})$, find $c \in X^{\mathrm{space}}$ such that

$$S^{\mathrm{t}} c = u.$$

More details about the system kernel $s(x, t)$

$$s(x,t) = -\frac{d}{dt}\bar{\mathbf{m}}(H(x,t)) = -\frac{d\bar{\mathbf{m}}}{dH}\frac{d}{dt}H(x,t).$$

Generate the following magnetic field:

$$H(x,t) = \underbrace{Gx}_{H_S(x)} - \underbrace{A\cos(\omega_0 t)}_{H_D(t)}.$$

- Selection field $H_S(x)$ with gradient $G > 0$.
- Drive field $H_D(t)$ with amplitude $A > 0$ and frequency ω_0.

Then,

$$s(x,t) = -\bar{\mathbf{m}}'(Gx - A\cos\omega_0 t)A\omega_0 \sin\omega_0 t.$$

Now, introduce new space variable $-\frac{A}{G} \leq y \leq \frac{A}{G}$ by

$$y = \frac{A}{G}\cos\omega_0 t, \quad t \in \left[0, \frac{T}{2}\right].$$

then

$$s(x,y) = -\bar{\mathbf{m}}'(G(x-y))\; G\omega_0\sqrt{\tfrac{A^2}{G^2} - y^2}.$$

Define function $\bar{\mathbf{m}}_G(x) := \bar{\mathbf{m}}(Gx)$, then

$$s(x,y) = -\bar{\mathbf{m}}_G'(x-y)\,\omega_0\sqrt{\tfrac{A^2}{G^2} - y^2}$$

and we can write for $-\frac{A}{G} \leq y \leq \frac{A}{G}$:

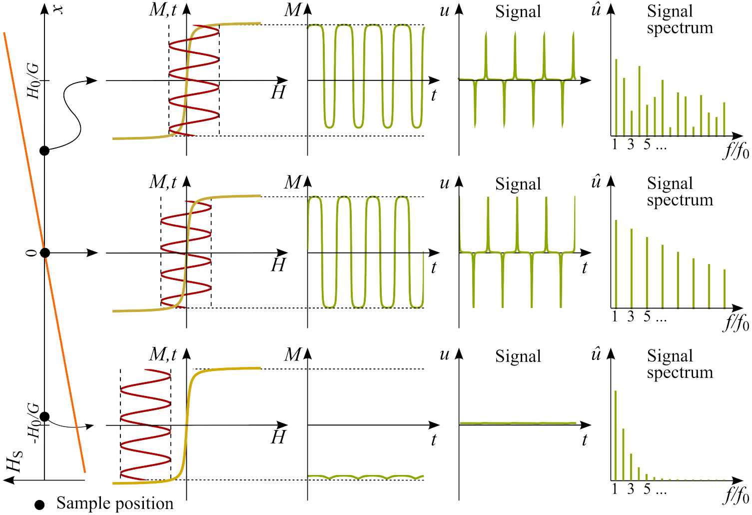
$$S^{\mathrm{y}} c(y) = -\omega_0\sqrt{\tfrac{A^2}{G^2} - y^2}\int_{\mathbb{R}} c(x)\bar{\mathbf{m}}_G'(x-y)dx$$

If we assume that $\bar{\mathbf{m}}'$ is even, then for $-\frac{A}{G} \leq y \leq \frac{A}{G}$ we have :

$$\begin{aligned} S^{\mathrm{y}} c(y) &= -\omega_0 \sqrt{\frac{A^2}{G^2} - y^2} \int_{\mathbb{R}} c(x) \bar{\mathbf{m}}'_G(y-x) dx \\ &= -\omega_0 \sqrt{\frac{A^2}{G^2} - y^2} \, (c * \bar{\mathbf{m}}'_G)(y), \end{aligned}$$

or, equivalently, in the time domain

$$S^{\mathrm{t}} c(t) = -\omega_0 \frac{A}{G} \sin(\omega_0 t) \, (c * \bar{\mathbf{m}}'_G)\left(\frac{A}{G} \cos \omega_0 t \right).$$



With kind permission of M. Ahlborg, C. Kaethner (IMT Lübeck)

