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Study of systems of Hammerstein integral equations of the first kind

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Abstract

The article addresses systems of Hammerstein integral equations of the first kind, in which the determinant of the non-diagonal matrix-kernel is identically equal to zero.The class of problems under consideration has fundamental differences from standard cases: the solution may not exist, be non-unique, or depend on high derivatives of the input data. In terms of matrix pencils, sufficient conditions for the local existence of a unique solution in the class of continuous functions are formulated. Illustrative examples are given. Difficulties arising in the construction of numerical methods for solving these problems are discussed.

Keywords: systems of integral equations of the first kind, Hammerstein equations, matrix pencils, quadrature formulas, implicit function

AMS subject classification: 45D05, 45G15

1 Introduction

The article is devoted to the study of the existence and uniqueness of solutions to systems of integral equations of the first kind

$$\int_0^t K(t,\tau)G(x(\tau),\tau)\,d\tau = f(t), \quad 0 \le \tau \le t \le 1, \tag{1}$$

where $K(t,\tau)$ is a given $(n \times n)$ matrix, $f(t)$ and $x(t)$ are given and unknown n-dimensional vector functions. It is assumed that the initial condition x_0 is given for this system. Vector function $G(.) : D \longrightarrow R^n$, $D \subset R^{n+1}$. Such problem statements are usually called systems of Hammerstein integral equations of the first kind. By the solution we understand any vector $x(t)$ that turns (1) into an identity and satisfies the condition

$$x(0) = x_0. \tag{2}$$

Nowadays, a qualitative theory (conditions for the existence of a unique solution in various classes of functions) has been developed for various classes of integral equations of the second kind and some classes of linear equations of the first kind. A historical overview and an extensive bibliography can be found in the monographs [1],[2],[3],[4],[5].

If the system (1) is linear, i.e., has the form

$$\int_0^t K(t,\tau)x(\tau)\,d\tau = f(t), \quad 0 \le \tau \le t \le 1, \tag{3}$$

where $K(t,\tau), f(t)$ are differentiable with respect to t r times and

$$K^{(j)}_{t^j}(t,\tau)\,|_{\tau=t} \equiv 0, f^{(r)}(t)\,|_{t=0} = 0, j = 0,1,\cdots,r-1,$$

$$\det K^{(r)}_{t^r}(t,\tau)\,|_{\tau=t} \ne 0 \ \forall t \in [0,1],$$

then, differentiating (3) with respect to t r times and multiplying the resulting system by $(K^{(r)}_{t^r}(t,\tau)\big|_{\tau=t})^{-1}$, we obtain an system of integral equations of the second kind. Note that we do not specify initial conditions (2) in this case.

For $n = 1$ and for the conditions

$$K(t,t) \ne 0 \ \forall t \in [0,1], f(0) = 0,$$

it is well known (see, for example, [6]) that there is a unique continuous solution to this problem when $K(t,t)$, $K'_t(t,\tau)$, $f'(t)$ are continuous functions.

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Approaches to the numerical solution of such problems can be found in the monographs [1],[2],[3] (collocation and multi-step methods), [4] (block methods), PhD thesis [7]. Also the monographs [5] and [8] present results on this topic.

In the article [9] sufficient conditions for the existence of a unique solution to the problem (3) are formulated under certain requirements on the matrix pencil $\lambda K(t,\tau)+K'_t(t,\tau)|_{\tau=t}$, the right-hand part $f(t)$ and $f'(t)$.

Differentiating (3) with respect to t, we have

$$A(t)x(t)+\int_0^t B(t,\tau)x(\tau)\,d\tau=\phi(t),\quad 0\le\tau\le t\le 1, \tag{4}$$

where $A(t)=K(t,t), B(t,\tau)=K'_t(t,\tau)|_{\tau=t}, \phi(t)=f'(t)$. When $\det A(t)\equiv 0$, but $A(t)$ is not identically zero matrix, we have systems of Volterra integral equations that are called equations of the fourth kind [10], [11], integral analogs of singular systems of ordinary differential equations [12], degenerate systems of integral equations of Volterra type [13], integral-algebraic equations [14], [15].

