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Generalized k-Cesàro Polynomials

Yahya Çin [a] · Nejla Özmen [b]

Abstract

The main purpose of this study is to introduce k-analogue of the generalized Cesàro polynomials and give their some properties. Firstly, generalized k-Cesàro polynomials are defined with the help of a generating function relation. After getting the explicit form of these polynomials, various generating function relations, the Mellin transform formula, expressions in terms of k-hypergeometric function and some integral representations are obtained.

Keywords: Generalized k-Cesàro polynomials, Generating function, k-hypergeometric function, Integral representations.

2010 AMS subject classification: 33C05, 33C45, 33C50.

1 Introduction

In 2011, Lin et. al. introduced the generalized Cesàro polynomials like follows [14]:

$$\mathfrak{g}_n^{(s)}(\lambda,x)=\binom{s+n}{n}\,{}_2F_1\left[-n,\lambda;-s-n;x\right],$$

where ${}_2F_1$ indicates Gauss's hypergeometric series whose familiar generalization of an random number of ξ numerator and χ denominator constants ($\xi,\chi\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$) is labeled and indicated by the generalized hypergeometric series ${}_\xi F_\chi$ defined by

$$\begin{aligned}{}_\xi F_\chi\left[\begin{array}{c}\rho_1,\cdots,\rho_\xi;\\ \tau_1,\cdots,\tau_\chi;\end{array}\ \mathfrak{z}\right] &= \sum_{\nu=0}^{\infty}\frac{(\rho_1)_\nu\cdots(\rho_\xi)_\nu}{(\tau_1)_\nu\cdots(\tau_\chi)_\nu}\frac{\mathfrak{z}^\nu}{\nu!}\\ &= {}_\xi F_\chi\left[\rho_1,\cdots,\rho_\xi;\ \tau_1,\cdots,\tau_\chi;\ \mathfrak{z}\right].\end{aligned}$$

Here $(\mathfrak{w})_\mathfrak{r}$ indicates the Pochhammer symbol defined by

$$\begin{aligned}(\mathfrak{w})_\mathfrak{r} &= \frac{\Gamma(\mathfrak{w}+\mathfrak{r})}{\Gamma(\mathfrak{w})}\ \left(\mathfrak{w}\in\mathbb{C}\setminus\mathbb{Z}_0^-\right)\\ &= \begin{cases}1, & \text{if }\ \mathfrak{r}=0\ ;\ \ \mathfrak{w}\in\mathbb{C}\setminus\{0\}\\ \mathfrak{w}(\mathfrak{w}+1)\cdots(\mathfrak{w}+\mathfrak{n}-1), & \text{if }\ \mathfrak{r}=\mathfrak{n}\in\mathbb{N};\ \ \mathfrak{w}\in\mathbb{C}\end{cases}\end{aligned}$$

and $\mathbb{Z}_0^-$ indicates the suite of nonpositive integers and $\Gamma(\mathfrak{w})$ acquainted Gamma function.

Moreover, they do the trick the generating functions [15]:

$$\sum_{n=0}^{\infty}\mathfrak{g}_n^{(s)}(\lambda,x)\omega^n=(1-\omega)^{-s-1}(1-x\omega)^{-\lambda}$$

and

$$\sum_{\nu=0}^{\infty}\binom{u+\nu}{\nu}\mathfrak{g}_{\nu+u}^{(s)}(\lambda,x)\omega^\nu=(1-\omega)^{-s-u-1}(1-x\omega)^{-\lambda}\mathfrak{g}_u^{(s)}\left(\lambda,\frac{x(1-\omega)}{1-x\omega}\right)$$

where $u=0,1,2,\ldots$.

[a] Department of Mathematics, University of Düzce, Türkiye.
[b] Department of Mathematics, University of Düzce, Türkiye

For $u \in \mathbb{C}$, $k \in \mathbb{R}^+$ the k-gamma function Γ_k is defined by

$$\Gamma_k(u) = \int_0^\infty \sigma^{u-1} e^{-\frac{\sigma^k}{k}} d\sigma$$

where $0 < Re(u)$ [6].

For $\tau, \phi \in \mathbb{C}$, $k \in \mathbb{R}^+$, the k-beta function B_k is defined by

$$B_k(\tau,\phi) = \frac{1}{k}\int_0^1 \sigma^{\frac{\tau}{k}-1}(1-\sigma)^{\frac{\phi}{k}-1} d\sigma \tag{1}$$

where $0 < Re(\tau)$ and $0 < Re(\phi)$ [4].

Lemma 1.1. *Let $k \in \mathbb{R}^+$, $\nu \in \mathbb{C}$ the k-beta function B_k and the k-gamma function Γ_k satisfy the following properties [4].*

$$\begin{aligned} \Gamma_k(\nu+k) &= \nu\Gamma_k(\nu), \\ \Gamma_k(\nu) &= k^{\frac{\nu}{k}-1}\Gamma_k\left(\frac{\nu}{k}\right), \\ B_k(\nu,\phi) &= \frac{1}{k}\, B_k\left(\frac{\nu}{k},\frac{\phi}{k}\right), \\ B_k(\nu,\phi) &= \frac{\Gamma_k(\nu)\Gamma_k(\phi)}{\Gamma_k(\nu+\phi)}. \end{aligned} \tag{2}$$

Looking at it the other way, permit $u \in \mathbb{C}$, $k \in \mathbb{R}^+$ and $\tau \in \mathbb{N}^+$. Later, the Pochhammer k-symbol is defined in [6] by

$$(u)_{\tau,k} = u(u+k)(u+2k)\ldots(u+(\tau-1)k)$$

especially, we indicate $(u)_{0,k} := 1$.

Definition 1.1. For every $\nu \in \mathbb{N}$, $k > 0$, we define (ν,k) factorial as [3]

$$\begin{aligned} (\nu,k)! &= \nu k(\nu k-k)(\nu k-2k)(\nu k-3k)\ldots 3k.2k.k \\ &= k^\nu \nu(\nu-1)(\nu-2)(\nu-3)\ldots 3.2.1 = k^\nu \nu! \end{aligned} \tag{3}$$

from the above definition of $(\nu,k)!$, we can conveniently prove that

$$\begin{aligned} (\nu k,k)! &= k^\nu(\nu k)! \\ (\nu+a,k)! &= k^\nu(\nu+a)!,\ a\in\mathbb{R},\ \nu\in\mathbb{N} \\ [(\nu+b)k,k]! &= k^\nu[(\nu+b)k]!,\ b\in\mathbb{R},\ \nu\in\mathbb{N}. \end{aligned}$$

Using the above results, we see that

$$(0,k)! = k^0 0! = 1, \quad (-\nu,k)! = \infty, \quad \nu\in\mathbb{N},\ k>0.$$

By Pochhammer k—symbol and relation (3), we get

$$(1)_{\nu,k} = \left(\frac{1}{k}+\nu-1,k\right)!.$$

Lemma 1.2. *If $\nu \in \mathbb{C}$ and $p, q \in \mathbb{N}^+$ later for $k \in \mathbb{R}^+$, we have*

$$(\nu)_{p,k} = \frac{\Gamma_k(\nu+pk)}{\Gamma_k(\nu)}, \tag{4}$$

$$(\nu)_{p,k} = k^p\left(\frac{\nu}{k}\right)_p, \tag{5}$$

$$(\nu)_{q+p,k} = (\nu)_{q,k}(\nu+qk)_{p,k} \tag{6}$$

where $(\nu)_{p,k}$ and $(\nu)_p$ indicate the Pochhammer k-symbol and Pochhammer symbol, respectively [6].

Lemma 1.3. *For every $\tau \in \mathbb{C}$ and $k \in \mathbb{R}^+$, the allowing identity holds*

$$\sum_{\nu=0}^{\infty}(\tau)_{\nu,k}\frac{x^\nu}{\nu!} = (1-kx)^{-\frac{\tau}{k}} \tag{7}$$

where $|x| < \frac{1}{k}$ [4].

