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Appell-Type Changhee Polynomials in the Framework of Fibonomial Calculus

Barış Can Çakır[a] · Esra Erkuş-Duman[b]

Abstract

In this paper, Appell-type Changhee F-polynomials, which correspond of Appell-type Changhee polynomials in Fibonomial Calculus, are introduced. Furthermore, the Appell-type Changhee F-polynomial matrix is defined. Some relations and identities involving these polynomials and matrices are established.

Keywords: Changhee polynomials, Appell-type Changhee polynomials, generating function, generalized Pascal matrix.

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1 Introduction

In the last decades, there has been considerable research on Changhee polynomials and their properties, generalizations and applications in several areas. Changhee polynomials and numbers are introduced in 2013 by Kim et al. [1]. In later years, higher-order Changhee polynomials and numbers, n-th twisted Changhee polynomials and numbers, differential equations for Changhee polynomials and their applications are derived [2, 3, 4, 5]. Furthermore, Changhee polynomials and numbers have been used in various fields, including mathematics, mathematical physics, computer science, engineering sciences, and real-world problems. These polynomials and numbers have been used for solving problems in different areas, illustrating their versatility and importance in various other disciplines.

The Changhee polynomials are defined by

$$\frac{2}{s+2}(s+1)^x = \sum_{k=0}^{\infty} Ch_k(x)\frac{s^k}{k!}. \tag{1}$$

If $x=0$ in (1), then we arrive at the so-called Changhee numbers $Ch_k(0) = Ch_k = \frac{k!}{2^k}$. In 2016, Lee et al. [6] defined Appell-type Changhee polynomials and numbers by the generating function relation

$$\frac{2}{s+2}e^{xs} = \sum_{k=0}^{\infty} Ch_k^*(x)\frac{s^k}{k!}.$$

When $x=0$, the Appell-type Changhee numbers $Ch_k^* = Ch_k^*(0)$ are equal to the Changhee numbers Ch_k.

Moreover, to find the remarkable results in the next sections, we recall information about the Fibonomial calculus. Fibonomial calculus is an intriguing extension of the classical calculus, which intertwines mathematical analysis with the properties of Fibonacci numbers [7, 8]. At its core, this branch of mathematics explores calculus concepts, such as derivatives, integrals, and series, within the framework of Fibonacci sequences and their generalizations. The Fibonacci sequence, defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

with initial values $F_0 = 0$ and $F_1 = 1$, has fascinated mathematicians for centuries due to its appearance in diverse fields like biology, physics, and art. Fibonomial calculus builds upon this foundation by defining operations and functions based on Fibonacci numbers, leading to novel interpretations of familiar mathematical structures. One of the key features of Fibonomial calculus is the use of Fibonomial coefficients, which generalize binomial coefficients by incorporating Fibonacci numbers. These coefficients play a central role in constructing Fibonacci-based versions of factorials, combinations, and summation formulas. Additionally, Fibonomial calculus introduces differential operators and integral transforms tailored to Fibonacci-related functions, creating

[a]Gazi University, Graduate School of Natural and Applied Sciences, Teknikokullar TR-06500, Ankara, Türkiye; bariso7978@gmail.com
[b]Gazi University, Faculty of Science, Department of Mathematics, Teknikokullar TR-06500, Ankara, Türkiye, eerkusduman@gmail.com

a rich framework for exploring patterns and relationships that extend beyond standard calculus. Applications of Fibonomial calculus range from combinatorics and number theory to advanced modeling in natural phenomena where Fibonacci-like growth patterns are prevalent. This field provides new tools for researchers to analyze sequences, series, and functions from a fresh perspective, while also deepening our understanding of the mathematical beauty inherent in the Fibonacci sequence.

