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Spectral Method for a Particular Case of the Heat Convection-Diffusion Equation

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Abstract

The purpose of this paper is to study the Legendre spectral method for solving a particular case of the heat convection-diffusion equation, wich is formulated as a mixed initial boundary value problem within the finite regular domain $\Lambda = (-1, 1)$. To tackle this problem, we employ certain techniques to transform it into a system of ordinary differential equations. Through matrix analysis, we derive a general term that characterizes all the ordinary differential equations in this system. solving this general term, provides the desired approximate solution, and we also present the error estimation.

Keywords: Heat convection-diffusion equation, Spectral method, Orthogonal polynomials, Error estimate.
MSC 2020 Classifications: 80A19, 65M70.

1 Introduction

The primary motivation of this work is the numerical analysis of discretization of a particular case of the heat convection-diffusion equation with a source term, formulated by mixed initial-boundary value problem using Legendre spectral method, which involves representing the solution in terms of truncation series. see also [1, 2, 5, 8, 10, 17, 18, 20, 21]. For more details and a comprehensive analysis of spectral methods, we refer to [9, 11, 12, 13, 14, 15].

The problem under consideration is governed by the following equation:

$$\begin{cases} \partial_t w(t,x) - \partial_x^2 w(t,x) + \partial_x w(t,x) + w(t,x) = \psi(t,x), & x \in \Lambda, t > 0 \\ w(t,-1) = w(t,1) = 0, & t > 0 \\ w(0,x) = g(x), & x \in \Lambda \end{cases}, \tag{1}$$

where $w(t,x)$ represents the temperature at the position x at time t, while $\psi(t,x)$ and $g(x)$ are known functions.

In this work, we propose an approximate solution to the inhomogeneous problem **(1)** using a truncated series expansion of the form:

$$w_N(t,x) = \sum_{n=1}^{N-1} \alpha_n(t)\ell_n(x) \tag{2}$$

with

$$\alpha_n(t) = \sum_{m=1}^{N-1} u_{nm}\ell_m(t),$$

Where $\ell_n(x)$ are the Lagrangian interpolating polynomials defined at the points x_j, with $-1 \le x_j \le 1$ and $0 \le j \le N$. These polynomials satisfy the property $\ell_i(x_j) = \delta_{ij}$ with $0 \le i, j \le N$. The discretization points $x_j, 0 \le j \le N$ are chosen as the collocation points on the Gauss-Lobatto Legendre grid, and the grid formed by x_j is denoted by Σ_{N+1}.

With this choice of solution form **(2)**, along with some additional techniques, we obtain a linear system which can be expressed in matrix form as $MD\alpha - B\alpha = MG$, where M is a diagonal invertible matrix, B is a square, positive-definite matrix, and $D = \frac{d}{dt}$ is the time derivative operator. To simplify the system, we define $\alpha = F\nu$ where F is an orthogonal matrix such that $F^{-1}\left(M^{-1}B\right)F = \Gamma$, with Γ being a diagonal matrix. This results in a system of $N-1$ ordinary differential equations.

We apply Lagrange's method of undetermined coefficients to solve for each component $\nu_i(t)$ of ν. Finally, we express the functions $\alpha_n(t)$ and and compute the coefficients u_{nm} with $1 \le n, m \le N-1$, yielding the desired approximate solution, see also [5, 6, 19, 22, 23, 25].

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2 Orthogonal polynomials

2.1 Properties

The Legendre polynomials L_n, where $n \geq 0$,are polynomials of degree n. These polynomials are orthogonal to each other in the space:

$$L^2(\Lambda) = \left\{ \psi : \Lambda \to \mathbb{R}, \text{measurable } / \int_\Lambda \psi^2(x)\, dx < +\infty \right\}.$$

The following properties are satisfied:

$$\int_\Lambda L_m(x) L_n(x) dx = \frac{1}{n+\frac{1}{2}} \delta_{mn},$$

where δ_{mn} is the Kronecker delta.

$$h_n(x) = \frac{n(n+1)}{2n+1} \left(L_{n-1}(x) - L_{n+1}(x) \right), \; n > 0, \tag{3}$$

$$L_n(x) = \frac{-1}{n(n+1)} h'_n(x), \; h_n(x) = \left(1 - x^2\right) L'_n(x), \; n \geq 0, \tag{4}$$

$$\int_\Lambda h_n^2(x)\, dx = \frac{[2n(n+1)]^2}{(4n^2-1)(2n+3)}. \tag{5}$$

3 Continuous problem

The problem **(1)** formulated in $L^2(\Lambda)$ as the pivot space, with the variational space defined as:

$$H^1(\Lambda) = \left\{ u \in L^2(\Lambda) \; / \; \partial_x u \in L^2(\Lambda) \right\}.$$