Decomposition into basic operators

Define $X^{\mathrm{fov}} = L^2([-\frac{A}{G}, \frac{A}{G}])$ and the operators

$$\begin{aligned}
Q^{\mathrm{conv}} &: X^{\mathrm{space}} \to X^{\mathrm{space}}, & Q^{\mathrm{conv}} f(x) &= (f * \bar{\mathbf{m}}'_G)(x) \\
Q^{\mathrm{fov}} &: X^{\mathrm{space}} \to X^{\mathrm{fov}}, & Q^{\mathrm{fov}} f(y) &= f(y) \\
Q^{\mathrm{time}} &: X^{\mathrm{fov}} \to X^{\mathrm{time}}, & Q^{\mathrm{time}} f(t) &= -\omega_0 \tfrac{A}{G} \sin(\omega_0 t)\, f(\tfrac{A}{G} \cos \omega_0 t)
\end{aligned}$$

Then

$$S^{\mathrm{t}} = Q^{\mathrm{time}} \circ Q^{\mathrm{fov}} \circ Q^{\mathrm{conv}}$$

and

- Q^{conv} is a bounded operator with $\|Q^{\mathrm{conv}}\| = \|\bar{\mathbf{m}}'_G\|_1$.
- Q^{fov} is a projection operator with $\|Q^{\mathrm{fov}}\| = 1$.
- Q^{time} is a bounded operator with $\|Q^{\mathrm{time}}\| = 1$.

Frequency measurements

Sometimes: frequency information of u is given as measured data. Introduce Hilbert space $X^{\mathrm{freq}} = l^2(\mathbb{N})$ and a new operator

$$Q^{\mathrm{fft}} : X^{\mathrm{time}} \to X^{\mathrm{freq}}, \quad Q^{\mathrm{fft}}(f)(n) = \frac{2}{T} \int_0^{\frac{T}{2}} f(t) \sin(n\omega_0 t) dt.$$

Then, we get a new forward operator and a new imaging model based on measured frequencies:

$$S^{\mathrm{f}} : X^{\mathrm{space}} \to X^{\mathrm{freq}}, \quad S^{\mathrm{f}} = Q^{\mathrm{fft}} \circ Q^{\mathrm{time}} \circ Q^{\mathrm{fov}} \circ Q^{\mathrm{conv}}.$$

$$S^{\mathrm{f}} c(n) = -\frac{A}{G} \frac{2\omega_0}{T} \int_0^{\frac{T}{2}} \sin(\omega_0 t) \, (c * M_G') \left(\frac{A}{G} \cos \omega_0 t \right) \sin(n\omega_0 t) dt$$

Alternative description of $S^{\mathrm{f}}c(n)$ with substitution $y = \frac{A}{G}\cos\omega_0 t$:

$$\begin{aligned} S^{\mathrm{f}}c(n) &= -\frac{2}{T}\int_{-\frac{A}{G}}^{\frac{A}{G}} (c * M'_G)(y)\sin\left(n \arccos \frac{G}{A} y\right) dy, \\ &= -\frac{2}{T}\int_{-\frac{A}{G}}^{\frac{A}{G}} (c * \bar{\mathbf{m}}'_G)(y) U_{n-1}\left(\tfrac{G}{A} y\right)\sqrt{1-\tfrac{G^2}{A^2}y^2}\, dy \end{aligned}$$

$S^{\mathrm{f}}c(n)$ can be interpreted as Chebyshev transform (with the Chebyshev polynomials of second kind) of the function $c * \bar{\mathbf{m}}'_G$.

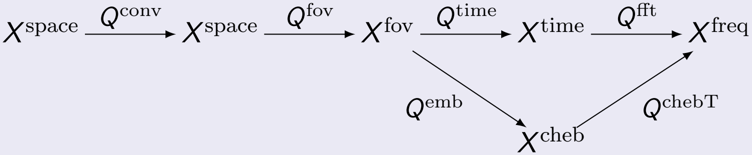
$$X^{\mathrm{cheb}} = L^2\left([-\tfrac{A}{G}, \tfrac{A}{G}], \sqrt{1-\tfrac{G^2}{A^2}y^2}\right),$$

$$Q^{\mathrm{chebT}} : X^{\mathrm{cheb}} \to X^{\mathrm{freq}},$$

$$Q^{\mathrm{chebT}} f(n) = -\frac{2}{T}\int_{-\frac{A}{G}}^{\frac{A}{G}} f(y) U_{n-1}\left(\tfrac{G}{A} y\right)\sqrt{1-\tfrac{G^2}{A^2}y^2}\, dy$$

Summary

$$S^{\mathrm{f}} = Q^{\mathrm{chebT}} \circ Q^{\mathrm{emb}} \circ Q^{\mathrm{fov}} \circ Q^{\mathrm{conv}} = Q^{\mathrm{fft}} \circ Q^{\mathrm{time}} \circ Q^{\mathrm{fov}} \circ Q^{\mathrm{conv}}$$



The Hilbert spaces X^{time}, X^{cheb} and X^{freq} are isometric isomorphic, the respective reconstructions equivalent..

