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Complex Generalized Stancu Operators Depending on Three Parameters

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Dedicated to the memory of Professor Sorin G. Gal (1953-2024)

Abstract

In this paper, we consider a new generalization of complex Stancu operators. We obtain quantitative upper estimates for the convergence, lower estimates from a qualitative Voronovskaja-type theorem and then the exact degree of simultaneous approximation by these operators attached to analytic functions in a disk centered at the origin with radius greater than 1. Also, we give some graphical and numerical examples.

Keywords : Perturbed Bernstein operator, complex Stancu operator, simultaneous approximation, exact degree of approximation.

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1 Introduction

In [13], with the help of a probabilistic method, Stancu modified the classical Bernstein operators

$$B_\mu(\varphi,\zeta)=\sum_{j=0}^{\mu}p_{\mu,j}(\zeta)\varphi\left(\frac{j}{\mu}\right),\quad \mu\in\mathbb{N}, \tag{1}$$

in which

$$p_{\mu,j}(\zeta)=\binom{\mu}{j}\zeta^j(1-\zeta)^{\mu-j}, \tag{2}$$

depending upon non-negative integer parameter s, as

$$L_{\mu,s}(\varphi,\zeta)=\sum_{j=0}^{\mu}w_{\mu,j,s}(\zeta)\varphi\left(\frac{j}{\mu}\right),\quad \zeta\in[0,1], \tag{3}$$

for $\varphi\in C[0,1]$, μ is any natural number such that $\mu>2s$, where

$$w_{\mu,j,s}(\zeta)=\begin{cases}(1-\zeta)p_{\mu-s,j}(\zeta); & 0\le j<s\\ (1-\zeta)p_{\mu-s,j}(\zeta)+\zeta p_{\mu-s,j-s}(\zeta); & s\le j\le\mu-s\\ \zeta p_{\mu-s,j-s}(\zeta); & \mu-s<j\le\mu\end{cases},$$

which is a generalization of Bernstein's fundamental functions $p_{\mu,j}(\zeta)$ given by (2). We declare that for $p_{\mu,j}(\zeta)$ in (2) the next recursive formula is realizable

$$\begin{aligned} p_{\mu,j}(\zeta) &= (1-\zeta)p_{\mu-1,j}(\zeta)+\zeta p_{\mu-1,j-1}(\zeta),\quad 1\le j\le\mu-1,\\ p_{\mu,0}(\zeta) &= (1-\zeta)p_{\mu-1,0}(\zeta)=(1-\zeta)^{\mu},\quad p_{\mu,\mu}(\zeta)=\zeta p_{\mu-1,\mu-1}(\zeta)=\zeta^{\mu}.\end{aligned} \tag{4}$$

The operator $L_{\mu,s}$ is termed as Stancu operator in the literature and from the description of $w_{\mu,j,s}(\zeta)$ the operator $L_{\mu,s}$ can be also embodied as

$$L_{\mu,s}(\varphi,\zeta)=\sum_{j=0}^{\mu-s}p_{\mu-s,j}(\zeta)\left[(1-\zeta)\varphi\left(\frac{j}{\mu}\right)+\zeta\varphi\left(\frac{j+s}{\mu}\right)\right].$$

Clearly, for $s=0$ and $s=1$, the operator $L_{\mu,s}$ becomes the well-known Bernstein operators. On the other hand, in [14], the author built a linear-positive polynomial operator $L_{\mu,s}^{a,b}$ of Bernstein type, depending upon a non-negative integer parameter s and two real parameters a,b providing the condition $0\le a\le b$, and investigated its some approximation features.

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Recently, employing a new and simple approach to develop the order of approximation of the Bernstein operators due to the use as starting point the recursive formula given by (4), Khosravian-Arab et al. [12] introduced a new modification of Bernstein operators below

$$B_{\mu}^{1}(\varphi,\zeta)=\sum_{j=0}^{\mu}p_{\mu,j}^{1}(\zeta)\varphi\left(\frac{j}{\mu}\right),\ \zeta\in[0,1], \tag{5}$$

where $\varphi\in C[0,1]$,

$$\begin{aligned} p_{\mu,j}^{1}(\zeta) &= u(\zeta,\mu)p_{\mu-1,j}(\zeta)+u(1-\zeta,\mu)p_{\mu-1,j-1}(\zeta), \quad 1\le j\le \mu-1, \\ p_{\mu,0}^{1}(\zeta) &= u(\zeta,\mu)(1-\zeta)^{\mu-1}, \quad p_{\mu,\mu}^{1}(\zeta)=u(1-\zeta,\mu)\zeta^{\mu-1}, \end{aligned} \tag{6}$$

and

$$u(\zeta,\mu)=u_1(\mu)\zeta+u_0(\mu),\quad \mu=0,1,...,$$

with two unknown real sequences $u_1(\mu)$ and $u_0(\mu)$. We state that for $u_1(\mu)=-1$ and $u_0(\mu)=1$, the operator B_{μ}^{1} gives the classical Bernstein operators given by (1). The operator B_{μ}^{1} is named as perturbed Bernstein-type operator. In view of the technique suggested by Khosravian-Arab et al. [12], some generalizations of the perturbed Bernstein operators defined by (5) were proposed in the real and complex cases and their some approximation properties were scrutinized. We may refer to a few (see, e.g., [1, 2, 4, 8, 9]). In [3], Acu and Başcanbaz-Tunca proposed complex form of the perturbed Bernstein-type operators by simply taking z instead of ζ in (5) for any complex-valued function φ that is analytic in an open disk, centered at the origin with a radius greater than 1. The authors got quantitative estimates of the convergence in compact disks and exact degree of simultaneous approximation by means of upper estimates in quantitative form, and of lower estimates derived from a qualitative Voronovskaja type result due to the methods asserted by Gal [11].