At present, the theory and numerical methods for solving such equations are at the very beginning of their development. The number of publications on this topic is only a few dozen. We note both the first articles [12], [14], [13] and publications that contain results obtained quite recently [11], [16], [17]. It is also worth noting that the importance of qualitative research and the construction of numerical methods for solving systems (4) is emphasized in the fifth chapter of [1].

The special case (1) with $n=1, K(t,t)\neq 0, \forall t\in[0,1]$ and $G'_x(x,t)\neq 0$ in the neighborhood of the point $(x_0,0)$ was considered in [1] (Chapter 3).

The authors are not aware of the results of the study of (1) with condition $\det K(t,t)\equiv 0$, where $K(t,t)$ is not identically zero matrix. It is this fact that motivated the investigation presented below.

2 Statement of the problem and its properties

Consider problem (1) under the assumption that the input elements have the smoothness required for performing the calculations and the initial conditions are specified correctly. Let us start with the characteristic features of problem (1) with condition (2). Even in the linear case, these systems may have no solution, have several solutions, and are unstable to perturbations of the input data, i.e., they belong to the class of ill-posed problems.

Let us give examples illustrating these facts.

Example 2.1.

$$\int_0^t \begin{pmatrix} a(t-\tau)^p & b(t-\tau)^{p+1} \\ c(t-\tau)^{p+1} & d(t-\tau)^{p+2}\end{pmatrix}\begin{pmatrix} x_1(\tau) \\ x_2(\tau)\end{pmatrix} d\tau=\begin{pmatrix} f_1(t) \\ f_2(t)\end{pmatrix}, \tag{5}$$

where p is a natural number.

Differentiating (5) i times and substituting $t=0$, we obtain that

$$f_1^{(i)}(t)|_{t=0}=f_2^{(i)}(t)|_{t=0}=0, i=0,1,\cdots,p.$$

After $p+1$ differentiations we have

$$ax_1(0)=f_1^{(p+1)}(0)/p!,\ f_2^{(p+1)}(0)=0.$$

and after $p+1$ differentiations of the second equation

$$ax_1(0)=f_2^{(p+2)}(0)/(p+2)!$$

is obtained. After $p+2$ differentiations of the first equation and $p+3$ differentiations of the second equation, we have differential-algebraic equation (DAEs) of the form

$$\begin{pmatrix} a & 0 \\ c & 0\end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}'+\begin{pmatrix} 0 & b(p+1) \\ 0 & d(p+2)\end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix}d\tau=\begin{pmatrix} f_1^{(p+2)}(t)/p! \\ f_2^{(p+3)}(t)/(p+1)!\end{pmatrix}.$$

For the existence of a unique solution to this DAE, the condition $ad(p+2)-bc(p+1)\neq 0$ is necessary. Otherwise, even the homogeneous DAE, and therefore the homogeneous system (5) have a set of solutions of the form $x_1(t)=v(t)$, $x_2(t)=-av'(t)/(b(p+1))$, where $v(t)$ is an arbitrary function from the class C^1.

Examples illustrating the instability of Volterra integral equations of the first kind to perturbations of the input data can be found in [8].

Since system (1) is nonlinear, its study for the existence and uniqueness of a solution in the neighborhood of the initial point $(x_0,0)$ is much more difficult than that of system (3). In particular, (1) can have multiple solutions. This makes it impossible to apply standard reasoning (the implicit function theorem [18]).

Let's give a fairly simple example.