Notation for reconstruction in MPI

x-**space reconstruction** (according to Goodwill & Conolly)

Given $u \in X^{\mathrm{space}}$, find $c \in X^{\mathrm{fov}}$ such that

$$S^{\mathrm{conv}} c = Q^{\mathrm{fov}} \circ Q^{\mathrm{conv}} c = u.$$

Independent of used time trajectory.

x-**space reconstruction** (language of Knopp & Buzug)

Given $u \in X^{\mathrm{time}}$, find $c \in X^{\mathrm{fov}}$ such that

$$S^{\mathrm{t}} c = Q^{\mathrm{time}} \circ Q^{\mathrm{fov}} \circ Q^{\mathrm{conv}} c = u.$$

Chebyshev reconstruction

Given $\hat{u} \in X^{\mathrm{freq}}$, find $c \in X^{\mathrm{fov}}$ such that

$$S^{\mathrm{f}} c = Q^{\mathrm{chebT}} \circ Q^{\mathrm{emb}} \circ Q^{\mathrm{fov}} \circ Q^{\mathrm{conv}} c = \hat{u}.$$

S^{f} modelling with Chebyshev polynomials

$$S^{\mathrm{f}} c(n) = -\frac{2}{T} \int_{-\frac{A}{G}}^{\frac{A}{G}} (c * \bar{\mathbf{m}}'_G)(y) U_{n-1}\left(\tfrac{G}{A} y\right) \sqrt{1 - \tfrac{G^2}{A^2} y^2}\, dy$$

$$= -\frac{2}{T} \int_{\mathbb{R}} (c * \bar{\mathbf{m}}'_G)(y) \mathbb{1}_{[-\frac{A}{G}, \frac{A}{G}]}(y) U_{n-1}\left(\tfrac{G}{A} y\right) \sqrt{1 - \tfrac{G^2}{A^2} y^2}\, dy$$

$$= -\frac{2}{T} \int_{\mathbb{R}} c(y) \left(\bar{\mathbf{m}}'_G * \left(\mathbb{1}_{[-\frac{A}{G}, \frac{A}{G}]} U_{n-1}\left(\tfrac{G}{A} \cdot\right) \sqrt{1 - \tfrac{G^2}{A^2} (\cdot)^2} \right) \right)(y)\, dy$$

Define

$$s_n^{\mathrm{f}}(x) = -\frac{2}{T} \left(\bar{\mathbf{m}}'_G * \left(\mathbb{1}_{[-\frac{A}{G}, \frac{A}{G}]} U_{n-1}\left(\frac{G}{A} \cdot\right) \sqrt{1 - \frac{G^2}{A^2} (\cdot)^2} \right) \right)(x).$$

Then

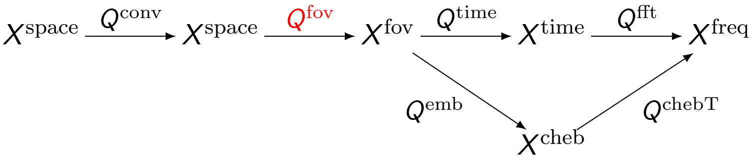
$$S^{\mathrm{f}} c(n) = \langle s_n^{\mathrm{f}}, c \rangle.$$

Pre- and postprocessing operators

Beside the mentioned operators, additional pre- and post processing operators are involved in the reconstruction process, depending on the used scanners.

- A transfer function (preprocessing) to fit the rows s_n^f to a measured system matrix.
- A symmetrization operator (preprocessing).
- Merging operators (postprocessing) that glue together reconstructions from several fields of view.
- Beautification operators, mild deconvolutions ... (postprocessing)

The operator Q^{fov}

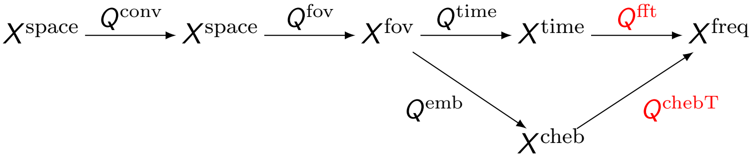


The operator Q^{fov} is a projection operator.