The corresponding norms for these spaces are defined as follows:

$$\|u\|^2_{L^2(\Lambda)} = \int_\Lambda u^2 dx,$$

$$\|u\|^2_{H^1(\Lambda)} = \int_\Lambda (u^2 + (\partial_x u)^2) dx.$$

3.1 Variational formulation

To derive the variational formulation for the continuous problem **(1)**, we define the subspace of the variational space with zero Dirichlet boundary conditions as:

$$V_0(\Lambda) = \left\{ w \in H^1(\Lambda) : \; w = 0 \text{ on } \partial\Lambda, \, t > 0, \, w = \mathrm{g}, \text{ at } t = 0 \right\}. \tag{6}$$

Next, we introduce the scalar product in $L^2(\Lambda)$ as follows:

$$(\psi, \phi) = \int_\Lambda \psi(t,x)\phi(t,x) dx. \tag{7}$$

The variational formulation of the problem **(1)** is then stated as:
Find $w \in V_0(\Lambda)$ such that for all $v \in V_0(\Lambda)$,

$$\alpha(w, v) = \langle \psi, v \rangle, \tag{8}$$

where the bilinear form $\alpha(.,.)$ is given by:

$$\alpha(w, v) = \int_\Lambda \left(\partial_t w - \partial_x^2 w + \partial_x w + w \right) v\, dx. \tag{9}$$

By integrating by parts, this becomes:

$$\alpha(w, v) = \int_\Lambda [\partial_t w v + \partial_x w \partial_x v + \partial_x w v + w v]\, dx. \tag{10}$$

4 Discrete form and space

Let N denote the discretization parameter for the problem **(1)**. In the context of spectral methods, N represents the degree of the polynomials used. The approximate space is generated by a finite-dimensional subspace of of $L^2(\Lambda)$, $\mathbb{P}_N^{\triangle}(\Lambda)$ is the approximate space corresponding to $V_0(\Lambda)$, where

$$\mathbb{P}_N^{\triangle}(\Lambda) = \left\{ q_n \in \mathbb{P}_N(\Lambda) : q_n(t,x) = 0 \text{ on } \partial\Lambda,\ q_n(x,0) = \sum_{k=1}^{N-1} g(x_k)\ell_k(x) \right\}.$$

Here, $\mathbb{P}_N(\Lambda)$ represents the set of polynomials of degree less than or equal to N. Additionally, we consider the exact quadrature formula and introduce a bilinear form α_N which approximates the form α. We also approximate the scalar product $(.,.)$ by the discrete forme $(.,.)_N$, see also [1, 3, 4, 6, 7, 9, 11, 19, 24].

4.1 Variational formulation of the discrete Problem

Firstly, we observe that the Lagrange polynomials $\ell_k(x)$, where $0 \le k \le N$, form a basis for $\mathbb{P}_N^{\triangle}(\Lambda)$. Accordingly, the exact solution w to the problem **(1)** is approximated by the solution w_N^I that belongs to $\mathbb{P}_N^{\triangle}(\Lambda)$, with $\left(w_N^I - g_N\right)$ belonging to $\mathbb{P}_N^{\nabla}(\Lambda)$, where

$$\mathbb{P}_N^{\nabla}(\Lambda) = \{q_n \in \mathbb{P}_N(\Lambda) :\ q_n(t,x) = 0 \text{ on } \partial\Lambda\}.$$

The variational problem is:

$$\begin{cases} \text{find } w_N^I \in \mathbb{P}_N^{\triangle}(\Lambda), \text{ s.t} \\ \forall v_N \in \mathbb{P}_N^{\nabla}(\Lambda), \alpha_N(w_N^I, v_N) = (\psi_N, v_N)_N \end{cases}, \tag{11}$$

where

$$\alpha_N(w_N^I, v_N) = \sum_{k=0}^{N} \left(\partial_t w_N^I v_N + \partial_x w_N^I \partial_x v_N + \partial_x w_N^I v_N + w_N^I v_N\right)(t, x_k)\rho_k, \tag{12}$$

with x_k and ρ_k defined in proposition **(4.1)**, and $w_N^I = w_N + g_N$, where $w_N \in \mathbb{P}_N^{\nabla}(\Lambda)$. The problem **(11)** is equivalent to the following problem: Find w_N^I in $\mathbb{P}_N^{\triangle}(\Lambda)$ such that $w_N = w_N^I - g_N$ in $\mathbb{P}_N^{\nabla}(\Lambda)$, and for all $v_N \in \mathbb{P}_N^{\nabla}(\Lambda)$:

$$\alpha_N(w_N, v_N) = \beta_N(g_N, v_N), \tag{13}$$

where:

$$\beta_N(g_N, v_N) = (\psi_N, v_N)_N - \alpha_N(g_N, v_N). \tag{14}$$

4.2 Existence and uniqueness of the solution

4.2.1 Gauss-Lobatto-Legendre quadrature

Proposition 4.1. *There exists a unique set of $N-1$ nodes $x_{k'}$, where $1 \le k' \le N-1$, within the interval Λ, with boundary conditions $x_0 = -1$ and $x_N = 1$. Additionally, there are $N+1$ positive weights $\rho_{k'}$, for $0 \le k' \le N$, such that the following exactness property holds:*

$$\forall \psi \in \mathbb{P}_{2N-1}(\Lambda), \int_{-1}^{1} \psi(x)\,dx = \sum_{k'=0}^{N} \psi(x_{k'})\rho_{k'}, \tag{15}$$

where $x_{k'}$, for $1 \le k' \le N-1$, represent the roots of the polynomial L_N', and the corresponding weights $\rho_{k'}$ are defined as follows:

$$\begin{cases} \rho_0 = \rho_N = \frac{2}{N(N+1)} \\ \rho_{k'} = \frac{\rho_0}{L_N^2(x_{k'})}, 1 \le k' \le N-1 \end{cases}.$$

