PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 18 · 2025 · Pages 85–96

Approximate solutions of a boundary value problem for delay nonlinear difference equations with computer realization

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Abstract

A unique class of difference equations with non-instantaneous impulses is examined. These impulses begin abruptly at specified initial time points and continue over given finite discrete intervals. Additionally, the current state appears on the right side of the equation, preventing recursive solutions and necessitating approximate methods. This paper proposes an algorithm that constructs two monotone sequences of successive approximations toward the solution of the given problem. It is theoretically demonstrated that both sequences converge to the solution. This algorithm has been implemented through a C# program using a procedural programming approach and is applied to approximate specific examples, including a modified Ricker model with non-instantaneous impulses.

Keywords: Difference equations, delay, boundary value problem, approximate solutions, computer realization.
MSC: 39A27, 65Q10.

1 Introduction

In modeling, a key challenge often revolves around the concept of time. Some researchers view real systems as continuous-time, using differential equation- based tools for simulation, while others consider real systems as discrete-time and therefore opt for discrete-time simulation tools. In [1], the author highlights that unlike continuous-time first-order systems, which cannot produce oscillatory behavior, discrete-time first-order systems can oscillate or even exhibit chaotic behavior. A discrete-time version of an epidemic model is developed in [2], while pattern formation in a discrete-time predator-prey model is explored in [3]. Interest in studying difference equations has grown recently, as seen in works like [4] and [5]. For instance, oscillation and stability in first-order difference equations are examined in [6], while [7] investigates positive solutions for second-order difference equations without impulses, and [4]demonstrates positive periodic solutions for second-order difference equations.

One of the challenges in difference equations lies in obtaining solutions, particularly when the current state appears non-linearly on both sides of the equation ([8],[9]). An approximate solution method involves the approach of upper and lower solutions combined with a monotone-iterative technique, used to construct two sequences of upper and lower solutions for nonlinear, non- instantaneous impulsive difference equations. This method is applied to difference equations in [10], [11], and impulsive difference equations in [12]

The main contributions of this paper could be summarized as follows:

- We examine a nonlinear boundary value problem for delay difference equations with non-instantaneous impulses, where the current state is involved on both sides of the equation, preventing recursive solutions;
- We propose an algorithm to obtain two monotone sequences that approximate the solution: one increasing sequence of lower solutions and one decreasing sequence of upper solutions;
- We prove the convergence of both sequences to the solution of the studied problem;
- We implement the proposed algorithm using a C# program developed in a procedural programming paradigm;
- We apply this algorithm to approximate solutions for specific examples, including a modified Ricker model with impulses.

2 Statement of the problem

Let the increasing sequence $\{\nu_i\}_{i=0}^{p+1}: \ \nu_i \in \mathbb{Z}_+, \ \nu_i \geq \nu_{i-1}+3, i=1,2,\dots,p$ and the sequence $\{d_i\}_{i=1}^{p}: \ d_i \in \mathbb{Z}_+, \ 1 \leq d_i \leq n_{i+1}-n_i-2, \ i=1,2,\dots,p$ be given where $\mathbb{Z}_+$ is the set of all nonegative integers.

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We will define some discrete sets:

- $\mathbb{Z}[a,b]=\{z\in\mathbb{Z}_+ : a\le z\le b\},\ a,b\in\mathbb{Z}_+,\ a<b$;
- $\mathcal{I}_k=\mathbb{Z}[\nu_k+d_k,\nu_{k+1}-2],\ k\in\mathbb{Z}[0,p-1],\ \mathcal{I}_p=\mathbb{Z}[\nu_p+d_p,\nu_{p+1}]$;
- $\mathcal{J}_k=\mathbb{Z}[\nu_k+1,\nu_k+d_k-1],\ k\in\mathbb{Z}[1,p]$ where $d_0=0$.

Consider the *boundary value problem (BVP)* for the nonlinear *non-instantaneous impulsive difference equation (DE)*

$$\begin{aligned}
&y(\nu+1)=f(\nu,y(\nu),y(\nu+1)) \text{ for } \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\
&y(\nu_k)=F(k,y(\nu_k-1)),\ k\in\mathbb{Z}[1,p],\\
&y(\nu)=g(\nu,y(\nu),y(\nu_k)) \text{ for } \nu\in\mathcal{J}_k,\ k\in\mathbb{Z}[1,p],\\
&y(\nu_0)=G(y(\nu_{p+1}),y(\xi)),
\end{aligned}\tag{1}$$

where $f:\bigcup_{k=0}^{p}\mathcal{I}_k\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\ F:\mathbb{Z}[1,p]\times\mathbb{R}\to\mathbb{R},\ g:\bigcup_{k=1}^{p}\mathcal{J}_k\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\ \xi:\ \nu_0+1\le\xi\le\nu_p$ is an integer, $G:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

3 Preliminaries

Definition 1. The function $\alpha:\mathbb{Z}[\nu_0,\nu_{p+1}]\to\mathbb{R}$ is called lower (upper) solutions of BVP for DE (1), if:

$$\begin{aligned}
&\alpha(\nu+1)\le(\ge)f(\nu,\alpha(\nu),\alpha(\nu+1)),\ \text{for } \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\
&\alpha(\nu_k)\le(\ge)F(k,\alpha(\nu_k-1)),\ k\in\mathbb{Z}[1,p],\\
&\alpha(\nu)\le(\ge)g(\nu,\alpha(\nu),\alpha(\nu_k)),\ \text{for } \nu\in\mathcal{J}_k,\ k\in\mathbb{Z}[1,p],\\
&\alpha(\nu_0)\le(\ge)G(\alpha(\nu_{p+1}),\alpha(\xi)).
\end{aligned}$$

Consider the linear difference equations with non-instantaneous impulses

$$\begin{aligned}
&u(\nu+1)=\mathcal{Q}_\nu u(\nu)+\sigma_n,\quad \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\
&u(\nu_k)=\mathcal{T}_k u(\nu_k-1)+\mu_k,\ k\in\mathbb{Z}[1,p],\\
&u(\nu)=\mathcal{L}_\nu u(\nu_k)+\gamma_{\nu,k},\ \text{for } n\in\mathcal{J}_k,\ k\in\mathbb{Z}[1,p],
\end{aligned}\tag{2}$$

with the initial value condition

$$u(\nu_0)=x_0,\tag{3}$$

where $u,x_0\in\mathbb{R}$, $\mathcal{Q}_\nu,\sigma_\nu$ for $\nu\in\bigcup_{k=0}^{p}\mathcal{I}_k$, $\mathcal{L}_\nu$, $\nu\in\bigcup_{k=1}^{p}\mathcal{J}_k$, $\gamma_{\nu,k}$ for $n\in\mathcal{J}_k, k\in\mathbb{Z}[1,p]$ and $\mathcal{T}_k,\mu_k:\ k\in\mathbb{Z}[1,p]$, are given real constants.

Note the initial value problem (2), (3) is in form of recurrence formulas and it is solved recursively in [13].

Lemma 1. ([13]) The initial value problem (2), (3) has an unique solution $u(\nu)$, $\nu\in\mathbb{Z}[\nu_0,\nu_{p+1}]$ such that

$$\begin{aligned}
u(\nu)=\mathcal{N}(\nu)\sum_{j=\nu_0-1}^{\nu-1}\Big(\prod_{i=j+1}^{\nu}R(i)\Big)\mathcal{S}_j\prod_{i=j+1}^{\nu-1}\mathbb{Q}_i+N(\nu)\sum_{j=\nu_0}^{n}\Big(\prod_{i=j+1}^{\nu}R(i)\Big)\zeta(j)\prod_{i=j}^{\nu-1}\mathcal{Q}_i+\tau(\nu)\\
\text{for } \nu\in\mathbb{Z}[\nu_0,\nu_{p+1}]
\end{aligned}\tag{4}$$

where

$$\mathbb{Q}_\nu=\begin{cases}\mathcal{Q}_\nu & \text{for } \nu\in\bigcup_{k=0}^{p}I_k,\\ 1 & \text{otherwise,}\end{cases}\qquad \mathcal{S}_\nu=\begin{cases}x_0 & \text{for } \nu=\nu_0-1,\\ \sigma_\nu & \text{for } \nu\in\bigcup_{k=0}^{p}I_k,\\ 0 & \text{otherwise,}\end{cases}\tag{5}$$