Lemma

The restriction operator $Q^{\mathrm{fov}} : X^{\mathrm{space}} \to X^{\mathrm{fov}}$ given by $Q^{\mathrm{fov}} f = f|_{[-\frac{A}{G},\frac{A}{G}]}$ is an orthogonal projection onto the subspace X^{fov} with operator norm equal to 1.

The operators Q^{fft} and Q^{chebT}.



The operator Q^{fft} and Q^{chebT} simply perform the sine transform and the Chebyshev transform, respectively.

Lemma

The operators $Q^{\mathrm{fft}} : X^{\mathrm{time}} \to X^{\mathrm{freq}}$ *and* $Q^{\mathrm{chebT}} : X^{\mathrm{cheb}} \to X^{\mathrm{freq}}$ *are isometries.*

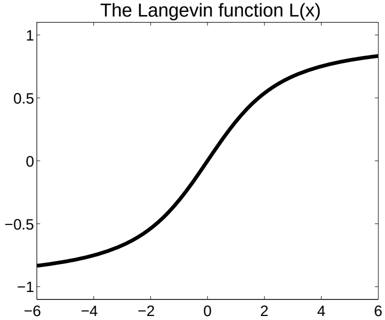
The MPI core operator Q^{conv}

$$Q^{\mathrm{conv}} : X^{\mathrm{space}} \to X^{\mathrm{space}}, \quad Q^{\mathrm{conv}} f(x) = (f * \bar{\mathbf{m}}_G')(x).$$

The magnetization $\bar{\mathbf{m}}_G$ can be modeled with the Langevin function

$$\bar{\mathbf{m}}_G(x) = \alpha L(\beta G x) = \alpha \left(\coth(\beta G x) - \frac{1}{\beta G x} \right)$$

where α and β are parameters describing the physical system.



The MPI core operator Q^{conv}

The convolution kernel is given by

$$\bar{\mathbf{m}}'_G(x) = \alpha\beta G L'(\beta G x),$$

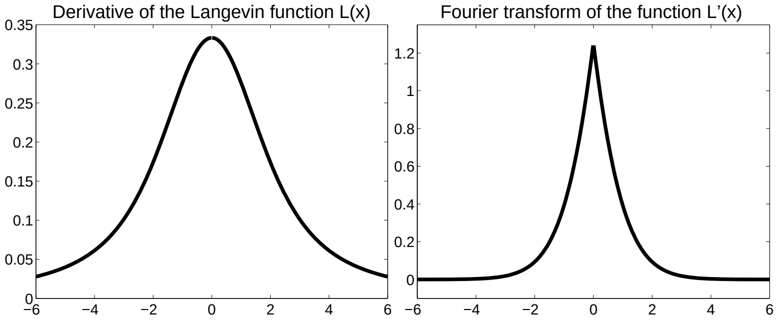
where

$$L'(x) = \begin{cases} \frac{1}{x^2} - \frac{1}{\sinh(x)^2}, & x \neq 0 \\ \frac{1}{3}, & x = 0 \end{cases}$$

The Fourier transform of L' is given by

$$\mathcal{F}(L')(\omega) = \sqrt{\frac{\pi}{2}} \left(|\omega| - \omega \coth \frac{\pi}{2}\omega \right).$$

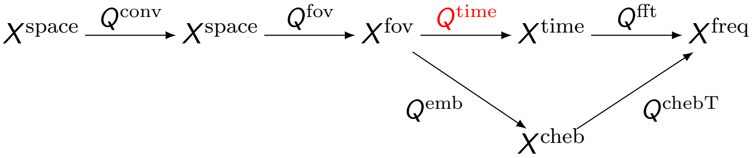
$\mathcal{F}(L')(\omega)$ decays exponentially as $\omega \to \infty$.



Lemma

The convolution kernel $\bar{\mathbf{m}}'_G$ *is analytic and the image* $R(Q^{\mathrm{conv}})$ *is contained in every Sobolev space* $H^s(\mathbb{R})$, $s \geq 0$. *In particular, the inverse problem* $Q^{\mathrm{conv}} f = g$ *is ill-posed.*

