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A New Approach for Solving Minimax Problems Using New Generation Smoothing Techniques

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Abstract

This study focuses on solving finite minimax problems. A new reformulation for the minimax problems is established based on indicator functions. The relations between the original and reformulated problems are investigated. Based on the new formulation of minimax problems, a new smoothing approach is proposed via the approximation of the indicator functions. A new algorithm is developed to solve the reformulated and smoothed problems. Finally, the performance of the algorithm is illustrated on some test problems, and the comparison of the obtained numerical results with the other methods is presented.

Keywords: Minimax problems, smoothing, non-smooth optimization.

MSC(2010): 90C33, 65K05,65D10.

1 Introduction

The minimax problem is a prominent non-smooth optimization problem that finds extensive application areas such as data fitting [1], engineering design [2], vehicle routing [3], portfolio selection [4], structural optimization [5], resource allocation [6], and others [7, 8, 9, 10, 11]. On the other hand, minimax problems are often used to address situations where uncertainty exists. The goal in uncertainty problems is to minimize the impact of the worst-case scenario and thus ensure robustness against unknown or unpredictable factors. Minimax problems are a subset of uncertainty problems that focus on worst-case optimization. When the decision maker exhibits risk aversion or the measurement of uncertainty is imprecise, they serve as valuable tools. Minimax techniques can be used to better balance robustness and performance in larger uncertainty problems [12, 13, 14, 15, 16, 17].

The minimax problem is stated as follows:

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1}$$

where

$$f(x) = \max_{j\in J} f_j(x) \tag{2}$$

and $f_j : \mathbb{R}^n \to \mathbb{R}$, $j \in J = \{1, 2, \dots, m\}$ are continuously differentiable. The problem (1) is difficult to solve since the objective function defined in (2) may be non-differentiable [18]. The invention of an efficient algorithm to address the finite minimax problem is of fundamental importance. In [19], a highly efficient approach for addressing minimax problems with reference to the aggregate method is introduced. An analogous method has been outlined in [20] in which a penalty function is constructed. Many algorithms have been developed in order to solve the problem (1), such as sub-gradient-based methods [21], bundle-methods [22], homotopy methods [23], and smoothing methods [24, 25, 26]. In recent years, many interesting methods have been developed to solve various variants of the problem (1) in [27, 28, 29, 30].

In this article, it is focused on the application of smoothing techniques specifically designed for non-smooth functions. Smoothing techniques enable the application of existing gradient-based methods for addressing finite minimax problems [31]. The concept of smoothing approaches involves the approximation of original, non-smooth functions through the application of smooth functions [32, 33, 34]. The approximation is regulated by adjustable parameters. A smoothing function $\tilde{f}_\varepsilon(x)$ is applied to $f(x)$, utilizing a smoothing parameter $\varepsilon > 0$. This approach transforms the minimax problems into a sequence of smooth optimization problems represented as $\min_{x\in\mathbb{R}^n} \tilde{f}_\varepsilon(x)$ through the implementation of smoothing techniques. Given specific appropriate assumptions, the solutions to $\min_{x\in\mathbb{R}^n} \tilde{f}_\varepsilon(x)$ converge to a solution of the original minimax problem as the smoothing parameter tends towards 0. Two significant categories of smoothing techniques exist. The initial technique is referred to as local smoothing, which involves the process of smoothing the original function within an appropriate vicinity of the kink points. The second technique, referred to as global smoothing, involves constructing smooth functions that serve as approximations of the

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original function throughout the entire domain. It is a challenging endeavor to create a smoothing function for the function $f(x)$ that is described in equation (2) since it contains a great deal of kink points. There have been suggestions made for alternate formulations in order to overcome these issues. Here is a list of some of them in chronological order. This is the restated version of the function $f(x)$ that can be found in [35] by

$$f(x) = f_1(x) + \max\{f_2(x) - f_1(x) + \max\{ \\ \dots \max\{f_{m-1}(x) - f_{m-2}(x) + \max\{f_m(x) - f_{m-1}(x), 0\}, 0\} \dots, 0\}, 0\}, \tag{3}$$

and a smoothing approach is proposed for addressing minimax problems for the first time. The initial local smoothing technique is introduced in [36] for addressing minimax problems through the formulation (3). Nonetheless, the aforementioned formula is beneficial; however, implementing it through computer programming becomes complex when m is large. The alternative penalty form with a smooth approximation is presented in [37] as follows:

$$F(x,\varepsilon) = \varepsilon \ln \sum_{j=1}^{m} \exp\left(\frac{f_j(x)}{\varepsilon}\right), \tag{4}$$

where ε is a smoothing parameter. The formula (4) has the important advantage of including both the penalty term and the smoothing term at the same time. It is effectively employed in numerous gradient-based algorithms [38]. However, when ε is too small, the numerical stability becomes uncontrollable due to an exponential term. A different intriguing representation of $f(x)$ is provided as

$$F(x,t) = t + \sum_{j=1}^{m} \max\{f_j(x) - t, 0\} \tag{5}$$

by adding a new variable t and the relation

$$f(x) = \min_{t\in\mathbb{R}} F(x,t)$$

is proved by [39, 40, 41]. Furthermore, the hyperbolic smoothing method introduced by [42, 43] is utilized to address min-max problems in [40, 41] by referencing formula (5). The smoothing variant of $F(x,t)$ incorporates two factors: the variable t and the smoothing factor ε, which could increase the computing cost as these parameters are modified. In recent years, much focus has been directed on smoothing approaches, with innovative smoothing techniques being developed and effectively implemented for numerous non-smooth issues [44, 45, 46, 47]. However, min-max problems have not been investigated with these new generation smoothing techniques. This study reformulates the function $f(x)$ in (2) to facilitate the application of new generation smoothing techniques for solving problem (1). The new formula is simple to comprehend; it contains no complex terms that would hinder numerical calculations, and it does not require additional parameters for adjustment. This study investigates the relationships between the original and reformulated functions concerning optimal points to demonstrate the equivalence of the original and proposed formulas. We present a novel smoothing technique, drawing inspiration from the work of [46, 48, 49], and elucidate its beneficial properties when applied to the reformulated function. We propose a novel algorithm for the numerical solution of the reformulated and smoothed problem. We analyze several numerical examples to demonstrate the algorithm's efficiency.The subsequent section presents foundational information regarding smoothing techniques. In Section 3, a novel formulation of the min-max problem is proposed. In Section 4, we introduce a new generation smoothing technique derived from the new formulation and examine the convergence properties of this technique. In Section 5, the minimization algorithm designed to identify an approximate solution for the problem (1) is presented. In Section 6, the application of the algorithms to significant test problems is demonstrated. The conclusion section provides final remarks.

2 Preliminaries

In this paper, the notation $\|x\| = \left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}}$ denotes the Euclidean norm in $\mathbb{R}^n$. Let f be an integrable function then, $L^1[a,b]$—norm of f is defined by

$$\|f\|_{L^1[a,b]} = \int_a^b |f(t)| dt.$$

The sub-differential of the function f at the point x_0 is defined as $\partial f(x_0) = \text{conv}\left\{\nabla f_j(x_0) : j \in \{j \in \mathbb{N} : f_j(x_0) = f(x_0)\}\right\}$ where conv is a convex hull of a set. A point $x_0 \in \mathbb{R}^n$ is called a stationary point of f if $0 \in \partial f(x_0)$. Additionally, x_k^* represents the k—th, while x^* denotes the global minimizer. The definition of smoothing function function is give as follows:

Definition 2.1. [31] Let h be a continuous function defined on $\mathbb{R}^n$ to R. The function $\tilde{h} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $h(x)$, if $\tilde{h}(\cdot,\varepsilon)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed β, and for any $x \in \mathbb{R}^n$,

$$\lim_{y\to x, \varepsilon\to 0} \tilde{h}(y,\varepsilon) = h(x).$$

The smoothing functions are generally focus on “max” function

$$q(t) = \max\{t, 0\}$$

for $t \in \mathbb{R}$ due to existence of “max” function at the core of the problem. The first smoothing function (defined by Bertsekas [35]) is the following:

$$\tilde{q}(t,\varepsilon,\beta) = \begin{cases} t - \frac{(1-\varepsilon)^2}{2\beta}, & \frac{(1-\varepsilon)}{\beta} \leq t, \\ yt + \frac{1}{2}\beta t^2, & \frac{-\varepsilon}{\beta} \leq t \leq \frac{(1-\varepsilon)}{\beta}, \\ \frac{-\varepsilon^2}{2\beta}, & t \leq \frac{-\varepsilon}{\beta}, \end{cases} \tag{6}$$

where ε and β parameters with $0 < \varepsilon < 1$ and $\beta > 0$. Zang [36] proposed the following smoothing function

$$\tilde{q}(t,\varepsilon) = \begin{cases} q(t), & |t| \geq \varepsilon, \\ \rho(t,\varepsilon), & |t| \leq \varepsilon, \end{cases} \tag{7}$$

where $\varepsilon > 0$ and $\rho(t,\varepsilon) = \frac{1}{4\varepsilon}t^2 + \frac{1}{2}t + \frac{1}{4}\varepsilon$. The smoothing functions (6) and (7) are used with formula (3). The exponential smoothing is defined as

$$\tilde{q}(t,\varepsilon) = \varepsilon \ln(1 + e^{\frac{x}{\varepsilon}}), \tag{8}$$

where $\varepsilon > 0$ and it is used with the formula (4). In [42], the hyperbolic smoothing technique is defined as

$$\tilde{q}(t,\varepsilon) = \frac{x + \sqrt{x^2 + \varepsilon^2}}{2}, \tag{9}$$

where $\varepsilon > 0$ and it is used with the formula (5) in [40]. In [46], a new generation smoothing technique is introduced, for the first time. In the beginning, the max function re-stated as

$$q(t) = \max\{t, 0\} = t\chi_A(t), \tag{10}$$

where

$$\chi_A(t) \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

The function $\chi_A(t)$ is smoothed by the help of smoothing techniques given in [46] and applied to minimax problem by considering the formula (5) in [49].