$$\mathcal{N}(\nu)=\begin{cases}\mathcal{L}_\nu & \text{for } \nu\in\bigcup_{k=1}^{p}\mathcal{J}_k,\\ 1 & \text{otherwise,}\end{cases}\qquad \tau(\nu)=\begin{cases}\gamma_{\nu,k} & \text{for } \nu\in J_k,\ k\in\mathbb{Z}[1,p],\\ 0 & \text{otherwise,}\end{cases}\tag{6}$$

$$\mathcal{R}(\nu)=\begin{cases}\mathcal{N}(\nu_k+d_k)=\mathcal{L}_{\nu_k+d_k} & \text{for } \nu=\nu_k+d_k,\quad k\in\mathbb{Z}[1,p],\\ \mathcal{T}_k & \text{for } \nu=\nu_k,\quad k\in\mathbb{Z}[1,p],\\ 1 & \text{otherwise,}\end{cases}\tag{7}$$

$$\zeta(n)=\begin{cases}\tau(\nu_k+d_k)=\gamma_{\nu_k+d_k,k} & \text{for } \nu=\nu_k+d_k+1,\ k\in\mathbb{Z}[1,p],\\ \mu_k & \text{for } \nu=\nu_k,\quad k\in\mathbb{Z}[1,p],\\ 0 & \text{otherwise.}\end{cases}\tag{8}$$

Lemma 2. ([13]) Let $\mu : \mathbb{Z}[\nu_0, \nu_{p+1}] \to \mathbb{R}$ satisfy the linear difference inequalities

$$\begin{aligned}
&\mu(\nu+1) \le \mathcal{Q}_\nu \mu(\nu), \quad \nu \in \bigcup_{k=0}^{p} \mathcal{I}_k,\\
&\mu(\nu_k) \le \mathcal{T}_k \mu(\nu_k - 1), \quad k \in \mathbb{Z}[1,p],\\
&\mu(\nu) \le \mathcal{M}_\nu \mu(\nu) + \mathcal{L}_\nu \mu(\nu_k), \qquad \nu \in \mathcal{J}_k, \ k \in \mathbb{Z}[1,p],\\
&\mu(\nu_0) \le 0,
\end{aligned}$$

where $\mathcal{Q}_\nu > 0$, $\left(\nu \in \bigcup_{k=0}^{p} \mathcal{I}_k\right)$, $\mathcal{T}_k > 0$, $\left(k \in \mathbb{Z}[1,p]\right)$ and $\mathcal{L}_\nu > 0$, $\mathcal{M}_\nu < 1$, $\left(\nu \in \bigcup_{k=0}^{p} \mathcal{J}_k\right)$.

Then $\mu(\nu) \le 0$ for every $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$.

4 Main results

Let the functions $\alpha, \beta : \mathbb{Z}[\nu_0, \nu_{p+1}] \to \mathbb{R}$ be such that $\alpha(\nu) \le \beta(\nu)$ for $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$. We define the following sets

$$\begin{aligned}
S(\alpha,\beta) &= \{u : \mathbb{Z}[\nu_0, \nu_{p+1}] \to \mathbb{R} : \alpha(\nu) \le u(\nu) \le \beta(\nu), \quad \nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]\}\\
\Omega_1(\alpha,\beta) &= \{u \in \mathbb{R} : \min_{\nu \in \bigcup_{k=0}^{p} \mathcal{I}_k} \alpha(\nu) \le u \le \max_{\nu \in \bigcup_{k=0}^{p} \mathcal{I}_k} \beta(\nu)\}\\
\Omega_2(\alpha,\beta) &= \{u \in \mathbb{R} : \min_{\nu \in \bigcup_{k=0}^{p-1} \mathcal{I}_k} \alpha(\nu+1) \le u \le \max_{\nu \in \bigcup_{k=0}^{p-1} \mathcal{I}_k} \beta(\nu+1)\}\\
\Lambda(\alpha,\beta) &= \{u \in \mathbb{R} : \min_{\nu \in \bigcup_{k=1}^{p} \mathcal{J}_k} \alpha(\nu) \le u \le \max_{\nu \in \bigcup_{k=1}^{p} \mathcal{J}_k} \beta(\nu)\}\\
\Gamma(\alpha,\beta) &= \{y \in \mathbb{R} : \min_{k \in \mathbb{Z}[1,p]} \alpha(\nu_k) \le y \le \max_{k \in \mathbb{Z}[1,p]} \beta(\nu_k)\}\\
\Upsilon(\alpha,\beta) &= \{z \in \mathbb{R} : \min_{k \in \mathbb{Z}[1,p]} \alpha(\nu_k - 1) \le z \le \max_{k \in \mathbb{Z}[1,p]} \beta(\nu_k - 1)\}
\end{aligned}$$

Theorem 1. Let:

1. The functions $\alpha, \beta : \mathbb{Z}[\nu_0, \nu_{p+1}] \to \mathbb{R}$ are lower and upper solutions of the BVP for DE (1), respectively, and the following inequalities $\alpha(\nu) \le \beta(\nu)$ hold for $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$.
2. The function $f \in C(\bigcup_{k=0}^{p} \mathcal{I}_k \times \Omega_1(\alpha,\beta) \times \Omega_2(\alpha,\beta), \mathbb{R})$ and there exist functions $K \in C((\bigcup_{k=0}^{p} \mathcal{I}_k, (-\infty, 1))$ and $P \in C(\bigcup_{k=0}^{p} \mathcal{I}_k, (0,\infty))$ such that for any $\nu \in \bigcup_{k=0}^{p} \mathcal{I}_k$ and $\xi_1, \xi_2 \in \Omega_1(\alpha,\beta)$, with $\xi_1 \le \xi_2$, and $\eta_3, \eta_4 \in \Omega_2(\alpha,\beta)$, with $\eta_3 \le \eta_4$ the inequality
$$f(\nu, \xi_1, \eta_3) - f(\nu, \xi_2, \eta_4) \le P(\nu)(\xi_1 - \xi_2) + K(\nu)(\eta_3 - \eta_4)$$
holds.
3. The function $F \in C(\mathbb{Z}[1,p] \times \mathbb{R}, \mathbb{R})$ and there exists a function $T \in C(\mathbb{Z}[1,p], (0,\infty))$ such that for any $\nu \in \mathbb{Z}[1,p]$ and $\xi_1, \xi_2 \in \Upsilon(\alpha,\beta)$ with $z_1 \le z_2$
$$F(\nu, \xi_1) - F(\nu, \xi_2) \le T(\nu)(\xi_1 - \xi_2).$$
4. The function $g \in C(\bigcup_{k=1}^{p} J_k \times \Lambda(\alpha,\beta) \times \Gamma(\alpha,\beta), \mathbb{R})$ and there exist functions $M \in C(\bigcup_{k=1}^{p} \mathcal{J}_k, (-\infty, 1))$ and $L \in C(\bigcup_{k=1}^{p} \mathcal{J}_k, (0,\infty))$ such that for any $\nu \in \bigcup_{k=1}^{p} \mathcal{J}_k$ and $\xi_1, \xi_2 \in \Lambda(\alpha,\beta)$, with $\xi_1 \le \xi_2$, and $\eta_3, \eta_4 \in \Gamma(\alpha,\beta)$, with $\eta_3 \le \eta_4$ the inequality
$$g(\nu, \xi_1, \eta_3) - g(\nu, \xi_2, \eta_4) \le M(\nu)(\xi_1 - \xi_2) + L(\nu)(\eta_3 - \eta_4)$$
holds.
5. The function $G \in C(S(\alpha,\beta) \times S(\alpha,\beta), \mathbb{R})$.
6. The problem (1) has an unique solution $u(n) \in S(\alpha,\beta)$.

Then we could construct two sequences of functions $\{\alpha^{(j)}(\nu)\}_0^\infty$ and $\{\beta^{(j)}(\nu)\}_0^\infty$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$ with $\alpha^{(0)} = \alpha$ and $\beta^{(0)} = \beta$, any element of these sequences is a lower/upper solution of (1), the inequalities

$$\alpha(\nu) \le \alpha^{(j)}(\nu) \le \alpha^{(j+1)}(\nu) \le \beta^{(j+1)}(\nu) \le \beta^{(j)}(\nu) \le \beta(\nu), \text{ for } \nu \in \mathbb{Z}[\nu_0, \nu_{p+1}], \ \ j \in \mathbb{Z}$$

hold and both sequences are convergent to the solution of BVP for DE (1).