Presume that $x\in\mathbb{C}$, $k\in\mathbb{R}^{+}$ and $0<Re(\nu)<Re(\delta)$, later the k-hypergeometric function is defined by [5],

$$
\begin{aligned}
{}_2F_{1,k}\left[\begin{array}{cc}(\sigma,k), & (\nu,k)\\ & (\delta,k)\end{array};x\right] &= \sum_{u=0}^{\infty}\frac{(\sigma)_{u,k}(\nu)_{u,k}}{(\delta)_{u,k}}\frac{x^{u}}{u!}\\
&= \frac{\Gamma_k(\delta)}{k\Gamma_k(\nu)\Gamma_k(\delta-\nu)}\int_0^1 \omega^{\frac{\nu}{k}-1}(1-\omega)^{\frac{\delta-\nu}{k}-1}(1-kx\omega)^{-\frac{\sigma}{k}}\,d\omega. \qquad (8)
\end{aligned}
$$

Presume that $x\in\mathbb{C}$, $k\in\mathbb{R}^{+}$ and $0<Re\,(\delta-\nu)$, later

$$
{}_2F_{1,k}\left[\begin{array}{cc}(\sigma,1), & (\nu,k)\\ & (\delta,k)\end{array};1\right]:=\frac{\Gamma_k(\delta)\Gamma_k(\delta-\nu-k\sigma)}{\Gamma_k(\delta-\nu)\Gamma_k(\delta-k\sigma)}. \qquad (9)
$$

For the special case $\sigma=-u$ in (9)

$$
{}_2F_{1,k}\left[\begin{array}{cc}(-u,1), & (\nu,k)\\ & (\delta,k)\end{array};1\right]=\frac{(\delta-\nu)_{u,k}}{(\delta)_{u,k}}. \qquad (10)
$$

In [9], let $k\in\mathbb{R}^{+}$, χ, $\omega\in\mathbb{C}$, ξ, σ, σ', $\phi\in\mathbb{C}$ and $\delta\in\mathbb{N}^{+}$. Later, the $F_{1,k}$ function with the parameters ξ, σ, σ', ϕ is accustomed by

$$
F_{1,k}\left(\xi,\sigma,\sigma';\phi;\chi,\omega\right)=\sum_{\kappa,\delta=0}^{\infty}\frac{(\xi)_{\kappa+\delta,k}(\sigma)_{\kappa,k}(\sigma')_{\delta,k}}{(\phi)_{\kappa+\delta,k}}\frac{\chi^{\kappa}}{\kappa!}\frac{\omega^{\delta}}{\delta!}
$$

where $\phi\neq 0,-1,-2...$ and $|\chi|<\frac{1}{k}$, $|\omega|<\frac{1}{k}$.

In [9], presume that $k\in\mathbb{R}^{+}$, χ, $\omega\in\mathbb{C}$, $0<Re(\xi)<Re(\phi)$, later the integral representation of the k-Appell function is like follows:

$$
F_{1,k}\left(\xi,\sigma,\sigma';\phi;\chi,\omega\right)=\frac{\Gamma_k(\phi)}{k\Gamma_k(\xi)\Gamma_k(\phi-\xi)}\int_0^1\kappa^{\frac{\xi}{k}-1}(1-\kappa)^{\frac{\phi-\xi}{k}-1}(1-k\chi\kappa)^{-\frac{\sigma}{k}}(1-k\omega\kappa)^{-\frac{\sigma'}{k}}\,d\kappa.
$$

Fractional calculus and its applications have been studied and intensively researched by many people for a long time. The interest in this subject has increased tremendously by researchers in many fields. Such as engineering, physics, space science, logical computation, biology, economics, statistics. One of the generalizations of fractional equations is the Riemann-Liouville k-fractional derivative operator, whose derivatives are studied in [7, 8, 10]. On the hand, we recall the definition of the Riemann-Liouville fractional derivative and its k-generalization, which we use in this work, as pursues:

In [1], the famous Riemann-Liouville fractional derivative of order κ is described by

$$
\mathfrak{D}_u^{\kappa}\{f(u)\}=\frac{1}{\Gamma(-\kappa)}\int_0^u f(\sigma)(u-\sigma)^{-\kappa-1}\,d\sigma, \qquad (11)
$$

where $Re(\kappa)<0$, especially, for the case $n-1<Re(\kappa)<n$, where $n=1,2,...$ (11) is written by

$$
\mathfrak{D}_u^{\kappa}\{f(u)\}=\frac{d^n}{du^n}\mathfrak{D}_u^{\kappa-n}\{f(u)\}=\frac{d^n}{du^n}\left\{\frac{1}{\Gamma(-\kappa+n)}\int_0^u f(\sigma)(u-\sigma)^{-\kappa+n-1}\,d\sigma\right\}.
$$

In [8], the k-analogue of Riemann-Liouville fractional derivative of order κ is defined by

$$
{}_k\mathfrak{D}_u^{\kappa}\{f(u)\}=\frac{1}{k\Gamma_k(-\kappa)}\int_0^u f(\sigma)(u-\sigma)^{-\frac{\kappa}{k}-1}\,d\sigma, \qquad (12)
$$

where $Re(\kappa)<0$ and $k\in\mathbb{R}^{+}$.

Especially, for the case $n-1<Re(\kappa)<n$, where $n=1,2,...$ (12) can be written by

$$
{}_k\mathfrak{D}_u^{\kappa}\{f(u)\}=\frac{d^n}{du^n}{}_kD_u^{\kappa-nk}\{f(u)\}=\frac{d^n}{du^n}\left\{\frac{1}{k\Gamma_k(-\kappa+nk)}\int_0^u f(\sigma)(u-\sigma)^{-\frac{\kappa}{k}+n-1}\,d\sigma\right\}. \qquad (13)
$$

In this investigation, we intention to graph the generalized k-Cesàro polynomials introduced on the hand by giving some generating functions and some values of this polynomial. We as well as present some special cases of the main results of this investigation.

2 Generating Functions For The Generalized k-Cesàro Polynomials

In this piece, we introduce the generalized k-Cesàro polynomials via generating function relation of them.

Definition 2.1. The generalized k-Cesàro polynomials are defined by

$$\sum_{n=0}^{\infty} \mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^n = (1-\omega)^{-\frac{s}{k}-1}(1-kx\omega)^{-\frac{\lambda}{k}}, \tag{14}$$

where $|\omega| < \min\left\{1, \left|\frac{1}{kx}\right|\right\}$.

Some values of the generalized k–Cesàro $\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)$ can be given as follows:

$$\begin{aligned}
\mathfrak{g}_{0,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) &= 1,\\
g_{1,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) &= \frac{s}{k}+1+\lambda x,\\
\mathfrak{g}_{2,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) &= \frac{1}{2k^2}\left[2k^2+3sk+s^2+\left(2ks\lambda+2k^2\lambda\right)x+\left(\lambda^2k^2+\lambda k^3\right)x^2\right],\\
\mathfrak{g}_{3,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) &= \frac{1}{6k^3}\left[6k^3+s^3+11sk^2+6ks^2+\left(6\lambda k^3+3k\lambda s^2+9s\lambda k^2\right)x\right]\\
&\quad+\frac{1}{6k^3}\left[\left(3\lambda k^4+3\lambda^2k^3+3\lambda sk^3+3sk^2\lambda^2\right)x^2+\left(2\lambda k^5+3\lambda^2k^4+\lambda^3k^3\right)x^3\right],\\
\mathfrak{g}_{4,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) &= \frac{1}{24k^4}\left[24k^4+s^4+10ks^3+50sk^3+35s^2k^2+\left(4\lambda ks^3+24\lambda k^4+24\lambda s^2k^2+44s\lambda k^3\right)x\right]\\
&\quad+\frac{1}{24k^4}\left[\left(12\lambda k^5+18s\lambda k^4+12\lambda^2k^4+18s\lambda^2k^3+6\lambda s^2k^3+6\lambda^2s^2k^2\right)x^2\right]\\
&\quad+\frac{1}{24k^4}\left[\left(8\lambda k^6+8s\lambda k^5+12\lambda^2k^5+12s\lambda^2k^4+4\lambda^3k^4+4s\lambda^3k^3\right)x^3\right]\\
&\quad+\frac{1}{24k^4}\left[\left(\lambda^4k^4+6\lambda^3k^5+11\lambda^2k^6+6\lambda k^7\right)x^4\right].
\end{aligned}$$

It is recognized that the special case $k=1$ of (14) lowers instantaneously to the generalized k-Cesàro polynomials $\mathfrak{g}_n^{(s)}(x,\lambda)$ in $\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)$.

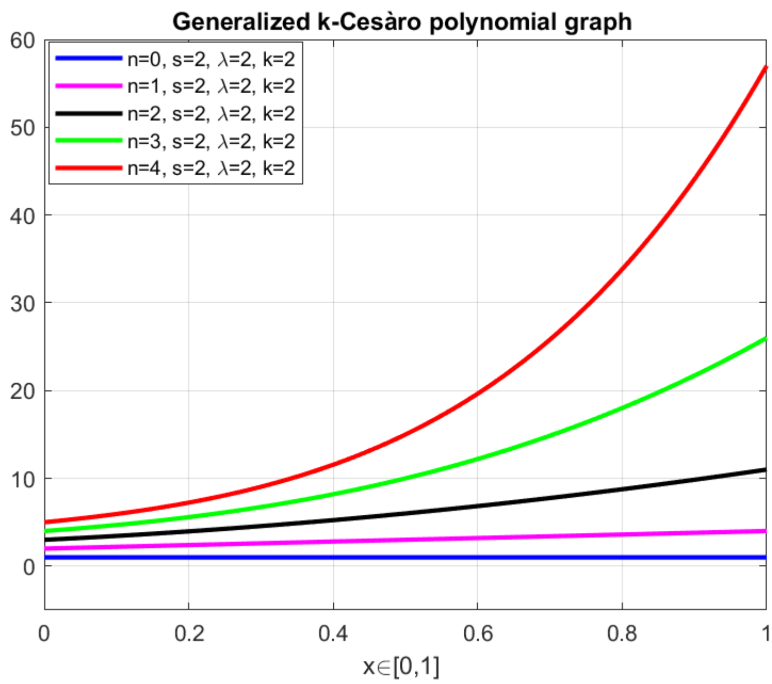

Figure 1:

It can be given graphically for some values of $\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)$ for $n=0,1,2,3,4$ and $s=2$, $\lambda=2$, $k=2$.