3 A New Formulation for the Minimax Problems

In this section, we first present a new reformulation of the problem (1) with the help of indicator functions as an alternative to the formulations (3), (4) and (5). Let us define the following function:

$$F(x) = \sum_{j=1}^{m} f_j(x)\chi_{A_j}(x), \tag{11}$$

where $\chi_{A_j}(x)$ function is the indicator function of the set A_j defined by

$$\chi_{A_j}(x) = \begin{cases} 0, & x \notin A_j, \\ 1, & x \in A_j, \end{cases}$$

where

$$\begin{aligned} A_1 &= \{x \in \mathbb{R}^n : f_1(x) \geq f_i(x), i = 2, 3, \ldots, m\}, \\ A_2 &= \{x \in (\mathbb{R}^n \setminus A_1) : f_2(x) \geq f_i(x), i = 1, 3, 4, \ldots, m\}, \\ &\vdots \\ A_m &= \left\{x \in \left(\mathbb{R}^n \setminus (\cup_{k=1}^{m-1} A_k)\right) : f_m(x) \geq f_i(x), i = 1, 2, \ldots, m-1\right\}. \end{aligned}$$

Remark 1. The collection $\{A_j\}_{j=1}^m$ is defined as $A_j \cap A_i = \emptyset$ for $i, j = 1, 2, \ldots, m$ and $j \neq i$. For any kink point $x_0 \in \mathbb{R}^n$ such that $f_j(x_0) = f_i(x_0)$, if the point $x_0 \in A_j$ then $x_0 \notin A_i$.

Based on the Remark 1, the set A_j is also defined as

$$A_j = \bigcap_{\substack{i=1 \\ i\neq j}}^{m} B_{ji},$$

where $B_{ji} = \{x \in \mathbb{R}^n : f_j(x) > f_i(x)\} \cup \{x_0 \in \mathbb{R}^n : f_j(x_0) = f_i(x_0) \text{ and } j < i\}$. Therefore, we have

$$\chi_{A_j}(x) = \prod_{\substack{i=1 \\ i\neq j}}^{m} \chi_{B_{ji}}(x), \tag{12}$$

where

$$\chi_{B_{ji}}(x) = \begin{cases} 0, & x \notin B_{ji}, \\ 1, & x \in B_{ji}. \end{cases}$$

The function $\chi_{B_{ji}}(x)$ is called as Improved Indicator Function (IIF). Note that once the function $\chi_{B_{ji}}(x)$ is defined, the IIF of B_{ij} is defined as $\chi_{B_{ij}}(x) = 1 - \chi_{B_{ji}}(x)$. Finally, we obtain

$$F(x) = \sum_{j=1}^{m} f_j(x) \prod_{\substack{i=1 \\ i\neq j}}^{m} \chi_{B_{ji}}(x). \tag{13}$$

The formula (13) is called as Improved Indicator Function Model (IIFM) at the rest of the paper. Now, we investigate the equivalence of the formulation IIFM to $f(x)$ in (2). Let $m = 2$, it is easy to see that $f(x) = F(x)$ for any point $x \in \mathbb{R}^n$ such that $f_1(x) \neq f_2(x)$. Let $x_0 \in \mathbb{R}^n$ be a kink point such that $f_1(x_0) = f_2(x_0) = f(x)$. According to formulation (12), $\chi_{A_1}(x) = \chi_{B_{12}}(x)$ and $\chi_{A_2}(x) = \chi_{B_{21}} = 1 - \chi_{B_{12}}(x)$ are obtained. Therefore, we have $\chi_{A_1}(x_0) = 1$ and $\chi_{A_2}(x_0) = 0$, and $f(x_0) = f_1(x_0) = F(x_0)$. The following lemma states equivalence of $f(x)$ and $F(x)$ for $m \geq 2$.

Lemma 3.1. *Assume that the functions $f(x)$ is defined as in (2) and $F(x)$ is defined as in (13). Then*

$$f(x) = F(x),$$

for all $x \in \mathbb{R}^n$.

Proof. For any $x_0 \in \mathbb{R}^n$, there exists an index j_0 such that $f(x_0) = f_{j_0}(x_0)$. That means, $f_{j_0}(x_0) \geq f_j(x_0)$ for all $j = 1, 2, \ldots, m$. Therefore, we have $x_0 \in A_{j_0} = \cap_{\substack{i=1 \\ i\neq j_0}}^{m} B_{j_0 i}$. By considering IIF, we obtain

$$\chi_{A_{j_0}}(x_0) = \prod_{\substack{i=1 \\ i\neq j_0}}^{m} \chi_{B_{j_0 i}}(x_0) = 1,$$

and $\chi_{A_j}(x_0) = 0$ for $j \neq j_0$. It can be easily concluded that

$$\begin{aligned} F(x_0) &= f_{j_0}(x_0)\chi_{A_{j_0}}(x_0) + \sum_{\substack{j\neq j_0 \\ j=1}}^{m} f_j(x_0)\chi_{A_j}(x_0) \\ &= f_{j_0}(x_0). \end{aligned}$$

□

Theorem 3.2. *Assume that the functions $f(x)$ is defined as in (2) and $F(x)$ is defined as in (13). A point $x^* \in \mathbb{R}^n$ is a stationary point of f if and only if it is a stationary point of $F(x)$.*

Proof. By considering the Lemma 3.1, the proof is obtained easily. □

Theorem 3.3. *Assume that the functions $f(x)$ is defined as in (2) and $F(x)$ is defined as in (13). A point $x^* \in \mathbb{R}^n$ is a local minimizer of f if and only if it is a local minimizer of $F(x)$.*

Proof. By considering the Lemma 3.1, the proof is obtained easily. □

The Theorems 3.2 and 3.3 states the equivalence of the formulas $f(x)$ and IIFM and given in (2) and (13), respectively.

4 A New Smoothing Approach

In this section, we propose a new smoothing approach for the function described in (13). It is easy to see that the function $F(x)$ may have non-smooth structure. Indeed, the non-smoothness of $F(x)$ is originated from the existence of the $\chi_{B_{ji}}(x)$ since $f_j(x)$ are continuously differentiable for $j = 1, \ldots, m$. The idea for eliminating this lack is that if the IIF $\chi_{B_{ji}}(x)$ is smoothed, then the function $F(x)$ becomes smooth. First, we define the smoothing function for indicator functions.

Definition 4.1. [49] Let h be a semi-continuous function (upper or lower) defined on $\mathbb{R}$ to R. The function $\tilde{g} : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is called a smoothing function of $g(t)$, if $\tilde{h}(\cdot, \varepsilon)$ is continuously differentiable in $\mathbb{R}^n$ for any fixed ε, and for any $t \in \mathbb{R}$,

$$\lim_{z \to t, \beta \to 0} \tilde{g}(z, \varepsilon) = h(t).$$

New generation smoothing techniques have been studied in [46, 48]. For the first time, they are applied to solve minimax problems in [49]. We now re-define the IIF as

$$\chi_{B_{ji}}(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0, \end{cases}$$

where $t = f_j(x) - f_i(x)$. In the following, we propose a new smoothing function of IIF as

$$\tilde{\chi}_{B_{ji}}(t, \varepsilon) = \begin{cases} 0, & t \leq -\varepsilon, \\ R(t, \varepsilon), & -\varepsilon \leq t \leq \varepsilon, \\ 1, & t \geq \varepsilon, \end{cases} \tag{14}$$

where $R(t, \varepsilon) = \frac{1}{2\pi} \sin(\frac{\pi}{\varepsilon} t) + \frac{t}{2\varepsilon} + \frac{1}{2}$ and $\varepsilon > 0$. The function $R(t, \varepsilon)$ is called smooth transition function. It is designed in order to supply twice continuously differentiability between the pieces of IIF. Therefore, $\chi_{B_{ji}}(t, \varepsilon)$ is second-order continuously differentiable. We have

$$\tilde{\chi}'_{B_{ji}}(t, \varepsilon) = \begin{cases} 0, & t \leq -\varepsilon, \\ R'(t, \varepsilon), & -\varepsilon \leq t \leq \varepsilon, \\ 0, & t \geq \varepsilon, \end{cases} \tag{15}$$

where $R'(t, \varepsilon) = \frac{1}{2\varepsilon} \left(\cos\left(\frac{\pi t}{\varepsilon} \right) + 1 \right)$ and

$$\tilde{\chi}''_{B_{ji}}(t, \varepsilon) = \begin{cases} 0, & t \leq -\varepsilon, \\ R''(t, \varepsilon), & -\varepsilon \leq t \leq \varepsilon, \\ 0, & t \geq \varepsilon, \end{cases} \tag{16}$$

where $R''(t, \varepsilon) = -\frac{\pi}{2\varepsilon^2} \sin\left(\frac{\pi t}{\varepsilon} \right)$.