Proof. See [2, 10, 11]. □

Definition 4.1. [1, 10]. We define the discrete inner product for all polynomials w_N and v_N in $\mathbb{P}_N^{\triangle}(\Lambda)$ as:

$$(w_N, v_N)_N = \sum_{k'=0}^{N} u_N(t, x_{k'}) v_N(t, x_{k'})\rho_{k'}. \tag{16}$$

Lemma 4.2. *The polynomial $h_{N-1} \in \mathbb{P}_N^0(\Lambda)$ verifies the double inequality:*

$$\|h_{N-1}\|_{L^2(\Lambda)}^2 \le (h_{N-1}, h_{N-1})_N \le \frac{3}{2}\|h_{N-1}\|_{L^2(\Lambda)}^2. \tag{17}$$

Here, the subspace $\mathbb{P}_N^0(\Lambda)$ is defined as:

$$\mathbb{P}_N^0(\Lambda) = \{p_n \in \mathbb{P}_N(\Lambda)\ /\ p_n(x) = 0 \text{ on } \partial\Lambda\}.$$

Proof. Using equations **(4)** and **(5)** in **(16)** and applying the exact quadrature formula,we can derive the desired result. For further details, see references [1, 6]. □

Proposition 4.3. *For any polynomial h_n in the space $\mathbb{P}_n^0(\Lambda)$, the following inequality holds:*

$$\|h_n\|_{L^2(\Lambda)} \leq \frac{1}{n}\left\|h_n^{'}\right\|_{L^2(\Lambda)} \leq 3\,\|h_n\|_{L^2(\Lambda)}. \tag{18}$$

Proof. By using **(3)**, **(4)** and **(5)**, and applying integration by parts along with the Cauchy-Schwarz inequality, we can derive the desired result. □

The Lagrange polynomials $\ell_j(x)$, $j = 1, ..., N-1$, can be expressed in the following form:

$$\ell_j(x) = \sum_{n=0}^{N-1} \gamma_{nj} h_n(x),$$

and using equation **(4)**, we obtain:

$$\ell_j(x) = \sum_{n=0}^{N-1} \lambda_{nj} L_n(x).$$

Proposition 4.4. *The set of Legendre polynomials $\{L_n(x)\}$ for $n = 0, ..., N$ forms a basis for the space $\mathbb{P}_N(\Lambda)$. Therefore, any polynomial $\psi_N \in \mathbb{P}_N(\Lambda)$ can be expressed as: $\psi_N(x) = \sum_{n=0}^{N} \alpha_n L_n(x)$, and the following inequality holds:*

$$c_3 \log(2N+1) \leq \|\psi\|^2_{L^2(\Lambda)} \leq c_4 \log(\exp(2)(2N+1)), \tag{19}$$

where $(c_3, c_4) = \left(\min(\alpha_n^2), \max(\alpha_n^2)\right)$.

Proof. See [2, 6]. □

Proposition 4.5. *For a positive integer m, the Sobolev space $H^s(\Lambda)$ is defined by:*

$$H^s(\Lambda) = \left\{\psi \in L^2(\Lambda) \,/\, 1 \leq k' \leq s, \frac{d^{k'}}{dx^{k'}}\psi \in L^2(\Lambda)\right\},$$

with the corresponding norm given by:

$$\|\psi\|^2_{H^s(\Lambda)} = \int_\Lambda \sum_{k'=0}^{s}\left(\frac{d^{k'}}{dx^{k'}}\psi\right)^2(x)\,dx.$$

Proposition 4.6. *The bilinear form $\alpha_N(.,.)$ defined in **(13)** satisfies respectively the following properties of continuity and ellipticity:*

$$\textbf{\textit{For all}}\ \ w_N, v_N \in \mathbb{P}_N^{\nabla}(\Lambda), \quad |\alpha_N(w_N, v_N)| \leq \frac{3}{2}\max(1, C_2)\left(\|w_N\|_{H_0^1(\Lambda)} \cdot \|v_N\|_{H_0^1(\Lambda)}\right), \tag{20}$$

$$\forall w_N \in \mathbb{P}_N^{\nabla}(\Lambda), \quad |\alpha_N(w_N, w_N)| \geq \min(1, C_1)\left(\|w_N\|^2_{H_0^1(\Lambda)}\right). \tag{21}$$

Proof. The continuity: We assume that the solution and its derivatives are bounded. Then, there exist two positive real constants C_1 and C_2 such that:

$$C_1\,|w_N(t, x_k)| \leq |\partial_t w_N(t, x_k)| \leq C_2\,|w_N(t, x_k)|. \tag{22}$$

Using lemma **(4.2)**, the exact quadrature formula, and the Cauchy-Schwarz inequality, we obtain the desired results. For further details, see also Bernardi et al. [12] and Boutaghou et al. [6].