Example 2.2.

$$\int_0^t \left(\begin{array}{c} x_1^2(\tau)+x_2^2(\tau) \\ (t-\tau)(-x_1^2(\tau)+x_2(\tau) \end{array} \right) d\tau = \left(\begin{array}{c} t \\ -t^2/2 \end{array} \right),$$

which is equivalent to a system of finite-dimensional equations

$$\left\{ \begin{array}{c} x_1^2(t)+x_2^2(t)=1 \\ -x_1^2(t)+x_2(t)=-1 \end{array} \right. . \tag{6}$$

This system has three solutions $(x_1(t), x_2(t))$ are equal to $(1,0)$, $(-1,0)$ and $(0,-1)$. For (6), the Jacobian matrix has the form

$$\left(\begin{array}{cc} 2x_1(t) & 2x_2(t) \\ -2x_1(t) & 1 \end{array} \right)$$

and is degenerate for the solution $(0,-1)$. This, in turn, does not make it possible to investigate the existence of a unique solution in the neighborhood of the initial point (to apply the implicit function theorem to study the original problem).

Before formulating sufficient conditions for the existence of a unique solution of (1), we present some auxiliary information.

3 Nonlinear integral equations and matrix pencils

Consider a system of integral equations of the form

$$U(x(t),t)+\int_0^t L(t,\tau)U(x(\tau,\tau)\,d\tau = g(t), \quad 0\le\tau\le t\le 1, \tag{7}$$

where $U(x(t),t) : D \to R^n, D \subset R^{n+1}$, $L(t,\tau)$ is an $(n\times n)$-matrix with continuous elements, $g(t)$ is a given, $x(t)$ is an unknown n-dimensional vector function. Initial conditions (2) are given for (7). We denote $W(x(t),t) = \frac{\partial}{\partial x}U(x,t)$ as the Jacobian matrix.

Proposition 3.1. *If the following conditions hold for problem (7):*

1. *The elements $g(t), W(x(t),t)$ and $L(t,\tau)$ are continuous functions in the neighborhood of the point $(x_0,0)$;*
2. *$U(x_0,0) = g(0)$;*
3. *$\det W(x_0,0) \neq 0$.*

Then problem (7) with condition (2) has a unique continuous solution in the neighborhood of $(x_0,0)$.

The proof is based on the fact that the vector function $U(x(t),t)$ is expressed uniquely by the elements $L(t,\tau)$ and $g(t)$ (see, for example, [6]) by virtue of the first condition of the proposition. The implicit function theorem ([18]), the second and third conditions guarantee us the existence of a unique solution of (7) in the neighborhood of the point $(x_0,0)$.

Next we need some information from the theory of matrix pencils.

Definition 3.1. [19] The expression $\lambda A+B$, where λ is a scalar, A and B are matrices of the same dimension is called a matrix pencil. If the matrices are square and $\det(\lambda A+B) \neq 0$, then the pencil is called regular. Otherwise, matrices A and B are rectangular, or $\det(\lambda A+B)\equiv 0$, the pencil is called singular.

Definition 3.2. [20] A pencil $\lambda A(t)+B(t)$ of variable matrices satisfies the "rank-degree" criterion on the interval $[0,1]$ if $rankA(t)=k=cons\,\forall t\in[0,1]$ and $\det(\lambda A(t)+B(t)) = a_0(t)\lambda^k + a_1(t)\lambda^{k-1}+\cdots+a_k(t)$, where the function $a_0(t)\neq 0\,\forall t\in[0,1]$.

Lemma 3.2. *[20] If the elements of the matrix $A(t)$ belong to the class $C^m_{[0,1]}$ and $rankA(t)=k=cons\,\forall t\in[0,1]$, then there exists an $(n\times n)$ non-singular for any $t\in[0,1]$ matrix $P(t)$ with elements from $C^m_{[0,1]}$ such that*

$$P(t)A(t) = \left(\begin{array}{c} A_1(t) \\ 0 \end{array} \right),$$

where $A_1(t)$ is a $(k\times n)$ matrix and $rankA(t)=k=cons\,\forall t\in[0,1]$.