Stimulating by the approach of Khosravian-Arab et al. [12] and the work of Stancu [13], we consider a new generalization of complex perturbed Bernstein-type operators studied in [3] as follows

$$\begin{aligned} & L_{\mu,s,1}^{a,b}(\varphi,z) \\ =\ & \sum_{j=0}^{\mu-s}p_{\mu-s,j}(z)\left[u(z,\mu)\varphi\left(\frac{j+a}{\mu+b}\right)+u(1-z,\mu)\varphi\left(\frac{j+s+a}{\mu+b}\right)\right], \end{aligned} \tag{7}$$

where a,b are two real parameters providing the condition $0\le a\le b$, $s\in\mathbb{Z},\mu\in\mathbb{N}$ such that $0\le s<\mu/2$, $z\in\mathbb{C}$, φ is a complex-valued analytic function in an open disk centered at the origin with radius greater than 1, $p_{\mu,j}(z)$ is given by (2) and

$$u(z,\mu)=u_1(\mu)z+u_0(\mu),\quad \mu=0,1,...,$$

with unknown real sequences $u_1(\mu)$ and $u_0(\mu)$. For these real sequences, we concern with the below cases:

(1) If $u_1(\mu)=0$, then $u_0(\mu)=\frac{1}{2}$.
(2) If $0<u_1(\mu)<1$, then $u_0(\mu)>0$.
(3) If $u_1(\mu)=1$, then $u_0(\mu)=0$.
(4) If $-1<u_1(\mu)<0$, then $u_0(\mu)>0$ and $u_1(\mu)+u_0(\mu)>0$.
(5) If $u_1(\mu)=-1$, then $u_0(\mu)=1$.

Obviously, for the special cases $u_1(\mu)=-1$ and $u_0(\mu)=1$ with $a=b=0$, the operator $L_{\mu,s,1}^{a,b}$ reduces to the complex Stancu operators investigated in [6]. When $u_1(\mu)=-1$ and $u_0(\mu)=1$, the operator $L_{\mu,s,1}^{a,b}$ gives the generalized complex Stancu operators studied in [7]. In the case of $s=0$ or $s=1$, the operator $L_{\mu,s,1}^{a,b}$ becomes the complex perturbed Bernstein-Stancu operators considered in [8]. For the cases $s=0$ or $s=1$ with $a=b=0$, the operator $L_{\mu,s,1}^{a,b}$ yields the complex perturbed Bernstein-type operators investigated thoroughly in [3]. Also, when $s=0$ or $s=1$ with $u_1(\mu)=-1$ and $u_0(\mu)=1$, these operators give the complex Bernstein-Stancu operators considered in [11]. In the present paper, firstly we get quantitative upper estimates for the generalized complex perturbed Bernstein operators $L_{\mu,s,1}^{a,b}$ and their derivatives on compact disks. Afterwards, we obtain lower estimates from a qualitative Voronovskaja type result. Lastly, by using obtained quantitative upper and qualitative lower estimates, we establish the exact order of simultaneous approximation for the new operator $L_{\mu,s,1}^{a,b}$.

2 Auxiliary Results

Before presenting main results, we remember some symbols which are used throughout the paper.

Let $H_R, R>1$, denote the open disk $H_R:=\{z\in\mathbb{C}:|z|<R\}$, and the closed disk included in H_R, $\overline{H}_{\rho}:=\{z\in\mathbb{C}:|z|\le\rho,\ 1\le\rho<R\}$ and furthermore $\|\varphi\|_{\rho}=\max\{|\varphi(z)|:|z|\le\rho\}$.

We shall use denotation $e_k(z)=z^k$, $k\in\mathbb{N}\cup\{0\}$, $z\in\mathbb{C}$ and assume that $L_{\mu,s,1}^{a,b}(e_0,z)=1$, namely

$$2u_0(\mu)+u_1(\mu)=1.$$

Obviously, we mention that the above sequences $u_i(\mu), i=0,1$, are bounded, that is $|u_i(\mu)|\leq K$, $K>0$.

In order to prove the main results, we need the ensuing results.

Lemma 2.1. *For the moments, the real form of the operator $L^{a,b}_{\mu,s,1}$ given by (7) verifies*

(i) $L^{a,b}_{\mu,s,1}(e_0,\zeta)=2u_0(\mu)+u_1(\mu)=1$,

(ii) $L^{a,b}_{\mu,s,1}(e_1,\zeta)=\frac{1}{\mu+b}\{(\mu-s-u_1(\mu)s)\zeta+a+(u_0(\mu)+u_1(\mu))s\}$,

(iii) $L^{a,b}_{\mu,s,1}(e_2,\zeta)=\frac{1}{(\mu+b)^2}\{(\mu-s)(\mu-s-1-2su_1(\mu))\zeta^2$
$+\left[(\mu-s)(1+2a+2su_1(\mu)+2su_0(\mu))-2sau_1(\mu)-s^2u_1(\mu)\right]\zeta$
$+a^2+(u_0(\mu)+u_1(\mu))\left(2sa+s^2\right)\}$,

(iv) $L^{a,b}_{\mu,s,1}(e_3,\zeta)=\frac{1}{(\mu+b)^3}\{(\mu-s)(\mu-s-1)(\mu-s-2-3su_1(\mu))\zeta^3$
$+3(\mu-s)[\mu(1+a+su_1(\mu)+su_0(\mu))-s(1+s)(2u_1(\mu)+u_0(\mu))$
$-s(1+a+2su_1(\mu))-1-a]\zeta^2+\left[\mu\left(1+3a+3a^2+3su_1(\mu)(1+2a+s)\right.\right.$
$+3su_0(\mu)(1+2a+s))-s^3(3u_0(\mu)+4u_1(\mu))$
$-s^2(3u_1(\mu)+3u_0(\mu)+9au_1(\mu)+6au_0(\mu))-s-3as-3a^2s-3a^2su_1(\mu)]\zeta$
$+a^3+s(u_0(\mu)+u_1(\mu))\left(3a^2+3sa+s^2\right)\}$,