The ellipticity: The bilinear form $\alpha_N(w_N, w_N)$ is given by:

$$\begin{aligned}\alpha_N(w_N, w_N) \;=\; & \sum_{k=0}^{N}\partial_t w_N(t, x_k) w_N(t, x_k)\rho_k + \sum_{k=0}^{N}\partial_x w_N(t, x_k)\partial_x w_N(t, x_k)\rho_k + \sum_{k=0}^{N}\partial_x w_N(t, x_k) w_N(t, x_k)\rho_k \\ & + \sum_{k=0}^{N} w_N(t, x_k) w_N(t, x_k)\rho_k.\end{aligned}$$

Using the exact quadrature formula, we can rewrite this as:

$$\begin{aligned}\alpha_N(w_N, w_N) \;=\; & \sum_{k=0}^{N}\partial_t w_N(t, x_k) w_N(t, x_k)\rho_k + \int_{-1}^{1}\partial_x w_N(t, x)\partial_x w_N(t, x)dx + \sum_{k=0}^{N}\partial_x w_N(t, x_k) w_N(t, x_k)\rho_k \\ & + \sum_{k=0}^{N} w_N(t, x_k) w_N(t, x_k)\rho_k,\end{aligned}$$

from (22) and the orthogonality properties, we obtain:

$|\alpha_N(w_N,w_N)| \geq C_1 \sum_{k=0}^{N} w_N(t,x_k)w_N(t,x_k)\rho_k + \int_{-1}^{1} \partial_x w_N(t,x)\partial_x w_N(t,x)dx + \sum_{k=0}^{N} w_N(t,x_k)w_N(t,x_k)\rho_k.$

Using inequality (17), we can express this as:

$$|a_N(w_N,w_N)| \geq \min(1,C_1)\left(||w_N||^2_{H_0^1(\Lambda)}\right).$$

Thus, this inequality yields the desired result.

□

Proposition 4.7. *("The inequality of stability") For any continuous function g defined on Λ, the problem (13) has a unique solution w_N in $\mathbb{P}_N^{\nabla}(\Lambda)$, and Moreover, this solution satisfies "the stability inequality"::*

$$||u_N(t,x)||_{H_0^1(\Lambda)} \leq \gamma\left(||\psi_N(t,x)||_{L^2(\Lambda)} + ||g_N(x)||_{L^2(\Lambda)}\right),$$

where γ is a constant.

Proof. By equation (13), we obtain:

$$\alpha_N(w_N,w_N) = (\psi_N,w_N)_N - \alpha_N(g_N,w_N) \leq |(\psi_N,w_N)_N| + |\alpha_N(g_N,w_N)|,$$

Next, applying inequality (17) and the Cauchy-Schwarz inequality, we obtain:

$$\begin{aligned} |(\psi_N,w_N)_N| + |\alpha_N(g_N,w_N)| \quad &\leq \quad \frac{3}{2}||\psi_N(t,x)||_{L^2(\Lambda)}.||w_N(t,x)||_{L^2(\Lambda)} + ||\partial_x g_N(x)||_{L^2(\Lambda)}.||\partial_x w_N(t,x)||_{L^2(\Lambda)} \\ &+ ||\partial_x g_N(x)||_{L^2(\Lambda)}.||w_N(t,x)||_{L^2(\Lambda)} + \frac{3}{2}||g_N(x)||_{L^2(\Lambda)}.||w_N(t,x)||_{L^2(\Lambda)}. \end{aligned}$$

The terms $||\partial_x g_N(x)||_{L^2(\Lambda)}$ and $||\partial_x w_N(t,x)||_{L^2(\Lambda)}$ are bounded.Therefore, there exists a positive constant γ such that:
$\alpha_N(w_N,w_N) \leq |(\psi_N,w_N)_N| + |\alpha_N(g_N,w_N)| \leq \gamma\left(||\psi_N(t,x)||_{L^2(\Lambda)} + ||g_N(x)||_{L^2(\Lambda)}\right)||w_N(t,x)||_{H_0^1(\Lambda)}.$
Finally, using inequality (21), we obtain the desired result. □

5 Numerical results

At the points x_k for $1 \leq k \leq N-1$, the problem (1) is equivalent to the following system of equations:

$$\begin{cases} \sum_{n=1}^{N-1}\ell_n(x_k)\alpha_n'(t) + \left[\ell_n(x_k) + \ell_n'(x_k) - \ell_n'(x_k)\right]\alpha_n(t) = \sum_{n=1}^{N-1}\psi_n(t)\ell_n(x_k) + g_N''(x_k) - g_N'(\mathbf{x}_k) - g_N(x_k) & \text{in } \Lambda \cap \Sigma_{N+1} \\ w_N(t,x_k) = 0 & \text{on } \partial\Lambda \cap \Sigma_{N+1} \\ \psi(t,x) = \sum_{n=1}^{N-1}\psi_n(t)\ell_n(x), \quad \psi_n(t) = \sum_{j=1}^{N-1}\psi_{jn}\ell_j(t), \quad \psi_{jn} = \psi\left(t_n,x_j\right) \end{cases}$$