In 2004, Krot defined factorial and binomial coefficients in Fibonomial calculus as

$$F_n! = F_n.F_{n-1}.F_{n-2}...F_1, \quad F_0! = 1$$

and

$$\binom{n}{k}_F = \frac{F_n!}{F_{n-k}!F_k!}, \quad (n \geq k \geq 1),$$

with $\binom{n}{0}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$. F-analogue of the binomial theorem and the exponential function $e_F(t)$ are defined by

$$(u +_F v)^n = \sum_{k=0}^{n} \binom{n}{k}_F u^k v^{n-k}.$$

and

$$e_F(t) = e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!},$$

respectively. Furthermore, the F-derivative operator D_x^F acts on the arbitrary function $f(x)$ according to the following formula

$$D_x^F(f(x)) = \frac{f(\alpha x) - f(\beta x)}{(\alpha - \beta) x},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ [8] (see also [9]).

In [10] (see also [9]), F-integral is defined by

$$\int_0^1 f(t) d_F(t) = (\alpha - \beta) \sum_{i=0}^{\infty} \frac{\beta^i}{\alpha^{i+1}} f\left(\frac{\beta^i}{\alpha^{i+1}}\right),$$

and especially,

$$\int_0^1 t^n d_F(t) = \frac{1}{F_{n+1}}.$$

In this paper, the corresponding analogue of Appell-type Changhee polynomials in Fibonomial calculus is defined. Some properties are obtained for these polynomials. A determinantal representation of Appell-type Changee F-polynomials is found and furthermore Appell-type Changee F-matrices are analyzed. Finally, a theorem giving relations involving mix generating functions is established and a few examples are given.

2 Appell-Type Changhee F-Polynomials

In this section, we define the Appell-type Changhee F-polynomials and study their fundamental properties.

Definition 2.1. Appell-type Changhee F-polynomials are defined by

$$\frac{2}{s+2} e_F^{xs} = \sum_{k=0}^{\infty} {}_F Ch_k^*(x) \frac{s^k}{F_k!}, \tag{2}$$

where ${}_F Ch_k^*(0) = {}_F Ch_k^*$ is the Appell-type Changhee F-numbers.

We first get the next result.

Theorem 2.1. *The explicit form of the Appell-type Changhee F-polynmials is as follows:*

$$ {}_F Ch_n^*(x) = \sum_{k=0}^{n} \binom{n}{k}_F \frac{(-1)^{n-k} F_{n-k}!}{2^{n-k}} x^k. \tag{3}$$

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty} {}_FCh_n^*(x)\frac{s^n}{F_n!} &= \frac{2}{s+2}e_F^{xs} \\ &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{s}{2}\right)^n\sum_{k=0}^{\infty}\frac{(xs)^k}{F_k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}x^k s^n}{2^{n-k}F_k!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_F\frac{(-1)^{n-k}F_{n-k}!x^k s^n}{2^{n-k}F_n!},\end{aligned}$$

which completes the proof. □

From (3), the first few values of Appell-type Changhee F-polynomials are provided as follows:

$$\begin{aligned}{}_FCh_0^*(x) &= 1 \\ {}_FCh_1^*(x) &= x-\frac{1}{2} \\ {}_FCh_2^*(x) &= x^2-\frac{1}{2}x+\frac{1}{4} \\ {}_FCh_3^*(x) &= x^3-x^2+\frac{1}{2}x-\frac{1}{4} \\ {}_FCh_4^*(x) &= x^4-\frac{3}{2}x^3+\frac{3}{2}x^2-\frac{3}{4}x+\frac{3}{8}\end{aligned}$$

The next result gives a recurrence relation for the Appell type Changhee F-polynomials.

Theorem 2.2. *The following recurrence relation for the Appell type Changhee F-polynomials holds:*

$$x^n = {}_FCh_n^*(x)+\frac{F_n}{2}\,{}_FCh_{n-1}^*(x)\ ; \qquad n\geq 1.$$

Proof. From the generating function relation for the Applell-type Changhee F-polynomials (2), we have