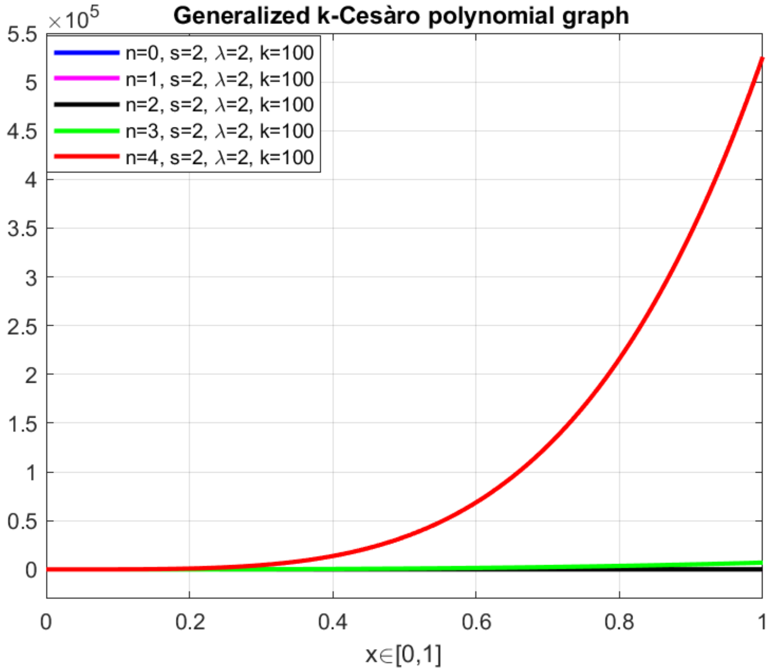

Figure 2:

It can be given graphically for some values of $\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)$ for $n=0,1,2,3,4$ and $s=2$, $\lambda=2$, $k=100$.

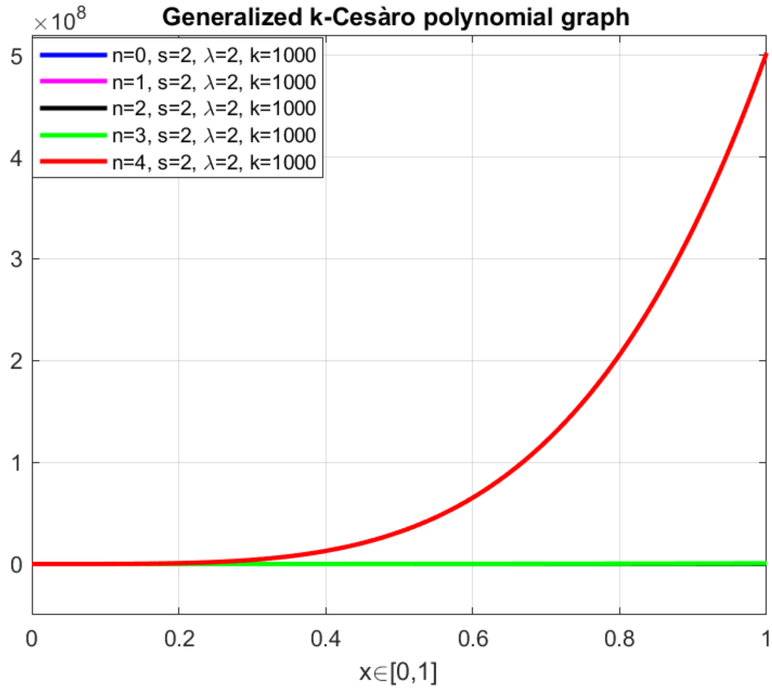

Figure 3:

It can be given graphically for some values of $\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)$ for $n=0,1,2,3,4$ and $s=2$, $\lambda=2$, $k=1000$.

Theorem 2.1. *The generalized k-Cesàro polynomials have the following explicit formula:*

$$\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)=\sum_{\tau=0}^{n}\left(\frac{s}{k}+1\right)_{n-\tau}\frac{(\lambda)_{\tau,k}\,x^{\tau}}{(n-\tau)!\tau!}. \tag{15}$$

Proof. From (12), we have

$$_{k}\mathfrak{D}_{\omega}^{\phi-æ}\left\{\omega^{\frac{\phi}{k}-1}\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^{n}\right\}={}_{k}\mathfrak{D}_{\omega}^{\phi-æ}\left\{\omega^{\frac{\phi}{k}-1}(1-\omega)^{-\frac{s}{k}-1}(1-kx\omega)^{-\frac{\lambda}{k}}\right\}$$

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right){}_{k}\mathfrak{D}_{\omega}^{\phi-æ}\left\{\omega^{\frac{\phi}{k}+n-1}\right\}={}_{k}\mathfrak{D}_{\omega}^{\phi-æ}\left\{\omega^{\frac{\phi}{k}-1}\sum_{n=0}^{\infty}\left(\frac{s}{k}+1\right)_{n}\frac{\omega^{n}}{n!}\sum_{\tau=0}^{\infty}(\lambda)_{\tau,k}\frac{(x\omega)^{\tau}}{\tau!}\right\}$$

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right){}_{k}\mathfrak{D}_{\omega}^{\phi-æ}\left\{\omega^{\frac{\phi+nk}{k}-1}\right\}=\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_{n}\frac{(\lambda)_{\tau,k}\,x^{\tau}}{n!\tau!}{}_{k}\mathfrak{D}_{\omega}^{\phi-æ}\left\{\omega^{\frac{\phi+\tau k+nk}{k}-1}\right\}$$

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\frac{1}{k\Gamma_{k}(æ-\phi)}\int_{0}^{\omega}t^{\frac{\phi+nk}{k}-1}(\omega-t)^{\frac{æ-\phi}{k}-1}dt\\=&\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_{n}\frac{(\lambda)_{\tau,k}\,x^{\tau}}{n!\tau!}\frac{1}{k\Gamma_{k}(æ-\phi)}\int_{0}^{\omega}t^{\frac{\phi+\tau k+nk}{k}-1}(\omega-t)^{\frac{æ-\phi}{k}-1}dt.\end{aligned} \tag{16}$$

Substituting $t=q\omega$ and by using (1), (2) in (16), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\frac{1}{k\Gamma_{k}(æ-\phi)}\int_{0}^{1}(q\omega)^{\frac{\phi+nk}{k}-1}(\omega-q\omega)^{\frac{æ-\phi}{k}-1}\omega dq\\=&\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_{n}\frac{(\lambda)_{\tau,k}\,x^{\tau}}{n!\tau!}\frac{1}{k\Gamma_{k}(æ-\phi)}\int_{0}^{1}(q\omega)^{\frac{\phi+\tau k+nk}{k}-1}(\omega-q\omega)^{\frac{æ-\phi}{k}-1}\omega dq\end{aligned}$$

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\frac{\omega^{\frac{\ae+nk}{k}-1}}{k\Gamma_k(\ae-\phi)}\int_0^1 q^{\frac{\phi+nk}{k}-1}(1-q)^{\frac{\ae-\phi}{k}-1}dq$$

$$=\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\frac{\omega^{\frac{\ae+\tau k+nk}{k}-1}}{k\Gamma_k(\ae-\phi)}\int_0^1 q^{\frac{\phi+\tau k+nk}{k}-1}(1-q)^{\frac{\ae-\phi}{k}-1}dq$$

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\frac{\omega^{\frac{\ae}{k}-1}\omega^{n}}{k\Gamma_k(\ae-\phi)}\int_0^1 q^{\frac{\phi+nk}{k}-1}(1-q)^{\frac{\ae-\phi}{k}-1}dq$$

$$=\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\frac{\omega^{\frac{\ae}{k}-1}\omega^{\tau+n}}{k\Gamma_k(\ae-\phi)}\int_0^1 q^{\frac{\phi+\tau k+nk}{k}-1}(1-q)^{\frac{\ae-\phi}{k}-1}dq$$

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\frac{\omega^{\frac{\ae}{k}-1}\omega^{n}}{\Gamma_k(\ae-\phi)}B_k(\phi+nk,\ae-\phi)$$

$$=\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\frac{\omega^{\frac{\phi}{k}-1}\omega^{\tau+n}}{\Gamma_k(\ae-\phi)}B_k(\ae+\tau k+nk,\ae-\phi)$$