(v) $L^{a,b}_{\mu,s,1}(e_4,\zeta)$
$=\frac{1}{(\mu+b)^4}\{(\mu-s)(\mu-s-1)(\mu-s-2)(\mu-s-3-4su_1(\mu))\zeta^4$
$+2(\mu-s)(\mu-s-1)[(\mu-s-2)(3+2a+2su_1(\mu)+2su_0(\mu))$
$-3su_1(\mu)(2+2a+s)]\zeta^3+(\mu-s)\left[(\mu-s-1)\left(7+12a+6a^2\right.\right.$
$+12su_1(\mu)(1+a)+12su_0(\mu)(1+a)+6s^2(u_1(\mu)+u_0(\mu)))$
$-2su_1(\mu)\left(2+6a+6a^2+3s+6as+2s^2\right)]\zeta^2+\left[(\mu-s)\left(1+4a+6a^2\right.\right.$
$+4a^3+2s(u_1(\mu)+u_0(\mu))\left(2+6a+6a^2+3s+6as+2s^2\right))-su_1(\mu)\left(4a^3\right.$
$+6sa^2+4s^2a+s^3)]\zeta+a^4+s(u_0(\mu)+u_1(\mu))\left(4a^3+6a^2s+4s^2a+s^3\right)\}$.

Lemma 2.2. *For the central moments, the real form of the operator $L^{a,b}_{\mu,s,1}$ given by (7) satisfies*

(i) $L^{a,b}_{\mu,s,1}(t-\zeta,\zeta)=\frac{1}{\mu+b}\{a+(u_0(\mu)+u_1(\mu))s-(s+u_1(\mu)s+b)\zeta\}$,

(ii) $L^{a,b}_{\mu,s,1}\left((t-\zeta)^2,\zeta\right)=\frac{1}{(\mu+b)^2}\{\left[-\mu+s^2+s+2s^2u_1(\mu)+2bs\right.$
$\left.+2bsu_1(\mu)+b^2\right]\zeta^2+\left[\mu-s-2as-2s^2u_1(\mu)-2s^2u_0(\mu)\right.$
$\left.-2asu_1(\mu)-s^2u_1(\mu)-2bsu_1(\mu)-2bsu_0(\mu)-2ba\right]\zeta+a^2$
$+(u_1(\mu)+u_0(\mu))\left(2sa+s^2\right)\}$,

(iii) $L^{a,b}_{\mu,s,1}\left((t-\zeta)^4,\zeta\right)=\frac{3\zeta^2(1-\zeta)^2\mu^2}{(\mu+b)^4}+\mathcal{O}\left(\frac{1}{(n+b)^3}\right)$.

Theorem 2.3. *If $\varphi\in C[0,1]$, then*

$$\lim_{\mu\to\infty}L^{a,b}_{\mu,s,1}(\varphi,\zeta)=\varphi(\zeta)$$

uniformly on $[0,1]$.

Proof. Since the sequences $u_0(\mu)$ and $u_1(\mu)$ verify the cases (1)-(5), the operator $L^{a,b}_{\mu,s,1}$ given by (7) in the real case is positive. Taking into consideration that the sequences $u_0(\mu)$, $u_1(\mu)$ are bounded, by making use of Korovkin theorem and Lemma (2.1) (i)-(iii), we obtain the uniform convergence of the operator $L^{a,b}_{\mu,s,1}$. □

Theorem 2.4. *Assume that $u_i(\mu)$, $i=0,1$, are convergent sequences and $L_1=\lim_{\mu\to\infty}u_1(\mu)$. If $\varphi\in C^2[0,1]$ and $\zeta\in[0,1]$, then*

$$\lim_{\mu\to\infty}(\mu+b)[L^{a,b}_{\mu,s,1}(\varphi,\zeta)-\varphi(\zeta)]$$
$$=\left[\frac{2a+s(1+L_1)}{2}-(s+sL_1+b)\zeta\right]\varphi'(\zeta)+\frac{\zeta(1-\zeta)}{2}\varphi''(\zeta).$$

Proof. From Taylor's formula,

$$\varphi(t)=\varphi(\zeta)+\varphi'(\zeta)(t-\zeta)+\frac{\varphi''(\zeta)}{2}(t-\zeta)^2+\theta(t,\zeta)(t-\zeta)^2$$

at the fixed point $\zeta\in[0,1]$, where θ is a continuous function on $[0,1]$ and $\lim_{t\to\zeta}\theta(t,\zeta)=\theta(\zeta,\zeta)=0$. By application of the operator $L^{a,b}_{\mu,s,1}$, one has

$$(\mu+b)[L^{a,b}_{\mu,s,1}(\varphi,\zeta)-\varphi(\zeta)]=\varphi'(\zeta)(\mu+b)L^{a,b}_{\mu,s,1}(t-\zeta,\zeta)$$
$$+\frac{\varphi''(\zeta)}{2}(\mu+b)L^{a,b}_{\mu,s,1}\left((t-\zeta)^2,\zeta\right)+(\mu+b)L^{a,b}_{\mu,s,1}\left(\theta(t,\zeta)(t-\zeta)^2,\zeta\right).$$