$$\begin{aligned}2e_F^{xs} &= (2+s)\sum_{n=0}^{\infty}{}_FCh_n^*(x)\frac{s^n}{F_n!} \\ 2\sum_{n=0}^{\infty}\frac{x^n s^n}{F_n!} &= 2\sum_{n=0}^{\infty}{}_FCh_n^*(x)\frac{s^n}{F_n!}+\sum_{n=0}^{\infty}{}_FCh_n^*(x)\frac{s^{n+1}}{F_n!} \\ 2+2\sum_{n=1}^{\infty}\frac{x^n s^n}{F_n!} &= 2{}_FCh_0^*(x)+2\sum_{n=1}^{\infty}{}_FCh_n^*(x)\frac{s^n}{F_n!}+\sum_{n=1}^{\infty}{}_FCh_{n-1}^*(x)\frac{s^n}{F_{n-1}!}.\end{aligned}$$

After some calculation, we arrive at the desired result. □

The next result is an easy consequence of Theorem 2.2.

Corollary 2.3. *The Applell-type Changhee F-numbers satisfy*

$${}_FCh_n^* = -\frac{F_n}{2}\,{}_FCh_{n-1}^* \quad \textit{with} \quad {}_FCh_0^* = 1.$$

Theorem 2.4. *The following summation formula for the Appell type Changhee F-polynomials holds:*

$${}_FCh_n^*(u+_F v) = {}_FCh_n^*(u)\sum_{k=0}^{n}\binom{n}{k}_F\left(\frac{u}{v}\right)^k.$$

Proof. From (3), we can write

$$\begin{aligned} {}_FCh_n^*(u+_F v) &= \sum_{k=0}^{n}\binom{n}{k}_F \frac{(-1)^{n-k}F_{n-k}!}{2^{n-k}}(u+_F v)^k \\ &= \sum_{k=0}^{n}\binom{n}{k}_F \frac{(-1)^{n-k}F_{n-k}!}{2^{n-k}}\sum_{l=0}^{k}\binom{k}{l}_F u^l v^{k-l} \\ &= \sum_{k=0}^{n}\binom{n}{k}_F \frac{(-1)^{n-k}F_{n-k}!}{2^{n-k}}v^k\sum_{l=0}^{k}\binom{k}{l}_F u^l v^{-l} \\ &= {}_FCh_n^*(v)\sum_{l=0}^{k}\binom{k}{l}_F\left(\frac{u}{v}\right)^l, \end{aligned}$$

which completes the proof. □

Theorem 2.5. *For $i \in \mathbb{N}_0$, we have*

$$\sum_{k=0}^{i} {}_FCh_k^*(x)S_2(k,i) = \sum_{k=0}^{i}\binom{i}{k}_F E_{k,F}\ {}_FBel_{i-k}(x),$$

where the Euler-Fibonacci numbers $E_{k,F}$ are defined by [7]

$$\sum_{k=0}^{\infty} E_{k,F}\frac{s^k}{F_k!} = \frac{2}{e_F^s+1},$$

and ${}_FBel_k(x)$ is the fibonomial version of the Bell polynomials and we can define these polynomials as

$$e_F^{x\left(e_F^s-1\right)} = \sum_{k=0}^{\infty} {}_FBel_k(x)\frac{s^k}{F_k!}.$$

Proof. If we substitute e_F^s-1 for s in (2), we get

$$\frac{2}{e_F^s+1}e_F^{x\left(e_F^s-1\right)} = \sum_{k=0}^{\infty} {}_FCh_k^*(x)\frac{\left(e_F^s-1\right)^k}{F_k!}.$$

Let $\mathcal{K}$ denote the each side of the last assertion. Then, we can write

$$\begin{aligned} \mathcal{K} &= \frac{2}{e_F^s+1}e_F^{x\left(e_F^s-1\right)} \\ &= \sum_{k=0}^{\infty} E_{k,F}\frac{s^k}{F_k!}\sum_{i=0}^{\infty} {}_FBel_i(x)\frac{s^i}{F_i!} \\ &= \sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}_F E_{k,F}\ {}_FBel_{i-k}(x)\frac{s^i}{F_i!}. \end{aligned}$$