Proof. Let $\eta \in S(\alpha,\beta)$ be an arbitrary fixed function. Consider the following initial value problem for the scalar linear difference equation

$$\begin{aligned} &u(\nu+1)=P(\nu)u(\nu)+K(\nu)u(\nu+1)+\psi(\nu,\eta(\nu),\eta(\nu+1)), \quad \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\ &u(\nu_k)=T(k)u(\nu_k-1)+\upsilon(k,\eta(\nu_k-1)), \quad k\in\mathbb{Z}[1,p],\\ &u(\nu)=M(\nu)u(\nu)+L(\nu)u(\nu_k)+\Xi(\nu,\eta(\nu),\eta(\nu_k)), \quad n\in\mathcal{J}_k,\ k\in\mathbb{Z}[1,p],\\ &u(\nu_0)=G(\eta(\nu_{p+1}),\eta(\xi)), \end{aligned} \tag{9}$$

where

$$\begin{aligned} &\psi(\nu,\mu,\rho)=f(\nu,\mu,\rho)-P(\nu)\mu-K(\nu)\rho, \quad \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\ &\upsilon(\nu,\mu)=F(\nu,\mu)-T(\nu)\mu, \ \ \nu\in\mathbb{Z}[1,p]\\ &\Xi(\nu,\mu,\rho)=g(\nu,\mu,\rho)-M(\nu)\mu-L(\nu)\rho, \quad \nu\in\bigcup_{k=1}^{p}\mathcal{J}_k. \end{aligned}$$

The problem (9) could be written in the following form

$$\begin{aligned} &u(\nu+1)=\frac{P(\nu)}{1-K(\nu)}u(\nu)+\frac{\psi(\nu,\eta(\nu),\eta(\nu+1))}{1-K(\nu)}, \quad \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\ &u(\nu_k)=T(k)u(\nu_k-1)+\upsilon(k,\eta(\nu_k-1)), \quad k\in\mathbb{Z}[1,p],\\ &u(\nu)=\frac{L(\nu)}{1-M(\nu)}u(\nu_k)+\frac{\Xi(\nu,\eta(\nu),\eta(\nu_k))}{1-M(]n)}, \quad \nu\in\mathcal{J}_k,\ k\in\mathbb{Z}[1,p],\\ &u(\nu_0)=G(\eta(\nu_{p+1}),\eta(\xi)). \end{aligned} \tag{10}$$

According to Lemma 1 with

$$\begin{aligned} &x_0=G(\eta(\nu_{p+1}),\eta(\xi)),\\ &\sigma_\nu=\frac{\psi(\nu,\eta(\nu),\eta(\nu+1))}{1-K(\nu)},\ \mathcal{Q}_\nu=\frac{P(\nu)}{1-K(\nu)}, \quad \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\ &\gamma_{\nu,k}=\frac{\Xi(\nu,\eta(\nu),\eta(\nu_k))}{1-M(\nu)},\ \mathcal{L}_\nu=\frac{L(\nu)}{1-M(\nu)}, \quad \text{for } \nu\in\mathcal{J}_k,\ k\in\mathbb{Z}[1,p],\\ &\mu_k=\upsilon(k,\eta(\nu_k-1)), \quad \mathcal{T}_k=T(k), \quad k\in\mathbb{Z}[1,p], \end{aligned}$$

the initial value problem (10) has an unique solution given by

$$\begin{aligned} u(\nu)=&N(\nu)\sum_{j=\nu_0-1}^{\nu-1}\Big(\prod_{i=j+1}^{\nu}R(i)\Big)\mathcal{S}_j\prod_{i=j+1}^{\nu-1}\mathcal{Q}_i+N(\nu)\sum_{j=\nu_0}^{\nu}\Big(\prod_{i=j+1}^{\nu}R(i)\Big)\zeta(j)\prod_{i=j}^{\nu-1}\mathcal{Q}_i+\tau(\nu)\\ &\text{for } \nu\in\mathbb{Z}[\nu_0,\nu_{p+1}], \end{aligned}$$

where

$$\mathcal{Q}_\nu=\begin{cases}\frac{P(\nu)}{1-K(\nu)} & \text{for } \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\ 1 & \text{otherwise,}\end{cases} \qquad \mathcal{S}_\nu=\begin{cases}G(\eta(\nu_{p+1}),\eta(\xi)) & \text{for } \nu=\nu_0-1,\\ \frac{\psi(\nu,\eta(\nu),\eta(\nu+1))}{1-K(\nu)} & \text{for } \nu\in\bigcup_{k=0}^{p}\mathcal{I}_k,\\ 0 & \text{otherwise,}\end{cases}$$

$$N(\nu)=\begin{cases}\frac{\mathcal{L}(\nu)}{1-M(\nu)}, & \nu\in\bigcup_{k=1}^{p}\mathcal{J}_k,\\ 1 & \text{otherwise,}\end{cases} \qquad \tau(\nu)=\begin{cases}\frac{\Xi(\nu,\eta(\nu),\eta(\nu_k))}{1-M(\nu)}, & \nu\in\mathcal{J}_k,\ k\in\mathbb{Z}[1,p],\\ 0 & \text{otherwise,}\end{cases}$$

$$R(\nu)=\begin{cases}\frac{\mathcal{L}(\nu_k+d_k)}{1-M(\nu_k+d_k)} & \text{for } \nu=\nu_k+d_k+1,\ k\in\mathbb{Z}[1,p],\\ \mathcal{T}(k) & \text{for } \nu=\nu_k, \quad k\in\mathbb{Z}[1,p],\\ 1 & \text{otherwise,}\end{cases}$$

$$\zeta(\nu)=\begin{cases}\frac{\Xi(\nu_k+d_k,\eta(\nu_k+d_k),\eta(\nu_k))}{1-M(\nu_k+d_k)} & \text{for } \nu=\nu_k+d_k+1,\ k\in\mathbb{Z}[1,p],\\ \upsilon(k,\eta(\nu_k-1)) & \text{for } \nu=\nu_k, \quad k\in\mathbb{Z}[1,p],\\ 0 & \text{otherwise.}\end{cases}$$

Let $\eta \in S(\alpha,\beta)$. Define the operator $\mathcal{W}: S(\alpha,\beta) \to S(\alpha,\beta)$ by the equality $\mathcal{W}\eta = u$, where u is the unique solution of IVP (9) for the function η.

We will prove that $\alpha \leq \mathcal{W}\alpha$, $\beta \geq \mathcal{W}\beta$. Let $\alpha^{(1)}$ is the unique solution of (9) with $\eta = \alpha$. Denote $W\alpha = \alpha^{(1)}$ and let $m(\nu) = \alpha(\nu) - \alpha^{(1)}(\nu)$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$.

Thus, $m(\nu_0) \leq G(\alpha(\nu_{p+1}), \alpha(\xi)) - G(\alpha(\nu_{p+1}), \alpha(\xi)) = 0$.

Let $\nu \in \bigcup_{k=0}^{p} \mathcal{I}_k$. Then we get

$$\begin{aligned} m(\nu+1) &= \alpha(\nu+1) - P(\nu)\alpha^{(1)}(\nu) - K(\nu)\alpha^{(1)}(\nu+1) - \psi(\nu,\alpha(\nu),\alpha(\nu+1)) \\ &\leq P(\nu)(\alpha(\nu)-\alpha^{(1)}(\nu)) + K(\nu)(\alpha(\nu+1)-\alpha^{(1)}(\nu+1)) = P(\nu)m(\nu) + K(\nu)m(\nu+1). \end{aligned}$$

Hence $m(\nu+1) \leq \frac{P(\nu)}{1-K(\nu)}\, m(\nu)$ for $\nu \in \bigcup_{k=0}^{p} \mathcal{I}_k$.