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\frac{\omega^{\frac{\ae}{k}-1}\omega^{n}}{\Gamma_k(\ae-\phi)}\frac{\Gamma_k(\phi+nk)\Gamma_k(\ae-\phi)}{\Gamma_k(\ae+nk)}$$

$$=\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\frac{\omega^{\frac{\ae}{k}-1}\omega^{\tau+n}}{\Gamma_k(\ae-\phi)}\frac{\Gamma_k(\phi+\tau k+nk)\Gamma_k(\ae-\phi)}{\Gamma_k(\ae+\tau k+nk)}$$

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^{\frac{\ae}{k}-1}\frac{\Gamma_k(\phi+nk)}{\Gamma_k(\ae+nk)}\frac{\Gamma_k(\phi)}{\Gamma_k(\phi)}\frac{\Gamma_k(\ae)}{\Gamma_k(\ae)}\omega^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\omega^{\frac{\ae}{k}-1}\frac{\Gamma_k(\phi+\tau k+nk)}{\Gamma_k(\ae+\tau k+nk)}\frac{\Gamma_k(\phi)}{\Gamma_k(\phi)}\frac{\Gamma_k(\ae)}{\Gamma_k(\ae)}\omega^{\tau+n}$$

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^{\frac{\ae}{k}-1}\frac{(\phi)_{n,k}}{(\ae)_{n,k}}\frac{\Gamma_k(\phi)}{\Gamma_k(\ae)}\omega^{n}$$

$$=\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s}{k}+1\right)_n\frac{(\lambda)_{\nu,k}x^{\tau}}{n!\tau!}\omega^{\frac{\ae}{k}-1}\frac{(\phi)_{\tau+n,k}}{(\ae)_{\tau+n,k}}\frac{\Gamma_k(\phi)}{\Gamma_k(\ae)}\omega^{\tau+n}. \tag{17}$$

Replacing n by $n-\tau$ in (17), we obtain

$$\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^{\frac{\ae}{k}-1}\frac{(\phi)_{n,k}}{(\ae)_{n,k}}\frac{\Gamma_k(\phi)}{\Gamma_k(\ae)}\omega^{n}=\sum_{n=0}^{\infty}\sum_{\tau=0}^{n}\left(\frac{s}{k}+1\right)_{n-\tau}\frac{(\lambda)_{\tau,k}x^{\tau}}{(n-\tau)!\tau!}\omega^{\frac{\ae}{k}-1}\frac{(\phi)_{n,k}}{(\ae)_{n,k}}\frac{\Gamma_k(\phi)}{\Gamma_k(\ae)}\omega^{n}.$$

From the coeffcients of ω^n on the one and the other sides of the last expression, the desired result can be achieved. □

Theorem 2.2. *The following formula holds true for the generalized k-Cesàro polynomials:*

$$\mathfrak{g}_{n,k}^{\left(\frac{s_1+s_2+k}{k}\right)}\left(\frac{\lambda_1+\lambda_2}{k},x\right)=\sum_{\tau=0}^{n}\mathfrak{g}_{n-\tau,k}^{\left(\frac{s_1}{k}\right)}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}_{\tau,k}^{\left(\frac{s_2}{k}\right)}\left(\frac{\lambda_2}{k},x\right). \tag{18}$$

Proof. Replacing s by s_1+s_2+k and λ by $\lambda_1+\lambda_2$ in (14), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s_1+s_2+k}{k}\right)}\left(\frac{\lambda_1+\lambda_2}{k},x\right)w^n &= (1-w)^{-\frac{s_1+s_2+k}{k}-1}(1-kxw)^{-\frac{\lambda_1+\lambda_2}{k}}\\ &= (1-w)^{-\frac{s_1}{k}-1}(1-kxw)^{-\frac{\lambda_1}{k}}(1-w)^{-\frac{s_2}{k}-1}(1-kxw)^{-\frac{\lambda_2}{k}}\\ &= \sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s_1}{k}\right)}\left(\frac{\lambda_1}{k},x\right)w^n\sum_{\tau=0}^{\infty}\mathfrak{g}_{\tau,k}^{\left(\frac{s_2}{k}\right)}\left(\frac{\lambda_2}{k},x\right)w^{\tau}\\ &= \sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s_1}{k}\right)}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}_{\tau,k}^{\left(\frac{s_2}{k}\right)}\left(\frac{\lambda_2}{k},x\right)w^{n+\tau}\\ &= \sum_{n=0}^{\infty}\sum_{\tau=0}^{n}\mathfrak{g}_{n-\tau,k}^{\left(\frac{s_1}{k}\right)}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}_{\tau,k}^{\left(\frac{s_2}{k}\right)}\left(\frac{\lambda_2}{k},x\right)w^{n}\end{aligned}$$

From the coeffcients of w^n on the one and the other sides of the last expression, the desired result can be achieved. □

Theorem 2.3. *The following generating function relation for the generalized k-Cesàro polynomials holds true:*

$$\sum_{n=0}^{\infty}\binom{n+u}{u}\mathfrak{g}_{n+u,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)z^{n}=(1-z)^{-\frac{s}{k}-u-1}(1-kxz)^{-\frac{\lambda}{k}}\mathfrak{g}_{u,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},\frac{x(1-z)}{1-kxz}\right),\tag{19}$$

where $|z|<\min\{1,|kx|\}$.

Proof. Replacing z by $w+z$ in (14), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)(z+w)^{n} &= (1-z-w)^{-\frac{s}{k}-1}(1-kxz-kxw)^{-\frac{\lambda}{k}}\\
\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)(z+w)^{n} &= (1-z)^{-\frac{s}{k}-1}\left(1-\frac{w}{1-z}\right)^{-\frac{s}{k}-1}(1-kxz)^{-\frac{\lambda}{k}}\left(1-\frac{kxw}{1-kxz}\right)^{-\frac{\lambda}{k}}\\
\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)(z+w)^{n} &= (1-z)^{-\frac{s}{k}-1}(1-kxz)^{-\frac{\lambda}{k}}\left(1-\frac{w}{1-z}\right)^{-\frac{s}{k}-1}\left(1-\frac{kx(1-z)}{1-kxz}\frac{w}{1-z}\right)^{-\frac{\lambda}{k}}\\
\sum_{n=0}^{\infty}\sum_{u=0}^{n}\binom{n}{u}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)z^{n}w^{n-u} &= (1-z)^{-\frac{s}{k}-1}(1-kxz)^{-\frac{\lambda}{k}}\sum_{u=0}^{\infty}\mathfrak{g}_{u,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},\frac{x(1-z)}{1-kxz}\right)\left(\frac{w}{1-z}\right)^{u}\\
\sum_{n=0}^{\infty}\sum_{u=0}^{\infty}\binom{n+u}{u}g_{n+u,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)z^{n}w^{u} &= \sum_{u=0}^{\infty}(1-z)^{-\frac{s}{k}-u-1}(1-kxz)^{-\frac{\lambda}{k}}g_{u,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},\frac{x(1-z)}{1-kxz}\right)w^{u}.
\end{aligned}$$

From the coeffcients of w^u on the one and the other sides of the last expression, the desired result can be achieved. □

3 Further properties and some applications

In this piece, we firstly give two explicit formula of the generalized k-Cesàro polynomials. Then we derive Mellin transform formula of these polynomials. End of this section, we obtain two different integral representations.

Theorem 3.1. *We have*

$$\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)=\frac{(s+k)_{n,k}}{(n,k)!}\,{}_{2}F_{1,k}\left[\begin{array}{cc}(-n,1), & (\lambda,k)\\ & (-s-kn,k)\end{array};xk\right],$$

where $x\in\mathbb{C}$, $k\in\mathbb{R}^{+}$ *and* $Re(-s-kn)>Re(\lambda)>0$.