On the other hand, we also can write

$$\begin{aligned} \mathcal{K} &= \sum_{k=0}^{\infty} {}_FCh_k^*(x)\frac{\left(e_F^s-1\right)^k}{F_k!} \\ &= \sum_{k=0}^{\infty} {}_FCh_k^*(x)\sum_{i=k}^{\infty} S_2(k,i)\frac{s^i}{F_i!} \\ &= \sum_{i=0}^{\infty}\sum_{k=0}^{i} {}_FCh_k^*(x)S_2(k,i)\frac{s^i}{F_i!}. \end{aligned}$$

Thus, the proof is complete. □

Theorem 2.6. *The F-derivative and F-integral relations for Appell-type Changhee F-polynomials, respectively, are as follows:*

(i) $D_x^F\left[{}_FCh_k^*(x)\right] = F_k\ {}_FCh_{k-1}^*(x)$

(ii) $\int_0^x {}_FCh_k^*(t)\,d_Ft = \frac{{}_FCh_{k+1}^*(x) - {}_FCh_{k+1}^*}{F_{k+1}}$.

Proof. The proof is clear from (3). □

Lemma 2.7 (see [11, 12]). *Let $u(s)$ and $v(s)$ be differentiable functions. Define $U_{(n+1)\times 1}(s)$ as a column matrix of size $(n+1)\times 1$, where the elements are given by $u_{k,1}(s)=u^{(k-1)}(s)$ for $1\le k\le n+1$. Similarly, let $V_{(n+1)\times n}(s)$ represent a matrix of dimensions $(n+1)\times n$, with entries defined as:*

$$v_{i,j}(s)=\begin{cases}\binom{i-1}{j-1}v^{(i-j)}(s), & if \quad i-j\ge 0,\\ 0, & if \quad i-j<0,\end{cases}$$

for $1\le i\le n+1$ and $1\le j\le n$. Denote by $\left|\mathbb{W}_{(n+1)\times(n+1)}(s)\right|$ the determinant of the lower Hessenberg matrix $\mathbb{W}_{(n+1)\times(n+1)}(s)$, where

$$\mathbb{W}_{(n+1)\times(n+1)}(s)=\left[U_{(n+1)\times 1}(s)\, V_{(n+1)\times n}(s)\right].$$

The $n-$th derivative of the ratio $\frac{u(s)}{v(s)}$ can then be computed as:

$$\frac{d^n}{ds^n}\left(\frac{u(s)}{v(s)}\right)=(-1)^n\frac{\left|\mathbb{W}_{(n+1)\times(n+1)}(s)\right|}{v^{n+1}(s)}. \tag{4}$$

Let $\{g_n(x)\}$ be a sequences of polynomials, and denote $g(x,s)=\sum_{n=0}^{\infty} g_n(x)\frac{s^n}{n!}$. If $g(x,s)$ can be written as

$$g(x,s)=\frac{u(x,s)}{v(x,s)},$$

then using Lemma 2.7, one can derive that $g_n(x)$ can be expressed as a determinant of order $n+1$, since

$$g_n(x)=\lim_{s\to 0}\frac{\partial^n}{\partial s^n}\left(\frac{u(x,s)}{v(x,s)}\right). \tag{5}$$

Theorem 2.8. *The Appell-type Changhee F-polynomials have the following determinantal representation:*

$$_FCh_n^*(x)=\left(-\frac{1}{2}\right)^n\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & \cdots & 0 & 0\\ x & 1 & 2\binom{1}{1}_F & 0 & 0 & \cdots & 0 & 0\\ x^2 & 0 & \binom{2}{1}_F & 2\binom{2}{2}_F & 0 & \cdots & 0 & 0\\ x^3 & 0 & 0 & \binom{3}{2}_F & 2\binom{3}{3}_F & \cdots & 0 & 0\\ x^4 & 0 & 0 & 0 & \binom{4}{3}_F & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ x^{n-1} & 0 & 0 & 0 & 0 & \cdots & \binom{n-1}{n-2}_F & 2\binom{n-1}{n-1}_F\\ x^n & 0 & 0 & 0 & 0 & \cdots & 0 & \binom{n}{n-1}_F \end{vmatrix}.$$