Since the functions $\ell_n(x)+\ell_n'(x)-\ell_n''(x)$ for $1 \leq n \leq N-1$ are polynomials with degree N, we multiply both sides of the equation by $\ell_m(x_k)\rho_k$ and apply the sum. Using the quadrature formula, and varying m from 1 to $N-1$, we obtain a linear system. This system can be expressed in matrix:

$$MD\alpha - B\alpha = MG, \tag{23}$$

where B is a square, positive-definite matrix of order $N-1$, with elements defined as:

$$\beta_{mn} = [-\ell_n(x_m) - \ell_n'(x_m) + \ell_n''(x_m)]\ell_m(x_k)\rho_m, \ n,m = 1,...,N-1,$$

M is a diagonal, invertible matrix with elements defined as:

$$\gamma_{mn} = \begin{cases} \rho_n \ , m = n \\ 0, \quad m \neq n \end{cases} \quad , \ m,n = 1,...,N-1.$$

G is a known vector given by:

$$\begin{aligned} G = (&\psi_1(t) + g_N''(x_1) - g_N'(x_1) - g_N(x_1), \psi_2(t) + g_N''(x_2) - g_N'(x_2) - g_N(x_2) \\ &,...,\psi_{N-1}(t) + g_N''(x_{N-1}) - g_N'(x_{N-1}) - g_N(x_{N-1}))^T, \end{aligned}$$

α is the unknown vector:

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), ..., \alpha_{N-2}(t), \alpha_{N-1}(t))^T,$$

$D = \frac{d}{dt}$ is the time derivative operator. Multiplying equation (23) by the invertible matrix M^{-1}, we obtain:

$$D\alpha - M^{-1}B\alpha = G\ , \tag{24}$$

the matrix $M^{-1}B$ has positive eigenvalues, and there exists an orthogonal, invertible matrix F such that:

$$F^{-1}\left(M^{-1}B\right)F=\Gamma,$$

where Γ is a diagonal matrix whose elements are the eigenvalues $\lambda_i=\beta_{ii}$ for $i=1,...,N-1$ of the matrix $M^{-1}B$, If we define the vector ν such that:

$$\alpha=F\nu,$$

then the system (24) becomes:

$$FD\nu-(M^{-1}B)F\nu=G, \tag{25}$$

Multiplying equation (25) by the the invertible matrix F^{-1}, we get:

$$D\nu-\Gamma\nu=F^{-1}G. \tag{26}$$

The matrix form (26) corresponds to $N-1$ linear ordinary differential equations of the form:

$$\nu_k'(t)-\lambda_k\nu_k(t)=h_k(t),\ 1\le k\le N-1, \tag{27}$$

where $h_k(t)$ is given by:

$$h_k(t)=\sum_{j=1}^{N-1}f_{k,j}^{-1}\left(\psi_j(t)+g_N''(x_k)-g_N'(x_k)-g_N(x_k)\right),\ 1\le k\le N-1, \tag{28}$$

Here, $f_{k,j}^{-1}$ are the elements of the inverse matrix F^{-1}. To solve the equations (27) using the Lagrange's method [25], The solution can be expressed in closed form as

$$\nu_k(t)=e^{\lambda_k t}\left(\int_0^t e^{-\lambda_k s}h_k(s)ds+d_k\right),$$

where d_k is a constant to be determined from the boundary conditions. Thus, equation (5) can be written as:

$$\nu_k(t)=e^{\lambda_k t}\left(\int_0^t e^{-\lambda_k s}h_k(s)ds+\sum_{j=1}^{N-1}f_{kj}^{-1}g_N(x_k)\right),$$

Finally, we obtain the functions:

$$\alpha_n(t)=\sum_{j=1}^{N-1}f_{nj}\nu_j(t),$$

where f_{nj} with $1\le j,n\le N-1$ are the elements of the matrix F, and the approximate solution is:

$$w(t,x)=\sum_{n=1}^{N-1}\sum_{j=1}^{N-1}f_{nj}\left(\int_0^t e^{-\lambda_k s}h_k(s)ds+\sum_{j=1}^{N-1}f_{kj}^{-1}g(x_k)\right)e^{\lambda_k t}\ell_n(x).$$

Given that t is in the interval $I=[0,T]$, we can consider the solution in the form:

$$w(t,x)=\sum_{n=1}^{N-1}\sum_{j=1}^{N-1}u_{nj}\ell_n(x)\ell_j(t),\quad \alpha_n(t)=\sum_{j=1}^{N-1}u_{nj}\ell_j(t),$$

where the coefficients u_{nj} are determined by:

$$u_{nj}=\sum_{j=1}^{N-1}f_{nj}\left(\int_0^{t_j} e^{-\lambda_k s}h_k(s)ds+\sum_{j=1}^{N-1}f_{kj}^{-1}g_N(x_k)\right)e^{\lambda_k t_j},$$