Proof. From (14), we have

$$\begin{aligned}
\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) &= \sum_{\xi=0}^{n}\left(\frac{s+k}{k}\right)_{n-\xi}\frac{(\lambda)_{\xi,k}\,x^{\xi}}{(n-\xi)!\xi!}\\
&= \sum_{\xi=0}^{n}\frac{(-1)^{\xi}\left(\frac{s+k}{k}\right)_{n}(\lambda)_{\xi,k}\,x^{\xi}}{\left(1-\frac{s+k}{k}-n\right)_{\xi}(-1)^{\xi}\frac{n!}{(-n)_{\xi}}\xi!}\\
&= \sum_{\xi=0}^{n}\frac{(-n)_{\xi}\left(\frac{s+k}{k}\right)_{n}(\lambda)_{\xi,k}\,x^{\xi}}{\left(-\frac{s}{k}-n\right)_{\xi}n!\xi!}.
\end{aligned}\tag{20}$$

By using (5) in (20), we get

$$\begin{aligned}
\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) &= \frac{\frac{(s+k)_{n,k}}{k^{n}}}{n!}\sum_{\xi=0}^{n}\frac{(-n)_{\xi,1}(\lambda)_{\xi,k}\,x^{\xi}}{\frac{(-s-kn)_{\xi,k}}{k^{\xi}}\xi!}\\
&= \frac{(s+k)_{n,k}}{(n,k)!}\sum_{\xi=0}^{\infty}\frac{(-n)_{\xi,1}(\lambda)_{\xi,k}\,(kx)^{\xi}}{(-s-kn)_{\xi,k}\,\xi!}.
\end{aligned}\tag{21}$$

By using (8) in (21), we obtain

$$\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)=\frac{(s+k)_{n,k}}{(n,k)!}\,{}_{2}F_{1,k}\left[\begin{array}{cc}(-n,1), & (\lambda,k)\\ & (-s-kn,k)\end{array};xk\right].$$

□

Theorem 3.2. *The following formula holds true:*

$$\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)=\frac{(s+k)_{n,k}}{(n,k)!}\left(1-k^{2}x\right)^{-\frac{\lambda}{k}}\,{}_{2}F_{1,k}\left[\begin{array}{cc}(-s-kn+n,1), & (\lambda,k)\\ & (-s-kn,k)\end{array};-\frac{xk}{1-k^{2}x}\right].$$

Proof. If we apply expressions (7) and (8) to the right side of the theorem, we get

$$\begin{aligned}
&\frac{(s+k)_{n,k}}{(n,k)!}\left(1-k^2x\right)^{-\frac{\lambda}{k}}\,{}_2F_{1,k}\left[\begin{array}{c}(-s-kn+n,1),\ (\lambda,k)\\ (-s-kn,k)\end{array};-\frac{xk}{1-k^2x}\right]\\
=\ &\frac{(s+k)_{n,k}}{(n,k)!}\sum_{\psi=0}^{\infty}\frac{(-1)^{\psi}(-s-kn+n)_{\psi,1}(\lambda)_{\psi,k}(xk)^{\psi}}{(-s-kn)_{\psi,k}\,\psi!}\sum_{\tau=0}^{\infty}\frac{(\lambda+k\psi)_{\tau,k}(xk)^{\tau}}{\tau!}\\
=\ &\frac{(s+k)_{n,k}}{(n,k)!}\sum_{\psi=0}^{\infty}\sum_{\tau=0}^{\infty}\frac{(-1)^{\psi}(-s-kn+n)_{\psi,1}(\lambda)_{\psi,k}(\lambda+k\psi)_{\tau,k}(xk)^{\psi+\tau}}{(-s-kn)_{\psi,k}\,\tau!\psi!}. \qquad (22)
\end{aligned}$$

If we apply the expression (6) to the expression (22)

$$\begin{aligned}
&\frac{(s+k)_{n,k}}{(n,k)!}\left(1-k^2x\right)^{-\frac{\lambda}{k}}\,{}_2F_{1,k}\left[\begin{array}{c}(-s-kn+n,1),\ (\lambda,k)\\ (-s-kn,k)\end{array};-\frac{xk}{1-k^2x}\right]\\
=\ &\frac{(s+k)_{n,k}}{(n,k)!}\sum_{\psi=0}^{\infty}\sum_{\tau=0}^{\infty}\frac{(-1)^{\psi}(-s-kn+n)_{\psi,1}(\lambda)_{\psi+\tau,k}(xk)^{\psi+\tau}}{(-s-kn)_{\psi,k}\,\psi!\tau!}. \qquad (23)
\end{aligned}$$

Replacing f by $\tau-\psi$ in (23), we obtain

$$\begin{aligned}
&\frac{(s+k)_{n,k}}{(n,k)!}\left(1-k^2x\right)^{-\frac{\lambda}{k}}\,{}_2F_{1,k}\left[\begin{array}{c}(-s-kn+n,1),\ (\lambda,k)\\ (-s-kn,k)\end{array};-\frac{xk}{1-k^2x}\right]\\
=\ &\frac{(s+k)_{n,k}}{(n,k)!}\sum_{\tau=0}^{\infty}\sum_{\psi=0}^{\tau}\frac{(-s-kn+n)_{\psi,1}(-\tau)_{\psi,1}(\lambda)_{\tau,k}(xk)^{\tau}}{(-s-kn)_{\psi,k}\,\psi!\tau!}\\
=\ &\frac{(s+k)_{n,k}}{(n,k)!}\sum_{\tau=0}^{\infty}\left(\sum_{\psi=0}^{\infty}\frac{(-s-kn+n)_{\psi,1}(-\tau)_{\psi,1}}{(-s-kn)_{\psi,k}\,\psi!}(1)^{\psi}\right)\frac{(\lambda)_{\tau,k}(xk)^{\tau}}{\tau!}\\
=\ &\frac{(s+k)_{n,k}}{(n,k)!}\sum_{\tau=0}^{\infty}{}_2F_{1,k}\left[\begin{array}{c}(-\tau,1),\quad(-s-nk+n,1)\\ (-s-kn,k)\end{array};1\right]\frac{(\lambda)_{\tau,k}(xk)^{\tau}}{\tau!}. \qquad (24)
\end{aligned}$$

Considering (10) in the expression (24), we attain

$$\begin{aligned}
&\frac{(s+k)_{n,k}}{(n,k)!}\left(1-k^2x\right)^{-\frac{\lambda}{k}}\,{}_2F_{1,k}\left[\begin{array}{c}(-s-kn+n,1),\ (\lambda,k)\\ (-s-kn,k)\end{array};-\frac{xk}{1-k^2x}\right]\\
=\ &\frac{(s+k)_{n,k}}{(n,k)!}\sum_{\tau=0}^{\infty}\frac{(-n)_{\tau,1}(\lambda)_{\tau,k}}{(-s-kn)_{\tau,k}}\frac{(xk)^{\tau}}{\tau!}\\
=\ &\frac{(s+k)_{n,k}}{(n,k)!}\,{}_2F_{1,k}\left[\begin{array}{c}(-n,1),\ (\lambda,k)\\ (-s-kn,k)\end{array};xk\right]\\
=\ &\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)
\end{aligned}$$

which completes the proof. □

Theorem 3.3. *The Mellin transformation relation below is for the generalized k-Cesàro polynomials holds true:*

$$\begin{aligned}
&M\left[e^{-\chi}\,{}_{k}\mathfrak{D}_{\omega}^{æ}\left\{\omega^{\frac{\phi}{k}}\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^{n}\right\};\Delta\right]\\
=\ &\frac{\Gamma(\Delta)\Gamma_k(\phi+k)}{\Gamma_k(\phi-æ+k)}\omega^{\frac{\phi-æ}{k}}F_{1,k}(\phi+k,\lambda,s+k;\phi-æ+k;x\omega,\omega/k),
\end{aligned}$$

where $Re(\phi)>Re(æ)>0,\ Re(s)>0,\ Re(\lambda)>0,\ Re(\Delta)>0,\ max\{|x\omega|,|\omega/k|\}<\frac{1}{k}$.

Proof. Applying the Mellin transform on definition (13), we have

$$
\begin{aligned}
& M\left[e^{-\chi}\ {}_k\mathfrak{D}_{\omega}^{æ}\left\{\omega^{\frac{\phi}{k}}\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^{n}\right\};\ \Delta\right] \\
= & M\left[e^{-\chi}\ {}_k\mathfrak{D}_{\omega}^{æ}\left\{\omega^{\frac{\phi}{k}}\sum_{n=0}^{\infty}\left(\frac{s+k}{k}\right)_n\frac{\omega^n}{n!}\sum_{\tau=0}^{\infty}(\lambda)_{\tau,k}\frac{(x\omega)^{\tau}}{\tau!}\right\};\ \Delta\right] \\
= & M\left[e^{-\chi}\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s+k}{k}\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\ {}_k\mathfrak{D}_{\omega}^{æ}\left\{\omega^{\frac{\phi+\tau k+nk}{k}}\right\};\ \Delta\right] \\
= & \int_0^{\infty}e^{-\chi}\chi^{\Delta-1}\left(\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s+k}{k}\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\ {}_k\mathfrak{D}_{\omega}^{æ}\left\{\omega^{\frac{\phi+\tau k+nk}{k}}\right\}\right)d\chi \\
= & \int_0^{\infty}e^{-\chi}\chi^{\Delta-1}\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s+k}{k}\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\frac{1}{k\Gamma_k(-æ)}\int_0^{\omega}\psi^{\frac{\phi+nk+\tau k+k}{k}-1}(\omega-\psi)^{-\frac{æ}{k}-1}d\psi d\chi. \qquad (25)
\end{aligned}
$$