From Lemma (2.2) (i)-(ii), we conclude that

$$\lim_{\mu\to\infty}(\mu+b)[L_{\mu,s,1}^{a,b}(\varphi,\zeta)-\varphi(\zeta)]=\left[a+\frac{(1+L_1)s}{2}-(s+sL_1+b)\zeta\right]\varphi'(\zeta)$$
$$+\frac{\zeta(1-\zeta)}{2}\varphi''(\zeta)+\lim_{\mu\to\infty}(\mu+b)L_{\mu,s,1}^{a,b}\left(\theta(t,\zeta)(t-\zeta)^2,\zeta\right).$$

Hence, it is sufficient to show that $\lim_{\mu\to\infty}(\mu+b)L_{\mu,s,1}^{a,b}\left(\theta(t,\zeta)(t-\zeta)^2,\zeta\right)=0$. For this purpose, applying Cauchy-Schwarz inequality, we have

$$(\mu+b)\left|L_{\mu,s,1}^{a,b}\left(\theta(t,\zeta)(t-\zeta)^2,\zeta\right)\right|$$
$$\leq\left(L_{\mu,s,1}^{a,b}\left(\theta^2(t,\zeta),\zeta\right)\right)^{\frac{1}{2}}\left((\mu+b)^2L_{\mu,s,1}^{a,b}\left((t-\zeta)^4,\zeta\right)\right)^{\frac{1}{2}}. \tag{8}$$

Because $\theta^2(\zeta,\zeta)=0$ and $\theta^2(\cdot,\zeta)\in C[0,1]$, from Theorem (2.3) it follows

$$\lim_{\mu\to\infty}L_{\mu,s,1}^{a,b}\left(\theta^2(t,\zeta),\zeta\right)=\theta^2(\zeta,\zeta)=0. \tag{9}$$

Also, by Lemma (2.2) (iii) we find

$$\lim_{\mu\to\infty}(\mu+b)^2L_{\mu,s,1}^{a,b}\left((t-\zeta)^4,\zeta\right)=3\zeta^2(1-\zeta)^2. \tag{10}$$

Consequently, using (8), (9) and (10), we conclude

$$\lim_{\mu\to\infty}(\mu+b)L_{\mu,s,1}^{a,b}\left(\theta(t,\zeta)(t-\zeta)^2,\zeta\right)=0.$$

□

Let Δ_h^j denote *the finite difference of order j with step h*, defined as

$$\Delta_h^j\varphi(\zeta)=j!h^j[\zeta,\zeta+h,\cdots,\zeta+jh;\varphi] \tag{11}$$

(see p. 121 of [10]). In [14] (formula 1.7), the author presented the below statement for the operator $L_{\mu,s}^{a,b}$

$$L_{\mu,s}^{a,b}(\varphi,\zeta) \tag{12}$$
$$=\sum_{j=0}^{\mu-s}\binom{\mu-s}{j}\left[(1-\zeta)\Delta_{1/(\mu+b)}^j\varphi\left(\frac{a}{\mu+b}\right)+\zeta\Delta_{1/(\mu+b)}^j\varphi\left(\frac{s+a}{\mu+b}\right)\right]\zeta^j,$$

where $\Delta_{1/\mu+b}^j$ is defined by (11) with $h=\frac{1}{\mu+b}$.

Therefore, we can represent the operators $L_{\mu,s,1}^{a,b}$ given by (7) as

$$L_{\mu,s,1}^{a,b}(\varphi,z)=\sum_{j=0}^{\mu-s}\binom{\mu-s}{j}\left[u(z,\mu)\Delta_{1/(\mu+b)}^j\varphi\left(\frac{a}{\mu+b}\right)\right.$$
$$\left.+u(1-z,\mu)\Delta_{1/(\mu+b)}^j\varphi\left(\frac{s+a}{\mu+b}\right)\right]z^j, \tag{13}$$

where for the special case $u_1(\mu)=-1$, $u_0(\mu)=1$, (13) yields (12).

In what follows, we give a relationship which will be used in the proof of the main results.

Lemma 2.5. *Assume that ρ and R are constants such that $1\leq\rho<R$, $0\leq a\leq b$ and φ is analytic in H_R with $\varphi(z)=\sum_{j=0}^{\infty}c_jz^j$.Then for all $z\in\overline{H}_\rho$ and $\mu\in\mathbb{N}$, we have*

$$L_{\mu,s,1}^{a,b}(\varphi,z)=\sum_{j=0}^{\infty}c_jL_{\mu,s,1}^{a,b}(e_j,z).$$

Proof. For any $\upsilon\in\mathbb{N}$, we possess $\varphi_\upsilon(z)=\sum_{j=0}^{\upsilon}c_jz^j$, $|z|\leq\rho$. Since $L_{\mu,s,1}^{a,b}$ is linear, we can obviously write down

$$L_{\mu,s,1}^{a,b}(\varphi_\upsilon,z)=\sum_{j=0}^{\upsilon}c_jL_{\mu,s,1}^{a,b}(e_j,z),$$

for all $|z| \le \rho$, $\rho \ge 1$. Therefore, it remains to confirm that $\lim_{\upsilon\to\infty} L^{a,b}_{\mu,s,1}(\varphi_\upsilon, z) = L^{a,b}_{\mu,s,1}(\varphi, z)$ for any fixed $\mu \in \mathbb{N}$ and $|z| \le \rho$. We state that for the sequences $u_0(\mu)$ and $u_1(\mu)$, by the possible cases (1)-(5) we have $u_0(\mu) \ge 0$ and $u_0(\mu) + u_1(\mu) \ge 0$. For all $|z| \le \rho$ with $\rho \ge 1$, from (7) one has