The trajectory operator Q^{time}



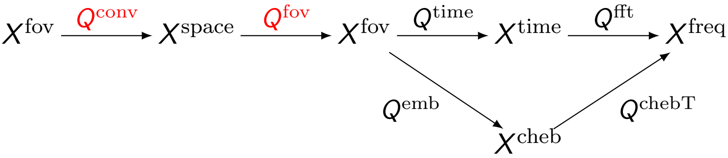
Lemma

The trajectory operator $Q^{\mathrm{time}} : X^{\mathrm{fov}} \to X^{\mathrm{time}}$ *given by*

$$Q^{\mathrm{time}} f(t) = -\omega_0 \frac{A}{G} \sin(\omega_0 t)\, f\left(\tfrac{A}{G} \cos \omega_0 t\right)$$

is bounded and injective. The inverse problem $Q^{\mathrm{time}} f = g$ *is ill-posed since the multiplier* $\sin(\omega_0 t)$ *vanishes at the boundary points.*

Main Theorem 1

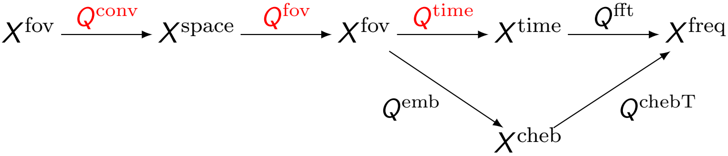


Theorem

The composed operator $S^{\text{conv}} = Q^{\text{fov}} \circ Q^{\text{conv}}$ *restricted to the space* X^{fov} *is self-adjoint, positive and its image* $R(S^{\text{conv}})$ *is contained in every Sobolev space* $H^s[-\frac{A}{G}, \frac{A}{G}]$. *As such,*

$$S^{\text{conv}} : L^2[-\tfrac{A}{G}, \tfrac{A}{G}] \to H^s[-\tfrac{A}{G}, \tfrac{A}{G}] \quad \textit{is compact and injective,}$$

Main Theorem 2



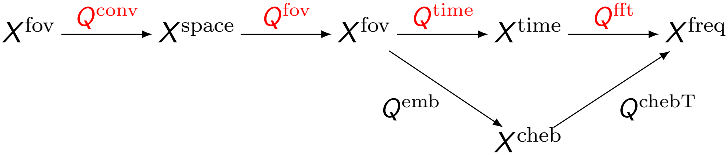
Theorem

The image $R(S^{\mathrm{t}})$ is contained in every Sobolev space $H^s[0, T/2]$, $s \geq 0$. Restricting S^{t} to the space X^{fov}, the operator

$$S^{\mathrm{t}} : X^{\text{fov}} \to H^s[0, T/2] \quad \textit{is compact and injective.}$$

In particular, the inverse problem $S^{\mathrm{t}} c = u$ is severely ill-posed.

Main Theorem 3



Theorem

The image $R(S^{\mathrm{f}})$ of the operator S^{f} is contained in the space c_∞ of rapidly decaying sequences. The operator

$$S^{\mathrm{f}} : X^{\mathrm{fov}} \to c_\infty \quad \text{is compact and injective.}$$

The inverse problem $S^{\mathrm{f}} c = \hat{u}$ is therefore also severely ill-posed.

Some numerical experiments

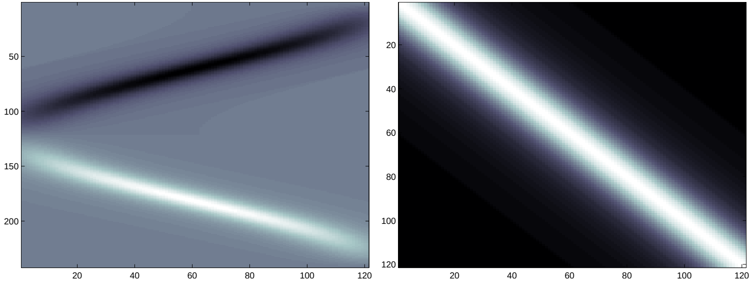


Figure: *Visualization of the discretization of the operators* S^{conv} *and* S^{t} *with* $G = 100$ *and* $A/G = 50$.
Left: $S^{\mathrm{t}} \in \mathbb{R}^{2N \times N}$ with $N = 120$.
Right: $S^{\mathrm{conv}} \in \mathbb{R}^{N \times N}$ with $N = 120$.

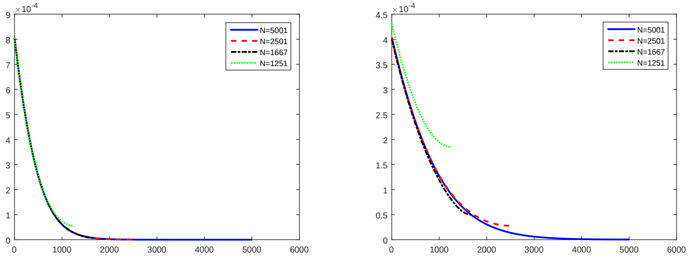


Figure: *Singular values of discretizations of* S^{conv}.