Another useful property of the smoothing function of the IIF is $\tilde{\chi}_{B_{ji}} = 1 - \tilde{\chi}_{B_{ij}}$. The relation between $\chi_{B_{ji}}(t)$ and its smoothing function $\tilde{\chi}_{B_{ji}}(t, \varepsilon)$ is investigated at the following lemmas.

Lemma 4.1. *Assume that $\chi_{B_{ji}}(t)$ is an IIF of the set $B_{ji} \subset \mathbb{R}^n$ and $\tilde{\chi}_{B_{ji}}(t, \varepsilon)$ is a smoothing function of $\chi_{B_{ji}}(t)$. Then, we have*

$$|\tilde{\chi}_{B_{ji}}(t, \varepsilon) - \chi_{B_{ji}}(t)| \leq \frac{1}{2},$$

for any $\varepsilon > 0$.

Proof. Since we have $\tilde{\chi}_{B_{ji}}(t, \varepsilon) = \chi_{B_{ji}}(t)$ for $t \leq -\varepsilon$ and $t \geq \varepsilon$, we discuss the cases $-\varepsilon \leq t \leq 0$ and $0 \leq t \leq \varepsilon$. For $-\varepsilon \leq t \leq 0$, we obtain

$$\left| \tilde{\chi}_{B_{ji}}(t, \varepsilon) - \chi_{B_{ji}}(t) \right| = |R(t, \varepsilon)| \leq \frac{1}{2},$$

and for $0 \leq t \leq \varepsilon$

$$\left| \tilde{\chi}_{B_{ji}}(t, \varepsilon) - \chi_{B_{ji}}(t) \right| = |R(t, \varepsilon) - 1| \leq \frac{1}{2}.$$

Therefore, the proof is completed. □

Lemma 4.2. *Assume that $\chi_{B_{ji}}(t)$ is an IIF of the set $B_{ji} \subset \mathbb{R}^n$ and $\tilde{\chi}_{B_{ji}}(t, \varepsilon)$ is the smoothing function. Then, we have*

$$\|\tilde{\chi}_{B_{ji}}(t, \varepsilon) - \chi_{B_{ji}}(t)\|_{L^1(\mathbb{R})} \leq \frac{\varepsilon}{2},$$

for any $\varepsilon > 0$.

Proof. Since we have $\tilde{\chi}_{B_{ji}}(t,\varepsilon)=\chi_{B_{ji}}(t)$ for $t\le-\varepsilon$ and $t\ge\varepsilon$, we deal with the case $-\varepsilon\le t\le\varepsilon$. For $-\varepsilon\le t\le\varepsilon$,

$$\begin{aligned}\left\|\tilde{\chi}_{B_{ji}}(t,\varepsilon)-\chi_{B_{ji}}(t)\right\|_{L^1(\mathbb{R})} &= \int_{-\varepsilon}^{\varepsilon}\left|\tilde{\chi}_{B_{ji}}(t,\varepsilon)-\chi_{B_{ji}}(t)\right|dt\\ &= \int_{-\varepsilon}^{0}|R(t,\varepsilon)|\,dt+\int_{0}^{\varepsilon}|R(t,\varepsilon)-1|\,dt\\ &= \left(\frac{\varepsilon}{4}-\frac{\varepsilon}{\pi^2}\right)+\left(\frac{\varepsilon}{4}-\frac{\varepsilon}{\pi^2}\right)\\ &< \frac{\varepsilon}{2}.\end{aligned}$$

Therefore, the proof is completed. □

Based on the new formulation and smoothing technique we define the smoothing function of the objective function $F(x)$ as

$$\tilde{F}(x,\varepsilon)=\sum_{j=1}^{m}f_j(x)\prod_{\substack{i=1\\ i\neq j}}^{m}\tilde{\chi}_{B_{ji}}(t_{ji},\varepsilon),$$

where $t_{ji}=f_j(x)-f_i(x)$ and the problem given in (1) is re-defined as

$$\min_{x\in\mathbb{R}^n}\tilde{F}(x,\varepsilon),\tag{17}$$

for $\varepsilon>0$. First, we introduce the case $m=2$ and obtain the following results.

Theorem 4.3. *Let $x\in\mathbb{R}^n$, $\varepsilon>0$*

$$|F(x)-\tilde{F}(x,\varepsilon)|\le\frac{\varepsilon}{2}.$$

Proof. Since $t_1=-t_2$, $\tilde{\chi}_{B_{12}}(t_1,\varepsilon)=\chi_{B_{12}}(t_1)$ and $\tilde{\chi}_{B_{21}}(t_2,\varepsilon)=\chi_{B_{21}}(t_2)$ for $t_1\le-\varepsilon$ and $t_1\ge\varepsilon$, we only concern with the case $-\varepsilon\le t_1\le\varepsilon$ for $\varepsilon>0$. For $-\varepsilon\le t_1\le\varepsilon$ we obtain

$$\begin{aligned}|F(x)-\tilde{F}(x,\varepsilon)| &= |f_1(x)\chi_{B_{12}}(t_1)+f_2(x)\chi_{B_{21}}(t_2)\\ &\quad-\big(f_1(x)\tilde{\chi}_{B_{12}}(t_1,\varepsilon)+f_2(x)\tilde{\chi}_{B_{21}}(t_2,\varepsilon)\big)|.\end{aligned}$$

Without loss of generality, we assume that $x\in B_{12}$ then we have $\chi_{B_{12}}(t_1)=1$ and $\chi_{B_{21}}(t_2)=0$ and

$$\begin{aligned}|F(x)-\tilde{F}(x,\varepsilon)| &= \left|f_1(x)\big(1-\tilde{\chi}_{B_{12}}(t_1,\varepsilon)\big)+f_2(x)\big(\tilde{\chi}_{B_{21}}(t_2,\varepsilon)\big)\right|\\ &= |f_1(x)-f_2(x)|\left|\tilde{\chi}_{B_{21}}(t_2,\varepsilon)\right|.\end{aligned}$$

Since $|f_1(x)-f_2(x)|\le\varepsilon$ and $\tilde{\chi}_{B_{21}}(t_2,\varepsilon)$ takes the value at most $\frac{1}{2}$, we have

$$|F(x)-\tilde{F}(x,\varepsilon)|\le\frac{\varepsilon}{2}.$$

Thus, the proof is completed. □

Theorem 4.4. *Let $\varepsilon>0$ and $x\in\mathbb{R}^n$*

$$\|\tilde{F}(x,\varepsilon)-F(x)\|_{L^1}\le\varepsilon^2.$$

Proof. Since $t_1=-t_2$ and $\tilde{\chi}_{B_{12}}(t_1,\varepsilon)=\chi_{B_{12}}(t_1)$ and $\tilde{\chi}_{B_{21}}(t_2,\varepsilon)=\chi_{B_{21}}(t_2)$ for $t_1\le-\varepsilon$ and $t_1\ge\varepsilon$, we need to consider the case $-\varepsilon\le t_1\le\varepsilon$ for $\varepsilon>0$. For $-\varepsilon\le t_1\le\varepsilon$, we obtain

$$\begin{aligned}\|\tilde{F}(x,\varepsilon)-F(x)\|_{L^1} &= \int_{-\varepsilon}^{\varepsilon}\Big|f_1(x)\tilde{\chi}_{B_{12}}(t_1,\varepsilon)+f_2(x)\tilde{\chi}_{B_{21}}(t_2,\varepsilon)\\ &\quad-\big(f_1(x)\chi_{B_{12}}(t_1)+f_1(x)\chi_{B_{21}}(t_2)\big)\Big|\,dt\\ &= \int_{-\varepsilon}^{\varepsilon}\Big|f_1(x)\big(\tilde{\chi}_{B_{12}}(t_1,\varepsilon)-\chi_{B_{12}}(t_1)\big)+f_2(x)\big(\tilde{\chi}_{B_{21}}(t_2,\varepsilon)-\chi_{B_{21}}(t_2)\big)\Big|\,dt\\ &= \int_{-\varepsilon}^{\varepsilon}|f_1(x)-f_2(x)|\left|\tilde{\chi}_{B_{12}}(t_1,\varepsilon)-\chi_{B_{12}}(t_1)\right|dt.\end{aligned}$$