$$\begin{aligned}
&\left|L^{a,b}_{\mu,s,1}(\varphi_\upsilon, z) - L^{a,b}_{\mu,s,1}(\varphi, z)\right| \\
\le\ & \sum_{j=0}^{\mu-s}\binom{\mu-s}{j}\left|z^j(1-z)^{\mu-s-j}\right|\left[\left|u_1(\mu)z + u_0(\mu)\right|\left|(\varphi_\upsilon-\varphi)\left(\frac{j+a}{\mu+b}\right)\right|\right. \\
&\left. + \left|u_1(\mu) - u_1(\mu)z + u_0(\mu)\right|\left|(\varphi_\upsilon-\varphi)\left(\frac{j+s+a}{\mu+b}\right)\right|\right] \\
\le\ & \sum_{j=0}^{\mu-s}\binom{\mu-s}{j}\rho^j(1+\rho)^{\mu-s-j}\left\|\varphi_\upsilon-\varphi\right\|_\rho\left[\left|u_1(\mu)z\right| + u_0(\mu)\right. \\
&\left. + \left|u_1(\mu) + u_0(\mu)\right| + \left|u_1(\mu)z\right|\right] \\
\le\ & \sum_{j=0}^{\mu-s}\binom{\mu-s}{j}(1+\rho)^{\mu-s}\left\|\varphi_\upsilon-\varphi\right\|_a\left[\left|u_1(\mu)\right|\rho + 2u_0(\mu) + u_1(\mu) + \left|u_1(\mu)\right|\rho\right] \\
\le\ & (1+\rho)^{\mu-s}2^{\mu-s}\left\|\varphi_\upsilon-\varphi\right\|_\rho\left(2\left|u_1(\mu)\right|\rho + \rho\right) \\
\le\ & (2+2\rho)^{\mu-s}(2K+1)\rho\left\|\varphi_\upsilon-\varphi\right\|_\rho,
\end{aligned}$$

for $|u_1(\mu)| \le K$, $K > 0$. Since $\sum_{j=0}^{\infty} c_j z^j$ is absolutely and uniformly convergent on each compact subset of H_R, we have $\lim_{\upsilon\to\infty}\|\varphi_\upsilon - \varphi\|_\rho = 0$ for $|z| \le \rho < R$, which completes the proof. □

3 Main Results

The first outcome of this part is quantitative upper estimates for simultaneous approximation.

Theorem 3.1. *Presuppose that s is a non negative integer, $\mu \in \mathbb{N}$ such that $\mu > 2s$, $0 \le a \le b$ and $\varphi : H_R \to \mathbb{C}$ is analytic in H_R, $R > 1$, with $\varphi(z) = \sum_{j=0}^{\infty} c_j z^j$.*

(i) Let $1 \le \rho < R$ be arbitrary fixed. For all $|z| \le \rho$, we have

$$\left|L^{a,b}_{\mu,s,1}(\varphi, z) - \varphi(z)\right| \le \frac{M^{1,b}_{s,\rho}(\varphi)}{\mu+b},$$

where

$$0 < M^{1,b}_{s,\rho}(f) = 2(K+1)\left\{\rho^2\sum_{j=2}^{\infty}\left|c_j\right| j(j-1)\rho^{j-2} + 2(s+b)\rho\sum_{j=1}^{\infty}\left|c_j\right| j a^{j-1}\right\} < \infty.$$

(ii) Moreover, if $1 \le \rho < \rho_1 < R$, then for all $|z| \le \rho$ and $\mu, p \in \mathbb{N}$, we have

$$\left|\left(L^{a,b}_{\mu,s,1}(\varphi, z)\right)^{(p)} - \varphi^{(p)}(z)\right| \le \frac{M^{1,b}_{s,\rho_1}(\varphi)\, p!\,\rho_1}{(\mu+b)(\rho_1-\rho)^{p+1}},$$

where $M^{1,b}_{s,\rho_1}(\varphi)$ is defined as in (i).

Proof. From Lemma (2.5), for all $|z| \le \rho$ we may write

$$\left|L^{a,b}_{\mu,s,1}(\varphi, z) - \varphi(z)\right| \le \sum_{j=0}^{\infty}\left|c_j\right|\left|L^{a,b}_{\mu,s,1}(e_j, z) - e_j(z)\right|.$$

Now, we will find upper bound for $\left|L^{a,b}_{\mu,s,1}(e_j, z) - e_j(z)\right|$. We observe the two cases; $0 \le j \le \mu - s$ and $j > \mu - s$.

Case 1: Because $L^{a,b}_{\mu,s,1}(e_0, z) - e_0(z) = 0$, we look at the case $1 \le j \le \mu - s$.
By (13) and (11), representing

$$T^{s,a,b,1}_{\mu,p,j} := \binom{\mu-s}{p}\frac{p!}{(\mu+b)^p}\left[\frac{a}{\mu+b}, \frac{a+1}{\mu+b}, \cdots, \frac{a+p}{\mu+b}; e_j\right], \tag{14}$$

and

$$T^{s,a,b,1}_{\mu,s+p,j} := \binom{\mu-s}{p}\frac{p!}{(\mu+b)^p}\left[\frac{s+a}{\mu+b}, \frac{s+a+1}{\mu+b}, \cdots, \frac{s+a+p}{\mu+b}; e_j\right] \tag{15}$$

for $0 \le p \le \mu - s$, $\mu \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$, making use of the trait of divided difference formula we may write

$$\begin{aligned} T^{s,a,b,1}_{\mu,j,j} &= T^{s,a,b,1}_{\mu,s+j,j} = \binom{\mu-s}{j} \frac{j!}{(\mu+b)^j} \\ &= \frac{(\mu-s)(\mu-s-1)\cdots(\mu-s-(j-1))}{(\mu+b)^j}. \end{aligned} \tag{16}$$