Let $\nu = \nu_k$. Then we have

$$m(\nu_k) \leq F(k,\alpha(\nu_k-1)) - \mathcal{T}(k)\alpha^{(1)}(\nu_k-1) - F(k,\alpha(\nu_k-1)) + \mathcal{T}(k)\alpha(\nu_k-1) = \mathcal{T}(k)m(\nu_k-1).$$

Let $\nu \in \bigcup_{k=1}^{p} \mathcal{J}_k$. Then we obtain

$$\begin{aligned} m(n) &\leq g(n,\alpha(n),\alpha(n_k)) - M(n)\alpha^{(1)}(n) - L(n)\alpha^{(1)}(n_k) \\ &\quad - g(n,\alpha(n),\alpha(n_k)) + M(n)\alpha(n) + L(n)\alpha(n_k) \\ &= M(n)m(n) + L(n)m(n_k). \end{aligned}$$

Therefore, the inequalities (9) hold for $m(\nu)$ with $\mathcal{Q}_\nu = \frac{P(n)}{1-K(n)}$, $\mathcal{T}_k = T(k)$, $\mathcal{M}_\nu = M(\nu)$, $\mathcal{L}_\nu = L(\nu)$. According to Lemma 2 $m(n) \geq 0$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$, i.e. $\alpha \leq \mathcal{W}\alpha$. Similarly, $\beta \geq \mathcal{W}\beta$.

We will prove that $\mathcal{W}$ is a nondecreasing operator in $S(\alpha,\beta)$. Let $\eta_1,\ \eta_2 \in S(\alpha,\beta:\ \ \eta_1(\nu) \leq \eta_2(\nu)$ for $\nu \in \mathbb{Z}[\nu_0, n_{p+1}]$. Denote $m = u^{(1)} - u^{(2)}$ where $u^{(1)} = \mathcal{W}\eta_1$ and $u^{(2)} = \mathcal{W}\eta_2$. It is easy to see that the function $m(\nu)$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$ is satisfying inequality 9 with $\mu(\nu) = m(\nu)$, $\mathcal{Q}_\nu = \frac{P(\nu)}{1-K(\nu)}$, $\mathcal{T}_\nu = T(k)$, $\mathcal{M}_\nu = M(\nu)$ and $\mathcal{L}_\nu = L(\nu)$. According to Lemma 2 $m(\nu) \leq 0$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$.

Consider a lower solution $\eta \in S(\alpha,\beta)$ of (1). Let $m = W\eta$. According to the above proved $\eta(\nu) \leq m(\nu)$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$. It is easy to show that the function m is a lower solution of (1). Analogously, if $\eta \in S(\alpha,\beta)$ is an upper solution of (1) then the function $m = \mathcal{W}\eta$ is an upper solution of (1).

Step by step we construct the sequences of functions $\{\alpha^{(j)}(\nu)\}_0^\infty$ and $\{\beta^{(j)}(\nu)\}_0^\infty$ by the equalities $\alpha^{(0)} = \alpha$, $\beta^{(0)} = \beta$, $\alpha^{(j)} = \mathcal{W}\alpha^{(j-1)}$, $\beta^{(j)} = \mathcal{W}\beta^{(j-1)}$. Also, any element $\alpha^{(j)}$ and $\beta^{(j)}$ is satisfying the equality 11 with $\eta = \alpha^{(j-1)}$ and $\eta = \beta^{(j-1)}$ respectively.

According to the proved above properties of the operator $\mathcal{W}$, the claims of Theorem 1 are satisfied. Both sequences being monotonic and bounded are convergent on $\mathbb{Z}[\nu_0, \nu_{p+1}]$. Denote the limits $A(\nu) = \lim_{s\to\infty} \alpha^{(s)}(\nu), B(\nu) = \lim_{s\to\infty} \beta^{(s)}(\nu)$.

By taking a limit in the equations for $\alpha^{(j)}$ corresponding to (11) for $s \to \infty$ we obtain

$$\begin{aligned} A(\nu) &= N(\nu) \sum_{j=\nu_0-1}^{\nu-1} \Big(\prod_{i=j+1}^{\nu} R(i)\Big) \frac{\psi(j,A(j),A(j+1))}{1-K(j)} \prod_{i=j+1}^{\nu-1} \frac{P(i)}{1-K(i)} \\ &+ \sum_{j=\nu_0}^{n} \Big(\prod_{i=j+1}^{\nu} R(i)\Big)\zeta(j) \prod_{i=j}^{\nu-1} \frac{P(i)}{1-K(i)} + \tau(\nu), \ \text{for}\ \nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]. \end{aligned} \tag{11}$$

Equality (11) show the function $A(n)$ is a solution of the boundary value problem for DE (1).

According to condition 5 $A(\nu) \equiv u(\nu) \equiv B(\nu)$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$.

□

5 Algorithm for construction the approximate solution

First we will construct an increasing sequence $\{\alpha^{(s)}\}_{s=0}^\infty$ of lower solutions of BVP for DE (1):

Step 1. Start with the chosen lower solution $\alpha^{(0)}(\nu)$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$ of (1).

Step 2. Obtain the next approximation $\alpha^{(s)}(\nu)$, $s = 1$, $\nu \in \mathbb{Z}[\nu_0, \nu_{p+1}]$ by the linear recurrence formulas

$$\begin{aligned} &\alpha^{(s)}(\nu+1) = \frac{P(\nu)}{1-K(\nu)}\alpha^{(s)}(\nu) + \frac{\psi(\nu,\alpha^{(s-1)}(\nu),\alpha^{(s-1)}(\nu+1))}{1-K(\nu)}, \quad \nu \in \bigcup_{k=0}^{p} \mathcal{I}_k, \\ &\alpha^{(s)}(\nu_k) = T(k)\alpha^{(s)}(\nu_k-1) + \upsilon(k,\alpha^{(s-1)}(\nu_k-1)), \quad k \in \mathbb{Z}[1,p], \\ &\alpha^{(s)}(\nu) = \frac{L(\nu)}{1-M(\nu)}\alpha^{(s)}(\nu_k) + \frac{\Xi(\nu,\alpha^{(s-1)}(\nu),\alpha^{(s-1)}(\nu_k))}{1-M(\nu)}, \quad n \in \mathcal{J}_k,\ k \in \mathbb{Z}[1,p], \\ &\alpha^{(s)}(\nu_0) = G(\alpha^{(s-1)}(\nu_{p+1}), \alpha^{(s-1)}(\xi)), \end{aligned} \tag{12}$$

Step 3. If $\max_{\nu\in\mathbb{Z}[\nu_0,\nu_{p+1}]}|\alpha^{(s)}(\nu)-\alpha^{(s-1)}(\nu)|<\varepsilon$ for an initially given number $\varepsilon>0$ we stop and the approximate solution of the BVP for DE (1) is $\alpha^{(s)}$, $s\in\mathbb{Z}[n_0,n_{p+1}]$. If the inequality is not satisfied, then we go to Step 2 with replacing s in the system (12) by $s+1$.

The construction of a decreasing sequence $\{\beta^{(s)}(\nu)\}_{s=0}^{\infty}$ of upper solutions of BVP for DE (1) is by the above algorithm replacing $\alpha^{(s)}(\nu)$ by $\beta^{(s)}(\nu)$ and starting by the chosen upper solution $\beta^{(0)}(\nu)$, $\nu\in\mathbb{Z}[\nu_0,\nu_{p+1}]$ of (1) in Step 1.

6 Computer realization of the algorithm

6.1 Example 1.

We will apply the above algorithm to a particular problem.

Let $\mathcal{I}_0=\mathbb{Z}[0,3]$, $\mathcal{I}_1=\mathbb{Z}[7,9]$, $\mathcal{I}_2=\mathbb{Z}[17,19]$, $\mathcal{I}_3=\mathbb{Z}[26,30]$, $\nu_k=5,11,20$, for $k=1,2,3$, $\mathcal{J}_1=\mathbb{Z}[6,7]$, $\mathcal{J}_2=\mathbb{Z}[12,16]$, $\mathcal{J}_3=\mathbb{Z}[21,25]$.

Consider the BVP for the following difference equation

$$\begin{aligned}
x(\nu)&=\frac{\sin(x(\nu))+0.9x(\nu-1)}{n}+0.1\\
&\quad\text{for } \nu\in\mathbb{Z}[1,4]\cup\mathbb{Z}[8,10]\cup\mathbb{Z}[17,19]\cup\mathbb{Z}[26,30],\\
x(5)&=\sqrt{x(4)+1},\quad x(11)=\sqrt{x(10)+1},\quad x(20)=\sqrt{x(19)+1},\\
x(\nu)&=x(5)e^{-2\nu}+e^{x(\nu)-\nu-3}\ \text{for } \nu\in\mathbb{Z}[6,7],\\
x(\nu)&=x(11)e^{-2\nu}+e^{x(\nu)-\nu-3}\ \text{for } \nu\in\mathbb{Z}[12,16],\\
x(\nu)&=x(20)e^{-2\nu}+e^{x(\nu)-\nu-3}\ \text{for } \nu\in\mathbb{Z}[21,25],\\
x(0)&=0.5\sqrt{|x(30)x(20|},
\end{aligned}\tag{13}$$

Note that (13) could not be considered as linear recurrence formulas which could be easily solved and obtained the solution.