Since $|f_1(x)-f_2(x)|\le\varepsilon$ and from Lemma 4.2, we have

$$\|\tilde{F}(x,\varepsilon)-F(x)\|_{L^1}\le\varepsilon^2.$$

Thus, the proof is completed. □

The Theorems 4.3 and 4.4 are verify theoretically that the proposed approach is a smoothing approach. In order to visualize the smoothing process we give the following example:

Example 4.1. Let the function f is defined as

$$f(x) = \max\{f_1(x), f_2(x)\},$$

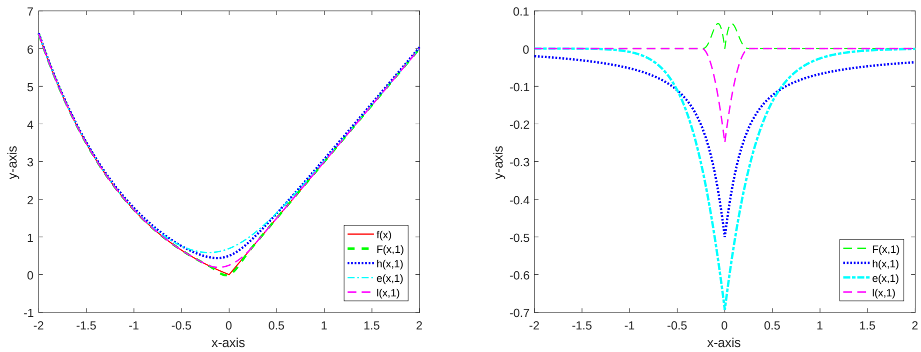
where $f_1(x) = e^{-x} - 1$ and $f_2(x) = 3x$. It can be easily computed that there is knot point at $x = 0$, the function f is continuous but non-differentiable and $\partial f(0) = [-1, 3]$. According to the concept of the sub-differential, the point $x_0 = 0$ is the stationary point. The graph of the function f can be imagined by considering the max function of f_1 and f_2 at Fig. 1 (a) (red and solid). By applying the above smoothing technique the smoothing function $\tilde{F}(x, \varepsilon)$ of f is obtained as

$$\tilde{F}(x, \varepsilon) = f_1(x)\tilde{\chi}_{B_{12}}(t_1, \varepsilon) + f_2(x)\tilde{\chi}_{B_{21}}(t_2, \varepsilon),$$

where $B_{12} = \{x \in \mathbb{R} : f_1(x) \geq f_2(x)\}$, $B_{21} = B_{12}^c$ and $t_1 = -t_2 = f_1(x) - f_2(x)$ for $x \in \mathbb{R}$. The graph of the function $F(x, \varepsilon)$ is illustrated in Fig. 1 (a) (green and dashed). In fact, we obtain an outer approximation to original function by the help of the above smoothing approach. We can deduce that for any function $f(x) = \max\{f_1(x), f_2(x)\}$, the inequality $f(x) = F(x) \geq \tilde{F}(x, \varepsilon)$ holds. Indeed, for $t_1 = f_1(x) - f_2(x) \geq 0$, we have

$$\begin{aligned} F(x) - F(x, \varepsilon) &= f_1(x) - (f_1(x)\tilde{\chi}_{B_{12}}(t_1, \varepsilon) + f_2(x)\tilde{\chi}_{B_{21}}(-t_1, \varepsilon) \\ &= f_1(x)(1 - \tilde{\chi}_{B_{12}}(t_1, \varepsilon)) - f_2(1 - \tilde{\chi}_{B_{12}}(t_1, \varepsilon)) \\ &= (f_1(x) - f_2(x))(1 - \tilde{\chi}_{B_{12}}(t_1, \varepsilon)). \end{aligned}$$

Since $(1 - \tilde{\chi}_{B_{12}}(t, \varepsilon)) \geq 0$ and $f_1(x) - f_2(x) \geq 0$, we obtain $F(x) \geq F(x, \varepsilon)$. The same result is obtained for $t_1 < 0$. In Fig. 2 , we



(a) (b)

Figure 1: (a) The red graph is the graph of $f(x)$, the green and dashed one is the graph of $\tilde{F}(x, 1)$, the blue and dotted one is the graph of hyperbolic smoothing function $\tilde{h}(x, 1)$, the cyan and dotted-dashed one is the graph of exponential smoothing function $\tilde{e}(x, 1)$ and the magenta and dashed one is the graph of local smoothing function $\tilde{l}(x, 1)$ and, (b) The graphs of difference functions are presented as $f(x) - \tilde{F}(x, 1)$, $f(x) - \tilde{f}(x, 1)$, $f(x) - \tilde{e}(x, 1)$ and $f(x) - \tilde{l}(x, 1)$, respectively.

illustrate the graph of smoothing functions that we mentioned in the introduction in a single framework in order to compare them visually. When the same value of $\varepsilon = 0.2$ is chosen for all smoothing approaches, the best approximation is achieved by our smoothing approach. Moreover, in Figure 1, we illustrate the other smoothing functions, such as hyperbolic [42], exponential [37], and the local smoothing approach in [36], by considering the same parameter value $\varepsilon = 1$ for all approaches. Moreover, in Figure 1 (a), we illustrate the other smoothing functions, such as hyperbolic [42], exponential [37], and the local smoothing approach in [36], by considering the same parameter value ε for all approaches.

According to Fig. 2 (a), choosing smaller ε values produces better approximations to the original function. The differences between the original function and the smoothing function with different parameter values are illustrated in Fig. 2 (a).

Let us continue giving the results about the degree of approximation of the the smoothing approach. Now, we present the convergence results for any finite value of m.

Theorem 4.5. *Let $x \in \mathbb{R}^n$, $\varepsilon > 0$*

$$\left| F(x) - \tilde{F}(x, \varepsilon) \right| \leq \varepsilon.$$

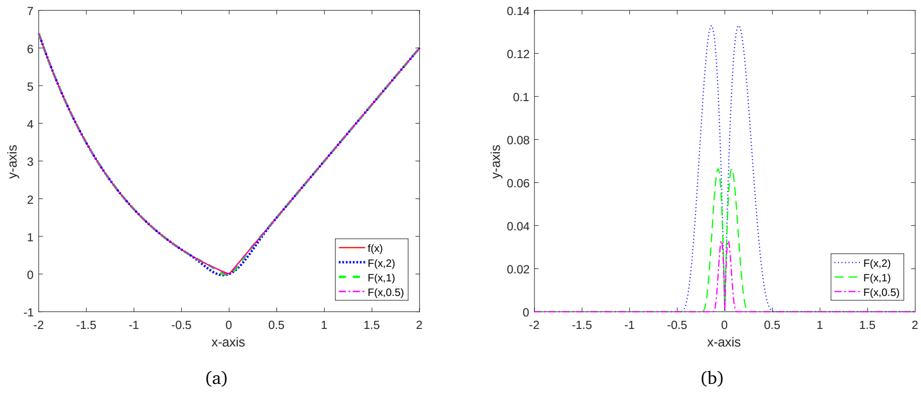


Figure 2: (a) The blue graph is the graph of $f(x)$, the red and dotted one is the graph of $\tilde{F}(x,1)$ and the green and dotted one is the graph of $\tilde{F}(x,1)$, and (b) The blue graph is the graph of $f(x)$, the red one is the graph of $\tilde{F}(x,0.2)$, the green one is the graph of exponential smoothing with $\varepsilon = 0.2$ and the yellow one is the graph of hyperbolic smoothing function with $\varepsilon = 0.2$.

Proof. For any $x \in \mathbb{R}^n$, we have

$$\left|F(x)-\tilde{F}(x,\varepsilon)\right| = \left|\sum_{j=1}^{m} f_j(x)\prod_{\substack{i=1\\ i\neq j}}^{m}\chi_{B_{ji}}(t_{ji}) - \sum_{j=1}^{m} f_j(x)\prod_{\substack{i=1\\ i\neq j}}^{m}\tilde{\chi}_{B_{ji}}(t_{ji},\varepsilon)\right|.$$

For a fixed $x \in \mathbb{R}^n$ there exists $j_0 \in \{1,2,\ldots,m\}$ such that $F(x) = f(x) = f_{j_0}(x)$ and $f_{j_0}(x) \geq f_j(x)$ for all $j \in \{1,2,\ldots,m\}$. We obtain

$$\begin{aligned}
\left|F(x)-\tilde{F}(x,\varepsilon)\right| &= \left|f_{j_0}(x) - \sum_{j=1}^{m} f_j(x)\prod_{\substack{i=1\\ i\neq j}}^{m}\tilde{\chi}_{B_{ji}}(t_{ji},\varepsilon)\right| \\
&= \left|f_{j_0}(x) - f_{j_0}(x)\prod_{\substack{i=1\\ i\neq j_0}}^{m}\tilde{\chi}_{B_{j_0 i}}(t_{j_0 i},\varepsilon) - \sum_{\substack{j=1\\ j\neq j_0}}^{m} f_j(x)\prod_{\substack{i=1\\ i\neq j}}^{m}\tilde{\chi}_{B_{ji}}(t_{ji},\varepsilon)\right| \\
&= \left|f_{j_0}(x)\left(1-\prod_{\substack{i=1\\ i\neq j_0}}^{m}\tilde{\chi}_{B_{j_0 i}}(t_{j_0 i},\varepsilon)\right) - \sum_{\substack{j=1\\ j\neq j_0}}^{m} f_j(x)\prod_{\substack{i=1\\ i\neq j}}^{m}\tilde{\chi}_{B_{ji}}(t_{ji},\varepsilon)\right|.
\end{aligned}$$