Thus, the approximate solution is:

$$w_N(t,x)=\sum_{n=1}^{N-1}\sum_{m=1}^{N-1}\left(\sum_{j=1}^{N-1}f_{nj}\left(\int_0^{t_j} e^{-\lambda_k s}h_k(s)ds+\sum_{j=1}^{N-1}f_{kj}^{-1}g_N(x_k)\right)e^{\lambda_k t_j}\right)\ell_n(x)\ell_m(t)+\sum_{n=1}^{N-1}g_N(x_n)\ell_n(x).$$

Using the expression (28), the approximate solution $w_N(t,x)$ can be written as:

$$\begin{aligned}w_N(t,x) &= \sum_{n=1}^{N-1}\sum_{m=1}^{N-1}\left(\sum_{j=1}^{N-1}f_{nj}\left(\int_0^{t_j} e^{-\lambda_k(s-t_j)}\sum_{j=1}^{N-1}f^{-1}(k,j)\left(\psi_j(s)+g_N''(x_k)-g_N'(\mathbf{x}_k)-g_N(x_k)\right)\right)ds\right.\\ &\quad \left.+\left(\sum_{j=1}^{N-1}f_{kj}^{-1}g_(x_k)\right)e^{\lambda_k t_j}\right)\ell_n(x)\ell_m(t)+\sum_{n=1}^{N-1}g_N(x_n)\ell_n(x),\end{aligned}$$

5.1 Error estimate

Definition 5.1. Let $\mathbb{P}_N^{\triangle}(\Lambda)$ be the polynomial space, which is dense in the space of continuous functions on Λ, and consequently in $V_0(\Lambda)$. This implies that any function $w \in V_0(\Lambda)$ admits the following expansion:

$$w(t,x) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \alpha(k,l) h_k(x) t_l(t).$$

We know

$$T_n(t) = \frac{n(n+1)}{2(2n+1)} \left(p_{n-1}(t) - p_{n+1}(t)\right), \tag{29}$$

where

$$p_n(t) = L_n\left(\frac{2}{T}t - 1\right), n \geq 0,$$

and using **(29)** then

$$w(t,x) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \gamma(k,l) h_k(x) p_l(t).$$

Proposition 5.1. *Let $winH_0^1(\Lambda)$ be the exact solution and $w_N \in \mathbb{P}_N^{\nabla}(\Lambda)$ be the approximate solution. The following error estimate holds between these solutions:*

$$\|w - w_N\|_{L^2(\Lambda)} \leq 3CN^{-1}\left(\|(g - g_N)\|_{L^2(\Lambda)} + \|\psi - \psi_N\|_{L^2(\Lambda)}\right),$$

C is a real positive constant.

Proof. We begin by using the ellipticity condition **(21)** and **(18)** to write the following expression:

$$\begin{aligned} N^2 \|w - w_N\|^2_{L^2(\Lambda)} &\leq \alpha(w - w_N, w - w_N) = (\psi - \psi_N, w - w_N)_N - \alpha(g - g_N, w - w_N), \\ &\leq C\left(\left|\int_\Lambda (\psi - \psi_N)(w - w_N)\,dx\right| + |\alpha(g - g_N, w - w_N)|\right). \end{aligned} \tag{30}$$

Where C is a real positive constant, applying the Cauchy-Schwarz inequality, we obtain:

$$\left|\int_\Lambda (\psi - \psi_N)(w - w_N)\,dx\right| \leq \|\psi - \psi_N\|_{L^2(\Lambda)} \|w - w_N\|_{L^2(\Lambda)}, \tag{31}$$

thanks to a triangle inequality, this yields

$$\begin{aligned} |\alpha(g - g_N, w - w_N)| &\leq \left|\int_\Lambda \partial_x(g - g_N)\partial_x(w - w_N)\,dx\right| + \left|\int_\Lambda \partial_t(g - g_N)(w - w_N)\,dx\right| \\ &+ \left|\int_\Lambda \partial_x(g - g_N)(w - w_N)\,dx\right| + \left|\int_\Lambda (g - g_N)(w - w_N)\,dx\right|, \end{aligned}$$

Note that since g does not depend on the time variable t the term involving $\partial_t(g - g_N)$ vanishes:

$$\int_\Lambda \partial_t(g - g_N)(w - w_N)\,dx = 0.$$

Now, we apply the Cauchy-Schwarz inequality to each of these integrals:

$$\left|\int_\Lambda (g - g_N)(w - w_N)\,dx\right| \leq \|(g - g_N)\|_{L^2(\Lambda)} \|(w - w_N)\|_{L^2(\Lambda)}. \tag{32}$$