A lower solution of (13) is $\alpha_0(\nu)=0$, $\nu\in\mathbb{Z}[0,29]$, because

$$\begin{aligned}
&0\le\frac{\sin(0)+0.9(0)}{\nu}+0.1\ \text{for } \nu\in\ \mathbb{Z}[1,4]\cup\mathbb{Z}[8,10]\cup\mathbb{Z}[17,19]\cup\mathbb{Z}[26,30],\\
&0\le\sqrt{1},\\
&0\le 0e^{-2\nu}+e^{0-\nu-3}\ \text{for } \nu\in\mathbb{Z}[6,7]\cup\mathbb{Z}[12,16]\cup\mathbb{Z}[21,25],\\
&x(0)=0.5\sqrt{|1|},
\end{aligned}$$

and an upper solution of (13) is $\beta_0(\nu)=3$, $\nu\in\mathbb{Z}[0,29]$, because

$$\begin{aligned}
&3\ge\frac{\sin(3)+0.9(3)}{\nu}+0.1\ \text{for } \nu\in\mathbb{Z}[1,4]\cup\mathbb{Z}[8,10]\cup\mathbb{Z}[17,19]\cup\mathbb{Z}[26,30],\\
&3\ge\sqrt{3+1},\\
&3\ge 3e^{-2\nu}+e^{-\nu}\ \text{for } \nu\in\mathbb{Z}[6,7]\cup\mathbb{Z}[12,16]\cup\mathbb{Z}[21,25],\\
&x(0)=0.5\sqrt{|9|},
\end{aligned}$$

The condition 2 of Theorem 1 is satisfied for the function $f(\nu,x,y)=\frac{0.9x+\sin(y)}{\nu}+0.1$ with $P(\nu)=\frac{0.9}{\nu}$ and $K(\nu)=-\frac{1}{\nu}$ because for any x_i, $k=1,2,3,4:\ 0\le x_1\le x_2\le 3,\ 0\le x_3\le x_4\le 3$ the inequality

$$\frac{0.9x_1+\sin(x_3)}{\nu}-\frac{0.9x_2+\sin(x_4)}{\nu}\le\frac{0.9}{\nu}(x_1-x_2)-\frac{1}{\nu}(x_3-x_4)$$

holds.

The condition 3 of Theorem 1 is satisfied for the function $F(k,x)=\sqrt{x+1}+0$ with $T(\nu)=0.25$ because for any $z_1,z_2:\ 0\le z_1\le z_2\le 3$ we have $F(k,z_1)-F(k,z_2)=\sqrt{z_1+1}-\sqrt{z_2+1}\le 0.25(z_1-z_2)$.

The condition 4 of Theorem 1 is satisfied for the function $g(\nu,x,y)=e^{x-\nu+1}+ye^{-2\nu}$ with $M(\nu)=e^{-\nu}<1$ and $L(\nu)=e^{-2\nu}$ because for any y_i, $i=1,2,3,4: 0\le y_1\le y_2\le 3$, and $0\le y_3\le y_4\le 3$ the inequality

$$g(\nu,y_1,y_3)-g(\nu,y_2,y_4)=e^{-\nu}(e^{y_1+1}-e^{y_2+1})+e^{-2\nu}(y_3-y_4)\le e^{-\nu}(y_1-y_2)+e^{-2\nu}(y_3-y_4)+$$

holds.

In this case the system (9) is reduced to the following system

$$
\begin{aligned}
u(\nu) &= \frac{0.9}{\nu+1}u(\nu-1) + \frac{\sin(\eta(\nu)) + 0.1\nu + \eta(\nu)}{\nu+1} \\
&\quad \text{for} \quad \nu \in \mathbb{Z}[1,4] \cup \mathbb{Z}[8,10] \cup \mathbb{Z}[17,19] \cup \mathbb{Z}[26,30], \\
u(5) &= \sqrt{\eta(4)+1}, \\
u(\nu) &= \frac{e^{-2\nu}}{1-e^{-\nu}}u(5) + \frac{e^{\eta(\nu)+1-\nu} - e^{-\nu}\eta(\nu)}{1-e^{-\nu}}, \qquad \nu \in \mathbb{Z}[6,7], \\
u(11) &= \sqrt{|\eta(10)|+1}, \\
u(\nu) &= \frac{e^{-2\nu}}{1-e^{-\nu}}u(11) + \frac{e^{\eta(\nu)+1-\nu} - e^{-\nu}\eta(\nu)}{1-e^{-\nu}}, \qquad \nu \in \mathbb{Z}[12,16], \\
u(20) &= \sqrt{|\eta(19)|+1} \\
u(\nu) &= \frac{e^{-2\nu}}{1-e^{-\nu}}u(20) + \frac{e^{\eta(\nu)+1-\nu} - e^{-\nu}\eta(\nu)}{1-e^{-\nu}}, \qquad \nu \in \mathbb{Z}[21,25], \\
u(0) &= 0.5\sqrt{|\eta(30)\eta(20))|}.
\end{aligned}
$$

6.1.1 Algorithm

First we will construct an increasing sequence $\{\alpha^{(s)}\}_{s=0}^{\infty}$ of lower solutions of BVP for NIDE (13):

Step 1. Start with $\alpha^{(0)}(\nu) = 0, \ \nu \in \mathbb{Z}[0,30]$.

Step 2. Obtain next approximation $\alpha^{(s)}(\nu)$, $s = 1, \ \ \nu \in \mathbb{Z}[0,30]$ by the linear recurrence formulas

$$
\begin{aligned}
\alpha^{(s)}(0) &= 0.5\sqrt{|\alpha^{(s-1)}(30)\alpha^{(s-1)}(20))|}, \\
\alpha^{(s)}(\nu) &= \frac{0.9}{\nu+1}\alpha^{(s)}(\nu-1) + \frac{\sin(\alpha^{(s-1)}(\nu)) + 0.1\nu + \alpha^{(s-1)}(\nu)}{\nu+1} \\
&\quad \text{for } \nu \in \mathbb{Z}[1,4] \cup \mathbb{Z}[8,10] \cup \mathbb{Z}[17,19] \cup \mathbb{Z}[26,30], \\
\alpha^{(s)}(5)&\sqrt{|\alpha^{(s-1)}(4)|+1}, \\
\alpha^{(s)}(\nu) &= \frac{e^{-2\nu}}{1-e^{-\nu}}\alpha^{(s)}(5) + \frac{e^{\alpha^{(s-1)}(\nu)+1-\nu} - e^{-\nu}\alpha^{(s-1)}(\nu)}{1-e^{-\nu}}, \qquad \nu \in \mathbb{Z}[6,7], \\
\alpha^{(s)}(11) &= \sqrt{|\alpha^{(s-1)}(10)|+1}, \\
\alpha^{(s)}(\nu) &= \frac{e^{-2\nu}}{1-e^{-\nu}}\alpha^{(s)}(11) + \frac{e^{\alpha^{(s-1)}(\nu)+1-\nu} - e^{-\nu}\alpha^{(s-1)}(\nu)}{1-e^{-\nu}}, \qquad \nu \in \mathbb{Z}[12,16], \\
\alpha^{(s)}(20) &= \sqrt{|\alpha^{(s-1)}(19)|+1}, \\
\alpha^{(s)}(\nu) &= \frac{e^{-2\nu}}{1-e^{-\nu}}\alpha^{(s)}(20) + \frac{e^{\alpha^{(s-1)}(\nu)+1-\nu} - e^{-\nu}\alpha^{(s-1)}(\nu)}{1-e^{-\nu}}, \qquad \nu \in \mathbb{Z}[21,25].
\end{aligned}
\tag{14}
$$

Step 3. If $\max_{\nu \in \mathbb{Z}[0,30]} |\alpha^{(s)}(\nu) - \alpha^{(s-1)}(\nu)| < \varepsilon$ for an initially given number $\varepsilon > 0$ we stop and the approximate solution of BVP for DE (13) is $\alpha^{(s)}$, $s \in \mathbb{Z}[0,30]$. If the inequality is not satisfied, then we go to Step 2 with replacing s in the system (14) by $s+1$.