Proof. If we use (4) taking $u(s)=2e_F^{xs}$ and $v(s)=s+2$, we can write

$$\frac{\partial^n}{\partial s^n}\left(\frac{2e_F^{xs}}{s+2}\right)=\frac{(-1)^n}{(s+2)^{n+1}}\left|\mathbb{W}(x,s)\right|,$$

where $\mathbb{W}(x,s)$ denotes the following matrix:

$$\begin{bmatrix} 2e_F^{xs} & s+2 & 0 & 0 & 0 & \cdots & 0 & 0\\ 2xe_F^{xs} & 1 & \binom{1}{1}_F(s+2) & 0 & 0 & \cdots & 0 & 0\\ 2x^2e_F^{xs} & 0 & \binom{2}{1}_F & \binom{2}{2}_F(s+2) & 0 & \cdots & 0 & 0\\ 2x^3e_F^{xs} & 0 & 0 & \binom{3}{2}_F & \binom{3}{3}_F(s+2) & \cdots & 0 & 0\\ 2x^4e_F^{xs} & 0 & 0 & 0 & \binom{4}{3}_F & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 2x^{n-1}e_F^{xs} & 0 & 0 & 0 & 0 & \cdots & \binom{n-1}{n-2}_F & \binom{n-1}{n-1}_F(s+2)\\ 2x^ne_F^{xs} & 0 & 0 & 0 & 0 & \cdots & 0 & \binom{n}{n-1}_F \end{bmatrix}.$$

Now, considering generating function relation (2) in (5), it follows that

$$_FCh_k^*(x)=\lim_{s\to 0}\frac{\partial^n}{\partial s^n}\left(\frac{2}{s+2}e_F^{xs}\right)$$

and so we arrive at the desired result. □

3 Appell-Type Changhee F-Polynomial Matrix

In this section, we introduce a new family of Appell-type Changhee F-matrices and give an example of them, including their factorization with the generalized Pascal matrix.

Definition 3.1. Let ${}_FCh_n^*(x)$ be the n-th Appell-type Changhee F-polynomial. The Appell-type Changhee F-polynomial matrix $\mathcal{C}_F^*(x)=\left[c_{ij}(x)\right]_{(n+1)\times(n+1)}$ for $i,j=0,1,2,...,n$ is defined by

$$c_{ij}(x)=\begin{cases}\binom{i}{j}_F\,{}_FCh_{i-j}^*(x), & \text{if } \ i\geq j,\\ 0, & \text{otherwise.}\end{cases}.$$

It should be noted that if $x=0$ is taken in this definition, the Appell-type Changhee number F-matrix $\mathcal{C}_F^*=\left[d_{ij}\right]$ can be defined by

$$c_{ij}=\begin{cases}\binom{i}{j}_F\,{}_FCh_{i-j}^*(0), & \text{if } \ i\geq j,\\ 0, & \text{otherwise.}\end{cases}.$$

Example 3.1. For $n=4$, the Appell-type Changhee F-polynomial matrix is as follows:

$$\mathcal{C}_F^*(x)=\begin{bmatrix} {}_FCh_0^*(x) & 0 & 0 & 0\\ \binom{1}{0}_F\,{}_FCh_1^*(x) & \binom{1}{1}_F\,{}_FCh_0^*(x) & 0 & 0\\ \binom{2}{0}_F\,{}_FCh_2^*(x) & \binom{2}{1}_F\,{}_FCh_1^*(x) & \binom{2}{2}_F\,{}_FCh_0^*(x) & 0\\ \binom{3}{0}_F\,{}_FCh_3^*(x) & \binom{3}{1}_F\,{}_FCh_2^*(x) & \binom{3}{2}_F\,{}_FCh_1^*(x) & \binom{3}{3}_F\,{}_FCh_0^*(x)\end{bmatrix}.$$