Lemma 3.3. *[13] Let $(n\times n)$ matrices $A(t)$ and $B(t)$ have block form*

$$A(t) = \left(\begin{array}{c} A_1(t) \\ 0 \end{array} \right), B(t) = \left(\begin{array}{c} B_1(t) \\ B_2(t) \end{array} \right),$$

where $A_1(t)$, $B_1(t)$ are $(k\times n)$ matrices, $B_2(t)$ is $((n-k)\times n)$ matrix and the matrix pencil $\lambda A(t)+B(t)$ satisfies the "rank-degree" criterion. Then

$$\det \left(\begin{array}{c} A_1(t) \\ B_2(t) \end{array} \right) \neq 0\,\forall t\in[0,1].$$

Using these results, we formulate sufficient conditions for the existence of a unique solution to system (1) in the neighborhood of the point $(x_0,0)$.

Proposition 3.4. *If the following conditions hold for system (1) with condition (2):*

1. *The elements* $K(t,\tau), K_t'(t,\tau), K_{tt}''(t,\tau), f(t), f'(t), f''(t)$ *are continuous functions;*
2. $f(0)=0$*;*
3. $K(0,0)G(x_0,0)=f'(0)$*;*
4. *the matrix pencil* $\lambda K(t,t)+K_t'(t,\tau)|_{\tau=t}$ *satisfies the "rank-degree" criterion;*
5. $\det(\partial/\partial x)G(x,t)\neq 0$ *in the neighborhood of the point* $(x_0,0)$*.*

Then this system of integral equations has a unique continuous solution.

Proof. Substituting $t=0$ into (1), we obtain $f(0)=0$ (condition 2).

By virtue of condition 1, we can act on (1) with the operator d/dt. As a result we have

$$K(t,t)G(x(t),t)+\int_0^t K_t'(t,\tau)G(x(\tau),\tau)\,d\tau=f'(t). \tag{8}$$

Substituting $t=0$ and x_0 into (8), we have $K(0,0)G(x_0,0)=f'(0)$. This identity holds according to the third condition.

We multiply (8) by a non-singular $\forall t\in[0,1]$ matrix $P(t)$, which vanishes the rows of the matrix $K(t,t)$ (such a matrix exists from Lemma 3.2). We will have a system of integral equations of block form

$$\begin{pmatrix} K_1(t) \\ 0 \end{pmatrix} G(x(t),t)+\int_0^t \begin{pmatrix} L_1(t,\tau) \\ L_2(t,\tau) \end{pmatrix} G(x(\tau),\tau)d\tau=\begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}, \tag{9}$$

where $K_1(t)$ and $L_1(t,\tau)$ are $(k\times n)$ matrices, $rank K_1=k=const\,\forall t\in[0,1]$, $L_2(t,\tau)$ is an $((n-k)\times n)$-matrix, $(\psi_1^\top(t),\psi_2^\top(t))^\top=P(t)f'(t)$.

According to the fourth condition of the proposition, the matrix pencil $\lambda\begin{pmatrix} K_1(t) \\ 0 \end{pmatrix}+\begin{pmatrix} L_1(t,t) \\ L_2(t,t) \end{pmatrix}$ satisfies the "rank-degree" criterion. Differentiating the second block row of (9) (this is possible by the first condition), we obtain

$$\begin{pmatrix} K_1(t) \\ L_2(t,t) \end{pmatrix} G(x(t),t)+\int_0^t \begin{pmatrix} L_1(t,\tau) \\ L_2'(t,\tau) \end{pmatrix} G(x(\tau),\tau)d\tau=\begin{pmatrix} \psi_1(t) \\ \psi_2'(t) \end{pmatrix}. \tag{10}$$

According Lemma 3.3,

$$\det\begin{pmatrix} K_1(t) \\ L_2(t,t) \end{pmatrix}\neq 0\,\forall t\in[0,1].$$

Therefore, (10) can be written as (7). From proposition 3.1 and the fifth condition it follows that (10), and therefore the original system has a unique solution. The proposition 3.4 is proved. □

Note that example 2.2 satisfies the conditions of the proposition 3.4 when choosing the initial conditions $x_0=(1,0)^\top$, $x_0=(-1,0)^\top$, but does not satisfy them when choosing the initial condition $x_0=(0,-1)^\top$ (the condition $\det(\partial/\partial x)G(x,t)\neq 0$ is violated).