As a result, from (13), we get

$$\begin{aligned} & L^{a,b}_{\mu,s,1}\left(e_j, z\right) \\ =& \sum_{p=0}^{\mu-s} \left\{ \left(u_1(\mu) z + u_0(\mu)\right) T^{s,a,b,1}_{\mu,p,j} + \left(u_1(\mu)(1-z) + u_0(\mu)\right) T^{s,a,b,1}_{\mu,s+p,j} \right\} e_p(z). \end{aligned} \tag{17}$$

Using convexity of all order of e_j, $T^{s,a,b,1}_{\mu,p,j}$ and $T^{s,a,b,1}_{\mu,s+p,j} \ge 0$. By (14), (15) and (17), for any $|z| \le \rho$, $1 \le \rho < R$, we have

$$\begin{aligned} & \left| L^{a,b}_{\mu,s,1}\left(e_j, z\right) - e_j(z) \right| \\ =& \left| \sum_{p=0}^{j} \left\{ \left(u_1(\mu) z + u_0(\mu)\right) T^{s,a,b,1}_{\mu,p,j} + \left(u_1(\mu)(1-z) + u_0(\mu)\right) T^{s,a,b,1}_{\mu,s+p,j} \right\} e_p(z) \right. \\ & \left. - e_j(z) \right| \\ =& \left| \left\{ \left(u_1(\mu) z + u_0(\mu)\right) T^{s,a,b,1}_{\mu,j,j} + \left(u_1(\mu)(1-z) + u_0(\mu)\right) T^{s,a,b,1}_{\mu,s+j,j} - 1 \right\} e_j(z) \right. \\ & \left. + \sum_{p=0}^{j-1} \left\{ \left(u_1(\mu) z + u_0(\mu)\right) T^{s,a,b,1}_{\mu,p,j} + \left(u_1(\mu)(1-z) + u_0(\mu)\right) T^{s,a,b,1}_{\mu,s+p,j} \right\} e_p(z) \right| \\ \le& \left| \left\{ \left(2u_0(\mu) + u_1(\mu)\right) \frac{(\mu-s)(\mu-s-1)\cdots(\mu-s-(j-1))}{(\mu+b)^j} - 1 \right\} e_j(z) \right| \\ & + \sum_{p=0}^{j-1} \left\{ \left|u_1(\mu) z + u_0(\mu)\right| T^{s,a,b,1}_{\mu,p,j} + \left|u_1(\mu) - u_1(\mu) z + u_0(\mu)\right| T^{s,a,b,1}_{\mu,s+p,j} \right\} \left| e_p(z) \right| \\ \le& \left| \frac{(\mu-s)(\mu-s-1)\cdots(\mu-s-(j-1))}{(\mu+b)^j} - 1 \right| \rho^j + \left(|u_1(\mu)| \rho + u_0(\mu) \right) \\ & \times \rho^{j-1} \sum_{p=0}^{j-1} T^{s,a,b,1}_{\mu,p,j} + \left(|u_1(\mu)| \rho + u_1(\mu) + u_0(\mu) \right) \rho^{j-1} \sum_{p=0}^{j-1} T^{s,a,b,1}_{\mu,s+p,j} \\ \le& \rho^j \left[1 - \frac{(\mu-s)(\mu-s-1)\cdots(\mu-s-(j-1))}{(\mu+b)^j} \right] \\ & + \left(|u_1(\mu)| + u_0(\mu) \right) \rho^j \left[\sum_{p=0}^{j} T^{s,a,b,1}_{\mu,p,j} - T^{s,a,b,1}_{\mu,j,j} \right] \\ & + \left(|u_1(\mu)| + u_1(\mu) + u_0(\mu) \right) \rho^j \left[\sum_{p=0}^{j} T^{s,a,b,1}_{\mu,s+p,j} - T^{s,a,b,1}_{\mu,s+j,j} \right] \\ \le& \rho^j \left[1 - \frac{(\mu-s)(\mu-s-1)\cdots(\mu-s-(j-1))}{(\mu+b)^j} \right] + \left(|u_1(\mu)| + u_0(\mu) \right) \\ & \times \left\{ \sum_{p=0}^{\mu-s} \binom{\mu-s}{p} \frac{p!}{(\mu+b)^p} \left[\frac{a}{\mu+b}, \frac{a+1}{\mu+b}, \cdots, \frac{a+p}{\mu+b}; e_j \right] - T^{s,a,b,1}_{\mu,j,j} \right\} \rho^j \\ & + \left(|u_1(\mu)| + u_1(\mu) + u_0(\mu) \right) \left\{ \sum_{p=0}^{\mu-s} \binom{\mu-s}{p} \frac{p!}{(\mu+b)^p} \left[\frac{s+a}{\mu+b}, \cdots, \frac{s+a+p}{\mu+b}; e_j \right] \right. \\ & \left. - T^{s,a,b,1}_{\mu,s+j,j} \right\} \rho^j. \end{aligned} \tag{18}$$