Definition 3.2. Generalized Fibo-Pascal matrix $U=\left[u_{ij}(x)\right]_{(n+1)\times(n+1)}$ is defined by [13]

$$u_{ij}(x)=\begin{cases}\binom{i}{j}_F x^{i-j} & \text{if } i\geqslant j\ ,\\ 0 & \text{otherwise.}\end{cases}.$$

Theorem 3.1. *Let $\mathcal{C}_F^*(x)$ be the Appell-type Changhee F-polynomial matrix and U be generalized Fibo-Pascal matrix, then we have*

$$\mathcal{C}_F^*(x)=U\cdot\mathcal{C}_F^*.$$

Proof. If the above definitions are substituted and then ${}_FCh_n^*(0)=\frac{(-1)^nF_n!}{2^n}$ is used, we can write

$$\begin{aligned}\left(U\cdot\mathcal{C}_F^*\right)_{ij} &= \sum_{k=j}^{i}u_{ik}(x)c_{kj}\\ &= \sum_{k=j}^{i}\binom{i}{k}_F x^{i-k}\binom{k}{j}_F\,{}_FCh_{k-j}^*(0)\\ &= \binom{i}{j}_F\sum_{k=0}^{i-j}\binom{i-j}{k}_F\,{}_FCh_k^*(0)\,x^{i-j-k}\\ &= \binom{i}{j}_F\sum_{k=0}^{i-j}\binom{i-j}{k}_F\frac{(-1)^kF_k!}{2^k}x^{i-j-k}\\ &= \binom{i}{j}_F\sum_{k=0}^{i-j}\binom{i-j}{k}_F\frac{(-1)^{i-j-k}F_{i-j-k}!}{2^{i-j-k}}x^{k}\\ &= \binom{i}{j}_F\,{}_FCh_{i-j}^*(x)\\ &= c_{ij}(x),\end{aligned}$$

which completes the proof. □

4 Generating Functions for the Appell-type Changhee F-Polynomials

In this section, we give some generating function relations for the Appell-type Changhee F-polynomials. We present some examples and special cases of our theorem.

Theorem 4.1. *Let $r\in\mathbb{N};\ \mu,\psi\in\mathbb{C};\ a_k\in\mathbb{C}\setminus\{0\}\ \ (k\in\mathbb{N}_0)$. Also let*

$$\Phi_\mu:\mathbb{C}^m\to\mathbb{C}\setminus\{0\}$$

be a bounded function.

$$\Theta_{\mu,\psi}\left[v_1,...,v_m;\eta\right] := \sum_{k=0}^{\infty} a_k \Phi_{\mu+\psi k}(v_1,...,v_m)\eta^k$$

and

$$\Xi_{n,r}^{\mu,\psi}(u;v_1,...,v_m;s) := \sum_{k=0}^{[n/r]} \frac{a_k}{F_{n-rk}!}\ {}_F Ch_{n-rk}^{*}(u)\,\Phi_{\mu+\psi k}(v_1,...,v_m)s^k, \tag{6}$$

where the notation $[n/r]$ *means the greatest integer less than or equal* n/r*. Then we have*

$$\sum_{n=0}^{\infty} \Xi_{n,r}^{\mu,\psi}(u;v_1,...,v_m;\frac{\eta}{s^p})s^n \tag{7}$$
$$= \frac{2}{s+2} e_F^{us}\ \Theta_{\mu,\psi}\left[v_1,...,v_m;\eta\right].$$

Proof. Let $\mathcal{M}$ denote the first member of the assertion (7). Using (6), we have

$$\mathcal{M} = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/r]} \frac{a_k}{F_{n-rk}!}\ {}_F Ch_{n-rk}^{*}(u)\,\Phi_{\mu+\psi k}(v_1,...,v_m)\eta^k s^{n-rk}.$$