We also apply the Cauchy-Schwarz inequality:

$$\left|\int_\Lambda \partial_x(g - g_N)\partial_x(w - w_N)\,dx\right| \leq \|\partial_x(g - g_N)\|_{L^2(\Lambda)} \|\partial_x(w - w_N)\|_{L^2(\Lambda)}. \tag{33}$$

We again apply the Cauchy-Schwarz inequality:

$$\left|\int_\Lambda \partial_x(g - g_N)(w - w_N)\,dx\right| \leq \|\partial_x(g - g_N)\|_{L^2(\Lambda)} \|(w - w_N)\|_{L^2(\Lambda)}. \tag{34}$$

using **(31)**, **(32)**, **(33)**, **(34)** and **(18)** into **(30)** , we get:

$$N^2 \|w - w_N\|^2_{L^2(\Lambda)} \leq 3CN\left(\|(g - g_N)\|_{L^2(\Lambda)} + \|\psi - \psi_N\|_{L^2(\Lambda)}\right) \|(w - w_N)\|_{L^2(\Lambda)}.$$

This yields the desired result. □

5.2 Condition number

Definition 5.2. The condition number of a non-singular $n \times n$ matrix B is defined as:

$$k_P(B) = \|B\|_P \left\|B^{-1}\right\|_P,$$

where $\|B\|_P$ is the spectral norm of B, which is given by: $\rho = \left(B^T B\right)^{\frac{1}{2}}$.

Remark 1. The condition number of a matrix A gives a measure of how sensitive systems of equations, with coefficients matrix A, are to small perturbations such as those caused by rounding. Then if the condition number of a matrix is large, the effect of rounding error in the solution process may be serious [25]. To compute the condition number of different order of these matrix we use the spectral norm, and all operations are made by the Maple, using [16].

5.3 Figure illustration

The Figures 1 and 2 present the behavior of the garithm of the condition number and the error as N vary from 3 to 10. In Figure 3, the variation of the functions $\alpha_n(t)$ is presented as n changes from 3 to 10. Figures 4 and 5, display the true and approximate solutions, w and w_N respectively, when $N = 10$. We consider the true explicit solution: $w(t,x) = \sin(\pi x)\exp(-(\frac{\pi}{10})^2 t)$ and $\psi(t,x) = \exp(-(\frac{\pi}{10})^2 t)\left((0.99\pi^2 + 1)\sin(\pi x) + \pi\cos(\pi x)\right)$.

Remark 2. This figure shows that the error decreases rapidly when N increass. Here we plot $\left(N, \|w - w_N\|_{L^2(\Lambda)}\right)$.

6 Conclusion

In this work, we proposed a numerical method for solving a specific case of the heat convection-diffusion equation, using the Legendre spectral method. The primary objective was to reduce the two-dimensional problem to a one-dimensional domain, leading to significant computational efficiency. As a result, the linear systems **(23)**,**(24)**,**(25)** and **(26)** are of size $(N-1)$, in contrast to other methods where the matrix order is $(N-1)^2$ (as seen in [10, 11]). This reduction in the size of the system allows for faster computations while maintaining high accuracy in the approximated solution.

The results obtained from the numerical experiments show that the proposed method provides an efficient and reliable solution to the problem, with the error estimates confirming the method's accuracy. The spectral approach, combined with Lagrange interpolation and Gauss-Lobatto nodes, demonstrates its capability to handle the problem efficiently, even in the presence of complex boundary and initial conditions.

Looking ahead, there are several avenues for future research that could further enhance the applicability and performance of this method. First, the proposed method could be extended to more complex, nonlinear convection-diffusion equations, which would broaden its scope. Additionally, the technique could be adapted to handle problems with time-dependent boundary conditions or higher-dimensional domains, which are common in real-world applications.

Another potential direction is the development of adaptive spectral methods, which could automatically refine the grid in regions where the solution exhibits sharp gradients or singularities, further improving accuracy. Furthermore, exploring parallelization strategies for solving the resulting linear systems could lead to significant computational savings, particularly in large-scale simulations.

Lastly, it would be interesting to investigate the hybridization of the Legendre spectral method with other numerical techniques, such as finite element or finite difference methods, to combine the strengths of each approach. This could provide more flexibility and improve the method's robustness in handling a wider range of physical problems.

In conclusion, the Legendre spectral method shows great promise for solving heat convection-diffusion equations, and its future development could play a key role in advancing numerical methods for more complex scientific and engineering applications.