Let us construct the set $K = \left\{k \in \{1,2,\ldots,m\} : |f_{j_0}(x) - f_k(x)| < \varepsilon, x \in \mathbb{R}^n\right\}$. It can be concluded that $\tilde{\chi}_{B_{ij_0}}(t_{ij_0},\varepsilon) \leq \frac{1}{2}$ and $\tilde{\chi}_{B_{j_0 i}}(t_{j_0 i},\varepsilon) \geq \frac{1}{2}$ for $i \in K$. Without loss of generality assume that $f_{j_0} \geq f_{k_1} \geq f_{k_2} \cdots \geq f_{k_p}$ such that $p = card(K)$. Then, we have $\prod_{\substack{i=1\\ k_i\neq j_0}}^{p}\tilde{\chi}_{B_{j_0 k_i}}(t_{j_0 k_i},\varepsilon) \geq \frac{1}{2^p}$. Therefore

$$\begin{aligned}
\left|F(x)-\tilde{F}(x,\varepsilon)\right| &\leq \Bigg|\left(1-\frac{1}{2^p}\right)f_{j_0}(x) - \left(\frac{1}{2}f_{k_1}(x) + \frac{1}{2^2}f_{k_2}(x)\right. \\
&\qquad \left. + \cdots + \frac{1}{2^p}f_{k_p}(x)\right)\Bigg| \\
&\leq \frac{1}{2}|f_{j_0}(x) - f_{k_1}(x)| + \frac{1}{2^2}|f_{j_0}(x) - f_{k_2}(x)| \\
&\qquad + \cdots + \frac{1}{2^p}|f_{j_0}(x) - f_{k_2}(x)| \\
&\leq \varepsilon.
\end{aligned}$$

Thus, the proof is completed. □

Theorem 4.6. *Suppose that the point x^* is an optimal solution for the problem (1) and $\overline{x}$ is an optimal solution for the problem (17). Then,*

$$|F(x^*) - \tilde{F}(\overline{x}, \varepsilon)| \leq \varepsilon.$$

Proof. Since $F(\bar{x}) \geq F(x^*) \geq \tilde{F}(\bar{x}, \varepsilon)$, we have

$$|F(x^*) - \tilde{F}(\bar{x}, \varepsilon)| \leq |F(\bar{x}) - \tilde{F}(\bar{x}, \varepsilon)|.$$

By the help of Theorem 4.3 and 4.5, we obtain

$$|F(\bar{x}) - \tilde{F}(\bar{x}, \varepsilon)| \leq \varepsilon.$$

It completes the proof. □

Theorem 4.7. *Let $\{\varepsilon_j\} \to 0$ and x^k be a solution of (17). Assume that $\overline{x}$ is an accumulation point of $\{x^k\}$. Then $\overline{x}$ is an optimal solution for (1).*

Proof. By considering the Theorem 4.6, the proof is obtained. □

5 Algorithm and Minimization Procedure

In this section the new algorithm is given to solve min-max problem defined in (1). We propose to use smoothed version of the problem (17) instead of the problem given in (1).

Algorithm 1 Improved Indicator Smoothing Algorithm (IISA)

1: Input: x^0, ε_0, q, $\tau \leftarrow 10^{-4}$, $k \leftarrow 0$ and $i = 0$.
2: **while** $\|\nabla \tilde{F}(x^k, \varepsilon_k)\| > \tau$ **do**
3: Solve the problem (17) by using gradient-based solver by considering x^k as an initial point. $x^{k+1} \leftarrow \operatorname{argmin} \tilde{F}(x, \varepsilon)$.
4: $x^k \leftarrow x^{k+1}$, $\varepsilon_k \leftarrow q\varepsilon_k$.
5: $i = i + 1$.
6: **end while**
7: Output: x^k, $f(x^k)$, i.

In the application process of IISA, the starting point x^0 is chosen randomly, the smoothing parameter $varepsilon_0 leq 1$, and the parameter q is selected, such as $0 < q < 1$. The variable i is used for counting the number of iterations. The convergence of IISA is stated by the following theorem:

Theorem 5.1. *Suppose the set*

$$\operatorname*{argmin}_{x \in \mathbb{R}^n} \tilde{F}(x, \varepsilon) \neq \emptyset,$$

for $\varepsilon \in (0, \varepsilon_0]$. Let x^k be generated by IISA. If $\{x^k\}$ has an accumulation point, then the accumulation point of $\{x^k\}$ is the solution for (1).

Proof. Let us define the set $\mathcal{L}(x_0) = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$ for starting point x_0. Let $\overline{x}$ be an accumulation point of $\{x^k\}$ and $\mathcal{L}(x_0)$ is bounded. We first show that $\overline{x} \in \mathcal{L}(x_0)$. Since

$$\tilde{F}(x_0, \varepsilon) \geq \tilde{F}(x^k, \varepsilon),$$

and according to Theorem 4.6, we have $f(x_0) \geq f(x^k)$ and $x^k \in \mathcal{L}(x_0)$. Since $\mathcal{L}(x_0)$ is bounded we obtain $\overline{x} \in \mathcal{L}(x_0)$. By the Theorem 4.7, $\overline{x}$ is the solution for (1). □

6 Numerical Examples

This section is devoted to present the numerical results of *IISA* on some test problems. We apply the *IISA* by using MATLAB on PC with configuration of Intel Core i5, 8GB RAM. We consider the BFGS method as a local search for *Algorithm I* and the parameters are selected as $\varepsilon_0 = 10^{-1}$ and $q = 10^{-1}$. It is accepted that the problem is solved, if the accuracy 10^{-4} with respect to function value is obtained.

We consider a class of 15 different test problems for numerical test procedure of the proposed and competing algorithms. The details of the test problems are given as follows:

Problem 1. [20] $f(x) = \max_{1 \le j < 2} f_j(x)$ where $f_j : \mathbb{R}^2 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= -x_1 - x_2, \\ f_2(x) &= -x_1 - x_2 + \left(x_1^2 + x_2^2 - 1\right), \end{aligned}$$

the global minimum value of the objective function f is $f^* = -\sqrt{2}$.

Problem 2. [20] $f(x) = \max_{1 \le j < 4} f_j(x)$ where $f_j : \mathbb{R}^2 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= 10(x_2 - x_1^2), \\ f_2(x) &= -f_1(x), \\ f_3(x) &= 1 - x_1, \\ f_4(x) &= -f_3(x), \end{aligned}$$

the global minimum value of the objective function f is $f^* = 0$.

Problem 3. [24] $f(x) = \max_{1 \le j < m} f_j(x)$ where $f_j : \mathbb{R}^3 \to \mathbb{R}$

$$\begin{aligned} f_j(x) &= \sin t_j - (x_3 t_j^2 + x_2 t_j + x_1), \quad j = 1, \dots, \frac{m}{2} \\ f_j(x) &= -f_{j-\frac{m}{2}}(x), \quad j = \frac{m}{2} + 1, \dots, m, \end{aligned}$$

and

$$t_j = \frac{j-1}{\left(\frac{m}{2} - 1\right)}, \quad j = 1, \dots, \frac{m}{2}.$$

The global minimum value of objective function f is $f^* = -4.50481 \times 10^{-3}$.

Problem 4. [24] $f(x) = \max_{1 \le j < 3} f_j(x)$ where $f_j : \mathbb{R}^2 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= (x_1)^2 + (x_2)^2, \\ f_2(x) &= (2 - x_1)^2 + (2 - x_2)^2, \\ f_3(x) &= 2\exp(-x_1 + x_2), \end{aligned}$$

the global minimum value of the objective function f is $f^* = 1.9522$.

Problem 5. [24] $f(x) = \max_{1 \le j < 2} f_j(x)$ where $f_j : \mathbb{R}^2 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= \left(x_1 - \sqrt{x_1^2 + x_2^2}\cos(x_1^2 + x_2^2)\right)^2 + 0.005(x_1^2 + x_2^2), \\ f_2(x) &= \left(x_1 - \sqrt{x_1^2 + x_2^2}\sin(x_1^2 + x_2^2)\right)^2 + 0.005(x_1^2 + x_2^2), \end{aligned}$$

the global minimum value of the objective function f is $f^* = 0$.

Problem 6. [24] $f(x) = \max_{1 \le j \le 50} f_j(x)$ where $f_j : \mathbb{R}^{200} \to \mathbb{R}$ and

$$f_j(x) = x_{4(j-1)+1}^2 + x_{4(j-1)+2}^2 + x_{4(j-1)+3}^2 + x_{4j}^2, \quad j = 1, \dots, 50.$$

the global minimum value of the objective function f is $f^* = 0$.