Since $j \leq \mu$, we get

$$\begin{aligned}
& \sum_{p=0}^{\mu-s}\binom{\mu-s}{p}\frac{p!}{(\mu+b)^p}\left[\frac{a}{\mu+b},\frac{a+1}{\mu+b},\cdots,\frac{a+p}{\mu+b};e_j\right] \\
\leq\ & \sum_{p=0}^{\mu}\binom{\mu}{p}\frac{p!}{(\mu+b)^p}\left[\frac{a}{\mu+b},\frac{a+1}{\mu+b},\cdots,\frac{a+p}{\mu+b};e_j\right] \\
=\ & B_\mu^{a,b}\left(e_j,1\right)=\left(\frac{\mu+a}{\mu+b}\right)^j \leq 1,
\end{aligned}$$

in which $B_\mu^{a,b}(\varphi,\zeta)=\sum_{j=0}^{\mu}p_{\mu,j}(\zeta)\varphi\left(\frac{j+a}{\mu+b}\right)$. Moreover, since $L_{\mu,s}^{a,b}(\varphi,1)=\varphi(\frac{\mu+a}{\mu+b})$ for the generalized Stancu operators in (12), we conclude that

$$\begin{aligned}
L_{\mu,s}^{a,b}(e_j,1) &= \sum_{p=0}^{\mu-s}\binom{\mu-s}{p}\frac{p!}{(\mu+b)^p}\left[\frac{s+a}{\mu+b},\frac{s+a+1}{\mu+b},\cdots,\frac{s+a+p}{\mu+b};e_j\right] \\
&= \left(\frac{\mu+a}{\mu+b}\right)^j \leq 1.
\end{aligned}$$

Therefore, from (16), (18) reduces to

$$\begin{aligned}
& \left|L_{\mu,s,1}^{a,b}\left(e_j,z\right)-e_j\left(z\right)\right| \\
\leq\ & 2\left(|u_1(\mu)|+1\right)\rho^j\left[1-\frac{(\mu-s)(\mu-s-1)\cdots(\mu-s-(j-1))}{(\mu+b)^j}\right].
\end{aligned} \tag{19}$$

Using the following inequality

$$1-\prod_{i=1}^{j}x_i \leq \sum_{i=1}^{j}(1-x_i), \quad 0\leq x_i\leq 1, i=1,\cdots,j,$$

(see, p. 69 of [11]), (19) follows that

$$\begin{aligned}
\left|L_{\mu,s,1}^{a,b}\left(e_j,z\right)-e_j\left(z\right)\right| &\leq 2\left(|u_1(\mu)|+1\right)\rho^j\sum_{i=1}^{j}\left[1-\frac{\mu-s-(i-1)}{\mu+b}\right] \\
&\leq 2(K+1)\rho^j\sum_{i=1}^{j}\frac{b+s-1+i}{\mu+b} \\
&\leq 2(K+1)\rho^j\frac{2j(s+b)+j(j-1)}{2(\mu+b)} \\
&\leq (K+1)\frac{2j(s+b)+j(j-1)}{\mu+b}\rho^j.
\end{aligned}$$

Case 2 : For $j>\mu-s$ and $|z|\leq\rho$ with $1\leq\rho<R$, from (17) we obtain that

$$\begin{aligned}
& \left|L_{\mu,s,1}^{a,b}\left(e_j,z\right)-e_j\left(z\right)\right| \\
\leq\ & \left|\sum_{p=0}^{\mu-s}\left\{(u_1(\mu)z+u_0(\mu))\,T_{\mu,p,j}^{s,a,b,1}+(u_1(\mu)(1-z)+u_0(\mu))\,T_{\mu,s+p,j}^{s,a,b,1}\right\}e_p\left(z\right)\right| \\
& +\left|e_j\left(z\right)\right| \\
\leq\ & (|u_1(\mu)|+u_0(\mu))\rho^\mu\sum_{p=0}^{\mu-s}T_{\mu,p,j}^{s,a,b,1}+(|u_1(\mu)|+u_1(\mu)+u_0(\mu))\rho^\mu\sum_{p=0}^{\mu-s}T_{\mu,s+p,j}^{s,a,b,1}+\rho^j \\
\leq\ & (|u_1(\mu)|+u_0(\mu))\rho^j\sum_{p=0}^{\mu-s}T_{\mu,p,j}^{s,a,b,1}+(|u_1(\mu)|+u_1(\mu)+u_0(\mu))\rho^j\sum_{p=0}^{\mu-s}T_{\mu,s+p,j}^{s,a,b,1}+\rho^j \\
\leq\ & \left[(|u_1(\mu)|+u_0(\mu))B_\mu^{a,b}\left(e_j,1\right)+(|u_1(\mu)|+u_1(\mu)+u_0(\mu))L_{\mu,s}^{a,b}\left(e_j,1\right)+1\right]\rho^j \\
\leq\ & 2\left(|u_1(\mu)|+1\right)\rho^j \leq 2\left(|u_1(\mu)|+1\right)\rho^j(\mu-s)\leq 2\left(|u_1(\mu)|+1\right)\rho^j j \\
\leq\ & 2\left(|u_1(\mu)|+1\right)\rho^j j\frac{j-1+s+b}{\mu+b} \\
\leq\ & 2(K+1)\rho^j\frac{j(j-1)+2j(s+b)}{\mu+b}.
\end{aligned}$$

In conclusion, combining Case 2 with the above Case 1, we arrive at the desired inequality.

(ii) For the simultaneous approximation, representing by γ the circle of radius $\rho_1 > \rho$ and center 0, since for any $|z| \leq \rho$ and $u \in \gamma$, we have $|u - z| \geq \rho_1 - \rho$, by Cauchy's formulas it follows that for all $|z| \leq \rho$ and $\mu \in \mathbb{N}$, we have

$$\begin{aligned} \left|\left(L_{\mu,s,1}^{a,b}(\varphi,z)\right)^{(p)} - \varphi^{(p)}(z)\right| &\leq \frac{p!}{2\pi}\int_{\gamma}\frac{\left|L_{\mu,s,1}^{a,b}(\varphi,u)-\varphi(u)\right|}{|u-z|^{p+1}}|du| \\ &\leq \frac{p!}{2\pi}\frac{M_{s,\rho_1}^{1,b}(\varphi)}{\mu+b}\frac{2\pi\rho_1}{(\rho_1-\rho)^{p+1}} \\ &= \frac{M_{s,\rho_1}^{1,b}(\varphi)}{\mu+b}\frac{p!\rho_1}{(\rho_1-\rho)^{p+1}}, \end{aligned}$$

which confirms (ii). □

To obtain the equivalence; as in [5] and [6], we present the qualitative Voronovskaja-type result for the new operator.