Interchanging the order of integrations (25) in above equation, we get

$$
\begin{aligned}
& M\left[e^{-\chi}\ {}_k\mathfrak{D}_{\omega}^{æ}\left\{\omega^{\frac{\phi}{k}}\sum_{n=0}^{\infty}g_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^{n}\right\};\ \Delta\right] \\
= & \int_0^{\infty}e^{-\chi}\chi^{\Delta-1}\left(\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s+k}{k}\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\frac{1}{k\Gamma_k(-æ)}\int_0^{1}(p\omega)^{\frac{\phi+nk+\tau k+k}{k}-1}(\omega-p\omega)^{-\frac{æ}{k}-1}\omega dp\right)d\chi \\
= & \int_0^{\infty}e^{-\chi}\chi^{\Delta-1}\left(\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s+k}{k}\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\frac{\omega^{\frac{\phi-æ}{k}+\tau+n}}{k\Gamma_k(-æ)}\int_0^{1}p^{\frac{\phi+nk+\tau k+k}{k}-1}(1-p)^{-\frac{æ}{k}-1}dp\right)d\chi \\
= & \int_0^{\infty}e^{-\chi}\chi^{\Delta-1}\left(\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\left(\frac{s+k}{k}\right)_n\frac{(\lambda)_{\tau,k}x^{\tau}}{n!\tau!}\frac{\omega^{\frac{\phi-æ}{k}+\tau+n}}{\Gamma_k(-æ)}B_k(\phi+\tau k+nk+k,-æ)\right)d\chi \\
= & \frac{\Gamma(\Delta)\Gamma_k(\phi+k)}{\Gamma_k(\phi-æ+k)}\omega^{\frac{\phi-æ}{k}}\sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}\frac{(s+k)_{n,k}(\lambda)_{\tau,k}(\phi+k)_{n+\tau,k}}{(\phi-æ+k)_{n+\tau,k}}\frac{(x\omega)^{\tau}(\omega/k)^{n}}{\tau!n!} \\
= & \frac{\Gamma(\Delta)\Gamma_k(\phi+k)}{\Gamma_k(\phi-æ+k)}\omega^{\frac{\phi-æ}{k}}F_{1,k}(\phi+k,\lambda,s+k;\phi-æ+k;x\omega,\omega/k)
\end{aligned}
$$

which completes the proof. □

Theorem 3.4. *The following integral representations of the generalized k-Cesàro polynomials can be given:*

$$
(a)\qquad \mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)=\frac{1}{\Gamma_k\left(\frac{s+k}{k}\right)\Gamma_k\left(\frac{\lambda}{k}\right)}\int_0^{\infty}\int_0^{\infty}e^{-(u_1+u_2)}\frac{\left(u_1+u_2kx\right)^n}{n!}u_1^{\frac{s}{k}}u_2^{\frac{\lambda}{k}-1}du_1du_2 \qquad (26)
$$

and

$$
(b)\qquad \mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)=\frac{(s+k)_{n,k}\Gamma_k(-s-kn)}{(n,k)!\Gamma_k(\lambda)\Gamma_k(-s-kn-\lambda)k}\int_0^{1}h^{\frac{\lambda}{k}-1}(1-h)^{-\frac{s+kn+\lambda}{k}-1}(1-kxh)^n dh. \qquad (27)
$$

Proof. (a) If we use the identity [2]

$$
z^{-u}=\frac{1}{\Gamma(u)}\int_0^{\infty}e^{-zt}t^{u-1}dt,\quad (Re(u)>0).
$$

If we use the generating function (15) on the right side of the expression (26), we arrive

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)t^{n} &= \frac{1}{\Gamma_k\left(\frac{s+k}{k}\right)}\int_0^{\infty} e^{-(1-t)u_1}u_1^{\frac{s}{k}}du_1\frac{1}{\Gamma_k\left(\frac{\lambda}{k}\right)}\int_0^{\infty} e^{-(1-kxt)u_2}u_2^{\frac{\lambda}{k}-1}du_2\\
&= \frac{1}{\Gamma_k\left(\frac{s+k}{k}\right)\Gamma_k\left(\frac{\lambda}{k}\right)}\int_0^{\infty}\int_0^{\infty} e^{-(u_1+u_2)}e^{t(u_1+u_2kx)}u_1^{\frac{s}{k}}u_2^{\frac{\lambda}{k}-1}du_1du_2\\
&= \frac{1}{\Gamma_k\left(\frac{s+k}{k}\right)\Gamma_k\left(\frac{\lambda}{k}\right)}\int_0^{\infty}\int_0^{\infty} e^{-(u_1+u_2)}\sum_{n=0}^{\infty}\frac{(u_1+u_2kx)^n}{n!}u_1^{\frac{s}{k}}u_2^{\frac{\lambda}{k}-1}du_1du_2t^n\\
&= \sum_{n=0}^{\infty}\frac{1}{\Gamma_k\left(\frac{s+k}{k}\right)\Gamma_k\left(\frac{\lambda}{k}\right)}\int_0^{\infty}\int_0^{\infty} e^{-(u_1+u_2)}\frac{(u_1+u_2kx)^n}{n!}u_1^{\frac{s}{k}}u_2^{\frac{\lambda}{k}-1}du_1du_2t^n.
\end{aligned}$$

The desired result is given by the coefficients t^n on both sides of the last equation.
(b) In view of the expression (8) and on the right side of (27), we have

$$\begin{aligned}
&\frac{(s+k)_{n,k}\,\Gamma_k(-s-kn)}{(n,k)!\Gamma_k(\lambda)\Gamma_k(-s-kn-\lambda)k}\int_0^1 h^{\frac{\lambda}{k}-1}(1-h)^{-\frac{s+kn+\lambda}{k}-1}(1-kxh)^n\,dh\\
&= \sum_{\xi=0}^{\infty}\frac{(-n)_{\xi,1}\,(s+k)_{n,k}\,\Gamma_k(-s-kn)\,(xk)^{\xi}}{(n,k)!\Gamma_k(\lambda)\Gamma_k(-s-kn-\lambda)\xi!}\frac{1}{k}\int_0^1 h^{\frac{\lambda+\xi k}{k}-1}(1-h)^{-\frac{s+kn+\lambda}{k}-1}\,dh. \qquad (28)
\end{aligned}$$

If we use (2) in the expression (28) and then consider (4), we have

$$\begin{aligned}
&= \sum_{\xi=0}^{\infty}\frac{(-n)_{\xi,1}\,(s+k)_{n,k}\,\Gamma_k(-s-kn)\Gamma_k(\lambda+\xi k)}{(n,k)!\Gamma_k(\lambda)\Gamma_k(\xi k-s-kn)}\frac{(xk)^{\xi}}{\xi!}\\
&= \frac{(s+k)_{n,k}}{(n,k)!}\sum_{\xi=0}^{\infty}\frac{(-n)_{\xi,1}\,\Gamma_k(-s-kn)\Gamma_k(\lambda+\xi k)}{\Gamma_k(\lambda)\Gamma_k(\xi k-s-kn)}\frac{(xk)^{\xi}}{\xi!}\\
&= \frac{(s+k)_{n,k}}{(n,k)!}\,{}_2F_{1,k}\left[\begin{matrix}(-n,1), & (\lambda,k)\\ & (-s-kn,k)\end{matrix};xk\right]\\
&= \mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)
\end{aligned}$$

which completes the proof. □

4 Multilinear and Multilateral Generating Functions

In this piece, we derive several families of multilinear and multilaretal generating functions for the generalized k-Cesàro polynomials $\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)$ which are generated by (14) and given explicitly by (15) using the similar method considered in [11, 12, 13, 17].

We begin by stating the following theorem.

Theorem 4.1. *Let*

$$\Xi_{\rho,\psi}\left[c_1,\dots,c_h;\iota\right] := \sum_{u=0}^{\infty}a_u\Delta_{\rho+\psi u}(c_1,\dots,c_h)\iota^u,\ \ (a_u\neq 0)$$

and

$$T_{n,p;k}^{\rho,\psi}\left(\frac{\lambda}{k},x;c_1,\dots,c_h;\iota\right) := \sum_{\tau=0}^{[n/p]}a_{\tau}\mathfrak{g}_{n-p\tau,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\Delta_{\rho+\psi\tau}(c_1,\dots,c_h)\iota^{\tau},$$

where the notation $[n/p]$ means the greatest integer less than or equal n/p. Then, for $p\in\mathbb{N}$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}T_{n,p;k}^{\rho,\psi}\left(\frac{\lambda}{k},x;c_1,\dots,c_h;\frac{\phi}{\omega^p}\right)\omega^n \qquad (29)\\
&= (1-\omega)^{-\frac{s}{k}-1}(1-kx\omega)^{-\frac{\lambda}{k}}\Xi_{\rho,\psi}\left[c_1,\dots,c_h;\phi\right]
\end{aligned}$$

provided that each member of (29) exists.