Problem 7. [51] $f(x) = \max_{1 \le j \le 2} f_j(x)$ where $f_j : \mathbb{R}^2 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= x_1^2 + (x_2 - 1)^2 + x_2 - 1, \\ f_2(x) &= -x_1^2 - (x_2 - 1)^2 + x_2 + 1, \end{aligned}$$

the global minimum value of the objective function f is $f^* = 0$.

Problem 8. [51] $f(x) = \max_{1 \le j \le 3} f_j(x)$ where $f_j : \mathbb{R}^2 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= \frac{1}{2}\left(x_1 + \frac{10x_1}{x_1 + 0.1} + 2x_2^2\right), \\ f_2(x) &= \frac{1}{2}\left(-x_1 + \frac{10x_1}{x_1 + 0.1} + 2x_2^2\right), \\ f_3(x) &= \frac{1}{2}\left(x_1 - \frac{10x_1}{x_1 + 0.1} + 2x_2^2\right), \end{aligned}$$

the global minimum value of the objective function f is $f^* = 0$.

Problem 9. [51] $f(x) = \max_{1\le j\le 3} f_j(x)$ where $f_j : \mathbb{R}^2 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= 5x_1 + x_2, \\ f_2(x) &= -5x_1 + x_2, \\ f_3(x) &= x_1^2 + x_2^2 + 4x_2, \end{aligned}$$

the global minimum value of the objective function f is $f^* = -3$.

Problem 10. [51] $f(x) = \max_{1\le j\le 6} f_j(x)$ where $f_j : \mathbb{R}^3 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= x_1^2 + x_2^2 + x_3^2 - 1, \\ f_2(x) &= x_1^2 + x_2^2 + (x_3 - 2)^2, \\ f_3(x) &= x_1 + x_2 + x_3 - 1, \\ f_4(x) &= x_1 + x_2 - x_3 + 1, \\ f_5(x) &= 2x_1^3 + 6x_2^2 + 2(5x_3 - x_1 + 1)^2, \\ f_6(x) &= x_1^2 - 9x_3, \end{aligned}$$

the global minimum value of the objective function f is $f^* = 3.5997$.

Problem 11. [51] $f(x) = \max_{1\le j\le 4} f_j(x)$ where $f_j : \mathbb{R}^4 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= x_1^2 + x_2^2 + 2x_3^2 + x_4^2 - 5x_1 - 5x_2 - 21x_3 + 7x_4, \\ f_2(x) &= f_1(x) + 10\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 - x_2 + x_3 - x_4 - 8\right), \\ f_3(x) &= f_1(x) + 10\left(x_1^2 + 2x_2^2 + x_3^2 + 2x_4^2 - x_1 - x_4 - 10\right), \\ f_4(x) &= f_1(x) + 10\left(2x_1^2 + 2x_2^2 + x_3^2 + 2x_1 - x_2 - x_4 - 5\right), \end{aligned}$$

the global minimum value of the objective function f is $f^* = -44$.

Problem 12. [51] $f(x) = \max_{1\le j\le 5} f_j(x)$ where $f_j : \mathbb{R}^7 \to \mathbb{R}$ and

$$\begin{aligned} f_1(x) &= (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 \\ &\quad +7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 + 8x_7, \\ f_2(x) &= f_1(x) + 10\left(2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 - 127\right), \\ f_3(x) &= f_1(x) + 10\left(7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 - 282\right), \\ f_4(x) &= f_1(x) + 10\left(23x_1 + x_2^2 + 6x_6^2 - 8x_7 - 196\right), \\ f_5(x) &= f_1(x) + 10\left(4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7\right), \end{aligned}$$

the global minimum value of the objective function f is $f^* = 680.63006$.

Problem 13. [52] $f(x) = \max_{1\le j<m} f_j(x)$ where $f_j : \mathbb{R}^2 \to \mathbb{R}$ and

$$f_j(x) = x_1^2 + 2x_1 t_j^2 + \exp(x_1 + x_2) - \exp(t_j),$$

$t_j = \frac{j}{(q-1)}$, $\quad j = 0, 1, \ldots, m-1$. The global minimum value of objective function f is $f^* = -1$.

Problem 14. [52] $f(x) = \max_{1\le i,j<m} f_{i,j}(x)$ where $f_{i,j} : \mathbb{R}^4 \to \mathbb{R}$

$$f_{i,j}(x) = \frac{(t_i - x_i)^2}{x_3^2} + \frac{(r_j - x_2)^2}{x_4^2} - 4,$$

and $t_i = \frac{i}{\sqrt{m}-1}$, $\quad t_j = \frac{j}{\sqrt{m}-1}$, $\quad i, j = 0, 1, \ldots, m-1$. The global minimum value of the objective function f is $f^* = -4$.

Table 1: The numerical results of *IISA* and comparison with other smoothing-based algorithms

			IISA				HSA				ESA				LSA			
Problem No.	n	m	$iter$	$feval$	$f.val$	Time	$iter$	$f.eval$	$f.val$	Time	$iter$	$f.eval$	$f.val$	Time	$iter$	$f.eval$	$f.val$	Time
1	2	2	17	105	-1.4142	0.012893	37	165	-1.4141	0.021611	23	96	-1.4017	0.011118	22	132	-1.4142	0.017186
2	2	4	13	54	2.064e-10	0.0073957	15	177	3.9525e-05	0.019457	12	63	0.22583	0.013994	19	138	0.00026929	0.014859
	3	50	48	384	4.5078e-03	0.078041	72	412	0.0045894	0.11084	18	92	15.932	0.010434	85	436	0.0047682	0.035897
3	3	102	42	452	4.5078e-03	0.25947	100	552	0.0046054	0.2563	12	76	0.75344	0.012377	72	424	0.0046188	0.036451
	3	202	24	396	4.5078e-03	0.78225	94	588	0.0048985	0.49994	18	92	21.618	0.015659	84	484	0.0048512	0.057917
4	2	3	24	132	1.9522	0.011758	24	150	1.9523	0.022188	40	178	1.9557	0.014741	24	141	1.9523	0.014551
5	2	2	6	27	1.1925e-06	0.0045825	98	486	0.11791	0.046196	181	831	0.68103	0.044065	130	651	0.11725	0.043132
6	200	50	31	7627	2.3018e-05	1.8152	38	9045	1.25e-05	2.3575	63	13869	1.5932	0.42376	105	22914	4.6415e-05	0.98944
7	2	2	12	93	-3.1668e-05	0.014047	30	144	3.2009e-05	0.015785	21	120	0.0028801	0.010926	10	117	2.4997e-05	0.0094584
8	2	3	9	93	-3.245e-06	0.013506	14	114	0.0089623	0.030249	13	103	0.006643	0.011662	13	129	1.4419e-05	0.012287
9	2	3	20	207	-2.9999	0.037632	15	123	-3	0.020384	16	150	-2.9944	0.017181	26	144	-2.9999	0.042089
10	3	6	26	292	3.5997	0.026583	92	628	3.6011	0.054323	46	308	3.6016	0.02028	22	220	3.5999	0.015759
11	4	4	56	605	-44	0.027176	139	995	-43.724	0.060258	87	520	-41.318	0.019938	71	545	-44	0.022625
12	7	5	19	992	678.38	0.031834	88	1568	699.36	0.079701	144	1395	678.87	0.039893	56	1008	678.39	0.032223
	2	10	16	69	-1	0.018347	57	198	-0.99997	0.028444	38	141	-0.5515	0.015934	35	114	-1	0.01382
13	2	100	12	66	-1	0.036296	69	264	-0.99997	0.14551	38	141	0.34575	0.019128	32	111	-1	0.015179
	2	1000	25	162	-1	0.8954	73	267	-0.9999	1.0409	37	138	1.2795	0.036004	33	120	-1	0.039464
	2	2000	23	156	-1	4.0921	76	279	-1	9.2256	40	147	1.5616	0.13991	33	120	-1	0.15908
	4	10	3	45	-4	0.012911	3	50	-3.9999	0.014949	5	35	-2.1245	0.012607	7	35	-3.9999	0.011161
14	4	100	4	29	-4	0.028304	6	35	-4	0.031669	5	25	-0.24906	0.011407	5	25	-3.9999	0.011997
	4	1000	3	26	-4	1.4376	8	105	-4	1.4704	6	35	1.6264	0.35892	6	38	-3.9999	0.33996
	4	2000	4	31	-4	5.739	10	120	-4	5.7936	4	40	2.191	1.583	4	35	-3.9999	1.3395
15	10	9	26	1397	24.302	0.042677	76	1782	59.329	0.13034	110	1719	25.54	0.050532	93	2321	24.307	0.057021

Problem 15. [53] $f(x) = \max\limits_{1\le j\le 9} f_j(x)$ where $f_j : \mathbb{R}^{10} \to \mathbb{R}$ and

$$
\begin{aligned}
f_1(x) &= x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 + 4(x_4-5)^2 + (x_5-3)^2 + 2(x_6-1)^2 \\
&\quad + 5x_7^2 + 7(x_8-11)^2 + 2(x_9-10)^2 + (x_{10}-7)^2 + 45, \\
f_2(x) &= f_1(x) + 10\big(3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4 - 120\big), \\
f_3(x) &= f_1(x) + 10\big(5x_1^2 + 8x_2 + (x_3-6)^2 - 2x_4 - 40\big), \\
f_4(x) &= f_1(x) + 10\big(0.5(x_1-8)^2 + 2(x_2-4)^2 + 3x_5^2 - x_6 - 30\big), \\
f_5(x) &= f_1(x) + 10\big(x_1^2 + 2(x_2-2)^2 - 2x_1x_2 + 14x_5 - 6x_6\big), \\
f_6(x) &= f_1(x) + 10\big(4x_1 + 5x_2 - 3x_7 + 9x_8 - 105\big), \\
f_7(x) &= f_1(x) + 10\big(10x_1 - 8x_2 - 17x_7 + 2x_8\big), \\
f_8(x) &= f_1(x) + 10\big(-3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_{10}\big), \\
f_9(x) &= f_1(x) + 10\big(-8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12\big),
\end{aligned}
$$

the global minimum value of the objective function f is $f^* = 24.31$.