Theorem 3.2. *Let ρ and R be constants such that $1 \leq \rho < R$. Also, φ is analytic in H_R with $\varphi(z) = \sum_{j=0}^{\infty} c_j z^j$, $0 \leq a \leq b$ and $L_1 = \lim_{\mu\to\infty} u_1(\mu)$. Then for all $\mu \in \mathbb{N}$, we have*

$$\begin{aligned} &\lim_{\mu\to\infty}(\mu+b)\left[L_{\mu,s,1}^{a,b}(\varphi,z)-\varphi(z)\right] \\ =\ &\left[\frac{2a+s(1+L_1)}{2}-(s+sL_1+b)z\right]\varphi'(z)+\frac{z(1-z)}{2}\varphi''(z) \end{aligned} \tag{20}$$

uniformly in $\overline{H}_\rho$.

Proof. Taking into account of Theorem 2.4, according to the Vitali theorem (see e.g., p. 1 of [11]), it is enough to demonstrate that the sequence

$$\left\{(\mu+b)\left[L_{\mu,s,1}^{a,b}(\varphi,z)-\varphi(z)\right]\right\}_{\mu\in\mathbb{N}}$$

of analytic functions in H_R is uniformly bounded in each $\overline{H}_\rho$, $1 \leq \rho < R$. In fact, the result is direct consequence of (i) of Theorem 3.1 and therefore, we find

$$\left|(\mu+b)\left[L_{\mu,s,1}^{a,b}(\varphi,z)-\varphi(z)\right]\right| \leq M_{s,\rho}^{1,b}(\varphi),$$

for all $\mu \in \mathbb{N}$ and $z \in \overline{H}_\rho$ with $1 \leq \rho < R$. This proves the theorem. Hence, there exist some constants $0 < C_1, C_2 < \infty$ such that

$$\frac{C_1}{\mu+b} \leq \left\|L_{\mu,s,1}^{a,b}(\varphi)-\varphi\right\|_\rho \leq \frac{C_2}{\mu+b}$$

on $\overline{H}_\rho$. □

Afterwards, in view of Theorem 3.1 and Theorem 3.2, we exhibit the followings.

Theorem 3.3. *In consideration of hypothesis of Theorem 3.1;*
(i) *If φ is not a polynomial of degree ≤ 0, then for all $1 \leq \rho < R$, we have*

$$\left\|L_{\mu,s,1}^{a,b}(\varphi)-\varphi\right\|_\rho \sim \frac{1}{\mu+b},\ \mu\in\mathbb{N},$$

where the constants in the equivalence depend on φ, s, ρ, b and $u_1(\mu)$.
(ii) *If $1 \leq \rho < \rho_1 < R$ and φ is not a polynomial of degree $\leq p-1$, $p \in \mathbb{N}$, we have*

$$\left\|\left(L_{\mu,s,1}^{a,b}(\varphi)\right)^{(p)}-\varphi^{(p)}\right\|_\rho \sim \frac{1}{\mu+b},\ \mu\in\mathbb{N},$$

where the constants in the equivalence depend on $\varphi, s, \rho, \rho_1, b,\ u_1(\mu)$ and p.

Later on, by use of MAPLE 24, let us see how the newly defined operator $L_{\mu,s,1}^{a,b}$ converges to a certain function.

Example 3.1. The convergence of the new operator $L_{\mu,s,1}^{a,b}(\varphi,z)$ to $\varphi(z) = z^5 - \frac{z}{6}$ ([violet, green, blue]) is illustrated in Figure 1 for $u_1(\mu) = \frac{1}{2\mu+1}$, $u_0(\mu) = \frac{\mu}{2\mu+1}$, $s = 2$, $a = 1$, $b = 1$ and different values $\mu = 10, 20, 30$ (in turn, [peru, red, orange], colorwheel and [yellow, pink, cyan]). In this example, one can see that the higher values of μ gives better approximation. In Figure 1, we plot the magnitudes of the the corresponding complex functions, where colors on the resulting surfaces represent the arguments of the functions.

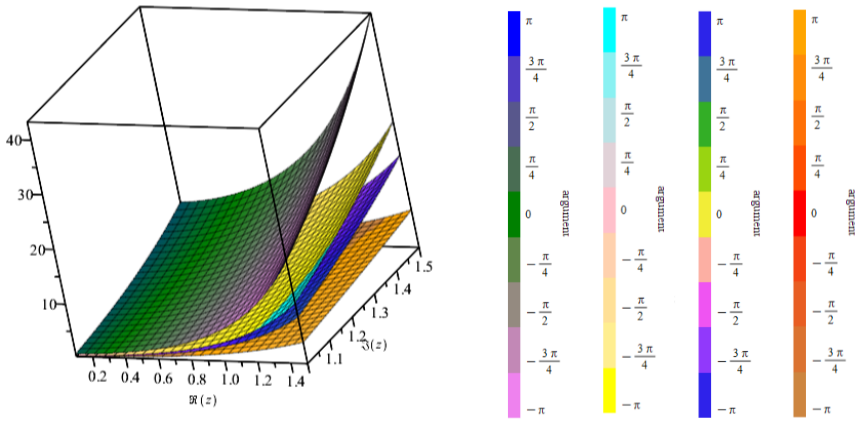

Figure 1: Approximation process of $L_{\mu,s,1}^{a,b}(\varphi,z)$ to $\varphi(z)=z^5-\frac{z}{6}$ for $\mu=10,20,30$.

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