Proof. Let S denote the first member of the assertion (29) of Theorem 4.1. Then,

$$S=\sum_{n=0}^{\infty}\sum_{\tau=0}^{[n/p]}a_{\tau}\mathfrak{g}_{n-p\tau,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\Delta_{\rho+\psi\tau}(c_1,\ldots,c_h)\phi^{\tau}\omega^{n-p\tau}.$$

Replacing n by $n+p\tau$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}a_{\tau}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\Delta_{\rho+\psi\tau}(c_1,\ldots,c_h)\phi^{\tau}\omega^{n} \\ &= \sum_{n=0}^{\infty}\sum_{\tau=0}^{\infty}a_{\tau}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\Delta_{\rho+\psi\tau}(c_1,\ldots,c_h)\phi^{\tau}\omega^{n} \\ &= \sum_{n=0}^{\infty}\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^{n}\sum_{\tau=0}^{\infty}a_{\tau}\Delta_{\rho+\psi\tau}(c_1,\ldots,c_h)\phi^{\tau} \\ &= (1-\omega)^{-\frac{s}{k}-1}(1-kx\omega)^{-\frac{\lambda}{k}}\Xi_{\rho,\psi}\left[c_1,\ldots,c_h;\phi\right] \end{aligned}$$

which completes the proof. □

By using a similar idea, we also get the next result immediately.

Theorem 4.2. *Let*

$$\Xi_{\rho,\psi,s_{1;k},s_{2;k}}^{n,p}\left[\frac{\lambda_1+\lambda_2}{k},x;c_1,\ldots,c_h;\delta\right]:=\sum_{\tau=0}^{[n/p]}a_{\tau}\mathfrak{g}_{n-p\tau,k}^{\left(\frac{s_1+s_2+k}{k}\right)}\left(\frac{\lambda_1+\lambda_2}{k},x\right)\Delta_{\rho+\psi\tau}(c_1,\ldots,c_h)\delta^{\tau}$$

where $a_\tau\neq 0$, $n,p\in\mathbb{N}$. Then, for $p\in\mathbb{N}$, we have

$$\begin{aligned} &\sum_{\tau=0}^{n}\sum_{r=0}^{[\tau/p]}a_r\mathfrak{g}_{n-\tau,k}^{\left(\frac{s_1}{k}\right)}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}_{\tau-pr,k}^{\left(\frac{s_2}{k}\right)}\left(\frac{\lambda_2}{k},x\right)\Delta_{\rho+\psi r}(c_1,\ldots,c_h)\delta^{r} \\ =\ &\Xi_{\rho,\psi,s_{1;k},s_{2;k}}^{n,p}\left[\frac{\lambda_1+\lambda_2}{k},x;c_1,\ldots,c_h;\delta\right] \end{aligned} \tag{30}$$

provided that each member of (30) exists.

Proof. Let E denote the first member of the assertion (30) of Theorem 4.2. Then, upon subsituting for the polynomials $\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)$ by (18) into the left-hand side of (30), we obtain

$$\begin{aligned} E &= \sum_{\tau=0}^{n}\sum_{r=0}^{[\tau/p]}a_r\mathfrak{g}_{n-\tau,k}^{\left(\frac{s_1}{k}\right)}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}_{\tau-pr,k}^{\left(\frac{s_2}{k}\right)}\left(\frac{\lambda_2}{k},x\right)\Delta_{\rho+\psi r}(c_1,\ldots,c_h)\delta^{r} \\ &= \sum_{r=0}^{[n/p]}\sum_{\tau=0}^{n-pr}a_r\mathfrak{g}_{n-\tau-pr,k}^{\left(\frac{s_1}{k}\right)}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}_{\tau,k}^{\left(\frac{s_2}{k}\right)}\left(\frac{\lambda_2}{k},x\right)\Delta_{\rho+\psi r}(c_1,\ldots,c_r)\delta^{r} \\ &= \sum_{r=0}^{[n/p]}a_r\left(\sum_{\tau=0}^{n-pr}\mathfrak{g}_{n-\tau-pr,k}^{\left(\frac{s_1}{k}\right)}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}_{\tau,k}^{\left(\frac{s_2}{k}\right)}\left(\frac{\lambda_2}{k},x\right)\right)\Delta_{\rho+\psi r}(c_1,\ldots,c_h)\delta^{r} \\ &= \sum_{r=0}^{[n/p]}a_r\mathfrak{g}_{n-pr,k}^{\left(\frac{s_1+s_2+k}{k}\right)}\left(\frac{\lambda_1+\lambda_2}{k},x\right)\Delta_{\rho+\psi r}(c_1,\ldots,c_r)\delta^{r} \\ &= \Xi_{\rho,\psi,s_{1;k},s_{2;k}}^{n,p}\left[\frac{\lambda_1+\lambda_2}{k},x;c_1,\ldots,c_h;\delta\right]. \end{aligned}$$

□

Theorem 4.3. *Let*

$$\Xi_{\rho,p,q;k}\left[\frac{\lambda}{k},x;c_1,\ldots,c_h;\delta\right]:=\sum_{u=0}^{\infty}a_u\mathfrak{g}_{m+uq,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\Delta_{\rho+pu}(c_1,\ldots,c_h)\delta^{u}$$

where $a_u\neq 0$ and

$$T_{\rho,p,q}(c_1,\ldots,c_h;\delta):=\sum_{\tau=0}^{[u/q]}\binom{m+u}{u-q\tau}a_{\tau}\Delta_{\rho+p\tau}(c_1,\ldots,c_h)\delta^{\tau}.$$

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Then, for $p,q \in \mathbb{N}$; we have

$$\begin{aligned}&\sum_{u=0}^{\infty} \mathfrak{g}_{m+u,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right) T_{\rho,p,q}(c_1,\ldots,c_h;\delta)\omega^u \\ =\ & (1-\omega)^{-\frac{s}{k}-m-1}(1-kx\omega)^{-\frac{\lambda}{k}}\Xi_{\rho,p,q;k}\left[\frac{\lambda}{k},\frac{x(1-\omega)}{1-kx\omega};c_1,\ldots,c_h;\delta\left(\frac{\omega}{1-\omega}\right)^q\right]\end{aligned} \tag{31}$$

provided that each member of (31) exists.

Proof. Let F denote the first member of the assertion (31) of Theorem 4.3. Then,

$$F=\sum_{u=0}^{\infty}\mathfrak{g}_{m+u,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\sum_{\tau=0}^{[u/q]}\binom{m+u}{u-q\tau}a_\tau\Delta_{\rho+p\tau}(c_1,\ldots,c_h)\delta^\tau\omega^u.$$

Replacing u by $u+q\tau$ and then using (19), we may write that

$$\begin{aligned}F &= \sum_{\tau=0}^{\infty}\sum_{u=0}^{\infty}a_\tau\binom{m+u+q\tau}{u}\mathfrak{g}_{m+u+q\tau,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\Delta_{\rho+p\tau}(c_1,\ldots,c_h)\delta^\tau\omega^{u+q\tau}\\ &= \sum_{\tau=0}^{\infty}a_\tau\left(\sum_{u=0}^{\infty}\binom{m+u+q\tau}{u}\mathfrak{g}_{m+u+q\tau,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)\omega^u\right)\Delta_{\rho+p\tau}(c_1,\ldots,c_h)(\delta\omega^q)^\tau\\ &= \sum_{\tau=0}^{\infty}a_\tau\left[(1-\omega)^{-\frac{s}{k}-m-q\tau-1}(1-kx\omega)^{-\frac{\lambda}{k}}\mathfrak{g}_{m+q\tau,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},\frac{x(1-\omega)}{1-kx\omega}\right)\right]\Delta_{\rho+p\tau}(c_1,\ldots,c_h)(\delta\omega^q)^\tau\\ &= (1-\omega)^{-\frac{s}{k}-m-1}(1-kx\omega)^{-\frac{\lambda}{k}}\Xi_{\rho,p,q;k}\left[\frac{\lambda}{k},\frac{x(1-\omega)}{1-kx\omega};c_1,\ldots,c_h;\delta\left(\frac{\omega}{1-\omega}\right)^q\right],\end{aligned}$$

which completes the proof. □

When multivariable function $\Delta_{\rho+\psi\tau}(c_1,\ldots,c_h)$ $\tau\in\mathbb{N}_0$, $h\in\mathbb{N}$, is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set $h=1$ and

$$\Delta_{\rho+\psi\tau}(c_1)=L_{\alpha,\beta,m,\rho+\psi\tau}(c_1)$$

in Theorem 4.1, where the modified Laguerre polynomials, denoted by $L_{\alpha,\beta,m,n}(x)$ [16], generated by

$$(1-\beta\omega)^{-m}\exp\left(\frac{\alpha x\omega}{\beta\omega-1}\right)=\sum_{n=0}^{\infty}L_{\alpha,\beta,m,n}(x)\omega^n \qquad (|\beta\omega|<1). \tag{32}$$

By the way, we give the following example:

Example 4.1. If

$$\Xi_{\rho,\psi}[c_1;\iota]:=\sum_{\tau=0}^{\infty}a_\tau\, L_{\alpha,\beta,m,\rho+\psi\tau}(c_1)\iota^\tau \quad (a_\tau\neq 0,\ \rho,\psi\in\mathbb{C})$$

then, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{\tau=0}^{[n/p]}a_\tau\mathfrak{g}_{n-p\tau,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)L_{\alpha,\beta,m,\rho+\psi\tau}(c_1)\frac{\iota^\tau}{\omega^{p\tau}}\omega^n\\ =\ &(1-\omega)^{-\frac{s}{k}-1}(1-kx\omega)^{-\frac{\lambda}{k}}\,\Xi_{\rho,\psi}[c_1;\iota].\end{aligned}$$

Using the generating relation (32) for the univariate polynomials $L_{\alpha,\beta,m,\rho+\psi\tau}(c_1)$ and getting $a_\tau=1$, $\rho=0$, $\psi=1$, we find that

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{\tau=0}^{[n/p]}\mathfrak{g}_{n-p\tau,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)L_{\alpha,\beta,m,\tau}(c_1)\iota^\tau\omega^{n-p\tau}\\ =\ &(1-\omega)^{-\frac{s}{k}-1}(1-kx\omega)^{-\frac{\lambda}{k}}(1-\beta\iota)^{-m}\exp\left(\frac{\alpha c_1\iota}{\beta\iota-1}\right).\end{aligned}$$

Thus, we obtain a result that provides a class of two-sided generating functions for the univariate extension of the modified Laguerre polynomials $L_{\alpha,\beta,m,\rho+\psi\tau}(c_1)$ and the generalized k-Cesàro polynomials $\mathfrak{g}_{n,k}^{\left(\frac{s}{k}\right)}\left(\frac{\lambda}{k},x\right)$.

If we set

$$h=2,\ c_1=\frac{\lambda_3}{k},\ \ c_2=x \text{ and } \Delta_{\rho+\psi\tau}\left(\frac{\lambda_3}{k},x\right)=\mathfrak{g}_{\rho+\psi\tau,k}^{\left(\frac{s_3}{k}\right)}\left(\frac{\lambda_3}{k},x\right)$$

in Theorem 4.2, we have the following relation for the generalized k-Cesàro polynomials.

Corollary 4.4. *If*

$$\Xi^{n,p}_{\rho,\psi,s_{1;k},s_{2;k}}[\frac{\lambda_1+\lambda_2}{k},x;\frac{\lambda_3}{k},x;\delta]:=\sum_{\tau=0}^{[n/p]}a_\tau\mathfrak{g}^{\left(\frac{s_1+s_2+k}{k}\right)}_{n-p\tau,k}\left(\frac{\lambda_1+\lambda_2}{k},x\right)\mathfrak{g}^{\left(\frac{s_3}{k}\right)}_{\rho+\psi\tau,k}\left(\frac{\lambda_3}{k},x\right)\delta^\tau$$
$$(a_\tau\neq 0,\ \ \rho,\ \psi\in\mathbb{C})$$

then, we have

$$\sum_{\tau=0}^{n}\sum_{r=0}^{[\tau/p]}a_r\mathfrak{g}^{\left(\frac{s_1}{k}\right)}_{n-\tau,k}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}^{\left(\frac{s_2}{k}\right)}_{\tau-pr,k}\left(\frac{\lambda_2}{k},x\right)\mathfrak{g}^{\left(\frac{s_3}{k}\right)}_{\rho+\psi r,k}\left(\frac{\lambda_3}{k},x\right)\delta^r$$
$$=\ \ \Xi^{n,p}_{\rho,\psi,s_{1;k},s_{2;k}}[\frac{\lambda_1+\lambda_2}{k},x;\frac{\lambda_3}{k},x;\delta] \tag{33}$$

provided that each of (33) exists.

Using (33) and $a_r=1$, $\rho=0$, $\psi=1$, $p=1$, $\delta^r=1$

$$\begin{aligned}&\sum_{\tau=0}^{n}\sum_{r=0}^{\tau}\mathfrak{g}^{\left(\frac{s_1}{k}\right)}_{n-\tau,k}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}^{\left(\frac{s_2}{k}\right)}_{\tau-r,k}\left(\frac{\lambda_2}{k},x\right)\mathfrak{g}^{\left(\frac{s_3}{k}\right)}_{r,k}\left(\frac{\lambda_3}{k},x\right)\\
=\ &\sum_{\tau=0}^{n}\mathfrak{g}^{\left(\frac{s_1}{k}\right)}_{n-\tau,k}\left(\frac{\lambda_1}{k},x\right)\sum_{r=0}^{\tau}\mathfrak{g}^{\left(\frac{s_2}{k}\right)}_{\tau-r,k}\left(\frac{\lambda_2}{k},x\right)\mathfrak{g}^{\left(\frac{s_3}{k}\right)}_{r,k}\left(\frac{\lambda_3}{k},x\right)\\
=\ &\sum_{\tau=0}^{n}\mathfrak{g}^{\left(\frac{s_1}{k}\right)}_{n-\tau,k}\left(\frac{\lambda_1}{k},x\right)\mathfrak{g}^{\left(\frac{s_2+s_3+k}{k}\right)}_{\tau,k}\left(\frac{\lambda_2+\lambda_3}{k},x\right)\\
=\ &\mathfrak{g}^{\left(\frac{s_1+s_2+s_3+2k}{k}\right)}_{n,k}\left(\frac{\lambda_1+\lambda_2+\lambda_3}{k},x\right).\end{aligned}$$

If we set

$$h=1,\ \Delta_{\rho+\psi\tau}(c_1)=\mathfrak{g}^{\left(\frac{s_1}{k}\right)}_{\rho+\psi\tau,k}\left(\frac{\lambda_1}{k},c_1\right)$$

in Theorem 4.3, we get a family of the bilinear generating functions for the generalized k-Cesàro polynomials as follows:

Corollary 4.5. *If*

$$\Xi_{\mu,p,q;k}[\frac{\lambda}{k},x;\frac{\lambda_1}{k},c_1;\delta]\ \ :\ \ =\sum_{u=0}^{\infty}a_u\mathfrak{g}^{\left(\frac{s}{k}\right)}_{m+uq,k}\left(\frac{\lambda}{k},x\right)\mathfrak{g}^{\left(\frac{s_1}{k}\right)}_{\rho+\psi u,k}\left(\frac{\lambda_1}{k},c_1\right)\delta^u$$
$$(a_u\neq 0,\ m\in\mathbb{N}_0,\ \ \rho,\psi\in\mathbb{C})$$

and

$$T_{\rho,p,q}\left(\frac{\lambda_1}{k},c_1;\delta\right):=\sum_{\tau=0}^{[u/q]}\binom{m+u}{u-q\tau}a_\tau\mathfrak{g}^{\left(\frac{s_1}{k}\right)}_{\rho+\psi\tau,k}\left(\frac{\lambda_1}{k},c_1\right)\delta^\tau$$

where $p,q\in\mathbb{N}$, then we have

$$\sum_{u=0}^{\infty}\mathfrak{g}^{\left(\frac{s}{k}\right)}_{m+u,k}(\frac{\lambda}{k},x)T_{\rho,p,q}\left(\frac{\lambda_1}{k},c_1;\delta\right)\omega^u \tag{34}$$
$$=(1-\omega)^{-\frac{s}{k}-m-1}(1-kx\omega)^{-\frac{\lambda}{k}}\Xi_{\rho,p,q;k}\left[\frac{\lambda}{k},\frac{x(1-\omega)}{1-kx\omega};\frac{\lambda_1}{k},c_1;\delta\left(\frac{\omega}{1-\omega}\right)^q\right]$$

provided that each member (34) exists.

5 Conclusion

In this study, we define the generalized k-Cesàro polynomials and we give many properties of these polynomial and also give some applications. For every suitable choice of the coefficients a_τ $(\tau\in\mathbb{N}_0)$, if the multivariable functions $\Delta_{\rho+\psi v}(c_1,...,c_h)$, $h\in\mathbb{N}$, expressed as an appropriate product of several simpler functions, the assertions of Theorem 4.1, Theorem 4.2 and Theorem 4.3 can be applied in order to derive verious families of multilinear and multilateral generating function for the family of the generalized k-Cesàro polynomials given explicity by (15).

If we choose $\lambda=1$ in (14), we arrive at the definition of the k-Cesàro polynomials as follows:

$$\sum_{n=0}^{\infty}\mathfrak{g}^{\left(\frac{s}{k}\right)}_{n,k}\left(\frac{1}{k},x\right)\omega^n=(1-\omega)^{-\frac{s}{k}-1}(1-kx\omega)^{-\frac{1}{k}}, \tag{35}$$

where, for $k=1$, $\mathfrak{g}^{(s)}_n(x)$ is the Cesàro polynomials ([1], p. 449, problem 20).

In sections 3 and 4, taking $\lambda=1$ in all Theorems, we get the similar results for the k-Cesàro polynomials defined by (35).

Finally, of course, if we take $\lambda=1$ and $k=1$ in the whole paper, all results in the paper reduce to the well-known classical version.

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