We presented some values of the sequence $\alpha^{(s)}(\nu)$ of (13) in Table 1 and their graphs - on Figures 1 (continuously) and Figure 2 (discretely). From both, the table and the graphs, it could be seen the sequence is increasing one.

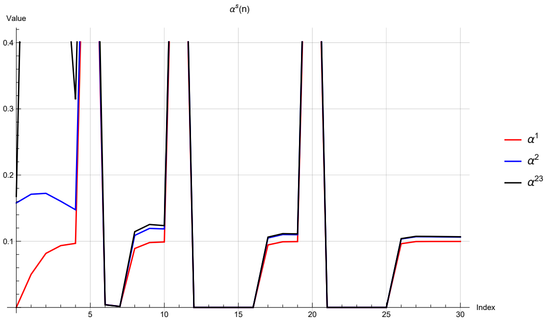

Figure 1. Graphs of some lower solutions of (13) *(lines).*

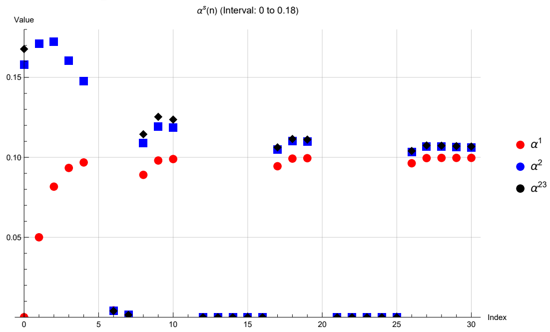

Figure 2. Graphs of some lower solutions of (13) *(discretely).*

The above described algorithm gives us a procedure to construction of a decreasing sequence $\{\beta^{(s)}(n)\}_{s=0}^{\infty}$ of upper solutions of BVP for DE (13) by $\alpha^{(s)}(\nu)$ by $\beta^{(s)}(\nu)$ and starting with $\beta^{(0)}(\nu) = 3, \ \nu \in \mathbb{Z}[0,30]$ in Step 1.

ν	α^1	α^2	α^3	α^4	...	α^{23}
$\nu=0$	0	0.15785	0.16692	0.16766	...	0.16773
$\nu=1$	0.05	0.17102	0.29572	0.41902	...	1.17079
$\nu=2$	0.08167	0.17239	0.27002	0.37130	...	1.06311
$\nu=3$	0.09337	0.16044	0.21580	0.26603	...	0.60935
$\nu=4$	0.09681	0.14757	0.17777	0.19880	...	0.31427
...	...	...	...	...	...	...
$\nu=26$	0.09630	0.10342	0.10395	0.10399	...	0.10399
$\nu=27$	0.09952	0.10686	0.10740	0.10743	...	0.10744
$n=28$	0.09964	0.10673	0.10724	0.10727	...	0.10728
$\nu=29$	0.09966	0.10651	0.10698	0.10701	...	0.10701
$\nu=30$	0.09967	0.10629	0.10673	0.10676	...	0.10676

Table 1: Values of lower solutions of (13).

Some values of the sequence of upper solutions $\beta^{(s)}(\nu)$ are given in Table 2 and graphs are on Figures 3 (continuously) and Figure 4 (discretly). From both, the table and the graphs, it could be seen the sequence $\beta^{(s)}(\nu)$ is decreasing one.

ν	β^1	β^2	β^3	β^4	...	β^{16}
$\nu=0$	1.50000	0.31948	0.17839	0.16867	...	0.16773
$\nu=1$	2.29556	1.71587	1.48296	1.36545	...	1.17477
$\nu=2$	1.80237	1.50666	1.34642	1.25009	...	1.06726
$\nu=3$	1.26581	0.96891	0.82624	0.74668	...	0.61206
$\nu=4$	0.93607	0.60267	0.46262	0.39619	...	0.31538
...	...	...	...	...	...	...
$\nu=26$	0.21263	0.11199	0.10458	0.10404	...	0.10399
$\nu=27$	0.21545	0.11536	0.10802	0.10748	...	0.10744
$\nu=28$	0.21155	0.11467	0.10780	0.10731	...	0.10728
$\nu=29$	0.20772	0.11390	0.10749	0.10704	...	0.10701
$n=30$	0.20413	0.11321	0.10719	0.10679	...	0.10676

Table 2: Values of upper solutions of (13).

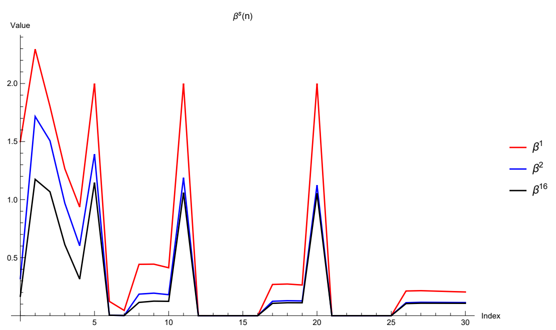

Figure 3. Graphs of some upper solutions of (13) *(lines).*

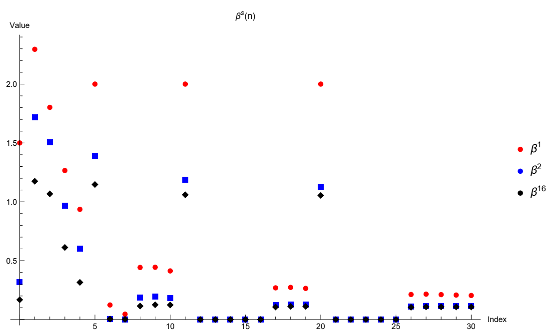

Figure 4. Graphs of some upper solutions of (13) *(discretely).*

Also, from both Table 1 and Table 2 it could be seen that both sequences of lower and upper solutions, respectively, have one and the same limit, which is the solution of the studied model.

The solution of (13), obtained by the suggested algorithm is

$$x(n) = (0.16773, 1.17079, 1.06311, 0.60935, 0.314266, 1.14631, 0.00394, 0, 00144,\\ 0.11443, 0.12533, 0.12361, 1.06001, 1.0189 * 10^{-05}, 3.7484 * 10^{-06},\\ 1.3789 * 10^{-06}, 5.0727 * 10^{-07}, 1.8661 * 10^{-07}, 0.10624,\\ 0.11149, 0.11112, 1.05410, 1.2619 * 10^{-09}, 4.6422 * 10^{-10}\\ 1.7078 * 10^{-10}, 6.2825 * 10^{-11}, 2.3112 * 10^{-11}, 0.10399\\ 0.10744, 0.10728, 0.10701, 0.10676).$$

The solutions of the appropriate linear recursive system (14), obtained by the suggested algorithm, was calculated using a C# program created through a procedural programming paradigm. The procedural paradigm is preferred when solving this type of linear equations due to its straightforward and efficient handling of step-by-step computations.

In the computation, the value of the error ε was taken as 0.001.

The program was run on an Intel platform supported by an Intel Core i5-1335U processor, requiring 19 ms to compute the sequence of lower solutions α and the sequence of upper solutions β. In the study we made attempted to rewrite the algorithm using an object-oriented programming paradigm. As a result, the CPU time increased nearly threefold, taking 58 milliseconds for the calculation of α on the same platform, and 52 milliseconds for β.

When increasing the accuracy to $\varepsilon = 0.000001$, reaching the final solution for α required 48 iterations and 33 ms of CPU time if procedural programming paradigm are used and 92 ms for object-oriented programming paradigm, while for β required 41 iterations and 37 ms of CPU time for procedural programming paradigm and 98 ms for object-oriented programming paradigm.

CAS Wolfram Mathematica was used to graph Figure 1 -Figure 4.

6.2 Example 2. Ricker model with impulses

Consider the Ricker population model which gives us the expected number $x(\nu+1)$ (or density) of the number of individuals in generation ν as a function of the number of individuals in the previous generation. The classical Ricker model was introduced in 1954 by Ricker [14] in the connection with the stock and the recruitment in fisheries. The classical model was used to predict the number of fish that will be present in a fishery. Recently it was generalized and applied for modeling various problems (see, for example [15] for modeling two species as a part of competition model, [16] for stochastic generalization).

We will modify this model, assuming that there will be external influence on some generations, i.e. non-instantaneous impulses will be applied.