Applying the double series manipulations

$$\sum_{i=0}^{\infty}\sum_{j=0}^{[i/r]} f(i,j) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} f(i+rj,j)$$

and using (2), we may write

$$\begin{aligned}\mathcal{M} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{a_k}{F_n!}\ {}_F Ch_n^{*}(u)\,\Phi_{\mu+\psi k}(v_1,...,v_m)\eta^k s^n \\ &= \sum_{n=0}^{\infty} {}_F Ch_n^{*}(u)\frac{s^n}{F_n!}\sum_{k=0}^{\infty} a_k \Phi_{\mu+\psi k}(v_1,...,v_m)\eta^k \\ &= \frac{2}{s+2} e_F^{us}\ \Theta_{\mu,\psi}\left[v_1,...,v_m;\eta\right],\end{aligned}$$

which is the right member of (7). □

It is possible to give many applications of Theorem 4.1 by making appropriate choices of the multivariable functions $\Phi_{\mu+\psi k}(v_1,...,v_m)$. Since this multivariable function is very general, we may deduce a number of particular formulas from this result. Now, we present the following two examples.

Example 4.1. The generalized F-Frobenius-Euler polynomials $H_{k,F}^{(\alpha)}(y;\lambda)$ are generated by (see [14])

$$\left(\frac{1-\lambda}{e_F^t-\lambda}\right)^{\alpha} e_F^{yt} = \sum_{k=0}^{\infty} H_{k,F}^{(\alpha)}(y;\lambda)\frac{t^k}{F_k!}, \tag{8}$$

where $|t| < \frac{\ln|\lambda|}{\ln|e_F|}$. If we take $m=1,\ v_1=y,\ a_k=\frac{1}{F_k!},\ \mu=0,\ \psi=1$ and replace the function $\Omega_{\mu+\psi k}$ in Theorem 4.1 with the generalized F-Frobenius-Euler polynomials, using the relation (8) and Theorem 4.1, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/r]} \frac{1}{F_{n-rk}!F_k!}\ {}_F Ch_{n-rk}^{*}(u)H_{k,F}^{(\alpha)}(y;\lambda)\eta^k t^{n-rk} = \frac{2}{s+2}e_F^{us+y\eta}\left(\frac{1-\lambda}{e_F^{\eta}-\lambda}\right)^{\alpha},$$

which is a class of bilateral generating functions for the Appell-type Changhee F-polynomials and the generalized F-Frobenius-Euler polynomials.

Example 4.2. Taking $m=1,\ v_1=v,\ a_k=\frac{1}{F_k!},\ \mu=0,\ \psi=1$ and taking the Appell-type Changhee F-polynomials instead of the function $\Phi_{\mu+\psi k}$ in Theorem 4.1 and also using (2), we get the following generating function relation, known bilinear, for the Appell-type Changhee F-polynomials:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/r]} \frac{1}{F_{n-rk}!F_k!}\ {}_F Ch_{n-rk}^{*}(u)\ {}_F Ch_k^{*}(v)\,\eta^k s^{n-rk} = \frac{2e_F^{us+v\eta}}{(s+2)(\eta+2)}.$$

5 Conclusion

In this study, we introduced the Appell-type Changhee F-polynomials as the corresponding analogues of the Appell-type Changhee polynomials in the context of Fibonomial calculus. Several fundamental properties of these polynomials were established, demonstrating their consistency with the underlying framework of Fibonomial calculus. Additionally, we derived a determinantal representation for the Appell-type Changhee F-polynomials and extended the analysis to include their matrix formulation, further enriching their structural understanding. The study also provided explicit generating functions for these polynomials, offering valuable tools for their computational and analytical applications.

This work not only expands the scope of Fibonomial calculus but also bridges classical polynomial theory and Fibonacci-related structures. The findings have potential applications in areas such as combinatorics, number theory, and the study of special functions.

Future research can explore several promising directions. First, the Appell-type Changhee F-polynomials could be studied in relation to other special polynomials and sequences within the Fibonomial framework, potentially revealing deeper interconnections. Second, their applications in solving differential equations or modeling phenomena exhibiting Fibonacci-like growth patterns could be investigated. Third, exploring multi-variable or matrix forms of these polynomials might lead to further generalizations and insights. Lastly, connections to q-calculus, p-adic analysis, or other advanced mathematical frameworks could open new avenues for theoretical development.

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