Since the matrix pencil $\lambda K(t,t)+K_t'(t,\tau)|_{\tau=t}$ does not satisfy the "rank-degree" criterion, the conditions of the proposition 3.4 do not hold for example 2.1.

4 Numerical method

Note that numerical methods for solving integral equations of the first kind ($n=1$) are currently much less developed than their qualitative research. Due to the fact that the differentiation operation is approximately defined in the C metric, the functions are related to ill-posed problems. The original equation is usually not differentiated to obtain an equation of the second kind, but methods are constructed that take into account their specificity.

The class of problems under consideration that satisfy the conditions of proposition 2 includes equations with a degree of instability [8] equal to one or two. We write it in the form of a system of linear integral equations

$$\int_0^t \begin{pmatrix} K_{11}(t,\tau) & 0 \\ 0 & K_{22}(t,\tau) \end{pmatrix}\begin{pmatrix} u(\tau) \\ v(\tau) \end{pmatrix} d\tau=\begin{pmatrix} \phi(t) \\ \psi(t) \end{pmatrix},\ 0\le\tau\le t\le 1, \tag{11}$$

$$x(t)=(u(t),v(t))^\top,\ f(t)=(\phi(t),\psi(t))^\top,$$

where the functions $K_{11}(t,\tau)$,$K_{22}(t,\tau)$, $\phi(t)$,$\psi(t)$ have the smoothness required to carry out all calculations and satisfy the conditions:

$$K_{11}(t,t)\neq 0\ \forall t\in[0,1],\ \phi(0)=0; \tag{12}$$

$$K_{22}(t,t) \equiv 0 \, (K_{22})_t^{'}(t,\tau)|_{\tau=t} \neq 0 \; \forall t \in [0,1], \; \psi(0) = \psi^{'}(0) = 0. \tag{13}$$

There are quite a lot of works on numerical methods for solving the first equation (11) with condition (12). An extensive bibliography is presented in [1],[2],[3],[5],[8]. The theory of numerical solution of the second equation (11) with condition (13) is less developed, algorithms can be found in [7], [4]. Note that many implicit multi-step methods are unstable for the first equation (11), while explicit methods are the opposite. The simplest methods based on the quadrature formula of the left (right) points do not converge to the exact solution of these equations. To illustrate this fact, it is enough to consider an algorithm based on the quadrature formula of the left point for the simplest equation $\int_0^t (t-\tau)v(\tau)d\tau = \exp(t) - 1 - t$, the exact solution of which is $v(t) = \exp(t)$. For this equation, this method gives an error $1/2 + O(h)$ at the first point, where h is the discretization step. Therefore, we will consider an algorithm based on the quadrature formula of the midpoint and linearization. We introduce on the segment $[0,1]$ the grid $t_i = ih, i = 1,2,...,N, h = 1/N$ and denote

$$K_{ij-1/2} = K(t_i, t_j - h/2), f_i = f(t_i), x_{j-1/2} \approx x(t_j - h/2),$$
$$G_{j-1/2} = G(x_{j-1/2}, t_j - h/2),$$
$$W_{i-1/2} = \frac{\partial}{\partial x} G(x_{i-1/2}, t_{i-1/2}) \Big|_{x = x_{i-1/2}}, i = 1,2,...,N, j = 1,2,...,i$$

A method based on the midpoint rule for (1) has the form:

$$h \sum_{j=1}^{i} K_{ij-1/2} G(x_{j-1/2}, t_{j-1/2}) = f_i. \tag{14}$$

A separate study will be devoted to convergence, software implementation and self-regularization property of this algorithm.

5 Concluding Remarks

The article singles out the class of systems of Hammerstein integral equations of the first kind. Sufficient conditions for the existence of a unique solution to such problems are given. In the future, it is planned to study the stability and convergence rate of both algorithm (14) and other numerical methods for solving such equations. Multi-step methods based on explicit quadrature formulas and block methods will be considered, some of which are given in [3].

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