Let the initial generation is $\nu_0 = 0$, and every fourth generation will be squared by an external perturbation and will continue to be the same number of individuals of as at the recently perturbed generation for three generations additionally, i.e. $\mathcal{I}_0 = \mathbb{Z}[1,3]$, $\mathcal{I}_1 = \mathbb{Z}[8,10]$, $\mathcal{I}_2 = \mathbb{Z}[15,17]$, $\mathcal{I}_3 = \mathbb{Z}[22,24]$, $n_k = 4, 11, 18$, for $k = 1,2,3$, $\mathcal{J}_1 = \mathbb{Z}[5,7]$, $\mathcal{J}_2 = \mathbb{Z}[12,14]$, $\mathcal{J}_3 = \mathbb{Z}[19,21]$.

Then we model the situation by

$$\begin{aligned}
&x(\nu) = x(\nu-1)e^{r}e^{-\frac{r}{L}x(\nu-1)} \quad \text{for} \quad \nu \in \mathbb{Z}[1,3]\cup\mathbb{Z}[8,10]\cup\mathbb{Z}[15,17]\cup\mathbb{Z}[22,24],\\
&x(4) = \sqrt{x(3)}, \quad x(11) = \sqrt{x(10)}, \quad x(18) = \sqrt{x(17)},\\
&x(\nu) = x(4) \text{ for } \nu \in \mathbb{Z}[5,7],\\
&x(n) = x(11) \text{ for } \nu \in \mathbb{Z}[12,14],\\
&x(\nu) = x(18) \text{ for } \nu \in \mathbb{Z}[19,21],\\
&x(0) = \sqrt{x(24)},
\end{aligned} \tag{15}$$

where r is the intrinsic growth rate and L as the carrying capacity of the environment.

Let for example $L = 10000$ and $r = 0.9$.

The function $x(\nu) = 1$, $\nu \in \mathbb{Z}[0,24]$ is a lower solution of (15) because

$$\begin{aligned}
&1 \le e^{0.9(1-\frac{1}{10000}1)} \quad \text{for} \quad \nu \in \mathbb{Z}[1,3]\cup\mathbb{Z}[8,10]\cup\mathbb{Z}[15,17]\cup\mathbb{Z}[22,24],\\
&1 = \sqrt{1}.
\end{aligned}$$

The function $x(\nu) = 10001$, $\nu \in \mathbb{Z}[0,24]$ is an upper solution of (15) because

$$\begin{aligned}
&10001 \ge 10001e^{0.9(1-\frac{1}{10000}10001)} \quad \text{for} \quad \nu \in \mathbb{Z}[1,3]\cup\mathbb{Z}[8,10]\cup\mathbb{Z}[15,17]\cup\mathbb{Z}[22,24],\\
&10001 \ge \sqrt{10001}.
\end{aligned}$$

Consider the function $f(\nu,x) = xe^{0.9(1-\frac{1}{10000}x)}$. For any x_i, $k = 1,2:$ $1 \le x_1 \le x_2 \le 10001$ we obtain the inequalities

$$\begin{aligned}
&f(\nu,x_1) - f(\nu,x_2) \le f'(\nu,\xi)(x_1 - x_2) = e^{0.9(1-\frac{1}{10000}\xi)}(1 - \frac{0.9}{10000}\xi)(x_1 - x_2)\\
&\le e^{0.9(1-\frac{1}{10000}10001)}(1 - \frac{0.9}{10000}10001)(x_1 - x_2).
\end{aligned}$$

Therefore, the condition 2 of Theorem 1 is satisfied for $f(\nu,x) = xe^{0.9}e^{-\frac{0.9}{10000}x}$ with $P(\nu) = e^{0.9(1-\frac{1}{10000}10001)}(1 - \frac{0.9}{10000}10001)$ and $K(\nu) = 0$.

6.2.1 Algorithm

First we will construct an increasing sequence $\{\alpha^{(s)}\}_{s=0}^{\infty}$ of lower solutions of BVP for NIDE (15):

Step 1. Start with $\alpha^{(0)}(\nu)=1,\ \nu\in\mathbb{Z}[0,24]$.

Step 2. Obtain next approximation $\alpha^{(s)}(\nu),\ s=1,\quad \nu\in\mathbb{Z}[0,24]$ by the linear recurrence formulas

$$\begin{aligned}
&\alpha^{(s)}(\nu)=e^{0.9(1-\frac{1}{10000}10001)}(1-\frac{0.9}{10000}10001)\Big(\alpha^{(s)}(\nu-1)-\alpha^{(s-1)}(\nu-1)\Big)\\
&\quad+\alpha^{(s-1)}(\nu-1)e^{0.9(1-\frac{1}{10000}\alpha^{(s-1)}(\nu-1))}\\
&\quad\text{for}\quad \nu\in\mathbb{Z}[1,3]\cup\mathbb{Z}[8,10]\cup\mathbb{Z}[15,17]\cup\mathbb{Z}[22,24],\\
&\alpha^{(s)}(4)=\sqrt{\alpha^{(s)}(3)},\quad x(11)=\sqrt{\alpha^{(s)}(10)},\quad x(18)=\sqrt{\alpha^{(s)}(17)},\\
&\alpha^{(s)}(\nu)=\alpha^{(s)}(4)\ \text{for}\ \nu\in\mathbb{Z}[5,7],\\
&\alpha^{(s)}(\nu)=\alpha^{(s)}(11)\ \text{for}\ \nu\in\mathbb{Z}[12,14],\\
&\alpha^{(s)}(\nu)=\alpha^{(s)}(18)\ \text{for}\ n\in\mathbb{Z}[19,21],\\
&\alpha^{(s)}(0)=\sqrt{\alpha^{(s-1)}(24)},
\end{aligned}\tag{16}$$

Step 3. If $\max_{\nu\in\mathbb{Z}[0,24]}|\alpha^{(s)}(\nu)-\alpha^{(s-1)}(\nu)|<\varepsilon$ for an initially given number $\varepsilon>0$ we stop and the approximate solution of the boundary value problem for the nonlinear difference equation with non-instantaneous impulses (15) is $\alpha^{(s)}$, $s\in\mathbb{Z}[0,24]$. If the inequality is not satisfied, then we go to Step 2 with replacing s in the system (16) by $s+1$.

Some values of the sequence of lower solutions $\alpha^{(s)}(n)$ are given in Table 3 and graphs are on Figure 5 (with lines) and Figures 5 (with points). From both, the table and the graphs it could be seen the sequence is increasing one.

ν	α^1	α^2	α^3	α^4	...	α^{58}
$\nu=0$	1	1.61875	2.60680	4.00183	...	14.69428
$\nu=1$	2.45938	2.52120	4.07961	6.54956	...	36.09426
$\nu=2$	2.60518	6.05394	6.35542	10.27730	...	88.48944
$\nu=3$	2.61974	6.75073	14.91230	16.01467	...	215.92181
$\nu=4$	1.61856	2.59822	3.86164	4.00183	...	14.69428
...	...	...	...	...	...	...
$\nu=20$	1.61875	2.60680	4.00183	5.08628	...	14.69430
$\nu=21$	1.61875	2.60680	4.00183	5.08628	...	14.69430
$\nu=22$	2.52120	4.07961	6.54956	9.94772	...	36.09433
$\nu=23$	2.61135	6.35542	10.27730	16.43930	...	88.48964
$\nu=24$	2.62036	6.79541	16.01467	25.87029	...	215.92243

Table 3: Values of lower solutions of (15)

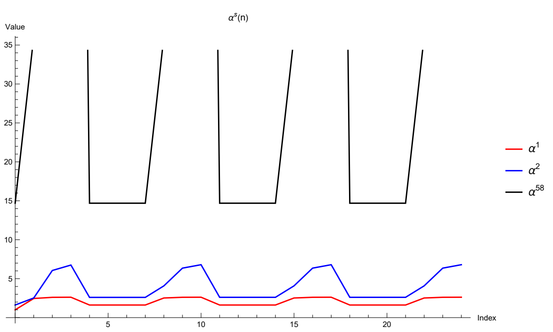

Figure 5. Graphs of some lower solutions of (15) *(lines).*

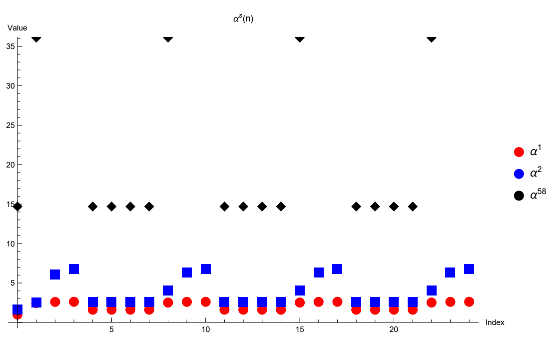

Figure 6. Graphs of some lower solutions of (15) *(discretely).*

The construction of a decreasing sequence $\{\beta^{(s)}(\nu)\}_{s=0}^{\infty}$ of upper solutions of BVP for DE (15) is by the above algorithm replacing $\alpha^{(s)}(\nu)$ by $\beta^{(s)}(\nu)$ and starting by $\beta^{(0)}(\nu)=10001,\ n\in\mathbb{Z}[0,24]$ in Step 1.