Based on the above test problems, 23 different test cases were created, varying in dimension (n) from 2 to 200 and in number of functions (m) from 2 to 2000. *IISA* has been applied to these test cases, and total iteration numbers "iter", total function evaluations "feval", function values "f.val", and the CPU time in seconds "Time" outputs are reported for each test case. The test results are illustrated in Tables 1 and 2.

In Table 1, the obtained numerical results from *IISA* are compared with *Algorithm I* with hyperbolic smoothing used in [40, 42] with min-max formula (5) called as (*HSA*), *Algorithm I* with exponential smoothing used in [37, 50] with min-max formula (4) called as (*ESA*) and *Algorithm I* with local smoothing used in [36] with min-max formula (3) called as (*LSA*). Moreover, we compare our numerical results with existing techniques in MATLAB, such as "fmincon" and "fminimax" in Table 2. We have considered randomly generated initial points for each test case.

It can be seen from the Table 1 that *IISA* presents better results than smoothing based algorithms such as *HSA*, *ESA* ans *LSA* at the rate of 70% considering all test cases in terms of total number of iterations. In terms of total function evaluations, *IISA* presents better results than the ESA and *HSA* at the rate of 50% considering all test problems. Moreover, by using *IISA* and *LSA* the correct solutions are obtained for all test problems but by using *HSA* and *ESA* the solutions are not close to desired results. Moreover, if anyone compares *IISA* with the *ESA* and *HSA* in terms of CPU time, it is seen that *IISA* is faster than than *ESA* and *HSA* at the rate of 50% considering all test cases. Conversely, the use of the *ESA* is complicated due to the presence of the exponential term. As the smoothing parameter $\varepsilon \to 0^+$, the exponential function $\exp\left(\frac{f_j(x)}{\varepsilon}\right)$ attains significantly large values. Therefore the function "fminunc" gives error and can not continue. The *HSA* is easy to control and it is possible to obtain the results with desired precision but it is slower than *IISA* and *LSA*.

It can be seen from the Table 2 that *IISA* presents better results than the existing algorithms *fmincon* and *fminimax* in MATLAB at the rate of 50% considering all test cases in terms of total number of iterations and total function evaluations. Moreover, if anyone compares *IISA* with the *fmincon* and *fminimax* in terms of CPU time, it is seen that *IISA* is advantageous than the competing algorithms.

Table 2: The comparison of the numerical results with “fmincon” and “fminimax”

			IISA				fmincon				fminimax			
Problem No.	n	m	$iter$	$feval$	$f.val$	Time	$iter$	$f.eval$	$f.val$	Time	$iter$	$f.eval$	$f.val$	Time
1	2	2	17	105	-1.4142	0.012893	75	138	-1.4142	0.018358	8	39	-1.4142	0.094444
2	2	4	13	54	2.064e-10	0.0073957	45	87	3.3295e-05	0.009955	3	14	0	0.015573
3	3	50	48	384	4.5078e-03	0.078041	83	149	0.004502	0.0039533	4	23	0.0044998	0.016172
	3	102	42	452	4.5078e-03	0.25947	331	597	0.0045078	0.025262	12	67	0.0045048	0.045712
	3	202	24	396	4.5078e-03	0.78225	133	240	0.0045107	0.017561	4	23	0.0045048	0.017019
4	2	3	24	132	1.9522	0.011758	44	82	1.9522	0.002018	6	30	1.9522	0.015722
5	2	2	6	27	1.1925e-06	0.0045825	1078	2000	0.1136	0.034323	262	1514	0.069661	0.22193
6	200	50	31	7627	2.3018e-05	1.8152	1.8759e+05	2e+05	3.9912	19.035	18	3653	7.4717e-07	1.4107
7	2	2	12	93	-3.1668e-05	0.014047	19	37	0	0.0009014	1	4	0	0.0083114
8	2	3	9	93	-3.245e-06	0.013506	22	45	0	0.0015607	1	4	0	0.009223
9	2	3	20	207	-2.9999	0.037632	56	101	-3	0.0034867	11	54	-3	0.039539
10	3	6	26	292	3.5997	0.026583	168	301	3.5997	0.0036002	11	65	3.5997	0.053047
11	4	4	56	605	-44	0.027176	274	463	-44	0.0034339	14	101	-44	0.017314
12	7	5	19	992	678.38	0.031834	3696	5682	691.97	0.049013	53	604	678.38	0.08168
13	2	10	16	69	-1	0.018347	96	198	-1	0.0048375	21	106	-1	0.055151
	2	100	12	66	-1	0.036296	96	198	-1	0.0088764	21	106	-1	0.081506
	2	1000	25	162	-1	0.8954	96	198	-1	0.039701	21	106	-1	0.22788
	2	2000	23	156	-1	4.0921	96	198	-1	0.16211	21	106	-1	1.0186
14	4	10	3	45	-4	0.012911	52	100	-4	0.0044919	5	34	-4	0.017266
	4	100	4	29	-4	0.028304	55	149	-4	0.017905	2	13	-4	0.012339
	4	1000	3	26	-4	1.4376	53	98	-4	0.81922	4	27	-4	0.24937
	4	2000	4	31	-4	5.739	52	100	-4	3.4243	4	27	-4	1.0047
15	10	9	26	1397	24.302	0.042677	4787	7219	29.366	0.079395	26	341	24.306	0.075346

7 Conclusion

In this study, a new reformulation of the finite minimax problems is presented based on the multiplication of the indicator functions for each variable. The equivalence between the original and reformulated problem is proved theoretically. A new smoothing function for the indicator function is introduced, and it is applied for the reformulated problem as a special case. By this reformulation and smoothing technique, the objective function of the minimax problem given in (2) has an alternative formulation to the literature. Moreover, a new smoothing approximation technique has been added to the literature for such a non-smooth problem. A new algorithm for solving reformulated and smoothed finite minimax problems is developed, and the convergence of the algorithm has been investigated theoretically. Moreover, the efficiency of the algorithm is demonstrated on some numerical examples, numerically. According to the comparison of the results with the other methods, it is shown that our approach is competitive with well-known prestigious approaches.

For future studies, the methodology can be adapted for penalty formulation of the constrained problem and the results can be compared with the well-known penalty forms. The methodology can be also extended to semi-infinite minimization process [54, 55].

References

[1] I. Al-Subaihi, G. A. Watson, Fitting parametric curves and surfaces by ℓ_∞ distance regression, *BIT Numer. Math.*, 45, 443–461, 2005.

[2] E. Polak, On the mathematical foundations of nondifferentiable optimization in engineering design. *SIAM Review*, 29(1): 21–89, 1987.

[3] E. M. Arkin, R. Hassin, A. Levin, Approximations for minimum and min-max vehicle routing problems. *J. Algorithm.* 59(1): 1–18, 2006.

[4] A. Nazemi, M. Mortezaee, A new gradient-based neural dynamic framework for solving constrained min-max optimization problems with an application in portfolio selection models. *Appl. Intell.* 49: 396–419, 2019.

[5] N. V. Banichuk, Minimax approach to structural optimization problems. *J. Optim. Theory Appl.* 20: 111-127, 1976.

[6] J. Du, L. Zhao, J. Feng,X. Chu, Computation offloading and resource allocation in mixed fog/cloud computing systems with min-max fairness guarantee. *IEEE Trans. Comm.* 66(4): 1594–1608, 2017.

[7] Z. Cobandag Guloglu, G. W. Weber, Risk modeling in optimization problems via value at risk, conditional value at risk, and its robustification. In: Modeling, Dynamics, Optimization and Bioeconomics II: DGS III, Porto, Portugal, February 2014, and Bioeconomy VII, Berkeley, USA, March 2014-Selected Contributions 3 (pp. 133–145). Springer International Publishing, 2017.