Left: For a field of view width $A/G = 125$, we discretized S^{conv} using N sample points. The singular values are displayed in decreasing order.

Right: For a field of view width $A/G = 250$, we discretized S^{conv} using N sample points. The corresponding singular values are displayed in decreasing order. We observe better approximation for smaller ratio A/G.

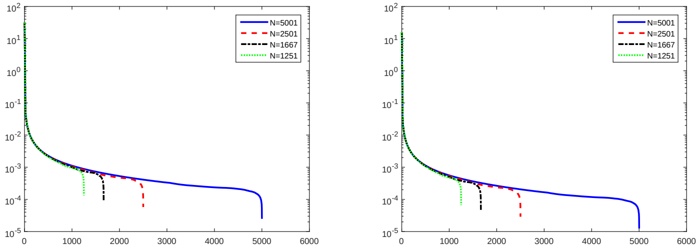


Figure: *Singular values of discretizations of S^{t}.*

Left: For a field of view width $A/G = 125$, we discretized S^{t} using N sample points. The singular values are displayed in decreasing order.

Right: For a field of view width $A/G = 250$, we discretized S^{t} using N sample points. The singular values are displayed in decreasing order. We observe that the singular values initially decrease fast, and that there are some almost vanishing singular values for each discretization.

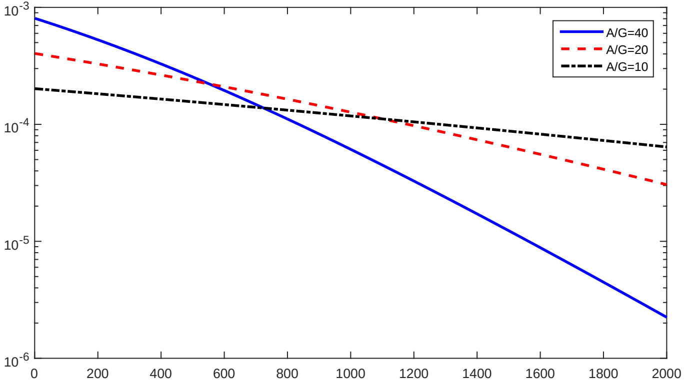


Figure: The first 2000 singular values of S^{conv} discretized using $N = 5001$ sampling points for different field of view widths A/G. We view these singular values as a good approximation of the singular values of S^{conv}. The semi-logarithmic plot suggests that the singular values decrease exponentially.

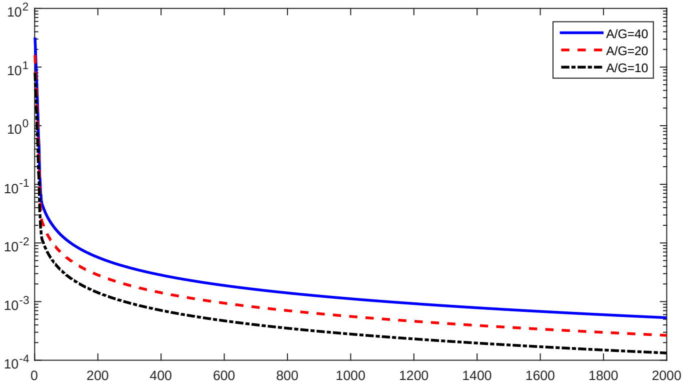


Figure: The first 2000 singular values of S^{t} discretized using $N = 5001$ sampling points for different field of view widths A/G. We view these singular values as a good approximation of the singular values of S^{t}. The semi-logarithmic plot shows that the singular values decrease very fast initially but then have a slower decay than in the previous figure.

Literature

[1] Erb W., Weinmann A., Ahlborg M., Brandt C.,Bringout G., Buzug T.M., Frikel J., Kaethner C., Knopp T., März T., Möddel M., Storath M., Weber A.
Mathematical Analysis of the 1D Model and Reconstruction Schemes for Magnetic Particle Imaging.
arXiv:1711.08074 [math.NA] (2017)

[2] Knopp T. and Buzug T.M.
Magnetic Particle Imaging: An Introduction to Imaging Principles and Scanner Instrumentation.
Springer, 2012

[3] Rahmer J., Weizenecker J., Gleich B. and Borgert J.
Signal encoding in magnetic particle imaging: properties of the system function.
BMC Medical Imaging 9:4 (2009)