Some values of the sequence of upper solutions $\beta^{(s)}(\nu)$ are given in Table 4 and graphs are on Figure 7 (with lines) and Figure 8 (with points). It could be seen the sequence is decreasing one.

ν	β^1	β^2	β^3	β^4	...	β^{57}
$\nu=0$	100.00500	99.95063	99.48368	94.89020	...	14.69451
$\nu=1$	9010.98056	243.76325	243.59070	242.05042	...	36.09500
$\nu=2$	9901.19601	8973.99042	586.53326	585.99054	...	88.49175
$\nu=3$	9990.12943	9897.00485	9004.20622	1368.40589	...	215.92872
$\nu=4$	99.95063	99.48369	94.89050	36.99197	...	14.69451
...	...	...	...	...	...	...
$\nu=20$	99.95063	99.48368	94.89020	36.96958	...	14.69449
$\nu=21$	99.95063	99.48368	94.89020	36.96958	...	14.69449
$\nu=22$	9010.97513	243.59070	242.05042	225.62119	...	36.09490
$\nu=23$	9901.19547	8973.97261	585.99054	580.87753	...	88.49142
$\nu=24$	9990.12938	9897.00307	9004.15002	1366.74986	...	215.92798

Table 4: Values of upper solutions of (15)

Also, from Table 3 and Table 4 it could be seen that both sequences of lower and upper solutions , respectively, have one and the same limit, which is the solution of the studied model.

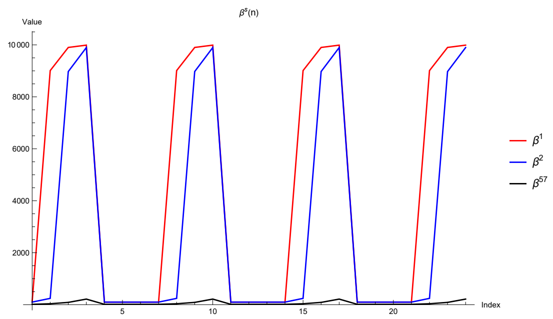

Figure 7. Graphs of some upper solutions of (15) *(lines).*

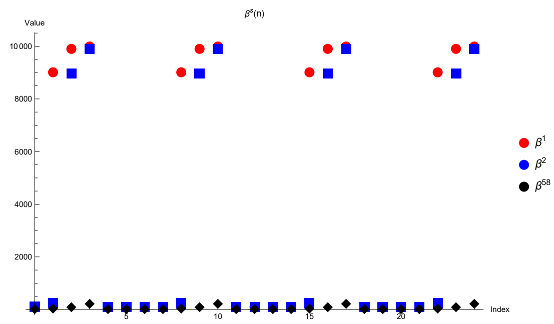

Figure 8. Graphs of some upper solutions of (15) *(discretely).*

The solution of (15), obtained by the suggested algorithm is

$$x(\nu) = (14.694, 36.094, 88.489, 215.922, 14.694, 14.694, 14.694, 14.694, 36.094, 88.489, 215.922, 14.694, 14.694, 14.694, 14.694, 36.094, 88.490, 215.922, 14.694, 14.694, 14.694, 14.694, 36.094, 88.490, 215.922).$$

The solution of (16), obtained by the suggested algorithm, was calculated using a C# program. In the computation, the value of the error was taken $\varepsilon = 0.001$. The program was run on an Intel platform supported by an Intel Core i5-1335U processor, requiring 70 ms in procedural programming paradigm, and 160 ms for object-oriented programming paradigm to compute the sequence of lower solutions α and the sequence of upper solutions β. When increasing the accuracy to $\varepsilon = 0.000001$, then the sequence of lower solutions α and the sequence of upper solutions β required 88 iterations and 125 ms of CPU time for procedural programming paradigm and 283 ms for object-oriented programming paradigm. CAS Wolfram Mathematica was used to plot Figure 5 - Figure 8.

7 Conclusions

The main purpose of this paper is suggesting an effective algorithm for approximate solving a nonlinear boundary value problem for a scalar difference equation with so called non-instantaneous impulses. These impulses are characterized that the duration of their acting is not negligible. The main characteristic of the considered discrete equation is the presence of the current state in both sides of the equations. It does not allow us to solve the equation recursively step by step. It requires to be build an approximate method for its solutions. The suggested algorithm is based on the application of the method of lower ad upper solution. Two monotone sequences of discrete valued functions are constructed. The convergence of these sequences, consisting of lower and upper solutions of the given problem, is proved theoretically. The effectiveness of the suggested algorithm is illustrated on two examples. They are solved by the application of C# program creating through a procedural programming paradigm. One of the examples is modification of Ricker model with non-instantaneous impulses. The computerized algorithm could be applied to study discrete models of some other processes and phenomena characterized by long lasting changes.

Acknowledgments. S.H. was partially supported by Fund Scientific Research FP23-FMI-002, University of Plovdiv "Paisii Hilendarski" and R.H. was partially supported by Fund Scientific Research MUPD23-FMI-009, University of Plovdiv "Paisii Hilendarski"; the reserch is partially supported by Bulgarian National Science Fund under Project KP-06-N62/1.

References

[1] J. Sterman, Business dynamics. Irwin-McGraw-Hill, 2000.

[2] L.J. Allen, Some discrete-time epidemic models. *Math.Biosci.* 124 (1): 83–105, 1994.

[3] M.G. Neubert, M. Kot, M.A. Lewis, M. A., Dispersal and pattern formation in a discrete-time predator-prey model. *Theor. Popul. Biol.* 48(1): 7–43, 1995.

[4] F. Merdivenci, G. Sh. Guseinov, Positive periodic solutions for nonlinear difference equations with periodic coefficients. *J. Math. Anal. Appl.* 232(1): 166–182, 1999.

[5] W.G. Kelley, A.C. Peterson, Difference equations, Harcourt/Academic Press, San Diego, CA, second edition, 2001.

[6] X.H. Tang, J.S. Yu, . Oscillation and stability for a system of linear impulsive delay difference equations; *Math. Appl. (Wuhan)* 14 (1): 28–32, 2001.

[7] F. Merdivenci, Two positive solutions of a boundary value problem for difference equations. *J. Differ. Equations Appl.* 1(3): 263–270, 1995.

[8] Agarwal, R. P., Difference equations and inequalities. National University of Singapore: Singapore, 2000.

[9] S. Elaydi, An introduction to difference equations. Dept. Math., Trinity University, 2005.

[10] C.V. Pao, Monotone iterative methods for finite difference system of reaction-diffusion equations. *Numerische Math.* 46(4): 571–586, 1985.

[11] P.Y.H. Pang, R.P. Agarwal, Monotone iterative methods for a general class of discrete boundary value problems. *Comput. Math. Appl.* 28 (1-3): 243–254, 1994.

[12] R. P. Agarwal, S. Hristova, A. Golev, K. Stefanova, Monotone-iterative method for mixed boundary value problems for generalized difference equations with maxima. *J. Appl. Math. Comput.* 43 (1):213–233, 2013.

[13] S. Hristova, K. Ivanova, Approximate solutions for difference equations with non-instantaneous impulses, *Num. Com. Meth. Sci. Eng.* 1(2): 67–75, 2019.

[14] W.E. Ricker, Stock and Recruitment, *J. Fisheries Research Board of Canada* 11(5): 559–623, 1954.

[15] M. Kulakov, G. Neverova, E. Frisman, The Ricker Competition Model of Two Species: Dynamic Modes and Phase Multistability. *Mathematics* 10, 1076, 2022. https:// doi.org/10.3390/math10071076

[16] T. Gadrich, G. Katriel, A mechanistic stochastic Ricker model: Analytical and numerical investigations. *Intern. J. Bifurc. Chaos* 26, (04), 1650067, 2016.