[8] D. Z. Du, P. M. Pardalos, Minimax and Applications. Kluwer Academic, Dordrecht, 1995.

[9] W. Fan, J. Liang, H. C. So, G. Lu, Min-max metric for spectrally compatible waveform design via log-exponential smoothing. *IEEE Trans.n Signal Process.* 68: 1075–1090, 2020.

[10] W. Fan, J. Liang, H. C. So, G. Lu, Min-max metric for spectrally compatible waveform design via log-exponential smoothing. *IEEE Trans. Signal Process.*, 68: 1075–1090, 2020.

[11] S. Mahmoudinazlou, C. Kwon, A hybrid genetic algorithm for the min—max multiple traveling salesman problem. *Comput. Oper. Res.* 162: 106455, 2024.

[12] E. Kropat, G. W. Weber, E. B. Tirkolaee, Foundations of semialgebraic gene-environment networks. *Journal of Dynamics and Games*, 7: 253–268, 2020.

[13] A. Ozmen, Y. Zinchenko, G. W. Weber,Robust multivariate adaptive regression splines under cross-polytope uncertainty: an application in a natural gas market. *Ann. Oper. Res.*, 324: 1337-1367, 2023.

[14] E. Savku, An approach for regime-switching stochastic control problems with memory and terminal conditions. *Optimization*, 1–18, 2024.

[15] E. B. Tirkolaee, A. Goli, S. Gutmen, G. W. Weber, K. Szwedzka, A novel model for sustainable waste collection arc routing problem: Pareto-based algorithms. *Ann. Oper. Res.* 324: 189–214, 2023.

[16] F. Yerlikaya-Ozkurt, P. Taylan, Enhancing classification modeling through feature selection and smoothness: A conic-fused lasso approach integrated with mean shift outlier modelling. *Journal of Dynamics and Games* 12: 1–23, 2025.

[17] T. Ewertowski, B. C. Guldogus, S. Kuter, S. Akyuz, G. W. Weber, J. Sadlowska-Wrzesinska, E. Racek, The use of machine learning techniques for assessing the potential of organizational resilience. *Cent. Eur. J. Oper. Res.* 32: 685-710, 2024.

[18] V. F. Demyanov, V. N. Malozemov, Introduction to Minimax. Wiley, New York,1974.

[19] L. Xingsi, An entropy-based aggregate method for minimax optimization. *Eng. Optim.* 18(4): 277-285, 1992.

[20] G. Di Pillo, L. Grippo, S. Lucidi, A smooth method for the finite minimax problem. *Math. Programm.*, 60(1): 187–214, 1993.

[21] M. Gaudioso, G. Giallombardo, G. Miglionico, An incremental method for solving convex finite min-max problems. *Math. Oper. Res.* 31(1): 173–187, 2006.

[22] A. Fuduli, M. Gaudioso, G. Giallombardo, G. Miglionico, A partially inexact bundle method for convex semi-infinite minmax problems. *Commun. Nonlinear Sci. Numer. Simul.* 21: 172–180, 2015.

[23] Z. Zhou, Y. Duan, An aggregate homotopy method for solving unconstrained minimax problems. *Optimization* 70: 1791–1808, 2021.

[24] E. Polak, J. O. Royset, R. S. Womersley, Algorithms with adaptive smoothing for finite minimax problems. *J. Optim. Theory Appl.* 119: 459–484, 2003.

[25] F. Guerra Vazquez, H. Gunzel, T. H. Jongen, On logarithmic smoothing of the maximum function. *Ann. Oper. Res.* 101: 209–220, 2001.

[26] Z. Zhou, Q. Yang, An active set smoothing method for solving unconstrained minimax problems. *Math. Probl. Engl.* 2020(1): 9108150, 2020.

[27] Y. Hao, S. Du, Y. Chen, A new three-term conjugate gradient method for solving the finite minimax problems. *Filomat*, 35(3): 737-758, 2021.

[28] Z. Zhou, X. Dai, X. An active set strategy to address the ill-conditioning of smoothing methods for solving finite linear minimax problems. *J. Global Optim.* 85(2): 421-439, 2023.

[29] J. He, H. Zhang, Z. Xu, An approximation proximal gradient algorithm for nonconvex-linear minimax problems with nonconvex nonsmooth terms. *J. Global Optim.* 90: 73–90, 2024.

[30] Y. Xu, Decentralized gradient descent maximization method for composite nonconvex strongly-concave minimax problems. *SIAM J. Optim.*, 34(1): 1006–1044, 2024.

[31] X. Chen, Smoothing methods for nonsmooth, nonconvex minimization. *Math. Programm.*, 134: 71–99, 2012.

[32] C. Chen, O. L. Mangasarian, A class of smoothing functions for nonlinear and mixed complementarity problems. *Comput. Optim. Appl.* 5: 97–138, 1996.

[33] C. Grossmann, Smoothing techniques for exact penalty methods. Contemporary Mathematics, In book Panaroma of Mathematics: Pure and Applied, Vol. 658, pp. 249-265, 2016.

[34] A. Sahiner, G. Kapusuz, N. Yilmaz, A new smoothing approach to exact penalty functions for inequality constrained optimization problems. *Numer. Algebra Control Optim.* 6(2): 161-173, 2016.

[35] D. P. Bertsekas, Nondifferentiable optimization via approximation. In: Balinski, M.L., Wolfe, P. (eds) Nondifferentiable Optimization. Mathematical Programming Studies (pp. 1-25), Springer, Berlin, 2009.

[36] I. Zang, A smoothing-out technique for min-max optimization. *Math. Programm.* 19: 61-77, 1980.

[37] S. Xu, Smoothing method for minimax problems. *Comput. Optim. Appl.* 20(3): 267–279, 2001.

[38] H. Yin, An adaptive smoothing method for continuous minimax problems. *Asia Pac. J. Oper. Res.* 32: 1540001, 2015.

[39] Y. Feng, L. Hongwei, Z. Shuisheng, L. Sanyang, A smoothing trust-region Newton-CG method for minimax problem. *Appl. Math. Comput.* 199(2): 581–589, 2008.

[40] A. M. Bagirov, A. Al Nuaimat, N. Sultanova, Hyperbolic smoothing function method for minimax problems. *Optimization* 62(6): 759–782, 2013.

[41] A. M. Bagirov, N. Sultanova, A. Al Nuaimat, S. Taheri, Solving minimax problems: Local smoothing versus global smoothing. In Numerical Analysis and Optimization: NAO-IV, Muscat, Oman, January 2017 NAO-IV (pp. 23-43). Springer International Publishing. (2018).

[42] A. E. Xavier, The hyperbolic smoothing clustering method. *Pattern Recogn.* 43: 731–737, 2010.

[43] M. Souza, A. E. Xavier, C. Lavor, N. Maculan, Hyperbolic smoothing and penalty techniques applied to molecular structure determination. *Oper. Res. Lett.* 39: 461–465, 2011.

[44] W. Sanchez, C. A. Arias, R. Perez, A Jacobian smoothing inexact Newton method for solving the nonlinear complementary problem. *Comput. Appl. Math.*, 43(5): 279, 2024.

[45] H. Wang, A decentralized smoothing quadratic regularization algorithm for composite consensus optimization with non-Lipschitz singularities. *Numer. Algorithms* 96(1): 369–396, 2024.

[46] N. Yilmaz, A. Sahiner, New smoothing approximations to piecewise smooth functions and applications. *Numer. Func. Anal. Optim.* 40(5):513–534, 2019.

[47] N. Yilmaz, A. Sahiner, Smoothing techniques in solving non-Lipschitz absolute value equations. *Int. J. Comput. Math.* 100(4): 867–879, 2023.

[48] N. Yilmaz, A. Kayacan, A new smoothing algorithm to solve a system of nonlinear inequalities. *Fundamental Journal of Mathematics and Applications* 6(3): 137–146, 2023.

[49] N. Yilmaz, A new smoothing function technique for solving minimax problems. *J. Appl. Anal. Comput.* 15(3): 1703-1718, 2025.

[50] J. K. Liu, L. Zheng, A smoothing iterative method for the finite minimax problem. *J. Comput. Appl. Math.* 374: 112741, 2020.

[51] L. Luksan, J. Vlcek, Test problems for nonsmooth unconstrained and linearly constrained optimization. Technical report, No:798, 2000.

[52] L. Dong, B. Yu, A spline smoothing Newton method for finite minimax problems. *J. Eng. Math.* 93: 145–158, 2015.

[53] J. Asaadi, A computational comparison of some non-linear programs. *Math. Programm.* 4(1): 144–154, 1973.

[54] H. T. Jongen, F. Twilt, G. W. Weber, Semi-infinite optimization: structure and stability of the feasible set. *J. Optim. Theory Appl.* 72:529–552, 1992.

[55] T. Ergenc, S. W. Pickl, N. Radde, G. W. Weber, Generalized semi-infinite optimization and anticipatory systems, *Int. J. Comput. Anticipatory Syst.* 15: 3–30, 2004.