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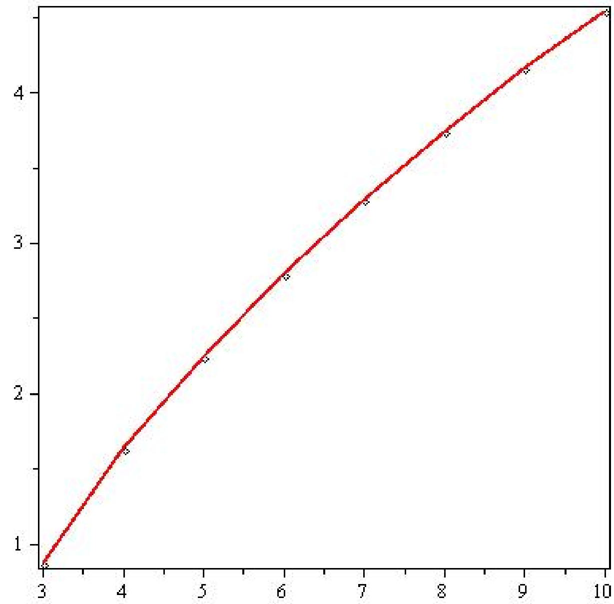

Figure 1: The condition number when N vary from 3 to 10

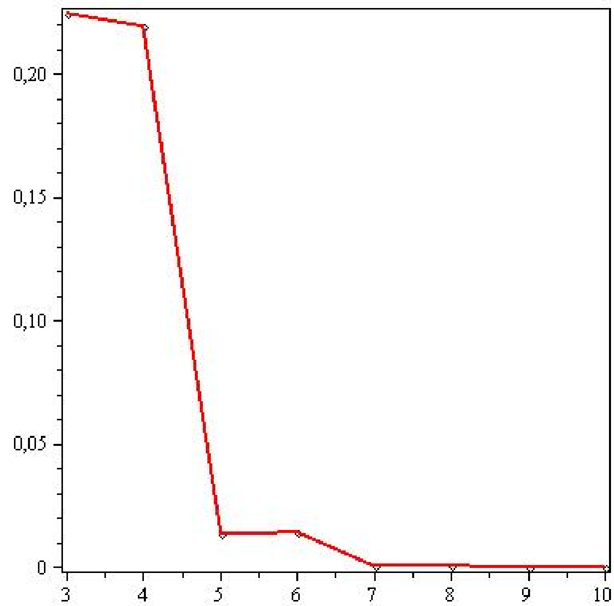

Figure 2: The behavior of the error when N vary from 3 to 10

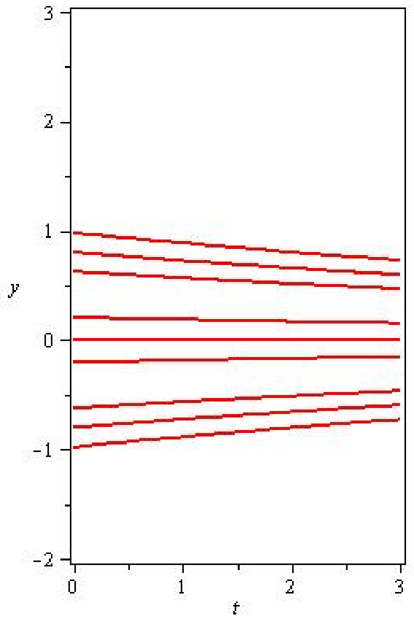

Figure 3: Plots of the functions $\alpha_n(t)$, n vary from 1 to 9

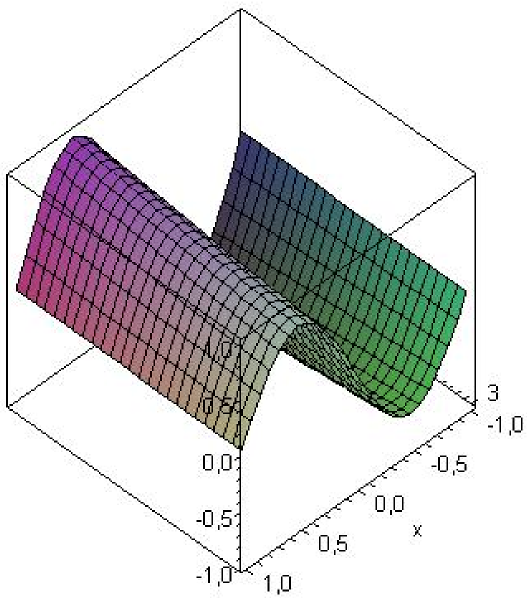

Figure 4: The true solution $w(t, x)$

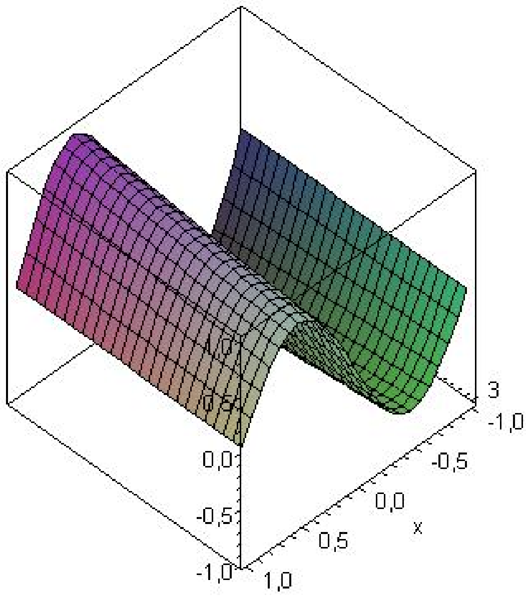

Figure 5: The approximate solution $w_N(t,x)$ when $N = 10$